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# FUZZY IDEALS AND FUZZY DOT IDEALS ON BH-ALGEBRAS

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**ABSTRACT**

*In this paper we introduce the notions of Fuzzy Ideals in BH-algebras and the notion of fuzzy dot Ideals of BH-algebras and investigate some of their results.*

**Keywords**: BH-algebras, BH- Ideals, Fuzzy Dot BH-ideal.

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## 1 INTRODUCTION

Y. Imai and K. Iseki [1, 2, and 3] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. K. Iseki and S. Tanaka, [7] are introduced Ideal theory of BCK-algebras P. Bhattacharya, N.P. Mukherjee and L.A. Zadeh [4] are introduced fuzzy relations and fuzzy groups. The notion of BH-algebras is introduced by Y. B Jun, E. H. Roh and H. S. Kim[9] Since then, several authors have studied BH-algebras. In particular, Q. Zhang, E. H. Roh and Y. B. Jun [10] studied the fuzzy theory in BH-algebras. L.A. Zadeh [6] introduced notion of fuzzy sets and A. Rosenfeld [8] introduced the notion of fuzzy group. O.G. Xi [5] introduced the notion of fuzzy BCK-algebras. After that, Y.B. Jun and J. Meng [10] studied Characterization of fuzzy sub algebras by their level sub algebras on BCK-algebras. J. Neggers and H. S. Kim[11] introduced on *d*-algebras, M. Akram [12] introduced on fuzzy *d*-algebras In this paper we classify the notion of Fuzzy Ideals on BH – algebras and the notion of Fuzzy dot

Ideals on BH – algebras. And then we investigate several basic properties which are related to fuzzy BH-ideals and fuzzy dot BH- ideals

## 2 PRELIMINARIES

In this section we cite the fundamental definitions that will be used in the sequel:

**Definition 2.1** [1, 2, 3]

Let X be a nonempty set with a binary operation $*$ and a constant 0. Then (X, $*$, 0) is called a BCK-algebra if it satisfies the following conditions

1. $((x * y) * (x * z)) * (z * y) = 0$
2. $(x * (x * y)) * y = 0$
3. $x * x = 0$
4. $x * y = 0, y * x = 0 \Rightarrow x = y$
5. $0 * x = 0$ $for\ \ all\, x, y, z \in X$

**Definition 2.2** [1, 2, 3]

Let X be a BCK-algebra and I be a subset of X, then I is called an ideal of X if

(I1) $0 \in I$

(I2) $y\ and\ x * y \in I \Rightarrow x \in I\ for\ all\, x, y \in I$

**Definition 2.3** [9, 10]

A nonempty set X with a constant 0 and a binary operation $*$ is called a BH-algebra, if it satisfies the following axioms

(BH1) $x * x = 0$

(BH2) $x * 0 = 0$

(BH3) $x * y = 0\ and\ y * x = 0 \Rightarrow x = y$ for all $x, y \in X$

Example 2.4

Let X= {0, 1, 2} be a set with the following cayley table

| $*$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 |
| 2 | 2 | 1 | 0 |

Then (X, $*$, 0) is a BH-algebra

**Definition 2.5**[9, 10]

Let X be a BH-algebra and I be a subset of X, then I is called an ideal of X if

(BHI1) $0 \in I$

(BHI2) $y\ and\ x * y \in I \Rightarrow x \in I$

(BHI3) $x \in I$ and $y \in X \Rightarrow x * y \in I\ for\ all\, x, y \in I$

A mapping $f: X \to Y$ of BH-algebras is called a homomorphism if $f\,(x * y) = f(x) * f(y)$ for all $x, y\, X \in X$. Note that if $f : X \to Y$ is homomorphism of BH-algebras, Then $f\,(0) = 0'$. We now review some fuzzy logic concepts. A fuzzy subset of a set X is a function $\mu : X \to [0, 1]$. For a fuzzy subset μ of X and t ∈ [0, 1], define U (μ; t) to be the set U (μ; t) = {x ∈ X | μ(x) ≥ t}. For any fuzzy subsets μ and ν of a set X, we define

$$(\mu \cap \nu)(x) = min\{\mu(x), \nu(x)\}\, for\ all\, x \in X.$$

Let $f : X \to Y$ be a function from a set X to a set Y and let μ be a fuzzy subset of X. The fuzzy subset ν of Y defined by $\nu(y) = \begin{cases} \sup\limits_{x\in f^{-1}(y)} \mu(x) & if \quad f^{-1}(y) \neq \phi, \quad \forall y \in Y \\ 0 & otherwise \end{cases}$

is called the image of μ under $f$, denoted by $f(\mu)$. If $\nu$ is a fuzzy subset of $Y$, the fuzzy subset μ of X given by $\mu(x) = \nu(f(x))$ for all x ∈ X is called the Preimage of ν under f and is denoted by $f^{-1}(v)$ . A fuzzy subset μ in X has the sup property if for any $T \subseteq X$ there exists $x_0$∈ T such that $\mu(x_0) = \sup\limits_{x\in f^{-1}(y)} \mu(z)$ . A fuzzy relation μ on a set X is a fuzzy subset of $X \times X$, that is, a map $\mu : X \times X \to [0, 1]$.

**Definition 2.6**[4, 6, 8]

Let X be a nonempty set. A fuzzy (sub) set $\mu$ of the set X is a mapping $\mu: X \to [0,1$

Definition 2.7[4, 6, 8]

Let $\mu$ be the fuzzy set of a set X. For a fixed s∈ [0,1], the set $\mu_s = \{x \in X: \mu(x) \geq s\}$is called an upper level of μ or level subset of $\mu$

Definition 2.8[5, 7]

A fuzzy set $\mu$ in X is called fuzzy BCK-ideal of X if it satisfies the following inequalities

1.$\mu(0) \geq \mu(x)$

2. $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$

**Definition 2.9** [11]

Let X be a nonempty set with a binary operation ∗ and a constant 0. Then (X, ∗, 0) is called a d - algebra if it satisfies the following axioms.

1.$x * x = 0$

2. $0 * x = 0$

3. $x * y = 0, y * x = 0 \Rightarrow x = y$ for all $x, y \in X$

**Definition 2.10** [12]

A fuzzy set $\mu$ in X is called fuzzy d-ideal of X if it satisfies the following inequalities

Fd1.$\mu(0) \geq \mu(x)$

Fd2$\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$

Fd3. $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ For all $x, y \in X$

**Definition 2.11**[12] A fuzzy subset $\mu$ of $X$ is called a fuzzy dot $d$-ideal of $X$ if it satisfies The following conditions:

1. $\mu(0) \geq \mu(x)$

2. $\mu(x) \geq \mu(x * y) . \mu(y)$

3. $\mu(x * y) \geq \mu(x). \mu(y)$for all $x, y \in X$

## 3. FUZZY IDEALS ON BH-ALGEBRAS

**Definition 3.1**

A fuzzy set $\mu$ in X is called fuzzy BH-ideal of X if it satisfies the following inequalities

1.$\mu(0) \geq \mu(x)$

2. $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$

3. $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$

Example 3.2

Let X= {0, 1, 2, 3} be a set with the following cayley table

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 |
| 2 | 2 | 2 | 0 | 0 |
| 3 | 3 | 3 | 3 | 0 |

Then (X, $*$ ,0) is not BCK-algebra. Since $\{(1* 3) * (1 * 2)\} * (2 * 3) = 1 \neq 0$

We define fuzzy set $\mu$ in X by $\mu(0) = 0.8$ and $\mu(x) = 0.01$ for all $x \neq 0$ in X. then it is easy to show that $\mu$ is a BH-ideal of X.

We can easily observe the following propositions

1. In a BH-algebra every fuzzy BH-ideal is a fuzzy BCK-ideal, and every fuzzy BCK-ideal is a fuzzy BH -Sub algebra
2. Every fuzzy BH-ideal of a BH- algebra is a fuzzy BH-subalgebra.

Example 3.3

Let X = {0, 1, 2} be a set given by the following cayley table

| * | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 |
| 2 | 2 | 2 | 0 |

Then (X, $*$ ,0) is a fuzzy BCK-algebra. We define fuzzy set $\mu$ in X by $\mu(0) = 0.7, \mu(1) = 0.5, \mu(2) = 0.2$

Then $\mu$ is a fuzzy BH-ideal of X.

**Definition 3.4**

Let $\lambda$ $and$ $\mu$ be the fuzzy sets in a set X. The Cartesian product $\lambda \times \mu: X \times X \to [0,1]$ is defined by $(\lambda \times \mu)(x,y) = \min\{\lambda(x), \mu(y)\}$ $\forall x, y \in X$.

**Theorem 3.5**

If $\lambda$ $and$ $\mu$ be the fuzzy BH-ideals of a BH- algebra X, then $\lambda \times \mu$ is a fuzzy BH-ideals of $X \times X$

**Proof**

For any $(x,y) \in X \times X,$ wehave

$(\lambda \times \mu)(0,0) = \min\{\lambda(0), \mu(0)\} \geq \min\{\lambda(x), \mu(y)\}$

$= (\lambda \times \mu)(x,y)$

That is $(\lambda \times \mu)(0,0) = (\lambda \times \mu)(x,y)$

Let $(x_1, x_2)$ $and$ $(y_1, y_2) \in X \times X$

Then, $(\lambda \times \mu)(x_1, x_2) = min\{\lambda(x_1), \mu(x_2)\}$

$\geq \min\{\min\{\lambda(x_1 * y_1), \lambda(y_1)\}, \min\{\mu(x_2 * y_2), \mu(y_2)\}$

$= \min\{\min \lambda(x_1 * y_1), \mu(x_2 * y_2), \min\{\lambda(y_1), \mu(y_2)\}\}$

$= \min\{(\lambda \times \mu)(x_1 * y_1, x_2 * y_2)), (\lambda \times \mu)(y_1, y_2)\}$

$= \min\{((\lambda \times \mu)(x_1, x_2) * (y_1, y_2)), (\lambda \times \mu)(y_1, y_2)\}$

That is $((\lambda \times \mu)(x_1, x_2)) = \min\{(\lambda \times \mu)(x_1, x_2) * (y_1, y_2)), (\lambda \times \mu)(y_1, y_2)\}$

And $(\lambda \times \mu)((x_1, x_2) * (y_1, y_2))$

$= (\lambda \times \mu)(x^1 * y^1, x^2 * y^2)$
$= \min\{\lambda(x_1 * y_1), \mu(x_2 * y_2)\}$
$\geq \min\{\min\{\lambda(x_1), \lambda(y_1)\}, \min\{\mu(x_2), \mu(y_2)\}\}$
$= \min\{\min(\lambda(x_1), \mu(x_2), \min\{(\lambda(y_1), \mu(y_2))\}$
$= \min\{(\lambda \times \mu)((x_1, x_2), (\lambda \times \mu)(y_1, y_2))$
That is $(\lambda \times \mu)((x_1, x_2) * (y_1, y_2))$
$= \min\{(\lambda \times \mu)((x_1, x_2), (\lambda \times \mu)(y_1, y_2)\}$
Hence $\lambda \times \mu$ is a fuzzy BH-ideal of $X \times X$

**Theorem 3.6**

Let $\lambda$ $and\, \mu$ be fuzzy sets in a BH-algebra such that $\lambda \times \mu$ is a fuzzy BH-ideal of $X \times X$.Then

i)Either $\lambda(0) \geq \lambda(x)$ $or$ $\mu(0) \geq \mu(x)$ $\forall x \in X$.
ii) If $\lambda(0) \geq \lambda(x)$ $\forall x \in X$, then either $\mu(0) \geq \lambda(x)$ or $\mu(0) \geq \mu(x)$
iii) If $\mu(0) \geq \mu(x)$ $\forall x \in X$, then either $\lambda(0) \geq \lambda(x)$ or $\lambda(0) \geq \mu(x)$

**Proof**

We use reduction to absurdity

i) Assume $\lambda(x) > \lambda(0)$ and $\mu(x) \geq \mu(0)$ for some $x, y \in X$.
Then $(\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(y)\}$
$> \min\{\{\lambda(0), \mu(0)\}$
$= (\lambda \times \mu)(0,0)$
$(\lambda \times \mu)(x, y) > (\lambda \times \mu)(0,0)$ $\forall x, y \in X$
Which is a contradiction to $(\lambda \times \mu)$ is a fuzzy BH-ideal of $X \times X$
Therefore either $\lambda(0) \geq \lambda(x) or\, \mu(0) \geq \mu(x)$ $\forall x \in X$.

ii) Assume $\mu(0) < \lambda(x)$ and $\mu(0) < \mu(y)$ for some $x, y \in X$.
Then $(\lambda \times \mu)(0,0) = \min\{\lambda(0), \mu(0)\} = \mu(0)$
And $(\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(y)\} > \mu(0)$
$= (\lambda \times \mu)(0,0)$
This implies $(\lambda \times \mu)(x, y) >$ and $(\lambda \times \mu)(0,0)$
Which is a contradiction to $\lambda \times \mu$ is a fuzzy BH-ideal of $X \times X$
Hence if $\lambda(0) \geq \lambda(x)$ $\forall x \in X, then$
Either $\mu(0) \geq \lambda(x)$ or $\mu(0) \geq \mu(x)$ $\forall x \in X$

iii) Assume $\lambda(0) < \lambda(x)$ or $\lambda(0) < \mu(y)$ $\forall x, y \in X$
Then $((\lambda \times \mu)(0, 0) = \min\{\lambda(0), \mu(0)\} = \lambda(0)$
And $(\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(y)\} > \lambda(0)$
$= (\lambda \times \mu)(0,0)$
This implies $(\lambda \times \mu)(x, y) > (\lambda \times \mu)(0,0)$
Which is a contradiction to $(\lambda \times \mu)$ is a fuzzy BH-ideal of $X \times X$
Hence if $\mu(0) \geq \mu(x)$ $\forall x \in X$ then either $\lambda(0) \geq \lambda(x)$ or $\lambda(0) \geq \mu(x)$
This completes the proof

**Theorem 3.7**

If $\lambda \times \mu$ is a fuzzy BH-idela of $X \times X$ then $\lambda$ $or$ $\mu$ is a fuzzy BH-ideal of X.

**Proof**

First we prove that $\mu$ is a fuzzy BH-ideal of X.

Given $\lambda \times \mu$ is a fuzzy BH-ideal of $X \times X$, then by theorem 3.6(i), either $\lambda(0) \geq \lambda(x)$ or $\mu(0) \geq \mu(x)\ \forall x \in X$.

Let $\mu(0) \geq \mu(x)$

By theorem 3.6(iii) then either $\lambda(0) \geq \lambda(x)$ or $\lambda(0) \geq \mu(x)$

Now $\mu(x) = \min\{\lambda(0), \mu(x)\}$

$= (\lambda \times \mu)(0, x)$

$\geq \min\{ ((\lambda \times \mu)(0, x) * (0, y)),\ (\lambda \times \mu)(0, y)\}$

$= \min\{ (\lambda \times \mu)(0 * 0), x * y), (\lambda \times \mu)(0, y)\}$

$= \min\{ (\lambda \times \mu)(0, x * y), (\lambda \times \mu)(0, y)\}$

$= \min\{ (\lambda \times \mu)(0 * 0), x * y), (\lambda \times \mu)(0, y)\}$

$= \min\{ \mu(x * y), (\mu)(y)\}$

That is $\mu(x) \geq \min\{ \mu(x * y), \mu(y)\}$

$\mu(x * y) = \min\{ \lambda(0), \mu(x * y)\}$

$= (\lambda \times \mu)(0, x * y)$

$= (\lambda \times \mu)(0 * 0, x * y)$

$= (\lambda \times \mu)(0, x) * (0, y)$

$\mu(x * y) \geq \min\{ (\lambda \times \mu)(0, x), (\lambda \times \mu)(0, y)\}$

$= \min\{\mu(x), \mu(y)\}$

That is, $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$

This proves that $\mu$ is a fuzzy BH-ideal of X.

Secondly to prove that $\lambda$ is a Fuzzy BH-ideal of X.

Using theorem 4.6(i) and (ii) we get

This completes the proof.

**Theorem 3.8**

If $\mu$ is a fuzzy BH-idela of $X$, then $\mu_t$ is a BH-idela of X for all $t \in [0,1]$

**Proof**

Let $\mu$ be a fuzzy BH-ideal of X,

Then By the definition of BH-ideal

$$\mu(0) \geq \mu(x)$$

$\mu(x) \geq \min\{ \mu(x * y), (\mu(y)\}$

$\mu(x * y) \geq \min\{ \mu(x), (\mu(y)\}\ \ \forall x, y \in X$

To prove that $\mu_t$ *is a* BH-ideal of x.

By the definition of level subset of $\mu$

$\mu_t = \{x / \mu(x) \geq t\}$

Let $x, y \in \mu_t$ and $\mu$ is a fuzzy BH-ideal of X.

Since $\mu(0) \geq \mu(x) \geq t$ implies $0 \in \mu_t$, for all $t \in [0,1]$

Let $x, y \in \mu_t$ and $y \in \mu_t$

Therefore $\mu(x * y) \geq t$ and $\mu(y) \geq t$

Now $\mu(x) \geq \min\{ \mu(x * y), (\mu(y)\}$

$\geq \min\{t, t\} \geq t$
Hence $(\mu(x) \geq t$

That is $x \in \mu_t$.

Let $x \in \mu_t$, $y \in X$

Choose y in X such that $\mu(y) \geq t$

Since $x \in \mu_t$ implies $\mu(x) \geq t$

We know that $\mu(x * y) \geq \min\{ \mu(x), (\mu(y)\}$

$\geq \min\{t, t\}$

$\geq t$

That is $\mu(x * y) \geq t$ implies $x * y \in \mu_t$

Hence $\mu_t$ is a BH-ideal of X.

**Theorem 3.9**

If X be a BH-algebra, $\forall t \in [0,1]$ and $\mu_t$ *is a BH* −ideal of X, then $\mu$ is a fuzzy BH-ideal of X.

**Proof**

Since $\mu_t$ *is a BH* −ideal of X

$0 \in \mu_t$

ii) $x * y \in \mu_t$ And $y \in \mu_t$ implies $x \in \mu_t$

iii) $x \in \mu_t$, $y \in X$ implies $x * y \in \mu_t$

To prove that $\mu$ *is a fuzzy* BH-ideal of X.

Let $x, y \in \mu_t$ then $\mu(x) \geq t$ and $\mu(y) \geq t$

Let $\mu(x) = t_1$ and $\mu(y) = t_2$

Without loss of generality let $t_1 \leq t_2$

Then $x \in \mu_{t_1}$

Now $x \in \mu_{t_1}$ and $y \in X$ *implies* $x * y \in \mu_{t_1}$

That is $\mu(x * y) \geq t_1$

$= \min\{t_1, t_2\}$

$= \min\{ \mu(x), \mu(y)\}$

$\mu(x * y) \geq \min\{ \mu(x), \mu(y)\}$

*ii) Let* $\mu(0) = \mu(x * x) \geq \min\{ \mu(x), \mu(y)\} \geq \mu(x)$ (by proof (i))

That is $\mu(0) \geq \mu(x)$ for all $x \in X$

iii) Let $\mu(x) = \mu(x * y) * (0 * y))$

$\geq \min\{ \mu(x * y), \mu(0 * y)\big)$ (By (i))

$\geq \min\{ \mu(x * y), \min\{\mu(0), \mu(y)\}$

$\geq \min\{ \mu(x * y), \{\mu(y)\big)$ (By (ii))

$\mu(x) \geq \min\{ \mu(x * y), \{\mu(y)\big)$

Hence $\mu$ is a fuzzy BH-ideal of X.

**Definition 3.10**

A fuzzy set $\mu$ in X is said to be fuzzy BH- $\boldsymbol{\chi}$ ideal if $\mu(x * u * v * y) \geq \min\{ \mu(x), \mu(y)\}$

**Theorem 3.11**

Every Fuzzy BH -ideal is a fuzzy BH- **χ**- ideal

**Proof**

It is trivial

**Remark**

Converse of the above theorem is not true. That is every fuzzy BH-**χ** -ideal is not true. That is every fuzzy BH -ideal. Let us prove this by an example

Let X = {0, 1, 2} be a set given by the following cayley table

| * | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 0 | 1 |
| 2 | 2 | 2 | 0 |

Then (X,∗ ,0) is a BH-algebra. We define fuzzy set: X→ $[0,1]$ by $\mu(0) = 0.8, \mu(x) = 0.2\ \forall x \neq 0$ clearly $\mu$ is a fuzzy BH-ideal of X But $\mu$ is not a BH-**χ**-ideal of X.

For Let $x = 0\ \ u = 1\ \ v = 1\ y = 1$

$\mu(x * u * v * y = \mu(0 * 1 * 1 * 1) = \mu(1) = 0.2$

$\min\{\ \mu(x),\ \mu(y)\} = \min\{\ \mu(0),\ \mu(0)\}$

$= \mu(0) = 0.8$

$\mu(x * u * v * y)\ \leq \min\{\ \mu(x), \mu(y)\}$

Hence $\mu$ is not a fuzzy BH-**χ**-ideal of X.

And $\mu(x * y) = \mu\left(f(a) * f(b)\right)$

$\geq \mu(f(a * b))$

$= \mu^f(a * b)$

$\geq \min\{\ \mu^f(a),\ \mu^f(b)\}$

$= \min\{\mu(f(a), \mu(f(b))\}$

$= \min\{\mu(x), \mu(y)\}$

Hence $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$

Hence $\mu$ is a fuzzy BH- **χ** ideal of Y.

## 4. FUZZY DOT *BH*-IDEALS OF *BH*-ALGEBRAS

**Definition 4.1.** A fuzzy subset $\mu$ of $X$ is called a fuzzy dot *BH*-ideal of $X$ if it satisfies

The following conditions:

(FBH1). $\mu(0) \geq \mu(x)$

(FBH2). $\mu(x) \geq \mu(x * y)\,.\,\mu(y)$

(FBH3). $\mu(x * y) \geq \mu(x).\mu(y)$ for all $x, y \in X$

**Example 4.2.** Let $X$ = {0, 1, *2, 3*} be a *BH*-algebra with Cayley table (Table 1) as follows:

**(Table 1)**

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 2 |
| 2 | 2 | 2 | 0 | 0 |
| 3 | 3 | 3 | 3 | 0 |

Define $\mu : X \rightarrow [0,1]$ by μ (0) =0.9, μ (*a*) = μ (*b)* =0.6, μ (*c)* =0.3 .It is easy to verify that μ is a Fuzzy dot $BH$-ideal of *X*.

**Proposition 4.3.** Every fuzzy *BH*-ideal is a fuzzy dot *BH*-ideal of a *BH*-algebra.

**Remark.** The converse of Proposition 4.3 is not true as shown in the following Example 4.2**.** Let $X = \{0, 1, 2, 3\}$ be a *BH*-algebra with Cayley table (Table 1) as follows:

**Example 4.4.** Let $X = \{0,1,2\}$ be a *BH*-algebra with Cayley table (Table 2) as follows:

**(Table 2)**

| * | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 2 |
| 2 | 2 | 1 | 0 |

Define μ: X → [0, 1] by μ (0) =0.8, μ (1) = 0.5, μ (2) =0.4. It is easy to verify that μ is a fuzzy Dot *BH*-ideal of *X,* but not a fuzzy *d*-ideal of *X* because

$\mu(x) \leq \min\{\mu(x * y), \mu(y)\}$

$\mu(1) = \min\{\mu(1 * 2), \mu(2)\}$

$= \mu(2)$

**Proposition 4.5.** Every fuzzy dot *BH*-ideal of a *BH*-algebra *X* is a fuzzy dot subalgebra of *X*.

**Remark.** The converse of Proposition 4.5 is not true as shown in the following Example:

**Example 4.6.** Let *X* be the BH-algebra in Example 4.4 and define μ: X → [0, 1] by μ (0)= μ (1)=0.9, μ (2)= 0.7. It is easy to verify that μ is a fuzzy dot sub algebra of *X,* but not a fuzzy dot *BH*-ideal of *X* because μ (2)=0.7 ≤ 0.81=μ (2 ∗1)· μ (1) .

**Proposition 4.7.** If μ and ν are fuzzy dot *BH*-ideals of a *BH*-algebra *X*, then so is μ ∩ ν.

Proof. Let *x, y* ∈ *X*. Then

$$\begin{aligned}(\mu \cap \nu)(0) &= min\{\mu(0), \nu(0)\} \\ &\geq min\{\mu(x), \nu(x)\} \\ &= (\mu \cap \nu)(x).\end{aligned}$$

Also, $(\mu \cap \nu)(x) = min\{\mu(x), \nu(x)\}$

$$\begin{aligned}&\geq min\{\mu(x * y) \cdot \mu(y), \nu(x * y) \cdot \nu(y)\} \\ &\geq (min\{\mu(x * y), \nu(x * y)\}) \cdot (min\{\mu(y), \nu(y)\}) \\ &= ((\mu \cap \nu)(x * y)) \cdot ((\mu \cap \nu)(y)).\end{aligned}$$

$$\begin{aligned}And, (\mu \cap \nu)(x * y) &= min\{\mu(x * y), \nu(x * y)\} \\ &\geq min\{\mu(x) \cdot \mu(y), \nu(x) \cdot \nu(y)\} \\ &\geq (min\{\mu(x), \nu(x)\}) \cdot (min\{\mu(y), \nu(y)\}) \\ &= ((\mu \cap \nu)(x)) \cdot ((\mu \cap \nu)(y)).\end{aligned}$$

Hence μ ∩ ν is a fuzzy dot *BH*-ideal of a *d*-algebra *X*.

**Theorem 4.8.** If each nonempty level subset $U(\mu; t)$ of $\mu$ is a fuzzy *BH*-ideal of *X* then μ is a fuzzy dot *BH*-ideal of *X,* where t ∈[0, 1].

**Definition 4.9**

Let σ be a fuzzy subset of X. The strongest fuzzy σ-relation on BH-algebra X is the fuzzy subset $\mu_\sigma$ of X ×X given by $\mu_\sigma$(x, y) = σ(x) · σ(y) for all x, y ∈ X. A fuzzy relation μ on BH-algebra X is called a Fuzzy σ-product relation if μ(x, y) ≥ σ(x) · σ(y) for all x, y ∈ X. A fuzzy relation μ on BH-algebra is called a left fuzzy relation on σ if μ(x, y) = σ(x) for all x, y ∈ X.

Note that a left fuzzy relation on σ is a fuzzy σ-product relation.

**Remark.** The converse of Theorem 4.8 is not true as shown in the following example:

**Example 4.10.** Let *X* be the *BH*-algebra in Example 4.4 and define μ: X → [0, 1] by

μ (0)=0.6, μ (1)=0.7, μ (2)= 0.8. We know that μ is a fuzzy dot *BH*-ideal of *X,* but

*U* (μ; 0.8) = {*x* ∈*X* | μ(*x*) ≥ 0.8} = {2, 2} is not *BH*-ideal of *X* since 0∉*U* (μ; 0.8).

**Theorem 4.11**. Let $f : X \to X'$ be an onto homomorphism of *BH*-algebras*,* ν be a fuzzy Dot BH-ideal of *Y.* Then the Preimage $f^{-1}(v)$ of ν under *f* is a fuzzy dot *BH*-ideal of *X.*

Proof. Let *x*∈ *X,*

$$f^{-1}(v)(0) = v(f(0)) = v(0')$$
$$\geq v(f(x)) = f^{-1}(v)(x)$$

For any *x, y*∈ *X,* we have

$f^{-1}(v)(x)$ = ν (f (x)) ≥ ν (*f* (*x)***f(y)*)) · ν (*f* (*y*))

= ν (*f* (*x** *y*)) · ν (*f* (*y*)) = $f^{-1}(v)(x * y).f^{-1}(v)(y)$

Also,

$f^{-1}(v)(x * y)$= ν (f (*x* **y*)) = ν (*f* (*x*) **f* (*y*))

≥ ν (*f* (*x*)) · ν(*f* (*y*)) = $f^{-1}(v)(x).f^{-1}(v)(y)$

Thus $f^{-1}(v)$is a fuzzy dot *BH*-ideal of *X.*

**Theorem: 4.12** An onto homomorphic image of a fuzzy dot *BH*-ideal with the sup Property is a fuzzy dot *BH*-ideal.

**Theorem 4.13.** If λ and μ are fuzzy dot *BH*-ideal of a *BH*-algebra *X*, then λ × μ is a fuzzy Dot *BH*-ideal of *X* × *X.*

Proof.

Let $x, y \in X$

λ × μ (0,0)=λ(0), μ(0)

≥ λ(x).μ(y)= (λ × μ) (x, y)

For any $x, x', y, y' \in X$ wehave

$(\lambda \times \mu)(x,y) = \lambda(x).\mu(y)$

$\geq \lambda(x * x').\lambda(x'))\mu(y * y').\mu(y')$

$= \lambda(x * x').\mu(y * y')).\lambda(x').\mu(y')$

$= (\lambda\times \mu)(x,y) * (x',y')).(\lambda\times \mu)(x',y')$

Also $(\lambda\times \mu)(x,y) * (x',y')) = (\lambda\times \mu)(x * x') * (y * y'))$.

$= \lambda(x * x').\mu(y * y')$

$\geq \lambda(x).\lambda(x')).(\mu(y).\mu(y'))$

$(\lambda(x).\mu(y)).(\lambda(x').\mu(y'))$

$= (\lambda\times \mu)(x,y).(\lambda \times \mu)(x',y')$

Hence $\lambda \times \mu$ is a fuzzy dot *BH*-ideal of $X \times X$.

**Theorem 4.14**. Let σ be a fuzzy subset of a *BH*-algebra *X* and $\mu_\sigma$ be the strongest fuzzy σ -relation on *BH*-algebra X. Then σ is a fuzzy dot *BH*-ideal of *X* if and only if $\mu_\sigma$ is a Fuzzy dot *BH*-ideal of $X \times X$.

**Proof**. Assume that σ is a fuzzy dot BH-ideal of X. For any x, y ∈ X we have

$\mu_\sigma(0, 0) = \sigma(0) \cdot \sigma(0) \geq \sigma(x) \cdot \sigma(y) = \mu_\sigma(x, y)$.

Let $x, x', y, y' \in X$. Then

$$\begin{aligned}
&\mu_\sigma((x,x') * (y,y')) \cdot \mu_\sigma(y,y') \\
&= \mu_\sigma(x * y, x' * y') \cdot \mu_\sigma(y,y') \\
&= (\sigma(x * y) \cdot \sigma(x' * y')) \cdot (\sigma(y) \cdot \sigma(y')) \\
&= (\sigma(x * y) \cdot \sigma(y)) \cdot (\sigma(x' * y') \cdot \sigma(y')) \\
&\leq \sigma(x) \cdot \sigma(x') = \mu_\sigma(x,x'). And,
\end{aligned}$$

$$\begin{aligned}
\mu_\sigma(x, x') \cdot \mu_\sigma(y, y') &= (\sigma(x) \cdot \sigma(x')) \cdot (\sigma(y) \cdot \sigma(y')) \\
&= (\sigma(x) \cdot \sigma(y)) \cdot (\sigma(x') \cdot \sigma(y')) \\
&\leq \sigma(x * y) \cdot \sigma(x' * y') \\
&= \mu_\sigma(x * y, x' * y') = \mu_\sigma((x,x') * (y,y')).
\end{aligned}$$

Thus $\mu_\sigma$ is a fuzzy dot *BH*-ideal of $X \times X$.

Conversely suppose that $\mu_\sigma$ is a fuzzy dot *BH*-ideal of $X \times X$. From (FBH1) we get

$(\sigma(0))^2 = \sigma(0) \cdot \sigma(0) = c\,(0, 0)$

And so $\sigma(0) \geq \sigma(x)$ for all $x \in X$. Also we have

$$\begin{aligned}
(\sigma(x))^2 &= \mu_\sigma((x, x) \\
&\geq \mu_\sigma((x, x)*(y, y)).\ \mu_\sigma(y, y) \\
&= \mu_\sigma((x*y), (x * y)).\ \mu_\sigma(y, y) \\
&= ((\sigma(x * y) \cdot \sigma(y)))^2
\end{aligned}$$

Which implies that $\sigma(x) \geq \sigma(x * y) \cdot \sigma(y)$ for all $x, y \in X$.

Also we have

$$\begin{aligned}
(\sigma(x * y)^2 &= \mu_\sigma((x*y), x * y) \\
&= \mu_\sigma((x,x) * (y,y)) \geq \mu_\sigma(x,x).\ \mu_\sigma(y,y) \\
&= (\sigma(x) \cdot \sigma(y))^2
\end{aligned}$$

So $\sigma(x * y) \geq \sigma(x) \cdot \sigma(y)$ for all $x, y \in X$.

Therefore σ is a fuzzy dot *BH*-ideal of *X*.

**Proposition 4.15**. Let μ be a left fuzzy relation on a fuzzy subset σ of a *BH*-algebra *X*. If μ is a fuzzy dot BH-ideal of $X \times X$, then σ is a fuzzy dot BH-ideal of a BH-algebra X.

Proof.

Suppose that a left fuzzy relation μ on σ is a fuzzy dot BH-ideal of $X \times X$.

Then

$\sigma(0) = \mu(0, z)\ \forall z \in X$

By putting $z=0$

$\sigma(0) = \mu(0,0) \geq \mu(x, y) = \sigma(x)$,

For all $x \in X$.

For any $x, x', y, y' \in X$

$$\sigma(x) = \mu(x,y) \geq \mu((x,y) * (x',y')).\mu(x',y')$$
$$= \mu((x * x'),(y * y')).\mu(y,y'))$$
$$= \sigma(x * x').\sigma(x')$$

Also

$$\sigma(x * x').= \mu(x * x', y * y') = \mu((x,y) * (x',y'))$$
$$\geq \mu(x,y).\mu(x',y')$$
$$=\sigma(x).\sigma(x')$$

Thus σ is a fuzzy dot BH-ideal of a BH-algebra X.

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