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# Computing Nash Equilibria in Probabilistic, Multiparty Spatial Models with Nonpolicy Components

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Although there exist extensive results concerning equilibria in spatial models of two-party elections with probabilistic voting, we know far less about equilibria in multiparty elections—i.e., under what conditions will equilibria exist, and what are the characteristics of equilibrium configurations? We derive conditions that guarantee the existence of a unique Nash equilibrium and develop an algorithm to compute that equilibrium in multiparty elections with probabilistic voting, in which voters choose according to the behaviorists' fully specified multivariate vote model. Previously, such computations could only be approximated by laborious search methods. The algorithm, which assumes a conditional logit choice function, can be applied to spatial competition for a variety of party objectives including vote-maximization and margin-maximization, and can also encompass alternative voter policy metrics such as quadratic and linear loss functions. We show that our conditions for an equilibrium are plausible given the empirically-estimated parameters that behaviorists report for voting behavior in historical elections. We also show that parties' equilibrium positions depend not only on the distribution of voters' policy preferences but also on their nonpolicy-related attributes such as partisanship and sociodemographic variables. Empirical applications to data from a recent French election illustrate the use of the algorithm and suggest that a unique Nash equilibrium existed in that election.

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## 1 Introduction

DETERMINISTIC MODELS OF spatial competition among three or more parties, i.e., multiparty competition, seldom lead to a stable configuration of strategies for vote-maximizing parties, even for a single issue-dimension (Eaton and Lipsey 1975; Cox 1990). Probabilistic models—which incorporate random variables to model unmeasured components of voter utility—represent voter utility more realistically and predict more accurately the stability of party locations that is empirically observed. Such models are investigated by Enelow and Hinich (1989), Coughlin (1992), Lin et al. (1999), and others. In a recent series of papers Adams and Merrill (1999a, b, 2000a) move a step farther by incorporating into the probabilistic model the measured nonpolicy variables that behaviorists find important, such as sociodemographic characteristics, party identification, and retrospective evaluations of incumbent performance (see, e.g., Campbell et al. 1960; Fiorina 1981).

Despite the extensive work on multiparty competition with probabilistic voting summarized above, there remain only limited theoretical or simulation results on the nature of multiparty equilibria. The unique theoretical contribution is Lin and co-workers' (1999) demonstration that as long as the policy component of utility is not large (relative to the random component), a unique, centralized, agglomerated equilibrium exists for all vote-maximizing parties. While this result is important, its practical significance is limited because it applies only when the policy component of utility approaches zero, a scenario that conflicts with the conclusion reported in empirical voting studies, that voters are significantly influenced by policies (see, e.g., Alvarez et al. 2000; Dow 1997; Pierce 1995; Whitten and Palmer 1996). Furthermore, this centralized equilibrium theorem does not square with empirical findings of dispersed policy positions in actual party systems (Budge 1994; Castles and Mair 1984; Huber and Inglehart 1995).

Simulation studies by de Palma et al. (1989, 1990) and Adams (1999) and empirical studies based on survey data from historical elections by Schofield et al. (1998) and Adams and Merrill (1999a, b) analyze multiparty equilibria in the more realistic situation where policies matter significantly to voters. The results reported in these papers suggest that for such elections, dispersed multiparty equilibria frequently exist. However, it is difficult to draw general conclusions from these studies because they address a limited set of election scenarios.

The Nash equilibria located in these studies have been found by a laborious search process. Under this procedure, optimal locations are estimated by a process in which the researcher successively adjusts the parties' locations—one party at a time—until no further increase in the expected vote share of any party can be observed. This brute-force method does not provide a clear measure of the proximity of the approximate locations obtained to the exact party optima. Furthermore, there is no guarantee that this method will locate existing equilibrium configurations, nor does it provide a reliable way of determining whether or not the equilibrium obtained is unique.

We provide an iterative algorithm based on the contraction mapping theorem. For *conditional logit* (CL) models, under appropriate conditions specified below, a unique Nash equilibrium exists and the algorithm converges to the vote-maximizing party locations constituting this equilibrium. The calculations determine the equilibrium locations to any degree of accuracy and can be implemented on a spreadsheet such as Excel. The behaviorists' fully specified vote model may be used. The algorithm employs the CL[1] format that has been

[1] Two alternatives for estimating model parameters are multinomial probit (MNP) and generalized extreme value (GEV), both of which permit correlated error terms. Although the inclusion of a correlation structure potentially alters the equilibrium configuration, identification of this structure is quite fragile (Keane 1992). Adams and

used to specify probabilistic voting in many multiparty spatial models and in empirical voting studies (de Palma et al. 1990; Adams 1999; Adams and Merrill 1999a, b, 2000a; Endersby and Galatas 1997; Lacy and Burden 1999; Whitten and Palmer 1996). It can be adapted to alternative voter policy metrics (e.g., quadratic losses versus linear losses) and to a variety of party motivations including vote maximization and margin maximization.

We believe that this algorithm makes two contributions to multiparty spatial modeling research, one practical and one theoretical. The practical advantage is that the speed of the algorithm facilitates Monte Carlo and bootstrapping analyses, which are essential to assess what factors affect the nature of equilibria and to measure the sensitivity of equilibrium outcomes to perturbations in the initial conditions. This offers an important advantage over the more time-consuming brute-force approach.

On the theoretical side, we derive conditions that guarantee a unique Nash equilibrium and that guarantee that the algorithm will converge to this equilibrium, and we show that these conditions are plausible given the empirically estimated parameters behaviorists report for fully specified models of the vote in historical elections. To our knowledge, this represents the first theoretical result on multiparty probabilistic spatial voting that is relevant to elections in which voters are significantly influenced by both policy and nonpolicy factors.[2]

Our motivation in developing the algorithm stems from our interest in applying spatial modeling techniques to explain the policies that political parties and candidates presented in historical elections. In previous papers we have explored the policy strategies of Norwegian parties (Adams and Merrill 1999a, b), British parties (Adams 2001b, Chap. 6), and presidential candidates in France and the United States (Adams and Merrill 2000a, b; Adams et al. 2001). Our results suggest that spatial modeling can indeed illuminate policy competition, in that the candidates and parties in these historical elections by and large took positions consistent with vote maximization. However, our ability to draw firm conclusions has been limited by the difficulty of locating candidate/party equilibria, by uncertainty over the sensitivity of these equilibria to perturbations of parameter values, and by concerns over whether our computed equilibria were unique. The algorithm we develop here can address these concerns and should, therefore, enhance our understanding of the policy strategies that candidates and parties present in historical elections.

For illustration purposes, we apply the algorithm to the 1988 French presidential election and we report theoretical and simulation results that illustrate the algorithm's potential uses. Our results suggest that multiparty spatial competition for fully specified vote models may commonly yield unique, dispersed equilibria, with many parties taking positions that are highly correlated with, but less extreme than, their partisans' mean preferred positions. We also find that parties' equilibrium positions under the fully specified vote model are robust to slight perturbations of the parameter values and that the nonpolicy variables that influence the vote strongly affect the nature of parties' optimal policy strategies. In contrast, we find that party competition under the simpler vote models explored in prior spatial modeling research usually fails to satisfy the conditions for existence of unique Nash equilibria.

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Merrill (2000a, note 12, 2000b, note 13) report empirical evidence that the equilibrium configuration need not be sensitive to the choice of model. Dow and Endersby (1998) compare MNP and CL both theoretically and via Monte Carlo studies. They conclude that the two methods yield comparable substantive results and that, because of the complexity and instability inherent in MNP, it may not be cost effective for the small gain achieved in specificity. They find that when—as is the case in our study—no entry or exit of candidates is considered, the independence-of-irrelevant-alternatives assumption inherent in CL is not as restrictive as it might appear.

[2] See Erikson and Romero (1990) for theoretical results on two-candidate competition for multivariate voting models.

We have devoted considerable attention in other papers (Adams and Merrill 1999a, b, 2000a) to assessing what models are preferable in describing voter and party behavior. Rather than repeat this discussion here, our primary purpose is to present conditions for the existence of a unique Nash equilibrium and to illustrate a methodological tool—our algorithm—to find such equilibria. This should facilitate both the further investigation of models and the determination of Nash equilibrium strategies in historical elections.

While we employ primarily the language of party competition in this paper, all of our results apply equally well to candidate-centered competition.

## 2 The Algorithm

We first develop the uniqueness theorem and the algorithm for locating Nash equilibria for the common spatial modeling problem in which elections involve vote-maximizing parties that compete for support from voters who experience quadratic policy losses. We then explore modifications for alternative models of party/voter motivations.

Under the CL model in an $n$-party election in which voters and parties are arrayed along a unidimensional policy continuum and utility is quadratic loss, the probability that voter $i$ chooses party $k$ is given by

$$P_{ik}(\mathbf{s}, a) = \frac{\exp[V_i(s_k, a)]}{\sum_j \exp[V_i(s_j, a)]} \tag{1}$$

where

$$V_i(s_k, a) = -a(x_i - s_k)^2 + \mathbf{B}_k \mathbf{t}_{ik} \tag{2}$$

Here, $a$ is the policy salience parameter, $x_i$ represents the voter's ideal location, $s_k$ is party $k$'s strategic location, $\mathbf{s}$ is the vector of all parties' strategic locations, $\mathbf{t}_{ik}$ is a vector of voter $i$'s individual, nonpolicy characteristics (such as class, religion, and party identification), and $\mathbf{B}_k$ represents a vector of salience parameters for these characteristics.[3] Equation (2) represents the behavioral researcher's general univariate specification, in which voter utility is a function of quadratic policy components and sociodemographic characteristics.

At a Nash equilibrium, each party $k$ cannot increase its vote share by changing strategy, i.e., its vote share is a maximum. The vote share of party $k$ is given by the expected value

$$EV_k(\mathbf{s}, a) = \sum_i P_{ik}(\mathbf{s}, a)$$

Hence, at a Nash equilibrium, the partial derivative of $EV_k(\mathbf{s}, a)$ with respect to $s_k$ is zero for all $k$, i.e., for all $k$,

$$\frac{\partial EV_k(\mathbf{s}, a)}{\partial s_k} = 0, \qquad \text{i.e.,}$$

[3] Under conditional logit, it is assumed that the utility $U_i(s_k, a)$ of each voter $i$ for each party $k$ is given by $V_i(s_k, a) + X_{ik}$, where the $X_{ik}$'s are random components that are generated independently from a type I extreme value distribution (see Train 1986).

$$\frac{\partial EV_k(\mathbf{s}, a)}{\partial s_k} = \sum \frac{\partial P_{ik}(\mathbf{s}, a)}{\partial s_k} = \sum_i \frac{\partial P_{ik}(\mathbf{s}, a)}{\partial V_i(s_k, a)} \frac{\partial V_i(s_k, a)}{\partial s_k}$$
$$= \sum_i P_{ik}(\mathbf{s}, a)[1 - P_{ik}(\mathbf{s}, a)](2a)(x_i - s_k) = 0 \tag{3}$$

Solving for the last occurrence of $s_k$, we obtain

$$s_k = \frac{\sum_i P_{ik}(\mathbf{s}, a)[1 - P_{ik}(\mathbf{s}, a)]x_i}{\sum_i P_{ik}(\mathbf{s}, a)[1 - P_{ik}(\mathbf{s}, a)]} \tag{4}$$

Define

$$g_k(\mathbf{s}) = \frac{\sum_i P_{ik}(\mathbf{s}, a)[1 - P_{ik}(\mathbf{s}, a)]x_i}{\sum_i P_{ik}(\mathbf{s}, a)[1 - P_{ik}(\mathbf{s}, a)]} \tag{5}$$

for each $k, k = 1, \ldots, n$. Let $\mathbf{I}$ be an interval including all voter positions, say with an appropriate change of coordinates, $[-M, M]$. We obtain the following theorem.

**Theorem 1.** If there exists a constant $K$ such that for all $\mathbf{s} \in \mathbf{I}^n$,

$$\max_k \sum_j \left| \frac{\partial g_k(\mathbf{s})}{\partial s_j} \right| \leq K < 1 \tag{6}$$

then there exists a unique Nash equilibrium $\mathbf{s} = (s_1, \ldots, s_n)$ of party strategies.

**Proof.** Note that each function $g_k$ is a function of all $s_1, \ldots, s_n$. Because

$$|g_k(\mathbf{s})| = \left| \frac{\sum_i P_i(\mathbf{s}, a)[1 - P_i(\mathbf{s}, a)]x_i}{\sum_i P_i(\mathbf{s}, a)[1 - P_i(\mathbf{s}, a)]} \right| \leq \left| \frac{\sum_i P_i(\mathbf{s}, a)[1 - P_i(\mathbf{s}, a)]}{\sum_i P_i(\mathbf{s}, a)[1 - P_i(\mathbf{s}, a)]} \right| M = M$$

together the functions $g_k, k = 1, \ldots, n$, define a vector-valued mapping $\mathbf{g}$ of $\mathbf{I}^n$ into $\mathbf{I}^n$. Condition (6) states that the supremum norm of the Jacobian matrix of this mapping is less than 1. Hence, we infer that $\mathbf{g} = (g_1, \ldots, g_n)$ is a contraction mapping on $\mathbf{I}^n$ whenever Eq. (6) holds for all $\mathbf{s}$ in $\mathbf{I}^n$. If this is the case, it follows (Ortega 1972, pp. 152–154) that there is a unique fixed point $\mathbf{s}$ of $\mathbf{g}$, i.e., such that $g_k(\mathbf{s}) = s_k$ for $k = 1, \ldots, n$. Thus, this $\mathbf{s}$ is the unique solution to equations of the form (3), $k = 1, \ldots, n$. Because, for each $k$, the expected vote $EV_k$ is positive-valued but approaches zero asymptotically, that is, $\lim_{s_k \to \pm\infty} EV_k(\mathbf{s}, a) = 0$, its unique critical point must be an absolute maximum.[4] Thus, for each party $k$, that party cannot increase its expected vote by deviation from its strategy $s_k$, so that $\mathbf{s}$ constitutes a Nash equilibrium. This Nash equilibrium is unique because, if it were not, $EV_k$ would have more than one critical point for some $k$, which is not the case. This completes the proof.

These same arguments lead us to a procedure by which this Nash equilibrium can be computed. Although $s_k$ appears implicitly on both sides of Eq. (4), we may use Eq. (4) as

[4] When Eq. (6) holds, a fortiori, the derivative of $g_k$ as a function of $s_k$ alone on the interval $\mathbf{I}$ is bounded by $K$. Hence, applying the contraction mapping theorem to $g_k$ on the interval $\mathbf{I}$, there is a unique point $s_k \in \mathbf{I}$ for which $g_k(s_k) = s_k$ or, equivalently, for which $\frac{\partial EV_k(\mathbf{s}, a)}{\partial s_k} = 0$. Thus, $EV_k$ has a unique critical point.

the basis of an iterative algorithm that determines a sequence of strategy sets that converge to a Nash equilibrium. Begin with a vector $\mathbf{s_0}$ of starting locations for the parties, e.g., let each coordinate of $\mathbf{s_0}$ be 0. For each party $k$, define the $k$th coordinate of $\mathbf{s}_1$ as

$$g_k(\mathbf{s_0})$$

and, in general, the $k$th coordinate of $\mathbf{s}_{m+1}$ as

$$g_k(\mathbf{s}_m) \tag{7}$$

**Theorem 2 (Algorithm).** If criterion (6) holds, then for starting values $\mathbf{s_0} \in \mathbf{I}^n$, the values $g_k(\mathbf{s}_m)$ defined by Eq. (7) converge to solutions of Eq. (3) that together constitute a unique Nash equilibrium.

**Proof.** By the contraction mapping theorem, the sequence $\mathbf{g}(\mathbf{s}_m)$ converges to the unique fixed point $\mathbf{s}$ of $\mathbf{g}$, which defines solutions to Eq. (3) and constitutes the unique Nash equilibrium.[5]

Note that the uniqueness of the critical points—and hence the uniqueness of the Nash equilibrium—is independent of any algorithm (or its starting points) by which they may be computed. Because of the continuity of the iteration functions $g_k$, if the algorithm converges, it must converge to a solution of Eq. (3). Because condition (6) is motivated by mathematical considerations, we present below empirical applications to election survey data, as well as Monte Carlo simulations, which suggest that Eq. (6) holds in many situations likely to occur in practice and, hence, that the algorithm frequently converges and converges to a unique equilibrium configuration.[6]

### 2.1 *Alternative Motivations*

In certain types of elections—particularly those involving candidates, not parties—some competitors may emphasize their expected vote margins relative to a particular opponent. Thus in the three-cornered 1992 American presidential contest, the competitive candidates,

[5]The precision of the last iterate is known not to exceed $K/(1-K)$ times the absolute difference between the last two iterates (Ortega 1972: 153). In practice, if $K$ is not near 1, one continues the iteration until the values do not change in at least the number of decimal places desired. Convergence can be speeded up considerably by use of the Aitken acceleration technique (Henrici 1964, Chap. 4). Implemented in Excel, the algorithm can determine equilibria to four significant digits for an election with several hundred voters and five candidates in about 2 sec. using a Pentium II processor; implemented in Fortran or C$^{++}$, the typical time required drops to less than a second. To obtain even three significant digits, a brute-force search by an experienced user requires about 10 min of user-computer interaction. An Excel spreadsheet implementing the algorithm is available at the *Political Analysis* web site and the web site http://course.wilkes.edu/merrill/.

[6]If quadratic utility is replaced with *linear* utility, given by $-|x_i - s_k|$, for each $k$, $k = 1, \ldots, n$, we may define $g_k(\mathbf{s}) = \tilde{s}_k$ where $\tilde{s}_k$ is the point such that

$$\sum_{x_i \le \tilde{s}_k} w_{ik} \ge \frac{1}{2} \qquad \text{and} \qquad \sum_{x_i \ge \tilde{s}_k} w_{ik} \ge \frac{1}{2}$$

where

$$w_{ik} = \frac{P_{ik}(\mathbf{s}, a)[1 - P_{ik}(\mathbf{s}, a)]}{\sum_i P_{ik}(\mathbf{s}, a)[1 - P_{ik}(\mathbf{s}, a)]}$$

The iteration function $g_k(\mathbf{s}) = \tilde{s}_k$ converges to a unique Nash equilibrium provided that the functions $g_k$ define a contraction mapping. Computer simulation and analysis of an historical election suggest that convergence often occurs and that, although multiple maxima may occur, the separation between maxima is far too small to be substantively significant. See the *Political Analysis* web site or the web site http://course.wilkes.edu/merrill/.

Bush and Clinton, may plausibly have focused on maximizing their vote margins relative to each other, rather than maximizing their expected vote shares (below we show that the 1988 French presidential election may have provided the candidates with similar motivations).

The algorithm can easily be modified to accommodate margin-maximizing motivations or a combination of vote and margin maximization. If party $k$ is motivated to maximize its margin relative to party $j$, the terms $P_{ik}[1 - P_{ik}]$ are replaced by the terms $P_{ik}[1 - P_{ik}] + P_{ik}P_{ij}$ in the definition of $g_k(\mathbf{s})$.[7] Alternatively, if party $k$ is motivated to maximize the total vote share of a coalition of parties consisting of a subset $C$ of parties (including party $k$), then the terms $P_{ik}[1 - P_{ik}]$ are replaced by the terms $P_{ik}[1 - \sum_{j \in C} P_{ij}]$. With either of these changes, Theorems 1 and 2 hold as before.

## 3 Nash Equilibria in an Historical Election

### 3.1 *The 1988 French Presidential Election*

As an example, we apply the algorithm to data on voter choices over the five major candidates (Lajoinie, Mitterrand, Barre, Chirac, and Le Pen) who competed in the first round of the 1988 French presidential election, using self-placement on the Left–Right scale for voter ideal points ($N = 748$) and respondents' mean candidate placements for the candidates' actual positions.[8] Previous voting research suggests that Left–Right ideology is the dominant dimension in French politics and has been found to reflect primarily economic preferences (see, e.g., Converse and Pierce 1986, 1993; Fleury and Lewis-Beck 1993). The nonpolicy component of the model was limited to party identification.[9] For this simple behavioral voting model, the policy salience parameter $a$ in Eq. (2) represents the importance voters attach to Left–Right ideology, $B$ represents the common value of the salience $B_k$ of partisanship over parties, and the party identification variable $t_{ik}$ is set at 1 if the voter identified with party $k$ and at 0 otherwise.

#### 3.1.1 Equilibrium Results for Vote-Maximizing Candidates

Using parameter estimates for $a$ and $B$ from maximum likelihood in a CL model (see Adams and Merrill 1999b), Nash equilibrium configurations for vote-maximizing candidates were calculated by the algorithm for both quadratic and linear utility and are listed in Table 1 and illustrated (for quadratic utility) in Fig. 1.

To assess the validity of criterion (6), the Jacobian norm was evaluated numerically at 100 randomly chosen configurations of potential strategies within the interval from 1 to 7 that encompasses the voter distribution. All but three values were found to be less than 1.0. These three were near the edge of the distribution, far away from likely locations of an equilibrium, suggesting that the criterion for convergence to a unique Nash equilibrium outlined in

[7] This follows by straightforward calculation because $\frac{\partial[P_{ik} - P_{ij}]}{\partial V_i(s_k, a)} = P_{ik}[1 - P_{ik}] - [-P_{ik}P_{ij}]$.

[8] The data are from Pierce's 1988 French Presidential Election Study (Pierce 1996). Our analysis focuses on the subsample of 748 respondents (of 1013) who reported voting for one of the five candidates in the first round and who could place themselves on the Left–Right dimension.

[9] Converse and Pierce (1993) argue that respondents' reported party identification is not equivalent to their current vote intention (but see also Fleury and Lewis-Beck 1993). In fact, 19% of the respondents who reported voting for one of the major candidates and reported a party identification voted for a candidate of a rival party. This is in addition to the 18% of the respondents in our data set who identified with other than one of the five major parties or reported no party identification at all. Adams and Merrill (2000a, Table 3A) show that additional nonpolicy variables (class, income, religion, economic perceptions, etc.) had relatively small effects on the 1988 French presidential vote, when controlling for partisanship. Equilibrium analysis based upon these more fully specified voting models produced results similar to those we report below.

**Table 1** Comparison of Nash equilibrium party strategies on the Left–Right scale with actual (perceived) positions and mean positions of partisans: 1988 French presidential election ($N = 748$)[a]

| | | | | *Equilibrium for vote-maximizing candidates* | | | |
|---|---|---|---|---|---|---|---|
| | | | | *Proximity model* | | *Mixed model* | |
| *Candidate* | *Proportion of partisans (1)* | *Actual placement (2)* | *Mean location of partisans (3)* | *Quadratic utility (4)* | *Linear utility (5)* | *Quadratic utility (6)* | *Linear utility (7)* |
| Lajoinie | 6.6% | 1.90 | 2.24 | 3.69 (0.05) | 3.55 | 3.32 | 3.14 |
| Mitterrand | 39.2% | 3.09 | 3.08 | 3.52 (0.03) | 3.35 | 3.19 | 3.18 |
| Barre | 13.5% | 4.81 | 4.85 | 4.24 (0.04) | 4.32 | 4.52 | 4.56 |
| Chirac | 18.0% | 5.55 | 5.36 | 4.44 (0.02) | 4.63 | 4.85 | 4.99 |
| Le Pen | 4.5% | 6.57 | 6.03 | 4.23 (0.05) | 4.15 | 4.54 | 4.34 |
| *Other* | 18.2% | — | 4.02 | — | — | — | — |
| $a$ (policy salience) | | | | 0.204 (0.023) | 0.693 (0.064) | 0.222 (0.024) | 0.599 (0.074) |
| $B$ (party i.d.) | | | | 2.14 (0.11) | 2.04 (0.11) | 2.22 (0.12) | 2.11 (0.12) |
| $\beta$ (mixing parameter for proximity/dir. model) | | | | — | — | 0.67 (0.12) | 0.88 (0.05) |
| log-likelihood | | | | −622.6 | −617.0 | −618.8 | −612.8 |

[a] Standard errors are in parentheses. Bootstrap standard errors for the equilibrium positions were computed only for quadratic utility and the proximity model. *Partisans* are defined as identifiers of the parties represented by the candidates indicated. The category *other* includes identifiers of other parties as well as respondents who did not identify with any party. See footnote 13 for the specification of the mixing parameter, $\beta$.

Theorem 1 appears to be satisfied in the region of interest.[10] Accordingly, for quadratic utility, we conclude that, *given the empirically estimated parameters for voting behavior in the 1988 French presidential election, it is almost certain that vote-seeking candidates had a unique policy Nash equilibrium*. The computed equilibrium configurations are given in column 4 (for quadratic losses) and column 5 (for linear losses). The candidates' optima are distinct from one another and the optima for the three major candidates—Mitterrand, Barre, and Chirac—are strongly correlated both with their actual (mean perceived) positions and with the mean positions of party identifiers.[11] This is consistent with previous work

[10] All second partial derivatives $\frac{\partial^2 EV}{\partial s_k{}^2} k$ (evaluated numerically) in each run reported in this paper were found to be negative, indicating that the critical points are maxima. Maxima were further confirmed by direct evaluation of the expected vote shares, as would be done in a brute-force search method. For linear policy losses, equilibrium configurations were found to be virtually unique. No party's equilibrium position varied by more than 0.03 for 100 sets of random starting positions.

[11] For quadratic utility, the correlation with the major candidates' actual positions is 0.996, and that with the mean positions of party identifiers, 1.000 (although the implications are limited because there are only three major candidates). For linear utility, the corresponding correlation coefficients are 0.998 and 0.9998. In the linear utility case, the voter distribution was smoothed by assuming, for instance, that the respondents who self-placed at 4.0 represent a group of voters distributed uniformly over the interval [3.5, 4.5].

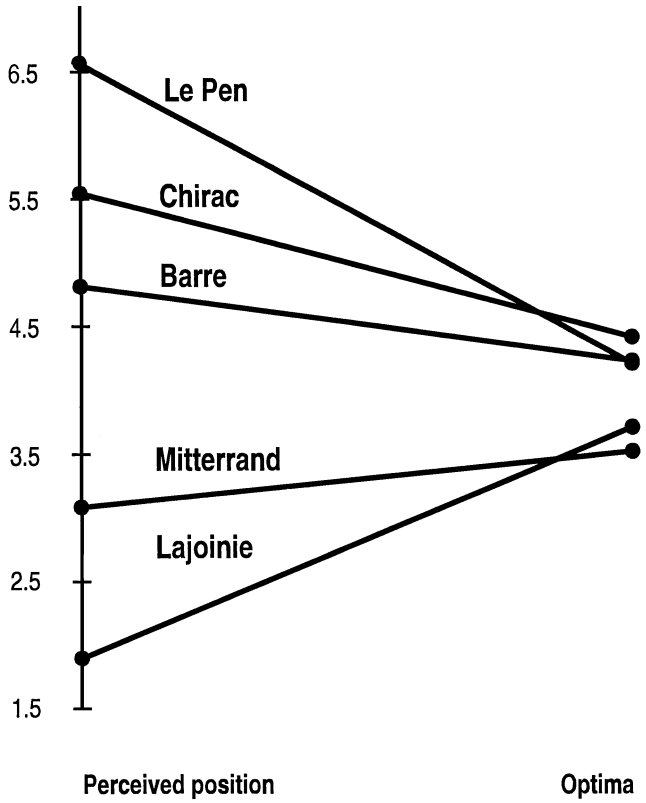


**Fig. 1** Optimal strategies computed from the algorithm for the 1988 French presidential election (quadratic utility).

(Adams and Merrill 1999a, b, 2000a; Adams 1998, 2000, 2001a), which concludes that vote-seeking candidates—such as the major candidates in the 1988 French presidential election—are motivated to present policies that appeal to voters biased toward them in part for nonpolicy reasons. The results imply, however, that the smaller, more extreme parties represented by Lajoinie and Le Pen do not assume the more moderate positions that would—according to the model—maximize their vote share. This likely occurs because of the loss of credibility that would entail and because of the policy orientation of the party elites. In addition, note that the equilibria obtained for quadratic and linear utilities are quite similar, so that the choice of metric does not greatly alter the predictions obtained from spatial modeling analysis.[12]

### 3.1.2 Equilibrium Results for Margin-Maximizing Candidates

In the first round of the French presidential election, it is crucial for a candidate to attain at least second place, because the two top candidates advance to a runoff. In the 1988 presidential election it was widely anticipated that the incumbent, the Socialist Francois Mitterrand, would easily advance to the runoff and that his opponent would be either

[12] In a purely spatial probabilistic model, quadratic utility leads to a convergent (but not necessarily unique) equilibrium at the voter mean, whereas linear utility implies an equilibrium at the voter median.

Jacques Chirac or Raymond Barre, two center–right candidates who were viewed as being almost equally competitive (see, e.g., Pierce 1995). These expectations proved well founded. Mitterrand eventually received 34.0% of the first-round vote, while the second-place Chirac received 19.8%, slightly outdistancing the third-place finisher Barre, who won a 16.5% vote share. It therefore appears plausible that Barre and Chirac each sought to maximize his *margin* with respect to the other (rather than each just maximizing his own vote). To analyze this scenario we reapplied the algorithm (for quadratic utility), using the assumption that Barre and Chirac maximized their relative vote margins (and that the remaining candidates maximized expected votes). Again, condition (6) is satisfied for convergence to a unique equilibrium configuration. This configuration is quite similar to that obtained assuming a purely vote-maximizing scenario, except that the optimal strategies for Barre and Chirac shift slightly to the right and closer to each other, as these two candidates focus on the competition between each other for the center–right vote.

### **3.1.3** The Effect of Alternative Voter Motivations on Equilibria

We note that the candidates' and parties' Nash equilibrium positions obtained for France for the partisan vote model are relatively centrist but are shaded in the direction of their partisans' policy preferences. Furthermore, the optimal locations are highly correlated with the mean placement of the major candidates and parties.

These optimal positions are, however, substantively closer to the center than are the actual (mean perceived) positions. Adams and Merrill (1999a, b) and Merrill and Grofman (1999) argue that models incorporating additional voter–candidate interactions better represent voter preferences and predict much more realistic optimal strategies. They consider a mixed policy voting model, in which voters' utilities for parties' policies are a function of both the proximity metric used here and the directional policy metric proposed by Rabinowitz and Macdonald (1989), in which voter utilities are a function of the direction and intensity of the parties' positions.[13] This model provides parties incentives to adopt more extreme strategies than does the pure proximity metric. Adaptation of the algorithm to a mixed proximity-directional model (whose parameters are estimated by maximum likelihood) is straightforward. Computations based upon a mixed proximity and directional model predict the actual party locations of the three major candidates in the 1988 French presidential election fairly accurately (see Table 1, columns 6 and 7) and predict the positions of seven Norwegian parties quite accurately (see Adams and Merrill 1999b). The latter values and those of Adams and Merrill (2000a) were computed by a brute, force search method.

### **3.1.4** Sensitivity Analysis

A limitation of equilibrium analyses based upon survey data is that both the voter distribution and the estimated vote model parameters may not accurately reflect the underlying voting population. Although ultimately there is no way to be certain that any sample faithfully reflects its population, we can address the following question: How sensitive are our conclusions to realistic changes in the underlying variables and to repeated sampling from the voter distribution determined by the survey?

[13]The (deterministic component of) the utility function for this mixed model (assuming quadratic loss) is given by $V_i(s_k, a) = a[2(1-\beta)(x_i - N)(s_k - N) - \beta(x_i - s_k)^2] + \mathbf{B}_k\mathbf{t}_{ik}$, where $\beta$ is a mixing parameter and $N$ is the neutral point.

Accordingly, we performed both standard and parametric bootstrap analyses. Bootstrap standard errors were estimated for the equilibrium positions under quadratic utility by resampling with replacement. These standard errors (reported in Table 1, column 4) for the candidates' equilibrium positions (computed relative to the voter mean) range from 0.02 (for Chirac) to 0.05 (for the two extreme candidates, Lajoinie and Le Pen).

For the parametric bootstrap we computed candidate equilibria for 100 sets of randomly generated voting parameters (for $a$ and $B$), each of which followed a normal distribution in which the mean and standard deviation were set equal to the corresponding values of the parameter estimates. This allowed us to estimate the standard errors associated with the candidates' equilibrium positions (under quadratic utility). These standard errors were all less than 0.025.

The relatively small size of the standard errors—for both standard and parametric bootstrapping—indicates that both the pattern of the equilibrium configuration and the fact that the equilibrium strategies are in general distinct from one another are robust. In addition, this exercise highlights the algorithm's value as a tool. Our bootstrap analysis runs took only a few minutes, while it would be prohibitively time-consuming to locate large numbers of candidate equilibria using the standard brute-force search method.

In summary, our analysis of candidate equilibrium in the 1988 French presidential election illustrates the advantages of employing the algorithm. The algorithm's flexibility permits analysis of alternative assumptions about voters' policy metrics (linear versus quadratic losses) and about candidate motivations (vote-maximizing versus margin-maximizing). The theorem derived from the algorithm permits a determination of uniqueness of the equilibrium. The algorithm's speed facilitates bootstrapping analyses.

## 4 Stability of Equilibria: A Basic Policy Voting Model Compared with the Behaviorists' Partisan Model

Unlike the above *partisan voting model*, which incorporates party i.d. in addition to policies, most multiparty spatial modeling studies employ a *basic model* which excludes measured nonpolicy voting influences. We have argued elsewhere that nonpolicy models such as the partisan model analyzed above are superior to the basic model in describing voting behavior and party strategies. Yet others (e.g., de Palma et al. 1989, 1990; Lomborg 1997; Schofield et al. 1998; Lin et al. 1999) have focused on the basic model. The questions therefore arise, Does the basic model imply multiparty equilibria? and How do these equilibria compare with those obtained for the partisan model? To address these issues we employed CL analysis to estimate the policy salience parameter $a$ for the basic model (the only parameter estimated for this model) for the 1988 French election data, using the quadratic policy loss function. We then applied the algorithm to compute Nash equilibrium configurations. The results are presented in Table 2A.

For vote-maximizing candidate competition under the basic model, the Jacobian norm substantially exceeds 1.0, indicating that the sufficient criterion given in Theorem 1 for convergence to a unique policy equilibrium is violated (see Table 2A, columns 1–3).[14] Nonetheless, for each set of candidate starting positions that we investigated—all candidates at the center (Table 2A, column 1), all candidates at their actual (mean perceived) positions (column 2), and the candidates at randomly chosen locations[15] (column 3)—the algorithm

[14] For each scenario, the Jacobian norm at the point of convergence is given in Table 2.

[15] For the random starting positions scenario the starting positions were chosen from a uniform distribution between points 1 unit below and 1 unit above the scale midpoint.

**Table 2** Equilibrium analysis for France, 1988 (using quadratic utility), varying candidates' starting positions

| | *Equilibrium for vote-maximizing candidates* | | |
|---|---|---|---|
| | *Starting position* | | |
| | *Center (1)* | *Actual locations (2)* | *Random locations (3)* |
| | (A) Basic model ($a = 0.414$, $B = 0$) | | |
| Lajoinie | 3.98 | 3.62 | 4.32[a] |
| Mitterrand | 3.98 | 3.62 | 3.50 |
| Barre | 3.98 | 3.62 | 3.50 |
| Chirac | 3.98 | 4.47 | 4.24 |
| Le Pen | 3.98 | 4.47 | 4.32 |
| Jacobian norm (at point of convergence) | 1.77 | 1.37 | 1.53 |
| | (B) Partisan model ($a = 0.204$, $B = 2.14$) | | |
| Lajoinie | 3.69 | 3.69 | 3.69 |
| Mitterrand | 3.52 | 3.52 | 3.52 |
| Barre | 4.24 | 4.24 | 4.24 |
| Chirac | 4.44 | 4.44 | 4.44 |
| Le Pen | 4.23 | 4.23 | 4.23 |
| Jacobian norm (at point (of convergence) | 0.64 | 0.64 | 0.64 |

[a] The values presented here for illustration under *random locations* are based on one particular set of candidates' random starting locations between 3.0 and 5.0. Other random starting locations led to the same equilibria for the partisan model but to a great variety of equilibria for the basic model.

converged to an equilibrium. A notable feature of these equilibria is that the parties tend to cluster into blocs, i.e., several parties have the same optimal strategy in equilibrium. This is consistent with previous simulation results reported by Schofield et al. (1998) for Israel and by de Palma et al. (1990) and Lomborg (1997) assuming the basic model on artificially generated voter distributions. However, as might be expected, the equilibrium configurations that we obtained for the basic model were not unique and, in fact, varied wildly depending on the starting values used in the algorithm. Indeed, a Monte Carlo simulation of 100 randomly chosen sets of starting values led to over 30 distinct equilibrium configurations, not even counting permutations.

We conclude that the basic model, if correct, may make the prediction of party strategies extremely difficult, because the nature of party equilibrium configurations for this model is not unique and may vary depending in part on parties' initial positions, on assumptions about party motivations (e.g., vote versus margin maximization), and on minor changes in the voting model parameters. Such sensitivity is serious because these characteristics cannot be known with certainty. This observation underlines the importance of distinguishing between the basic and the nonpolicy models, a distinction that cannot be pursued in this paper.

In contrast, the equilibria defined by the partisan vote model are independent of candidates' starting positions (see Table 2B). This is not surprising because—for the partisan model—the value of the Jacobian norm specified in criterion (6) is consistently below 1, i.e., the criterion for existence of a unique Nash equilibrium is satisfied.

Why is there a unique equilibrium under the partisan vote model but a plethora of wildly different equilibria for the basic model? We believe that the central reason is that in multicandidate elections, partisan voting provides candidates with strong electoral motivations to present policies that reflect the beliefs of their parties' supporters. This strategic incentive, which is analyzed extensively elsewhere (Adams and Merrill 1999a, b; Adams, 1998, 2000, 2001a), typically obtains regardless of the model of candidate motivations (vote- versus margin-maximizing) or the positioning of rival candidates. Hence the candidates are "pulled" forcefully in the direction of their partisans' mean preferences,[16] and the resulting equilibrium configuration does not depend on particulars of the analyst's specification. Under the basic model, in contrast, all voters are implicitly modeled as independent—i.e. they do not display nonpolicy-related biases—and hence they are all equally "in play" from the perspective of vote-seeking candidates.

Additional simulation analyses in which the parameters $a$ and $B$ were systematically varied over ranges observed in historical elections[17] suggest that, for realistic values of the voting parameters, there is a unique policy equilibrium under the partisan vote model (but not under the basic vote model). Numerical evaluation of the size of the Jacobian norm for the iteration mapping suggests that Eq. (6) holds in many situations likely to occur in practice—and hence that there is a unique equilibrium configuration and that the algorithm can indeed be expected to converge to it.

## 5 Discussion

We have developed an algorithm that can locate equilibria for quite general voting models, of the kind behaviorists use, as long as they are based on conditional logit assumptions and quadratic or linear loss functions. This has not been done before. This algorithm can locate equilibria to any degree of precision for a variety of candidate/party motivations including maximization of the vote of one's own party, the margin relative to another party, or the vote of one's coalition. The algorithm can also accommodate both the behaviorist's multivariate voting specification and the standard policy-based model typically employed in spatial modeling studies.

We have presented a theoretical sufficient condition that a unique Nash equilibrium exists and that the algorithm be guaranteed to converge to it. In historical elections, we show empirically that, given the fully specified voting models employed by behaviorists, the sufficient criteria for convergence are often met (and often convergence occurs without satisfaction of the criteria). In contrast, we find that the basic vote model that omits measured nonpolicy factors does not satisfy the convergence criteria, and indeed, multiple equilibria exist for the basic model that are quite different from one another.

The algorithm we have developed offers several advantages when applied to party competition under the behavioral vote model. First, it determines the equilibrium locations to a precision limited only by the precision of the computing equipment used and does so very rapidly, in contrast to a laborious search process requiring constant experimenter intervention.

[16] The candidates' optima are similar to, but less extreme than, their partisans' mean preferences (see Tables 1 and 2) because, while the candidates have electoral incentives to attach added weight to their partisans' policy beliefs, they also attach nonzero weights to the preferences of other members of the electorate (i.e., independent voters and rival parties' partisans).

[17] See the *Political Analysis* web site or the web site http://course.wilkes.edu/merrill/ for the details of the simulations.

Second, because of its speed, the algorithm allows extensive computer simulation of elections under a wide variety of conditions. This permits us not only to perform sensitivity analysis through bootstrapping but also to assess the relative influence of a number of factors on parties' policy positions at equilibrium. In fact, we have shown (Merrill and Adams 2001), that, ceteris paribus, larger parties maximize their vote share at positions farther from the center than smaller parties and that party policy eccentricity also increases with the salience of policy voting and with the degree of dispersion of the electorate's policy preferences. Although for illustrative purposes we have presented the algorithm in one dimension, its extension to multidimensional models or to models permitting abstention introduces no major difficulty.

The basic voting model (without nonpolicy variables) leads to multiple equilibria that are quite different from each other, and the particular equilibrium that is reached depends on such arbitrary factors as the parties' starting points. In contrast, fully specified voting models frequently lead to a unique equilibrium and thus to a unique prediction for the configuration of parties' policies. Furthermore, our applications to France suggest that equilibrium configurations under the fully specified model are quite similar for alternative voter policy metrics (e.g., linear versus quadratic losses) and for varying models of party motivations (i.e., vote versus margin maximization).

Given the difficult specification decisions that spatial modelers confront when analyzing party behavior in historical elections, the above results provide the following basis for optimism: that under the fully specified voting model, alternative specifications (about voters and parties) should not greatly affect model predictions and, moreover, that the algorithm often converges and a unique Nash equilibrium may exist. No doubt many hurdles remain in the quest to apply spatial insights to party and candidate behavior in historical elections, but this much at least augers well for the usefulness of spatial modeling.

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