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Supporting Information for

Bird's Decision to Shift the Direction of Migration Path Depends on the Position of Sun as well as Moon: A Directional Statistical Inference

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Supporting Information Text

1. Details about the Datasets

A. Dataset 1: American Region Bird path. We have compiled a data set of 620 observations from online resources (1), where we specifically focused on the. Birds of the American region. The dataset contains six bird species' names, migration path latitude, longitude, and observation date. The dataset contains different bird species' names, migration path latitude, longitude, and observation date.

- **Swainson's hawk**: The Swainson's hawk is a raptor with a thin body and narrow wings that migrates impressive distances. Swainson's hawks rely heavily on thermals and typically avoid crossing water from which thermals rarely rise. The entire population of Swainson's hawks must migrate through the narrow land bridge between North and South America, often in massive flocks of hundreds of thousands of birds. During their journey from North America to South America.

- **Black-Bellied Plover**: The Black-bellied Plover is Washington's largest plover. Their breeding habitat is Arctic islands and coastal areas throughout the northern coasts of Alaska, Canada, and Russia. They nest on the ground in a dry open tundra with good visibility; the nest is a shallow gravel scrape. They migrate to winter in coastal areas all through the world. In the New World, they winter from southwest British Columbia and Massachusetts south to Argentina and Chile.

- **Black-crowned night heron**: Black-crowned Night-Herons often spend their days perched on tree limbs or concealed among foliage and branches. These social birds roost and nest in groups, although they typically forage independently. Look for them in most wetland habitats across North America, including estuaries, marshes, streams, lakes, and reservoirs.

- **Brown pelican**: Brown Pelicans live along southern and western sea coasts and are rarely seen inland (except at the Salton Sea in California, where they are regular in large numbers). They nest in colonies, often on isolated islands free of land predators. On the Pacific Coast, Brown Pelican adults have red skin on their throats in the breeding season. On the Atlantic and Gulf Coasts, Brown Pelicans are slightly smaller, and their throat skin is greenish-black.

- **Long-billed curlew**: The long-billed curlew (Numenius americanus) is a large North American shorebird of the family Scolopacidae. This species was called the "sickle-bird" and the "candlestick bird." The species breeds in central and western North America, migrating southward and coastward for the winter.

- **Pacific loon**: Pacific Loons Nests in freshwater lakes in arctic and subarctic tundra and taiga. Winters in coastal ocean waters, bays, and estuaries. Occasionally, migrants appear in inland bodies of water, especially large reservoirs.

The dataset is taken from https://nationalzoo.si.edu/migratory-birds/migratory-birds-tracking-map

```
> head(bird_data)
                species birdID       date    longitude latitude
1 Black-bellied plover     19 07-07-2017     -166.3367  64.8885
2 Black-bellied plover     19 08-07-2017     -166.3169  64.8800
3 Black-bellied plover     19 10-07-2017     -166.3270  64.8595
4 Black-bellied plover     19 10-07-2017     -166.3195  64.8677
5 Black-bellied plover     19 12-07-2017     -166.3118  64.8656
6 Black-bellied plover     19 14-07-2017     -165.6876  61.5097
```

B. Dataset 2: American Region Bird path. One fascinating area of research uses GPS to track the movements of animals. It is now possible to manufacture a small GPS device that is solar-charged, so you don't need to change batteries and use it to track the flight patterns of birds.

The data for this case study comes from the LifeWatch INBO project. Several data sets have been released as part of this project. This small dataset consists of migration data for three gulls named Eric, Nico, and Sanne.

The file contains eight columns and includes variables like latitude, longitude, altitude, and time stamps.

The data is taken from https://www.kaggle.com/datasets/turhancankargin/bird-migration

```
  altitude                date_time  direction latitude longitude  speed_2d
1       71 2013-08-15 00:18:08+00 -150.46975 49.41986  2.120733 0.1500000
2       68 2013-08-15 00:48:07+00 -136.15114 49.41988  2.120746 2.4383601
3       68 2013-08-15 01:17:58+00  160.79748 49.42031  2.120885 0.5966574
4       73 2013-08-15 01:47:51+00   32.76936 49.42036  2.120859 0.3101612
5       69 2013-08-15 02:17:42+00   45.19123 49.42033  2.120887 0.1931321
6       54 2013-08-15 02:47:38+00  -46.34448 49.42037  2.120840 2.9047719
```

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2. Detail of Von Mises Fisher Mixture Model Fit for the Two Datasets

A mixture model is a probabilistic model for representing the presence of sub-populations within an overall population without requiring that an observed data set should identify the sub-population to which an individual observation belongs. However, while problems associated with "mixture distributions" relate to deriving the properties of the overall population from those of the sub-populations, "mixture models" are used to make statistical inferences about the properties of the sub-populations given only observations on the pooled population, without sub-population identity information.
the density for directional parametric mixture distribution $p(x|\boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_K)$ represent appropriate parameter set, as follows.

$$p(\boldsymbol{\theta}) = \sum_{i=1}^{K} \phi_i f(\vec{x}; \vec{\mu}, \kappa) \qquad [1]$$

$$\begin{array}{lcl} K & = & \text{number of mixture components} \\ N & = & \text{number of observations} \\ \theta_{i=1\ldots K} & = & \text{parameter of distribution of observation associated with component i} \\ \phi_{i=1\ldots K} & = & \text{mixture weight, i.e., prior probability of a particular component i} \\ \boldsymbol{\phi} & = & K\text{-dimensional vector set } \phi_{1\ldots K}\text{; sum to 1} \\ z_{i=1\ldots N} & = & \text{component of observation } i \\ x_{i=1\ldots N} & = & \text{observation } i \\ F(x|\theta) & = & \text{probability distribution of an observation, parametrized on } \theta \\ z_{i=1\ldots N} & \sim & \text{Categorical}(\boldsymbol{\phi}) \\ x_{i=1\ldots N}|z_{i=1\ldots N} & \sim & F(x_{i=1\ldots N}|\theta_{z_i}) \end{array} \qquad [2]$$

In our paper, the i th vector component is characterized by $f(\vec{x}; \vec{\mu}, \kappa) = C_3(\kappa)\exp(\kappa\vec{x}\cdot\vec{\mu})$ as von Mises-Fisher distribution with weights ϕ_i, parameter θ_i as means $\boldsymbol{\mu_i}$ and concentration matrices $\boldsymbol{\kappa_i}$. Here, $C_3(\kappa)$ is the normalization coefficient which can be shown to be dependent only on κ and the dimension (3, in the case of the sphere). For details, we refer to (2), (3).

For fitting the mixture Von Mises Fisher distribution to both datasets, we take the help of R package **movMF** of (4). We treat the circular uniform distribution as Von Mises Fisher with large means $\boldsymbol{\mu}$ and large (diluted) concentration matrices $\boldsymbol{\kappa}$. In "Statistical Analysis for Directional Mixture Distribution Model Fitting for Dataset 1" section in the original paper we used those table and gave the interpretation about how it is used.

Sl No.	Lat range	Est μ	Est κ	Dist
1	(33.71 to 70.27)	-2.978336	0.2750625	VonMises
2	(64.86 to 64.87)	2.032647	-	Uniform
3	(54.23 to 69.94)	2.825971	0.3877652	VonMises
4	(59.62 to 70.90)	-1.76427	1.506198	VonMises
5	(54.20 to 61.53)	-1.700339	3.581648	VonMises
6	(57.84 to 61.07)	3.08695	1.768419	VonMises
7	(56.89 to 62.96)	-0.4778354	0.4880568	VonMises
8	(54.60 to 70.37)	-0.08920665	0.7903575	VonMises
9	(54.80 to 71.36)	1.681203	0.8349311	VonMises
10	(35.20 to 39.23)	-2.319494	0.9116722	VonMises
11	(26.71 to 70.68)	0.7958838	-	Uniform
12	(61.20 to 61.22)	-1.626239	32752.33	VonMises
13	(26.73 to 51.14)	2.695495	1.500461	VonMises
14	(23.21 to 32.93)	1.236262	1.268279	VonMises
15	(38.89 to 38.94)	1.222303	3708.063	VonMises
16	(26.67 to 26.74)	1.582261	1380.315	VonMises
17	(26.73 to 51.14)	2.695495	1.500461	VonMises
18	(28.11 to 52.67)	1.933752	1.259675	VonMises
19	(38.88 to 38.93)	1.209689	1903.807	VonMises
20	(25.69 to 26.99)	1.159925	3.330345	VonMises
21	(-36.172 to 41.991)	1.256473	0.3443209	VonMises

Table S1. Latitude of all the Bird Species for Dataset 1. This table contains a serial number which is similar to the Longitude table S2. Together(According to the Serial Number) they give us the location of the bird parameter similar to our Dataset. In this table we estimated the circular mean and variance of each partitions we considered.

3. American region Code

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Sl No.	Lon range	Est μ	Est κ	Dist
1	(-236.3 to -166.3)	-2.766708	0.7512237	VonMises
2	(-166.3 to -166.3)	-2.925382	-	Uniform
3	(-166.3 to -165.9)	-2.744784	45.86884	VonMises
4	(-165.9 to -165.8)	-2.499111	738.0808	VonMises
5	(-165.8 to -165.5)	-2.29021	118.4591	VonMises
6	(-165.4 to -165.2)	-1.935756	174.6475	VonMises
7	(-165.1 to -164.4)	-1.401699	19.20709	VonMises
8	(-164.2 to -163.1)	-0.3995937	7.54067	VonMises
9	(-163.1 to -151.0)	0.06193641	1.021821	VonMises
10	(-75.76 to -75.43)	-0.2345965	68.17028	VonMises
11	(-151.0 to -113.2)	-0.03902648	-	Uniform
12	(-165.1 to -165.12)	-1.704596	3625.177	VonMises
13	(-113.1 to -111.6)	0.3206552	6.151091	VonMises
14	(-81.00 to -80.12)	0.9164418	15.45086	VonMises
15	(-77.15 to -77.05)	-1.714681	912.4339	VonMises
16	(-113.2 to -113.1)	-0.04765007	8712.64	VonMises
17	(-113.1 to -11.6)	0.3206552	6.151091	VonMises
18	(-111.6 to -111.1)	1.736291	58.35508	VonMises
19	(-77.05 to -77.00)	-1.641978	3131.77	VonMises
20	(-109.9 to - 109.3)	-2.692523	17.03175	VonMises
21	(-121.96 to -61.23)	-0.2878401	0.3737553	VonMises

Table S2. Longitude of all the Bird Species for Dataset 1. This table contains a serial number which is similar to the Latitude table S1. Together(According to the Serial Number) they give us the location of the bird parameter similar to our Dataset. In this table we estimated the circular mean and variance of each partitions we considered.

```
> s = cbind(bird_data$latitude , bird_data$longitude)
> set.seed(2023)
> vMFs <-
+   function(K){
+     movMF(s, k = K, control= list(nruns = 20))
+   }
> sd = lapply(18:22, vMFs)
> sd
[[1]]
theta:
              [,1]          [,2]
1  -7.145609e+00 -3.851465e+01
2   1.270413e+02 -3.873836e+02
3   3.907324e+05 -8.377703e+05
4   5.230866e+00 -7.508216e+01
5  -1.080639e+03 -2.023105e+03
6   3.097846e+03 -1.988368e+04
7   1.270788e+02 -3.676966e+02
8   2.164790e+02 -7.756841e+02
9   1.731468e+05 -4.681335e+05
10  1.702274e+02 -4.507122e+02
11  4.548514e+04 -1.176107e+05
12  8.665092e+06 -1.818446e+07
13  1.292443e+02 -3.708601e+02
14  4.572598e+02 -9.937912e+02
15  1.241151e+02 -3.738342e+02
16  1.294325e+05 -2.578079e+05
17  1.345268e+02 -3.799355e+02
18  2.521856e+01 -1.160795e+02
alpha:
 [1] 0.06599879 0.04518660 0.05057956 0.04735171 0.03479526 0.02836056 0.03614557
 [8] 0.19250215 0.05384421 0.01285201 0.03512889 0.02699677 0.06427562 0.05343906
[15] 0.03482576 0.11435695 0.04594304 0.05741751
L:
[1] 1972.222
```

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```
[[2]]
theta:
                [,1]           [,2]
1     3.095045e+06 -5.954813e+06
2     8.057747e+06 -1.690973e+07
3     8.524116e+01 -2.829592e+02
4     2.451440e+02 -7.658898e+02
5     1.474670e+02 -4.354116e+02
6    -1.145379e+03 -2.140655e+03
7     2.664590e+02 -9.769636e+02
8     4.585910e+04 -1.185841e+05
9     3.720016e+05 -7.976019e+05
10    2.428804e+01 -1.192141e+02
11   -1.623259e+01 -1.116864e+02
12    3.733990e+02 -8.571399e+02
13    1.229235e+05 -2.448357e+05
14   -2.745290e+02 -6.509499e+02
15    5.118403e+00 -1.022458e+02
16    1.495146e+02 -9.806753e+02
17    1.435514e+02 -4.127092e+02
18    1.435652e+02 -4.157034e+02
19    1.720128e+05 -4.650696e+05
alpha:
 [1] 0.005494583 0.027468901 0.025110064 0.088794084 0.007637632 0.035550840 0.167095016
 [8] 0.034989297 0.051674128 0.028267173 0.048850790 0.072220717 0.116715077 0.013802008
[15] 0.036190350 0.054475336 0.060210742 0.071192174 0.054261088
L:
[1] 1978.023

[[3]]
theta:
                [,1]           [,2]
1        204.25950     -639.4484
2        205.45128     -762.8477
3     315066.80019  -606990.8666
4      -1166.43628    -2177.6272
5        113.23400     -381.8232
6        164.72515     -534.7322
7        114.99824     -387.7821
8        228.08633     -536.2571
9      19551.42121   -41603.2304
10        28.96799     -233.1784
11   168909.32031  -456669.2218
12    38621.79718   -99886.8535
13   113741.43373  -226532.6982
14     -224.96802     -529.0647
15      -12.48868     -101.4856
16      140.24705     -462.8661
17      211.93685     -630.3590
18     7364.60242   -47319.7939
19     3507.48922    -8230.5473
20      150.45907     -434.3878
alpha:
 [1] 0.046714471 0.148931416 0.006789976 0.034694220 0.022343642 0.033312618 0.027442571
 [8] 0.021912766 0.096056412 0.064137065 0.055041890 0.039939782 0.119219502 0.014950208
[15] 0.060767532 0.026145783 0.075063275 0.024163819 0.033601051 0.048771999
L:
[1] 1937.451

[[4]]
theta:
```

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```
                [,1]           [,2]
1    1.249357e+02 -3.543863e+02
2   -3.601633e+01 -1.911704e+02
3    1.109391e+02 -3.309904e+02
4    1.086571e+02 -3.460510e+02
5    1.224085e+02 -3.473644e+02
6   -2.478876e+02 -5.906545e+02
7    1.227984e+02 -3.472832e+02
8    2.916692e+01 -1.635644e+02
9    4.290858e+02 -9.387569e+02
10   3.911803e+05 -8.387301e+05
11   2.638731e+02 -9.481338e+02
12   4.262776e+04 -2.725949e+05
13   1.292677e+05 -2.574796e+05
14   1.327016e+02 -4.146993e+02
15   1.151525e+02 -3.344207e+02
16  -1.147660e+03 -2.144609e+03
17   5.393152e+03 -1.440438e+04
18   2.361426e+01 -1.973910e+02
19   2.791902e+01 -1.501361e+02
20  -9.083905e+00 -1.343012e+02
21   8.660729e+06 -1.817531e+07
alpha:
 [1] 0.03339187 0.02499722 0.02901613 0.02462846 0.04055378 0.01477682 0.03862032
 [8] 0.03879298 0.05720840 0.05054998 0.16321725 0.01880017 0.11442242 0.06330500
[15] 0.03040900 0.03542639 0.09854276 0.02465134 0.03491192 0.03677785 0.02699993
L:
[1] 1959.354

[[5]]
theta:
                [,1]           [,2]
1    2.293824e+02     -570.35355
2    1.174510e+02     -343.59721
3    1.283064e+05 -255565.54136
4    1.639342e+00      -79.53284
5    1.770374e+05 -478652.60496
6   -1.143835e+03    -2137.91201
7    3.018913e+01     -123.85512
8    2.821293e+02    -1048.48233
9    2.001819e+02     -669.47020
10   4.863496e+04 -125734.81837
11   1.226834e+02     -351.02755
12   1.552279e+02    -1022.44664
13   5.236018e+02    -1109.12551
14   1.370275e+02     -377.49202
15   2.352550e+04   -50083.10040
16  -2.656161e+02     -632.43189
17   1.200715e+02     -346.87689
18   1.134026e+02     -338.21527
19   9.939248e+01     -307.95360
20  -2.912392e+01     -165.40101
21   1.373818e+02     -378.18280
22   3.867219e-01      -69.91516
alpha:
 [1] 0.009863943 0.038637967 0.114823254 0.029965138 0.052801889 0.035590538 0.007750685
 [8] 0.134008756 0.084625765 0.033145470 0.020190732 0.061359252 0.030607962 0.041616690
[15] 0.084122111 0.014282913 0.033435881 0.015533180 0.035955948 0.030557480 0.062314184
[22] 0.028810261
L:
[1] 1920.548
```

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```
> sapply(sd, BIC)
[1] -3520.002 -3472.904 -3532.709
-3457.356 -3508.564

> sd[[1]]$details # For K value 18
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
    100      100

$logLiks
 [1] 1797.497 1803.263 1972.222 1789.855 1793.386 1800.713 1806.563 1895.214 1807.846
[10] 1897.930 1898.224 1795.139 1807.922 1792.505 1801.124 1789.682 1798.363 1789.863
[19] 1933.492 1813.479

$E
[1] "softmax"

$kappa
NULL

$minalpha
[1] 0

$converge
[1] TRUE

> sd[[2]]$details # For K value 19
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
    100      100

$logLiks
 [1] 1824.323 1805.737 1978.023 1899.085 1805.190 1792.797 1791.965 1898.103 1794.830
[10] 1804.282 1906.931 1905.853 1821.932 1929.129 1796.334 1792.708 1804.677 1826.792
[19] 1915.927 1904.183

$E
[1] "softmax"

$kappa
NULL

$minalpha
[1] 0

$converge
[1] TRUE

> sd[[3]]$details # For K value 20
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
```

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```
    100      100

$logLiks
 [1] 1884.878 1790.687 1910.348 1788.199 1801.095 1910.100 1798.847 1788.891 1792.354
[10] 1930.457 1937.451 1908.702 1791.586 1797.613 1788.044 1802.188 1826.313 1801.884
[19] 1804.373 1918.972

$E
[1] "softmax"

$kappa
NULL

$minalpha
[1] 0

$converge
[1] TRUE

> sd[[4]]$details # For K value 21
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
    100      100

$logLiks
 [1] 1870.700 1801.047 1802.570 1795.897 1823.650 1801.235 1792.170 1959.354 1805.532
[10] 1793.185 1806.308 1898.207 1805.238 1790.805 1931.023 1907.803 1917.843 1927.847
[19] 1814.756 1802.283

$E
[1] "softmax"

$kappa
NULL

$minalpha
[1] 0

$converge
[1] TRUE

> sd[[5]]$details # For K value 22
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
    100      100

$logLiks
 [1] 1831.717 1880.203 1918.282 1800.204 1911.742 1918.168 1801.790 1792.626 1802.114
[10] 1796.603 1794.744 1802.607 1821.185 1795.286 1920.548 1809.030 1897.343 1805.040
[19] 1903.566 1799.476

$E
[1] "softmax"

$kappa
```

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```
NULL

$minalpha
[1] 0

$converge
[1] TRUE

> sapply(sd, BIC)
[1] -3597.011 -3588.947 -3488.135 -3512.276 -3414.998
```

4. European Region

```
> e = cbind(a1$latitude , a1$longitude)
> set.seed(2022)
> EvMFs <-
+   function(K){
+     movMF(e, k = K, control= list(nruns = 20))
+   }
> Esd
[[1]]
theta:
      [,1]       [,2]
1 6.55026 -2.942784
alpha:
[1] 1
L:
[1] 84531.98

[[2]]
theta:
        [,1]         [,2]
1 183.21675 -203.117053
2  18.89648   -2.786861
alpha:
[1] 0.3973311 0.6026689
L:
[1] 110557

[[3]]
theta:
         [,1]        [,2]
1  155.48147 -170.54229
2   68.53485  -23.65068
3 4724.75350  300.46946
alpha:
[1] 0.4137033 0.2897239 0.2965727
L:
[1] 145544.7

[[4]]
theta:
           [,1]          [,2]
1 1.606353e+01 -9.034498e+00
2 5.154084e+02 -5.958039e+02
3 4.831414e+03  3.076631e+02
4 7.473574e+07 -2.368913e+07
alpha:
[1] 0.2341941 0.3193796 0.2948623 0.1515639
L:
[1] 198373
```

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```
[[5]]
theta:
             [,1]           [,2]
1  1.405530e+02 -1.498642e+02
2  2.291350e+01 -9.555982e+00
3  4.078959e+06 -4.853311e+06
4  4.854890e+03  3.092664e+02
5  7.521198e+07 -2.384008e+07
alpha:
[1] 0.2649200 0.1571423 0.1319677 0.2944920 0.1514780
L:
[1] 220980.8

[[6]]
theta:
             [,1]           [,2]
1  1.410572e+02 -1.504514e+02
2  4.720996e+05  4.167295e+04
3  4.079829e+06 -4.854346e+06
4  2.192372e+01 -9.091303e+00
5  7.880985e+03  4.742698e+02
6  7.521436e+07 -2.384083e+07
alpha:
[1] 0.26407529 0.03813963 0.13196237 0.15947426 0.25487086 0.15147759
L:
[1] 223484.5

[[7]]
theta:
             [,1]           [,2]
1  7.768454e+03  4.673953e+02
2  7.599797e+07 -2.408921e+07
3  4.320274e+06 -5.140407e+06
4  4.595943e+05  4.057539e+04
5  1.045483e+02 -9.049909e+01
6  3.802452e+01 -1.240167e+01
7  3.177443e+02 -3.614085e+02
alpha:
[1] 0.25574642 0.15133646 0.13040385 0.03823432 0.11777899 0.12046265 0.18603731
L:
[1] 224391.6

[[8]]
theta:
             [,1]           [,2]
1  1.633109e+02 -1.031426e+02
2  7.584059e+07 -2.403932e+07
3  2.919718e+02 -3.324702e+02
4  4.261411e+06 -5.070378e+06
5  3.758480e+06 -3.891155e+06
6  4.778908e+03  3.041587e+02
7  2.580223e+03 -2.199959e+03
8  6.014632e+01 -1.710144e+01
alpha:
[1] 0.04683069 0.15136456 0.19000668 0.13079434 0.03187038 0.29594574 0.05562754 0.09756008
L:
[1] 227227.6

[[9]]
theta:
             [,1]           [,2]
```

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```
1  7.628722e+07  -2.418089e+07
2  4.932540e+02  -5.831950e+02
3  3.032197e+03   1.272199e+02
4  1.059237e+02  -9.338095e+01
5  1.374141e+04   1.097937e+03
6  8.194247e+05  -8.481291e+05
7  4.146385e+01  -1.396367e+01
8  3.009328e+04   1.687638e+03
9  4.355077e+06  -5.181811e+06
alpha:
[1] 0.15128461 0.13427352 0.01940869 0.12883317 0.10325505 0.03991329 0.11873265 0.17412022
[9] 0.13017880
L:
[1] 228259.9

[[10]]
theta:
             [,1]          [,2]
1   8.319235e+03   5.009535e+02
2   2.581177e+02  -6.823996e+01
3   4.452958e+05  -4.608733e+05
4   3.909663e+02  -4.549188e+02
5   8.337693e+02  -7.248919e+02
6   4.109150e+05   3.629825e+04
7   9.579057e+01  -5.558359e+01
8   4.340647e+06  -5.164645e+06
9   3.464089e+02  -4.464990e+00
10  7.717096e+07  -2.446099e+07
alpha:
 [1] 0.25233560 0.06550730 0.04387500 0.16491902 0.06908147 0.03914456 0.06920065
 [8] 0.13028752 0.01452099 0.15112790
L:
[1] 229573.1

[[11]]
theta:
             [,1]          [,2]
1   5.214142e+02  -5.253213e+02
2   8.084894e+02  -9.776349e+02
3   9.242520e+04   8.107817e+03
4   4.390754e+06  -5.224253e+06
5   9.613394e+03  -1.008952e+04
6   4.322195e+06  -4.474735e+06
7   2.354865e+03  -2.001942e+03
8   1.650351e+03  -1.089463e+03
9   7.599057e+07  -2.408686e+07
10  9.440098e+03   5.602314e+02
11  3.778011e+01  -1.222108e+01
alpha:
 [1] 0.03923287 0.11988789 0.04739072 0.12994472 0.03456095 0.03041099 0.05690963
 [8] 0.02370342 0.15133797 0.24609082 0.12053002
L:
[1] 230457.6

[[12]]
theta:
             [,1]          [,2]
1   2.014852e+07  -2.085688e+07
2   8.366047e+01  -4.901767e+01
3   7.714761e+07  -2.445359e+07
4   5.197906e+06  -6.184505e+06
```

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```
5   1.959115e+03 -1.679552e+03
6   7.740791e+03 -8.131614e+03
7   1.938966e+04 -1.895087e+04
8   4.869042e+03  3.104276e+02
9   4.995018e+02 -1.668651e+01
10  5.712492e+03 -6.757654e+03
11  1.775368e+04 -2.352431e+04
12  2.627467e+02 -6.927473e+01
alpha:
 [1] 0.02488825 0.07274872 0.15113203 0.12435533 0.05871018 0.05888056 0.01436941
 [8] 0.29524292 0.01079178 0.09650444 0.02771946 0.06465692
L:
[1] 232191.4

[[13]]
theta:
             [,1]          [,2]
1   1.766098e+06  1.551756e+05
2   1.389782e+02 -3.560498e+01
3   5.675790e+02  1.637928e+00
4   2.339896e+06 -7.418437e+05
5   2.128742e+08 -6.747162e+07
6   5.845110e+03  3.885628e+02
7   2.975118e+06 -3.080149e+06
8   5.071150e+04  2.800699e+03
9   2.944196e+02 -3.356242e+02
10  1.078816e+02 -6.435244e+01
11  2.596805e+03 -2.214437e+03
12  4.261375e+06 -5.070334e+06
13  3.548102e+03  1.653941e+02
alpha:
 [1] 0.029824921 0.063307066 0.010343478 0.026259663 0.135342474 0.147194544 0.032946397
 [8] 0.105683328 0.188331878 0.065541825 0.054870305 0.130794506 0.009559617
L:
[1] 234222

[[14]]
theta:
             [,1]          [,2]
1   4.391067e+05  3.877510e+04
2   8.716091e+05 -1.968566e+05
3   9.486646e+01 -7.790779e+00
4   6.865619e+01 -2.619730e+01
5   2.792648e+02 -7.351220e+01
6   8.065667e+02 -9.768151e+02
7   4.361936e+06 -5.189969e+06
8   1.127292e+02 -6.792079e+01
9   9.964331e+02 -1.034238e+03
10  2.544159e+03 -2.170876e+03
11  5.625625e+06 -5.824189e+06
12  2.127108e+08 -6.741983e+07
13  2.257234e+06 -7.156529e+05
14  8.003930e+03  4.814508e+02
alpha:
 [1] 0.03860525 0.00805271 0.02010620 0.02158656 0.02957412 0.11663720 0.13013575
 [8] 0.05861573 0.07419719 0.05640443 0.03005307 0.13542660 0.02636393 0.25424125
L:
[1] 234005.3

[[15]]
theta:
```

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```
              [,1]          [,2]
1   1.809356e+03 -1.540822e+03
2   6.719224e+05 -1.517912e+05
3   7.008996e+01 -2.326096e+01
4   1.327176e+06  1.167212e+05
5   4.258041e+06 -5.066368e+06
6   8.365837e+02 -4.098258e+02
7   6.919132e+05 -7.161105e+05
8   5.362458e+03  3.447067e+02
9   4.333147e+02 -5.392758e+00
10  2.184151e+06 -6.924971e+05
11  3.073211e+02 -3.523928e+02
12  2.312899e+02 -5.855561e+01
13  2.125014e+08 -6.735346e+07
14  6.173333e+04  3.415634e+03
15  1.230741e+03 -8.134469e+02
alpha:
 [1] 0.062490107 0.008273497 0.017992775 0.031575414 0.130813812 0.025909363 0.040445549
 [8] 0.167953527 0.012640516 0.026394453 0.179555600 0.037100609 0.135530369 0.092790395
[15] 0.030534013
L:
[1] 235382.2

> Esd[[1]]$details # For K value 1
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
      1     100

$logLiks
 [1] 84531.98 84531.98 84531.98 84531.98 84531.98 84531.98 84531.98 84531.98 84531.98
[10] 84531.98 84531.98 84531.98 84531.98 84531.98 84531.98 84531.98 84531.98 84531.98
[19] 84531.98 84531.98

$E
[1] "softmax"

$kappa
NULL

$minalpha
[1] 0

$converge
[1] TRUE

> Esd[[2]]$details # For K value 2
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
     26     100

$logLiks
 [1] 110557 110557 110557 110557 110557 110557 110557 110557 110557 110557 110557 110557
[13] 110557 110557 110557 110557 110557 110557 110557 110557

$E
```

Draft

```
[1] "softmax"

$kappa
NULL

$minalpha
[1] 0

$converge
[1] TRUE

> Esd[[3]]$details # For K value 3
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
     16      100

$logLiks
 [1] 145491.2 145491.2 135302.5 145491.2 145491.2 145491.2 145491.2 145544.7 145544.7
[10] 145544.7 145544.7 145544.7 145491.2 145491.2 145491.2 145491.2 145544.7 145544.7
[19] 145491.2 145491.2

$E
[1] "softmax"

$kappa
NULL

$minalpha
[1] 0

$converge
[1] TRUE

> Esd[[4]]$details # For K value 4
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
     43      100

$logLiks
 [1] 198373.0 171399.2 171399.2 198373.0 152394.5 198373.0 171399.2 198373.0 171399.2
[10] 152394.5 152394.5 171399.2 152394.5 152394.5 147289.4 198373.0 152394.5 152394.5
[19] 198373.0 152394.5

$E
[1] "softmax"

$kappa
NULL

$minalpha
[1] 0

$converge
[1] TRUE
```

Draft

```
> Esd[[5]]$details # For K value 5
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
     79      100

$logLiks
  [1] 220980.8 220980.8 220980.8 220980.8 173674.8 147770.6 154013.8 220980.8 220980.8
 [10] 220980.8 220980.8 220980.8 220980.8 220980.8 220980.8 220980.8 220980.8 220980.8
 [19] 154013.8 220980.8

$E
[1] "softmax"

$kappa
NULL

$minalpha
[1] 0

$converge
[1] TRUE

> Esd[[6]]$details # For K value 6
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
    100      100

$logLiks
  [1] 221331.7 221897.2 221909.9 221910.6 222611.2 221331.7 221910.6 221910.6 221907.4
 [10] 221871.8 223484.5 221014.4 221266.4 221910.6 221910.4 222625.0 221888.7 221907.9
 [19] 222861.9 221909.9

$E
[1] "softmax"

$kappa
NULL

$minalpha
[1] 0

$converge
[1] TRUE

> Esd[[7]]$details # For K value 7
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
    100      100

$logLiks
  [1] 223220.9 221453.3 222164.2 223224.1 221965.5 222142.9 223521.8 223533.4 224360.2
 [10] 222197.3 221962.7 222228.0 224391.6 223519.3 221813.7 222087.3 221747.6 221844.1
```

Draft

```
[19] 222327.3 224215.8

$E
[1] "softmax"

$kappa
NULL

$minalpha
[1] 0

$converge
[1] TRUE

> Esd[[8]]$details # For K value 8
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
    100     100

$logLiks
 [1] 223193.1 222152.0 223971.5 223647.9 225113.8 227227.6 223607.1 224886.3 224864.7
[10] 221879.1 223717.8 224120.1 223474.4 224338.5 223660.6 223469.0 222112.7 223543.9
[19] 222320.4 224065.8

$E
[1] "softmax"

$kappa
NULL

$minalpha
[1] 0

$converge
[1] TRUE

> Esd[[9]]$details # For K value 9
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
    100     100

$logLiks
 [1] 225472.1 225405.3 226166.9 222954.8 225018.0 223896.9 225274.2 228259.9 221960.6
[10] 224836.8 225187.3 222545.0 225565.6 224522.0 223936.4 224567.3 224706.4 226449.5
[19] 224285.6 223613.6

$E
[1] "softmax"

$kappa
NULL

$minalpha
[1] 0
```

Draft

```
$converge
[1] TRUE

> Esd[[10]]$details # For K value 10
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
    100      100

$logLiks
 [1] 225107.4 225114.9 222310.3 225917.2 223950.6 224129.0 225248.9 229573.1 225582.1
[10] 226230.5 224264.5 223453.6 225709.7 224840.0 224568.7 227745.6 225067.8 223599.3
[19] 226979.9 226970.7

$E
[1] "softmax"

$kappa
NULL

$minalpha
[1] 0

$converge
[1] TRUE

> Esd[[11]]$details # For K value 11
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
    100      100

$logLiks
 [1] 229290.5 230054.3 229456.7 226258.6 226476.4 226274.0 227224.4 224713.9 227352.4
[10] 227678.6 226046.0 230149.1 223595.5 228909.6 230457.6 226101.4 229925.6 225582.7
[19] 226054.3 222514.5

$E
[1] "softmax"

$kappa
NULL

$minalpha
[1] 0

$converge
[1] TRUE

> Esd[[12]]$details # For K value 12
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
    100      100
```

Draft

```
$logLiks
 [1] 232191.4 231777.5 226465.0 230863.4 226595.7 227411.7 229381.9 223822.1 227334.5
[10] 228571.0 226843.7 228938.2 227761.6 227019.7 229418.8 225768.8 229251.4 230498.8
[19] 227235.7 227396.9

$E
[1] "softmax"

$kappa
NULL

$minalpha
[1] 0

$converge
[1] TRUE

> Esd[[13]]$details # For K value 13
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
    100     100

$logLiks
 [1] 230415.9 231891.2 231709.4 226190.1 227811.1 223647.1 226596.3 228449.4 229543.9
[10] 227693.2 226052.4 234222.0 229681.6 228817.1 229656.1 226996.2 230706.4 229444.5
[19] 228463.4 225707.1

$E
[1] "softmax"

$kappa
NULL

$minalpha
[1] 0

$converge
[1] TRUE

> Esd[[14]]$details # For K value 14
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
    100     100

$logLiks
 [1] 230175.2 234005.3 231404.5 227845.3 233090.4 232604.7 227748.0 227981.7 231423.7
[10] 232084.6 232746.2 230534.6 227550.6 226598.9 231951.3 230865.0 230328.0 226475.3
[19] 231819.6 233935.7

$E
[1] "softmax"

$kappa
NULL
```

Draft

```
$minalpha
[1] 0

$converge
[1] TRUE

> Esd[[15]]$details # For K value 15
$reltol
[1] 1.490116e-08

$iter
   iter maxiter
    100     100

$logLiks
 [1] 227855.2 231942.6 231006.8 228274.3 231134.6 228822.7 232357.0 233525.0 232147.1
[10] 229930.0 228214.2 232635.4 231470.2 233444.2 231382.0 227496.8 228961.8 227490.4
[19] 235382.2 228398.6

$E
[1] "softmax"

$kappa
NULL

$minalpha
[1] 0

$converge
[1] TRUE

> gt
 [1] -169041.9 -221058.9 -291001.2 -396624.7 -441807.2 -446781.4 -448562.6 -454201.4
 [9] -456232.8 -458826.2 -460562.2 -463996.7 -468024.6 -467558.2 -470279.0
> min(gt)
[1] -470279
```

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