Online Appendix for the Paper

Existence of the Wealth–Consumption Ratio in Asset Pricing Models with Recursive Preferences

Walter Pohl
NHH Bergen

Karl Schmedders
IMD Lausanne

Ole Wilms
Universität Hamburg
and Tilburg University

Online Appendix

B Existence Condition of Borovička and Stachurski (2020)

Borovička and Stachurski (2020) derive a necessary and sufficient condition for asset pricing models with Epstein–Zin utility, which is given by

$$\begin{aligned} \Lambda &\equiv \delta M_C^{1-\frac{1}{\psi}} \\ M_C &= \lim_{n\to\infty} \left[E_0 \left(e^{(1-\gamma)\sum_{t=1}^{n} \Delta c_t} \right) \right]^{\frac{1}{(1-\gamma)n}}; \end{aligned} \tag{B.1}$$

see equation (4) of their paper. A solution then exists if and only if $\Lambda < 1$. To compute the bound, Monte Carlo simulations are used and the expectation is replaced by the sample mean. For the simulations, Borovička and Stachurski (2020) use m sample paths and argue that if m and n are sufficiently large, an accurate approximation of condition (B.1) is obtained. We test this claim in a simple long-run risk model for which (B.1) can be computed analytically. For this test, we use the

consumption dynamics as in (19) without jumps ($l_0 = 0$):

$$\begin{aligned} \Delta c_{t+1} &= \mu + x_t + \sigma \epsilon_{c,t+1} \\ x_{t+1} &= \rho x_t + \phi_x \sigma \epsilon_{x,t+1} \\ & \epsilon_{c,t+1}, \epsilon_{x,t+1} \sim i.i.d.\ N(0,1). \end{aligned}$$

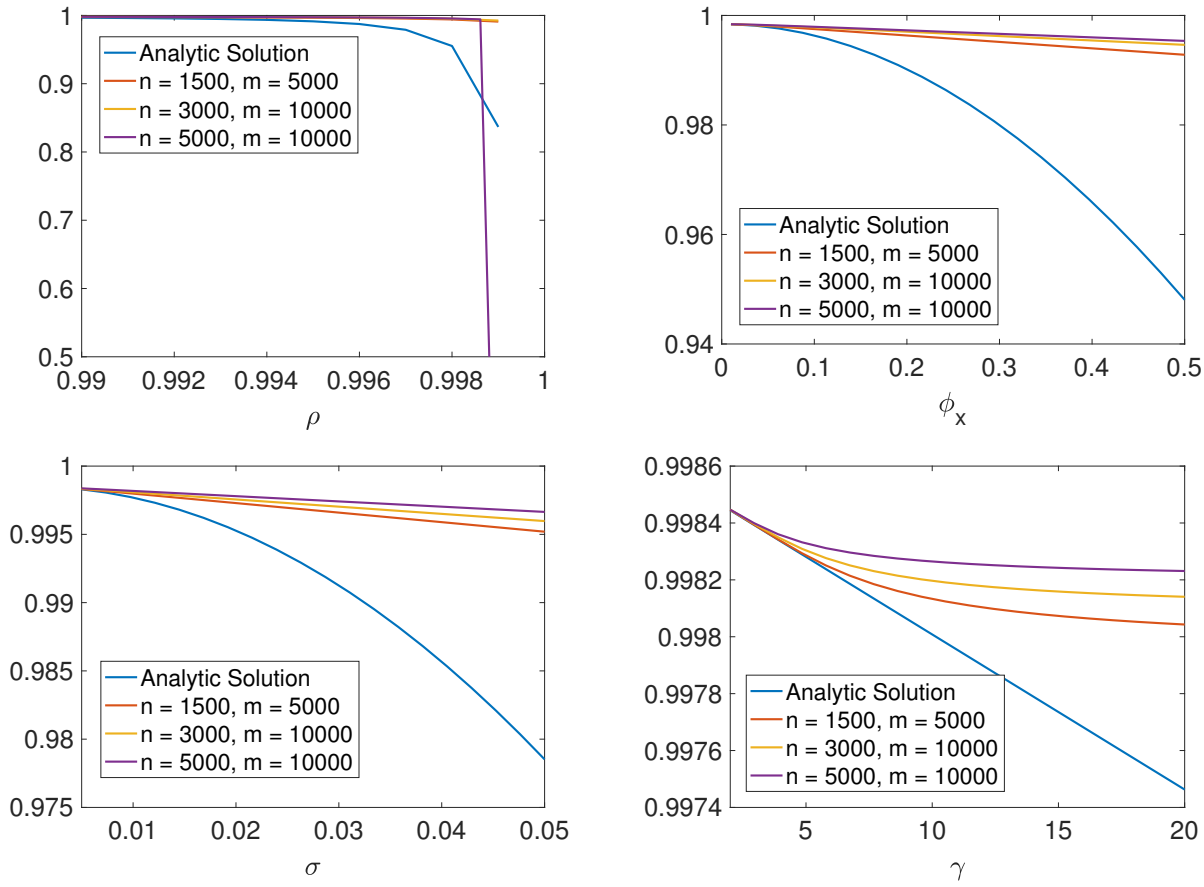
Borovička and Stachurski (2020) show that

$$M_C = e^{\mu + \frac{1}{2}(1-\gamma)\sigma^2 + \frac{1}{2}\frac{(1-\gamma)\sigma^2\phi_x^2}{(1-\rho)^2}},$$

which is the same as our condition (19). Hence, in this stylized example condition (19) is necessary as well as sufficient for the existence of a solution.

Figure 1 plots condition (B.1) using Monte Carlo simulations with different sample lengths, n, and different numbers of sample paths, m, as well as the closed-form solution for the simple long-run risk model (19). The model parameters are given by $\delta = 0.998, \psi = 1.5, \mu = 0.0015, \sigma = 0.0078, \rho_x = 0.979, \phi_x = 0.044$, and $l_0 = 0$. Each plot shows the bound as a function of one of the model parameters. Consider first the case for different values of ρ. Increasing ρ toward 1 yields a significant difference between the analytical value for the bound and its numerical counterpart. More importantly, increasing the number of sample paths, m, or the sample length, n, does not reduce this gap. Increasing the sample length beyond $5,000$ is not feasible as the exponential of the sum quickly reaches machine precision. This pattern also occurs for the other model parameters; see the plots for ϕ_x, σ, and γ, respectively. In none of the cases does increasing n and m lead to a convergence toward the bound. While in this example both the true value of the bound and the numerical value lie below 1, it is straightforward to construct cases where the two yield opposite conclusions, by, for example, increasing the discount factor or the mean growth rate in the model. Hence, the numerical computation of the bound can make false predictions about the existence of solutions. While in this stylized example the numerical solution can be compared to the analytic counterpart, such a comparison is impossible in models with more complex dynamics for consumption growth. In such a setup, it is therefore challenging to verify whether the bound is accurately approximated or not.

Figure 1: The Existence Condition of Borovička and Stachurski (2020)



Condition (B.1) using Monte Carlo simulations with different sample lengths, n, and different numbers of sample paths, m, as well as the closed-form solution for the simple long-run risk model (19). The model parameters are given by $\delta = 0.998, \psi = 1.5, \mu = 0.0015, \sigma = 0.0078, \rho_x = 0.979, \phi_x = 0.044$, and $l_0 = 0$. Each plot shows the bound as a function of one of the model parameters.

C Existence Conditions for Additional Models

In the following we derive our existence conditions for the volatility-of-volatility model of Bollerslev, Tauchen, and Zhou (2009), Tauchen (2011), and Bollerslev, Xu, and Zhou (2015) and analyze existence for the model of Drechsler and Yaron (2011), which includes three state processes to reflect variability in the mean growth rate of consumption, in stochastic volatility, and in mean volatility, as well as multiple Poisson jumps with time-varying intensities.

C.1 Volatility of Volatility (Bollerslev, Tauchen, and Zhou 2009)

In this section, we consider the influence of volatility of volatility, as in the work of Bollerslev, Tauchen, and Zhou (2009), Tauchen (2011), and Bollerslev, Xu, and Zhou (2015), on the existence of solutions. In these studies it is assumed that the agent has Epstein–Zin preferences (10) with $\psi > 1$ and a preference for the early resolution of risks, so we again focus on the case of $\theta < 0$. We use the parsimonious model formulation of Bollerslev, Tauchen, and Zhou (2009) without the long-run risk factor to separately analyze the influence of volatility of volatility on the existence of solutions:

$$\begin{aligned} \Delta c_{t+1} &= \mu + \sigma_t \epsilon_{c,t+1} \\ \sigma^2_{t+1} &= \bar{\sigma}^2(1-\rho_\sigma) + \rho_\sigma \sigma^2_t + \sqrt{q_t}\epsilon_{\sigma,t+1} \\ q_{t+1} &= \mu_q(1-\rho_q) + \rho_q q_t + \phi_q \sqrt{q_t}\epsilon_{q,t+1} \\ \epsilon_{c,t+1}, \epsilon_{\sigma,t+1}, \epsilon_{q,t+1} &\sim i.i.d.\ N(0,1). \end{aligned} \tag{C.1}$$

Section D.3 contains a derivation of the following existence condition for this process.

Proposition 1 (Existence for the volatility-of-volatility factor)**.** *The test value, $\Omega(\theta)$, to prove existence and non-existence for the consumption process* (C.1) *is given by*

$$\Omega(\theta) = \underbrace{\ln(\delta)}_{\text{discounting}} + \underbrace{\alpha\mu}_{\textit{mean growth}} + \underbrace{\frac{\theta\alpha^2\bar{\sigma}^2}{2}}_{\textit{c-shock}} + \underbrace{\frac{\mu_q(1-\rho_q)\left(1-\rho_q-\sqrt{(1-\rho_q)^2-\frac{\theta^4\alpha^4\phi_q^2}{4(1-\rho_\sigma)^2}}\right)}{\theta\phi_q^2}}_{\textit{σ- and q-shock}}.$$

The first three terms are well known from the previous sections; thus, we focus on the volatility-

of-volatility term here. Note that the influence of the σ- and of the q-shock cannot be separated as q_t follows a discrete-time version of a continuous-time square root–type process. For this specification, the model solution can become complex, a feature that has already been mentioned by Bollerslev, Tauchen, and Zhou (2009) and is extensively discussed by Pohl, Schmedders, and Wilms (2018). More precisely, the larger ρ_q, ρ_σ, ϕ_q, and $|1-\gamma|$, the more likely a complex solution becomes. Assume that $(1-\rho_q)^2 > \frac{\theta^4\alpha^4\phi_q^2}{4(1-\rho_\sigma)^2}$ so we focus on real solutions. How does the volatility-of-volatility factor, q_t, affect the existence of a solution? Note that $1-\rho_q \geq \sqrt{(1-\rho_q)^2 - \frac{\theta^4\alpha^4\phi_q^2}{4(1-\rho_\sigma)^2}}$. So adding volatility of volatility increases $\Omega(1)$. More precisely, the larger μ_q becomes, the larger is $\Omega(1)$, and hence a persistent and volatile q_t process makes the existence of a solution for the model with Epstein–Zin utility less likely; see Theorem 5. Furthermore, the σ- and q-shocks decrease $\Omega(\theta)$ if $\theta < 0$. Hence, from Theorem 5 it follows that the non-existence region for the model with Epstein–Zin preferences decreases when adding volatility of volatility. For the calibration used in the paper of Bollerslev, Tauchen, and Zhou (2009)—that is, $\delta = 0.997, \gamma = 10, \psi = 1.5, \mu = 0.0015, \rho_\sigma = 0.978, \bar{\sigma}^2 = 0.0078^2, \phi_q =$ 1e-3, $\rho_q = 0.8$, and $\mu_q =$ 1e-6—the existence condition $\Omega(\theta)$ is complex, and therefore no real solution exists. Bollerslev, Tauchen, and Zhou (2009) overcome this issue by linearizing the model and exogenously specifying the level of the wealth–consumption ratio.

C.2 Existence for General Affine Processes with Gaussian and Poisson Jump Shocks

We have demonstrated how Theorem 5 can be applied to analytically analyze the existence of solutions for different models that fall into the discrete-time affine class. Deriving these conditions in closed form can sometimes be tedious. An alternative is to simply compute $\Omega(\theta)$ and $\Omega(1)$ numerically. This yields an easy and efficient way to prove existence. The supplementary material includes a simple Matlab script that can be used to prove existence for any model that falls in the affine model class.

In Online Appendix Section D.4, we show how to compute $\Omega(\theta)$ for any discrete-time affine process. We use this approach to numerically prove existence for the model of Drechsler and Yaron (2011), which features a stochastic mean growth rate of consumption, stochastic volatility with a

time-varying mean, and jumps in the mean growth rate of consumption as well as in volatility. The model dynamics are given by

$$X_{t+1} = \mu + FX_t + G_t z_{t+1} + J_{t+1}, \tag{C.2}$$

where $X_t \in \mathbb{R}^n$ denotes the vector of state processes, $z_{t+1} \sim N(0, I)$ is a vector of Gaussian shocks, and $G_t G_t' = h + \sum_i H_i X_{i,t}$. J_{t+1} is the vector of compound Poisson jumps. Hence, the ith component of J_{t+1} is given by $J_{i,t+1} = \sum_{j=1}^{N_{i,t+1}} \xi_{i,j}$, where $N_{i,t+1}$ is a Poisson counting process and $\xi_{i,j}$ is the size of the jump that occurs upon the jth increment of $N_{i,t+1}$. Drechsler and Yaron (2011) assume

$$X_t = \begin{bmatrix} 1 \\ \Delta c_t \\ x_t \\ \bar{\sigma}_t^2 \\ \sigma_t^2 \end{bmatrix}, \mu = \begin{bmatrix} 1 \\ \mu_c \\ 0 \\ \mu_{\bar{\sigma}} \\ \mu_\sigma \end{bmatrix}, F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \rho_x & 0 & 0 \\ 0 & 0 & 0 & \rho_{\bar{\sigma}} & 0 \\ 0 & 0 & 0 & (1-\tilde{\rho}_\sigma) & \rho_\sigma \end{bmatrix} h = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \phi_c^2(1-\omega_c)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_{\bar{\sigma}}^2(1-\omega_{\bar{\sigma}})^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$l_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, l_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l_{1,x} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & l_{1,\sigma} \end{bmatrix} H_\sigma = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \phi_c^2\omega_c^2 & 0 & 0 & 0 \\ 0 & 0 & \phi_x^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_\sigma^2 \end{bmatrix}.$$

The intensity process of the $N_{i,t+1}$ can be time varying and is given by the ith component of $\lambda_t = l_0 + l_1 X_t$ with $l_0 \in \mathbb{R}^n$ and $l_1 \in \mathbb{R}^{n \times n}$. The jumps in x_t follow the negative of a demeaned gamma distribution, $\xi_x \sim -\Gamma(\nu_{\xi_x}, \frac{\mu_{\xi_x}}{\nu_{\xi_x}}) + \mu_{\xi_x}$, and the jumps in σ_t^2 follow a gamma distribution, $\xi_\sigma \sim \Gamma(\nu_{\xi_\sigma}, \frac{\mu_{\xi_\sigma}}{\nu_{\xi_\sigma}})$. The corresponding moment-generating functions of $J_{i,t+1}$ are then given by

$$M_{J_x}(k) = e^{\lambda_{x,t}\left(\left(1+\frac{\mu_{\xi_x}}{\nu_{\xi_x}}k\right)^{-\nu_{\xi_x}} e^{k\mu_{\xi_x}} - 1\right)}$$

and

$$M_{J_\sigma}(k) = e^{\lambda_{\sigma,t}\left(\left(1-\frac{\mu_{\xi_\sigma}}{\nu_{\xi_\sigma}}k\right)^{-\nu_{\xi_\sigma}}-1\right)}.$$

Drechsler and Yaron (2011) rewrite the model dynamics in terms of compensated jump shocks so that the model is easier to calibrate. They define $\tilde{J}_{t+1} = J_{t+1} - E_t(J_{t+1})$ such that

$$X_{t+1} = \tilde{\mu} + \tilde{F}X_t + G_t z_{t+1} + \tilde{J}_{t+1} \tag{C.3}$$

with

$$\tilde{F} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \rho_x & 0 & 0 \\ 0 & 0 & 0 & \rho_{\bar{\sigma}} & 0 \\ 0 & 0 & 0 & (1-\tilde{\rho}_\sigma) & \tilde{\rho}_\sigma \end{bmatrix},$$

$\tilde{\mu} = (I - \tilde{F})E(X_t)$, and I is the identity matrix. The advantage of the compensated notation is that the model can be calibrated in terms of the true persistence parameter, $\tilde{\rho}_\sigma$, and the unconditional means, $E(X_t)$. The parameters used by Drechsler and Yaron (2011) are given by $\psi = 2, \delta = 0.999, E(\Delta c_t) = 0.0016, E(\bar{\sigma}_t^2) = 1, E(\sigma_t^2) = 1, \rho_x = 0.976, \rho_{\bar{\sigma}} = 0.985, \tilde{\rho}_\sigma = 0.87, \phi_c = 0.0066, \omega_c = 0.5, \phi_x = 0.032\phi_c, \phi_{\bar{\sigma}} = 0.1, \phi_\sigma = 0.35, l_{1,x} = 0.8/12, l_{1,\sigma} = 0.8/12, \mu_{\xi_x} = 3.645\phi_x, \nu_{\xi_x} = 1, \mu_{\xi_\sigma} = 2.55,$ and $\nu_{\xi_\sigma} = 1$. Furthermore, they use $\gamma = 9.5$ so the agent prefers an early resolution of risks. Using the specification for the model dynamics together with the moment-generating functions, we can now compute the h_i functions—see (D.4) in Section D.4 of the Online Appendix—to numerically compute the limit of the A_i coefficients. Using the Matlab script with an iteration limit for i of 1,000,000, we obtain

$$\Omega(1) = -0.0093 < 0. \tag{C.4}$$

Further increasing the iteration limit for i to 2,000,000 does not change the first 12 digits of the limit. Hence, we can be assured of the existence of a solution for the model of Drechsler and Yaron (2011).

D Proofs for Section 2

In this section we present the derivations of the existence condition, $\Omega(\theta)$, that we provide for various log consumption growth processes from the asset pricing literature. All proofs follow the general approach illustrated in Section 1.4.1. We apply the general statements of Theorem 4 and Theorem 5 to concrete applications. For this purpose, we need to derive the function h and the sequence of vector coefficients $(A_i)_{i\in\mathbb{N}}$ for each specific model. This sequence is the solution to a recursive system of equations. The existence condition then follows from the limit condition of the theorem. In Section D.4 we show how Theorem 4 and Theorem 5 can be applied to general processes in the affine class.

D.1 Long-Run Risks and Stochastic Volatility (Bansal and Yaron 2004)

We derive the test function $\Omega(\theta)$ for the model with the long-run risk process of Bansal and Yaron (2004) with different kinds of shocks to stochastic volatility, $\epsilon_{\sigma,t+1}$; see Proposition 2.

Proof of Proposition 2. Set $X_t = (X_{0,t}, X_{1,t}, X_{2,t}, X_{3,t})'$ with $X_{0,t} = 1$, $X_{1,t} = \Delta c_t$, $X_{2,t} = x_t$, $X_{3,t} = \sigma_t^2$, and $k = (k_0, k_1, k_2, k_3)$. We obtain

$$\begin{aligned} E_t e^{k\cdot X_{t+1}} &= E_t e^{k_0 X_{0,t+1} + k_1 X_{1,t+1} + k_2 X_{2,t+1} + k_3 X_{3,t+1}} \\ &= e^{k_0 + k_1\mu + k_1 X_{2,t} + \frac{1}{2}k_1^2 X_{3,t} + k_2\rho X_{2,t} + \frac{1}{2}k_2^2\phi_x^2 X_{3,t} + k_3.\bar{\sigma}^2(1-\rho_\sigma) + k_3\rho_\sigma X_{3,t} + \ln(M_{\epsilon_\sigma}(\phi_\sigma k_3))}, \end{aligned}$$

where $M_{\epsilon_\sigma}(k)$ denotes the moment-generating function of $\epsilon_{\sigma,t+1}$. Using the notation $E_t(e^{k\cdot X_{t+1}}) = e^{h(k)\cdot X_t}$ with

$$h(k) = (k_0 + h_0(k_1, k_2, k_3), h_1(k_1, k_2, k_3), h_2(k_1, k_2, k_3), h_3(k_1, k_2, k_3))$$

leads to

$$\begin{aligned} k_0 + h_0(k_1, k_2, k_3) &= k_0 + k_1\mu + k_3\bar{\sigma}^2(1-\rho_\sigma) + \ln(M_{\epsilon_\sigma}(k_3)) \\ h_1(k_1, k_2, k_3) &= 0 \\ h_2(k_1, k_2, k_3) &= k_1 + k_2\rho \\ h_3(k_1, k_2, k_3) &= \frac{1}{2}k_1^2 + \frac{1}{2}k_2^2\phi_x^2 + k_3\rho_\sigma. \end{aligned}$$

We need to compute the bound $w_t(\theta)$ for $v_t = 1 + I_t(e^{\tau \cdot X_{t+1}} v_{t+1})$ with $\tau = (\ln\delta, \alpha, 0, 0)$ and $\alpha = 1 - \frac{1}{\psi}$. According to Corollary 3, the bound is then given by

$$w_t(\theta) = \sum_{i=0}^{\infty} e^{A_i(\theta)\cdot X_t}$$

with $A_0(\theta) = (0,0,0,0)$ and $A_i(\theta) = h(\theta(A_{i-1}(\theta) + \tau))/\theta$. Denote by $A_{j,i}$ the jth element of A_i. We then have

$$\begin{aligned} A_{0,i} &= A_{0,i-1} + \ln\delta + (A_{1,i-1} + \alpha)\mu + A_{3,i-1}\bar{\sigma}^2(1-\rho_\sigma) + \ln(M_{\epsilon_\sigma}(\theta\phi_\sigma A_{3,i-1}))/\theta \\ A_{1,i} &= 0 \\ A_{2,i} &= (A_{1,i-1} + \alpha) + \rho A_{2,i-1} \\ A_{3,i} &= \frac{1}{2}\theta(A_{1,i-1} + \alpha)^2 + \rho_\sigma A_{3,i-1} + \frac{1}{2}\theta(A_{2,i-1})^2\phi_x^2. \end{aligned}$$

Simplifying yields

$$\begin{aligned} A_{0,i} &= A_{0,i-1} + \ln\delta + \alpha\mu + A_{3,i-1}\bar{\sigma}^2(1-\rho_\sigma) + \ln(M_{\epsilon_\sigma}(\theta\phi_\sigma A_{3,i-1}))/\theta \\ A_{2,i} &= \alpha + \rho A_{2,i-1} \\ A_{3,i} &= \frac{1}{2}\theta\alpha^2 + \rho_\sigma A_{3,i-1} + \frac{1}{2}\theta(A_{2,i-1})^2\phi_x^2. \end{aligned}$$

Recall the sufficient condition (18) to show existence of the bound $w_t(\theta)$,

$$\lim_{i\to\infty} (A_{i+1}(\theta) - A_i(\theta)) \cdot X_t < 0.$$

Observe that $\lim_{i\to\infty} A_{2,i} = \frac{\alpha}{1-\rho}$ and so it must also hold that $\lim_{i\to\infty}(A_{2,i} - A_{2,i-1}) = 0$. Similarly,

$$\lim_{i\to\infty} A_{3,i} = \frac{1}{1-\rho_\sigma}\left(\frac{1}{2}\theta\alpha^2 + \frac{1}{2}\frac{\theta\alpha^2}{(1-\rho)^2}\phi_x^2\right)$$

and so $\lim_{i\to\infty}(A_{3,i} - A_{3,i-1}) = 0$. By continuity, it follows that $\lim_{i\to\infty} \ln(M_{\epsilon_\sigma}(\theta A_{3,i-1}))/\theta = \ln(M_{\epsilon_\sigma}(\theta \lim_{i\to\infty} A_{3,i-1}))/\theta$ and hence

$$\lim_{i\to\infty}(A_{0,i} - A_{0,i-1}) = \ln\delta + \alpha\mu + \lim_{i\to\infty} A_{3,i-1}\bar{\sigma}^2(1-\rho_\sigma) + \frac{1}{\theta}\ln(M_{\epsilon_\sigma}(\theta \lim_{i\to\infty} A_{3,i-1})).$$

Substituting $\lim_{i\to\infty} A_{3,i}$ into the last expression leads to Proposition 2. □

D.2 Consumption Disasters with Time-Varying Disaster Intensities (Wachter 2013)

We derive the test function $\Omega(\theta)$ for the model with consumption disasters and time-varying disaster intensities of Wachter (2013); see Proposition 3.

Proof of Proposition 3. For the jump process J_{t+1} in consumption growth it holds that

$$E_t e^{uJ_{t+1}} = e^{\lambda_t(M_\xi(u)-1)}$$

with

$$M_\xi(u) = E_t e^{u\xi_j} = e^{u\mu_\xi + \frac{1}{2}u^2\sigma_\xi^2}.$$

Set $X_t = (X_{0,t}, X_{1,t}, X_{2,t})'$ with $X_{0,t} = 1$, $X_{1,t} = \Delta c_t$, $X_{2,t} = \lambda_t$, and $k = (k_0, k_1, k_2)$. We obtain

$$\begin{aligned} E_t e^{k\cdot X_{t+1}} &= E_t e^{k_0 X_{0,t+1} + k_1 X_{1,t+1} + k_2 X_{2,t+1}} \\ &= e^{k_0 + k_1\mu + \frac{1}{2}k_1^2\sigma^2 + X_{2,t}(e^{k_1\mu_\xi + \frac{1}{2}k_1^2\sigma_\xi^2} - 1) - \frac{2\kappa\bar{\lambda}}{\phi_l^2}\log\left(1 - \frac{\phi_l^2}{2\kappa}(1-e^{-\kappa})k_2\right) + \frac{k_2 e^{-\kappa}}{1-\frac{\phi_l^2}{2\kappa}(1-e^{-\kappa})k_2}X_{2,t}}. \end{aligned}$$

Using the notation $E_t(e^{k\cdot X_{t+1}}) = e^{h(k)\cdot X_t}$ with

$$h(k) = (k_0 + h_0(k_1,k_2), h_1(k_1,k_2), h_2(k_1,k_2))$$

leads to

$$\begin{aligned} k_0 + h_0(k_1, k_2) &= k_0 + k_1\mu + \frac{1}{2}k_1^2\sigma^2 - \frac{2\kappa\bar{\lambda}}{\phi_l^2}\log\left(1 - \frac{\phi_l^2}{2\kappa}(1-e^{-\kappa})k_2\right) \\ h_1(k_1, k_2) &= 0 \\ h_2(k_1, k_2) &= (e^{k_1\mu_\xi + \frac{1}{2}k_1^2\sigma_\xi^2} - 1) + \frac{k_2 e^{-\kappa}}{1 - \frac{\phi_l^2}{2\kappa}(1-e^{-\kappa})\,k_2}. \end{aligned}$$

We need to compute the bound $w_t(\theta)$ for $v_t = 1 + I_t(e^{\tau \cdot X_{t+1}} v_{t+1})$ with $\tau = (\ln\delta, \alpha, 0, 0)$ and $\alpha = 1 - \frac{1}{\psi}$. According to Corollary 3, the bound is then given by

$$w_t(\theta) = \sum_{i=0}^{\infty} e^{A_i(\theta)\cdot X_t}$$

with $A_0(\theta) = (0,0,0,0)$ and $A_i(\theta) = h(\theta(A_{i-1}(\theta) + \tau))/\theta$. Denote by $A_{j,i}$ the jth element of A_i.

$$\begin{aligned} A_{0,i} &= A_{0,i-1} + \ln\delta + (A_{1,i-1} + \alpha)\mu + \frac{1}{2}\theta(A_{1,i-1}+\alpha)^2\sigma^2 - \frac{2\kappa\bar{\lambda}}{\theta\phi_l^2}\log\left(1 - \frac{\phi_l^2}{2\kappa}(1-e^{-\kappa})\theta A_{2,i-1}\right) \\ A_{1,i} &= 0 \\ A_{2,i} &= \frac{1}{\theta}(e^{\theta(A_{1,i-1}+\alpha)\mu_\xi + \frac{1}{2}\theta^2(A_{1,i-1}+\alpha)^2\sigma_\xi^2} - 1) + \frac{1}{\theta}\frac{\theta A_{2,i-1}e^{-\kappa}}{1 - \frac{\phi_l^2}{2\kappa}(1-e^{-\kappa})\,\theta A_{2,i-1}}. \end{aligned}$$

Simplifying yields

$$\begin{aligned} A_{0,i} &= A_{0,i-1} + \ln\delta + \alpha\mu + \frac{1}{2}\theta\alpha^2\sigma^2 - \frac{2\kappa\bar{\lambda}}{\theta\phi_l^2}\log\left(1 - \frac{\phi_l^2}{2\kappa}(1-e^{-\kappa})\theta A_{2,i-1}\right) \\ A_{2,i} &= \frac{1}{\theta}(e^{\alpha\theta\mu_\xi + \frac{1}{2}\theta^2\alpha^2\sigma_\xi^2} - 1) + \frac{A_{2,i-1}e^{-\kappa}}{1 - \frac{\phi_l^2}{2\kappa}(1-e^{-\kappa})\,\theta A_{2,i-1}}. \end{aligned}$$

The fixed-point equation based on the recursion for $A_{2,i}$ is a quadratic equation and so has (generically) two solutions. However, starting from $A_{2,0} = 0$, the recursion can reach only one of the two fixed points. Since the infinite recursion has a well-defined finite limit for $\phi_l \to 0$, we can identify the correct limit from the two solutions of the quadratic equation, since one of them does not remain finite for $\phi_l \to 0$. The limit is then

$$\lim_{i\to\infty} A_{2,i} = \frac{1 + K_0K_1 - e^{-\kappa} - \sqrt{(1+K_0K_1-e^{-\kappa})^2 - 4K_0K_1}}{2K_1},$$

where $K_0 = \frac{1}{\theta}(e^{\alpha\theta\mu_\xi + \frac{1}{2}\theta^2\alpha^2\sigma_\xi^2} - 1)$ and $K_1 = \frac{\phi_l^2}{2\kappa}(1 - e^{-\kappa})\,\theta$. So it must also hold that

$$\lim_{i\to\infty}(A_{2,i} - A_{2,i-1}) = 0.$$

We obtain

$$\begin{aligned}
\lim_{i\to\infty}(A_{0,i} - A_{0,i-1}) &= \ln\delta + \alpha\mu + \frac{1}{2}\theta\alpha^2\sigma^2 - \frac{2\kappa\bar{\lambda}}{\theta\phi_l^2}\log\left(1 - \frac{\phi_l^2}{2\kappa}(1-e^{-\kappa})\theta\lim_{i\to\infty}A_{2,i-1}\right) \\
&= \ln\delta + \alpha\mu + \frac{1}{2}\theta\alpha^2\sigma^2 \\
&- \frac{2\kappa\bar{\lambda}}{\theta\phi_l^2}\log\left(1 - \frac{1 + K_0K_1 - e^{-\kappa} - \sqrt{(1+K_0K_1-e^{-\kappa})^2 - 4K_0K_1}}{2}\right).
\end{aligned}$$

Substituting $\lim_{i\to\infty} A_{2,i}$ into the last expression leads to Proposition 3. □

D.3 Volatility of Volatility (Bollerslev, Tauchen, and Zhou 2009)

We derive the test function $\Omega(\theta)$ for the model with the volatility-of-volatility process of Bollerslev, Tauchen, and Zhou (2009); see Proposition 1.

Proof of Proposition 1. Set $X_t = (X_{0,t}, X_{1,t}, X_{2,t}, X_{3,t})'$ with $X_{0,t} = 1$, $X_{1,t} = \Delta c_t$, $X_{2,t} = \sigma_t^2$, $X_{3,t} = q_t$, and $k = (k_0, k_1, k_2, k_3)$. We obtain

$$\begin{aligned}
E_t e^{k\cdot X_{t+1}} &= E_t e^{k_0X_{0,t+1} + k_1X_{1,t+1} + k_2X_{2,t+1}} \\
&= e^{k_0 + k_1\mu + \frac{1}{2}k_1^2X_{2,t} + k_2\bar{\sigma}^2(1-\rho_\sigma) + k_2\rho_\sigma X_{2,t} + \frac{1}{2}k_2^2X_{3,t} + k_3\mu_q(1-\rho_q) + k_3\rho_qX_{3,t} + \frac{1}{2}.\phi_q^2k_3^2X_{3,t}}.
\end{aligned}$$

Using the notation $E_t(e^{k\cdot X_{t+1}}) = e^{h(k)\cdot X_t}$ with

$$h(k) = (k_0 + h_0(k_1,k_2,k_3), h_1(k_1,k_2,k_3), h_2(k_1,k_2,k_3), h_3(k_1,k_2,k_3))$$

leads to

$$\begin{aligned} k_0 + h_0(k_1, k_2, k_3) &= k_0 + k_1\mu + k_2\bar{\sigma}^2(1-\rho_\sigma) + k_3\mu_q(1-\rho_q) \\ h_1(k_1, k_2, k_3) &= 0 \\ h_2(k_1, k_2, k_3) &= \frac{1}{2}k_1^2 + k_2\rho_\sigma \\ h_3(k_1, k_2, k_3) &= \frac{1}{2}k_2^2 + k_3\rho_q + \frac{1}{2}\phi_q^2 k_3^2. \end{aligned}$$

We need to compute the bound $w_t(\theta)$ for $v_t = 1 + I_t(e^{\tau \cdot X_{t+1}} v_{t+1})$ with $\tau = (\ln\delta, \alpha, 0, 0)$ and $\alpha = 1 - \frac{1}{\psi}$. According to Corollary 3, the bound is then given by

$$w_t(\theta) = \sum_{i=0}^{\infty} e^{A_i(\theta)\cdot X_t}$$

with $A_0(\theta) = (0,0,0,0)$ and $A_i(\theta) = h(\theta(A_{i-1}(\theta) + \tau))/\theta$. Denote by $A_{j,i}$ the jth element of A_i. We then have

$$\begin{aligned} A_{0,i} &= A_{0,i-1} + \ln\delta + (A_{1,i-1} + \alpha)\mu + A_{2,i-1}\bar{\sigma}^2(1-\rho_\sigma) + A_{3,i-1}\mu_q(1-\rho_q) \\ A_{1,i} &= 0 \\ A_{2,i} &= \frac{1}{2}\theta(A_{1,i-1} + \alpha)^2 + \rho_\sigma A_{2,i-1} \\ A_{3,i} &= \frac{1}{2}\theta(A_{2,i-1})^2 + \rho_q A_{3,i-1} + \frac{1}{2}\theta(A_{3,i-1})^2\phi_q^2. \end{aligned}$$

Simplifying yields

$$\begin{aligned} A_{0,i} &= A_{0,i-1} + \ln\delta + \alpha\mu + A_{2,i-1}\bar{\sigma}^2(1-\rho_\sigma) + A_{3,i-1}\mu_q(1-\rho_q) \\ A_{2,i} &= \frac{1}{2}\theta\alpha^2 + \rho_\sigma A_{2,i-1} \\ A_{3,i} &= \frac{1}{2}\theta(A_{2,i-1})^2 + \rho_q A_{3,i-1} + \frac{1}{2}\theta(A_{3,i-1})^2\phi_q^2. \end{aligned}$$

Recall the sufficient condition (18) to show existence of the bound $w_t(\theta)$,

$$\lim_{i\to\infty} (A_{i+1}(\theta) - A_i(\theta)) \cdot X_t < 0.$$

Observe that $\lim_{i\to\infty} A_{2,i} = \frac{\theta\alpha^2}{2(1-\rho_\sigma)}$ and so it must also hold that $\lim_{i\to\infty}(A_{2,i} - A_{2,i-1}) = 0$. The fixed-point equation based on the recursion for $A_{3,i}$ is a quadratic equation and so has (generically) two solutions. However, starting from $(A_{2,0}, A_{3,0}) = (0,0)$, the recursion can reach only one of the two fixed points. Since the infinite recursion has a well-defined finite limit for $\phi_q \to 0$, we can identify the correct limit from the two solutions of the quadratic equation, since one of them does not remain finite for $\phi_q \to 0$. The limit is then

$$\lim_{i\to\infty} A_{3,i} = \frac{1-\rho_q - \sqrt{(1-\rho_q)^2 - \frac{\theta^4\alpha^4\phi_q^2}{4(1-\rho_\sigma)^2}}}{\theta\phi_q^2}.$$

Finally, we obtain

$$\lim_{i\to\infty}(A_{0,i} - A_{0,i-1}) = \ln\delta + \alpha\mu + \lim_{i\to\infty}\left(A_{2,i-1}\bar{\sigma}^2(1-\rho_\sigma) + A_{3,i-1}\mu_q(1-\rho_q)\right).$$

Substituting $\lim_{i\to\infty} A_{2,i}$ and $\lim_{i\to\infty} A_{3,i}$ into the last expression leads to Proposition 1. □

D.4 General Example for Affine Processes with Gaussian and Poisson Jump Shocks

In the following we show how to compute the test function $\Omega(\theta)$ for general processes that fall into the discrete-time affine class. For this purpose we consider the affine framework used by Eraker (2008) and Drechsler and Yaron (2011). The framework uses a VAR specification with both Gaussian shocks and Poisson jump shocks to model consumption dynamics (Eraker and Shaliastovich (2008) use a similar framework in continuous time):

$$X_{t+1} = \mu + FX_t + G_t z_{t+1} + J_{t+1}, \tag{D.1}$$

where $X_t \in \mathbb{R}^n$ denotes the vector of state processes, $z_{t+1} \sim N(0, I)$ is a vector of Gaussian shocks, and J_{t+1} is the vector of compound Poisson jumps.[1] Hence, the ith component of J_{t+1} is given by $J_{i,t+1} = \sum_{j=1}^{N_{i,t+1}} \xi_{i,j}$, where $N_{i,t+1}$ is a Poisson counting process and $\xi_{i,j}$ is the size of the jump that

[1]Note that the first element of X_t is 1 as defined above and the dynamics driving consumption growth follow thereafter.

occurs upon the jth increment of $N_{i,t+1}$. Following Drechsler and Yaron (2011), we let the $N_{i,t+1}$ be independent of each other conditional on time-t information and assume that the $\xi_{i,j}$ are i.i.d. The intensity process of the $N_{i,t+1}$ can be time varying and is given by the ith component of $\lambda_t = l_0 + l_1 X_t$ with $l_0 \in \mathbb{R}^n$ and $l_1 \in \mathbb{R}^{n\times n}$. To ensure that the dynamics fall in the affine class (Duffie, Pan, and Singleton 2000) we impose $G_t G_t' = h + \sum_i H_i X_{i,t}$ with $h \in \mathbb{R}^n$ and $H_i \in \mathbb{R}^{n\times n}$. For the application of Corollary 3 we need the MGF of X_t,

$$M_X(k) = E_t(e^{k\cdot X_{t+1}}) = e^{k\mu + kFX_t} E_t(e^{kG_t z_{t+1}}) E_t(e^{kJ_{t+1}}), \tag{D.2}$$

where we use the independence of z_{t+1} and J_{t+1}. Drechsler and Yaron (2011, Appendix A.1) show how to derive $M_X(k)$. In the following we provide a brief summary of their derivation.

As z_{t+1} are Gaussian, we have $E_t(e^{kG_t z_{t+1}}) = e^{\frac{1}{2}kG_tG_t'k'} = e^{\frac{1}{2}khk' + \frac{1}{2}\sum_i kH_ik'X_{i,t}}$. Furthermore, note that $E_t(e^{k_i J_{i,t+1}}) = e^{\lambda_{i,t}(M_{\xi_i}(k_i)-1)}$, where $M_{\xi_i}(k_i)$ is the MGF of the jump size ξ_i and $\Psi_{i,t}(k_i) = \lambda_{i,t}(M_{\xi_i}(k_i)-1)$ is the cumulant generating function of $J_{i,t+1}$. We denote by $M_\xi(k) = [M_{\xi_1}(k), M_{\xi_2}(k), \ldots]$ the vector of the MGFs of the jump sizes. We then obtain $E_t(e^{kJ_{t+1}}) = e^{(M_\xi(k)-1)\lambda_t} = e^{(M_\xi(k)-1)l_0 + (M_\xi(k)-1)l_1 X_t}$. Then $M_X(k)$ is given by

$$M_X(k) = e^{k\mu + kFX_t + \frac{1}{2}khk' + \frac{1}{2}\sum_i kH_ik'X_{i,t} + (M_\xi(k)-1)l_0 + (M_\xi(k)-1)l_1X_t}. \tag{D.3}$$

Using $M_X(k)$, we can now determine the vector function $h(k)$,

$$\begin{aligned} k_0 + h_0(k) &= k\mu + \frac{1}{2}khk' + (M_\xi(k)-1)l_0 \\ h_i(k) &= (kF)_i + \frac{1}{2}kH_ik' + ((M_\xi(k)-1)l_1)_i \quad \forall \quad i \geq 1, \end{aligned} \tag{D.4}$$

where $(y)_i$ denotes the ith element of the vector y.

Having the vector function $h(k)$, we can compute $A_i(\theta) = h(\theta(A_{i-1(\theta)} + \tau))/\theta$ with $A_0(\theta) = 0$. The calculations can be done in closed form for many model specifications that have appeared in the literature in recent years, as we demonstrate in Section 2. For complex models the derivations can become quite tedious. So instead—to prove existence in a specific parameterized application—the

expressions can simply be evaluated numerically. In Online Appendix C.2, we use this approach to show existence in the model of Drechsler and Yaron (2011) that includes three state processes, Gaussian shocks, and multiple Poisson jumps.

E Proofs for Section 3

The proof of Corollary 4 relies on the following standard inequalities. If $\theta < 1$, then for any random variable x,

$$\Big(E(x^\theta)\Big)^{1/\theta} \leq E(x),$$

while for $\theta > 1$,

$$\Big(E(x^\theta)\Big)^{1/\theta} \geq E(x).$$

(See parts (A) and (B) in the proof of Corollary 2 for a derivation.)

Proof of Corollary 4. We compare S_t from equation (24) with equation (25). Similar to the proof of Corollary 2, the proof splits into two cases. If $X_{t+1} \geq 0$, then $S_t X_{t+1} \geq 0$, $R_t X_{t+1} \geq 0$.

(A) Let $0 \neq \theta, \theta' \leq 1$. Then we obtain

$$\Big(E(e^{\alpha\theta\Delta c_{t+1}} v_{t+1}^\theta | s^t, \pi)\Big)^{\frac{1}{\theta}} \leq E(e^\alpha v_{t+1} | s^t, \pi).$$

Thus,

$$\begin{aligned}\left[E\left(\Big(E(e^{\alpha\theta\Delta c_{t+1}} v_{t+1}^\theta | s^t, \pi)\Big)^{\frac{\theta'}{\theta}} | s^t\right)\right]^{\frac{1}{\theta'}} &\leq \left[E\left(E(e^\alpha v_{t+1} | s^t, \pi)^{\theta'} | s^t\right)\right]^{\frac{1}{\theta'}} \\ &= \left[E\left(E(e^\alpha v_{t+1} | s^t, \pi) | s^t\right)\right].\end{aligned}$$

Hence, $S_t X_{t+1} \leq R_t X_{t+1}$.

(B) The proof is the same as above, but the inequalities reverse. □

Proof of Corollary 5. The proof just follows from applying the Minkowski inequality twice. For $\theta, \theta' < 1$,

$$\begin{aligned}
& H_t(x_t + y_t) \\
= \;& \delta \left[E \left(\left[E(e^{\alpha\theta\Delta c_{t+1}}(x_{t+1} + y_{t+1})^\theta | s^t, \pi) \right]^{\frac{\theta'}{\theta}} | s^t \right) \right]^{\frac{1}{\theta'}} \\
\leq \;& \delta \left[E \left(\left(\left[E(e^{\alpha\theta\Delta c_{t+1}} x_{t+1}^\theta | s^t, \pi) \right]^{\frac{1}{\theta}} + \left[E(e^{\alpha\theta\Delta c_{t+1}} y_{t+1}^\theta | s^t, \pi) \right]^{\frac{1}{\theta}} \right)^{\theta'} | s^t \right) \right]^{\frac{1}{\theta'}} \\
\leq \;& \delta \left[E \left(\left[E(e^{\alpha\theta\Delta c_{t+1}} x_{t+1}^\theta | s^t, \pi) \right]^{\frac{\theta'}{\theta}} | s^t \right) \right]^{\frac{1}{\theta'}} + \delta \left[E \left(\left[E(e^{\alpha\theta\Delta c_{t+1}} y_{t+1}^\theta | s^t, \pi) \right]^{\frac{\theta'}{\theta}} | s^t \right) \right]^{\frac{1}{\theta'}} \\
= \;& H_t x_{t+1} + H_t y_{t+1}.
\end{aligned}$$

Thus, H_t is subadditive, and we can apply Theorem 1. The argument for $\theta, \theta' > 1$ is identical, except the inequalities reverse. □

Proof of Proposition 4. By assumption, $G_T = 1$. Assume by induction

$$g_{t+1,T} = E_{t+1} \left(e^{\alpha\theta(c_T - c_{t+1})} \right)^{1/\theta}.$$

Then

$$\begin{aligned}
g_{t,T} &= H_t g_{t+1,T} \\
&= \delta E_t \left(e^{\alpha\theta\Delta c_{t+1}} g_{t+1,T}^\theta \right)^{1/\theta} \\
&= \delta E_t \left(e^{\alpha\theta\Delta c_{t+1}} \delta^{T-(t+1)} E_{t+1} \left(e^{\alpha\theta(c_T - c_{t+1})} \right) \right)^{1/\theta} \\
&= \delta^{T-t} E_t \left(e^{\alpha\theta(c_T - c_t)} \right)^{1/\theta},
\end{aligned}$$

completing the proof of the proposition. □

E.1 Markov-Switching with Learning and Smooth Ambiguity Aversion (Ju and Miao 2012)

Proof of Proposition 5. Let A be the matrix

$$A = \begin{pmatrix} \delta q_{11} e^{\alpha\mu_1 + 1/2\alpha^2\sigma^2} & \delta q_{12} e^{\alpha\mu_2 + 1/2\alpha^2\sigma^2} \\ \delta q_{21} e^{\alpha\mu_1 + 1/2\alpha^2\sigma^2} & \delta q_{22} e^{\alpha\mu_2 + 1/2\alpha^2\sigma^2} \end{pmatrix}.$$

Note that equation (28) is a recursive equation of the form

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

with the solution being the infinite sum

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + A \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \cdots.$$

If the absolute values of the eigenvalues of A are less than 1, then the infinite sum is equal to

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (1 - A)^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The entries of the matrix A are positive, so by the Perron–Frobenius theorem the largest eigenvalue in absolute value is positive (Horn and Johnson 2012). The eigenvalues of A are the roots of the characteristic polynomial

$$\lambda^2 - \operatorname{tr}(A)\lambda + \det(A).$$

For the given A,

$$\begin{aligned}
\operatorname{tr}(A) &= \delta e^{1/2\alpha^2\sigma^2}\left(e^{\alpha\mu_1}q_{11} + e^{\alpha\mu_2}q_{22}\right) \\
&= \delta e^{1/2\alpha^2\sigma^2} B_1 \\
\det(A) &= \delta^2 e^{\alpha(\mu_1+\mu_2)+\alpha^2\sigma^2}\left(q_{11}q_{22} - q_{12}q_{21}\right) \\
&= -\delta^2 e^{\alpha(\mu_1+\mu_2)+\alpha^2\sigma^2}\left(1 - q_{11} - q_{22}\right) \\
&= \delta^2 e^{\alpha^2\sigma^2} B_2,
\end{aligned}$$

where

$$\begin{aligned}
B_1 &= \left(e^{\alpha\mu_1}q_{11} + e^{\alpha\mu_2}q_{22}\right) \\
B_2 &= -e^{\alpha(\mu_1+\mu_2)}\left(1 - q_{11} - q_{22}\right).
\end{aligned}$$

By the quadratic formula, the roots of this equation are

$$\frac{\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}(A)^2 - 4\det(A)}}{2}.$$

The Perron–Frobenius theorem implies this equation has a real root, so both roots are real. Since the entries of A are positive, the trace is positive and the largest root,

$$\frac{\operatorname{tr}(A) + \sqrt{\operatorname{tr}(A)^2 - 4\det(A)}}{2},$$

is positive as well. For equation (28) to have a solution, this eigenvalue must be smaller than 1. If we factor out the common term of $\delta e^{1/2\alpha^2\sigma^2}$ and take the logarithm, the statement of the proposition follows. □

E.2 Random Walk with Learning (Collin-Dufresne, Johannes, and Lochstoer 2016)

Proof of Lemma 1. The inner expectation is of the form

$$\begin{aligned} E\left(e^{\alpha\theta\Delta c_{t+1}}x_{t+1}^{\theta}|\mu_t,\mu\right) &= E\left(e^{\theta k_0+\theta k_1\frac{1}{1+a_t}\mu_t+\theta\left(\alpha+k_1\frac{a_t}{1+a_t}\right)\Delta c_{t+1}}|\mu_t,\mu\right) \\ &= e^{\theta k_0+\theta k_1\frac{1}{1+a_t}\mu_t}E\left(e^{\theta\left(\alpha+k_1\frac{a_t}{1+a_t}\right)\Delta c_{t+1}}|\mu_t,\mu\right) \\ &= e^{\theta k_0+\theta k_1\frac{1}{1+a_t}\mu_t}e^{\theta\left(\alpha+k_1\frac{a_t}{1+a_t}\right)\mu+1/2\theta^2\left(\alpha+k_1\frac{a_t}{1+a_t}\right)^2\sigma^2}. \end{aligned}$$

Calculating the outer expectation,

$$\begin{aligned} H_t x_{t+1} &= \delta E\left(e^{\theta' k_0+\theta' k_1\frac{1}{1+a_t}\mu_t}e^{\theta'\left(\alpha+k_1\frac{a_t}{1+a_t}\right)\mu+1/2\theta\theta'\left(\alpha+k_1\frac{a_t}{1+a_t}\right)^2\sigma^2)}\right)^{\frac{1}{\theta'}} \\ &= \delta e^{k_0+k_1\frac{1}{1+a_t}\mu_t+1/2\theta\left(\alpha+k_1\frac{a_t}{1+a_t}\right)^2\sigma^2}E\left(e^{\theta'\left(\alpha+k_1\frac{a_t}{1+a_t}\right)\mu}\right)^{\frac{1}{\theta'}} \\ &= \delta e^{k_0+k_1\frac{1}{1+a_t}\mu_t+1/2\theta\left(\alpha+k_1\frac{a_t}{1+a_t}\right)^2\sigma^2}e^{\left(\alpha+k_1\frac{a_t}{1+a_t}\right)\mu_t+1/2\theta'\left(\alpha+k_1\frac{a_t}{1+a_t}\right)^2a_t\sigma^2} \\ &= \delta e^{k_0+(\alpha+k_1)\mu_t+1/2\left(\alpha+k_1\frac{a_t}{1+a_t}\right)^2(\theta+a_t\theta')\sigma^2}, \end{aligned}$$

we obtain the expression stated in the lemma. □

Proof of Proposition 6. Let $X_t=(X_{0,t},X_{1,t})'$, with $X_{0,t}=1$, $X_{1,t}=\mu_t$, and $k=(k_0,k_1)$. By Lemma 1, we obtain

$$H_t e^{k\cdot X_{t+1}}=e^{h(k)\cdot X_t},$$

where

$$h(k)=\left(k_0+h_0(k_1),h_1(k_1)\right).$$

The individual h_i are of the form

$$\begin{aligned} h_0(k_1) &= \ln\delta+1/2\left(\alpha+k_1\frac{a_t}{1+a_t}\right)^2(\theta+a_t\theta')\sigma^2 \\ h_1(k_1) &= \alpha+k_1. \end{aligned}$$

The Minkowski bound is then given by

$$\sum_{i=0}^{\infty} e^{A_i(\theta,\theta')\cdot X_t},$$

where $A_0(\theta,\theta') = (0,0)$, and $A_{i+1}(\theta,\theta') = h(A_i(\theta,\theta'))$. Denote by $A_{j,i}$ the jth element of A_i. We then have

$$\begin{aligned} A_{0,i} &= A_{0,i-1} + \ln\delta + 1/2\left(\alpha + A_{1,i}\frac{a_t}{1+a_t}\right)^2(\theta + a_t\theta')\sigma^2 \\ A_{1,i} &= \alpha + A_{1,i-1}. \end{aligned}$$

This implies that $A_{1,i} = i\alpha$ and

$$A_{0,i} = A_{0,i-1} + \ln\delta + 1/2\left(\alpha + i\alpha\frac{a_t}{1+a_t}\right)^2(\theta + a_t\theta')\sigma^2.$$

The Minkowski bound converges if

$$\limsup_{i\to\infty}(A_{i+1}(\theta,\theta') - A_i(\theta,\theta')) < 1,$$

but note that $A_{0,i+1} - A_{0,i} \to \infty$, except for the degenerate case, when $a_t = 0$. This case only occurs if $a_0 = 0$. Otherwise $a_t \neq 0$. □

References

Bansal, R. and A. Yaron (2004): "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *Journal of Finance*, 59, 1481–1509.

Bollerslev, T., G. Tauchen, and H. Zhou (2009): "Expected Stock Returns and Variance Risk Premia," *Review of Financial Studies*, 22, 4463–4492.

Bollerslev, T., L. Xu, and H. Zhou (2015): "Stock Return and Cash Flow Predictability: The Role of Volatility Risk," *Journal of Econometrics*, 187, 458–471.

Borovička, J. and J. Stachurski (2020): "Necessary and Sufficient Conditions for Existence and Uniqueness of Recursive Utilities," *The Journal of Finance*, 75, 1457–1493.

Collin-Dufresne, P., M. Johannes, and L. A. Lochstoer (2016): "Parameter Learning in General Equilibrium: The Asset Pricing Implications," *American Economic Review*, 106, 664–698.

Drechsler, I. and A. Yaron (2011): "What's Vol Got to Do with It," *Review of Financial Studies*, 24, 1–45.

Duffie, D., J. Pan, and K. J. Singleton (2000): "Transform Analysis and Asset Pricing for Affine Jump-diffusions," *Econometrica*, 68, 1343–1376.

Eraker, B. (2008): "Affine General Equilibrium Models," *Management Science*, 54, 2068–2080.

Eraker, B. and I. Shaliastovich (2008): "An Equilibrium Guide to Designing Affine Pricing Models," *Mathematical Finance*, 18, 519–543.

Horn, R. A. and C. R. Johnson (2012): *Matrix Analysis*, Cambridge University Press, 2nd ed.

Ju, N. and J. Miao (2012): "Ambiguity, Learning, and Asset Returns," *Econometrica*, 80, 559–591.

Pohl, W., K. Schmedders, and O. Wilms (2018): "Higher-Order Effects in Asset Pricing Models with Long-Run Risks," *Journal of Finance*, 73, 1061–1111.

Tauchen, G. (2011): "Stochastic Volatility in General Equilibrium," *Quarterly Journal of Finance*, 1, 707–731.

Wachter, J. A. (2013): "Can Time-Varying Risk of Rare Disasters Explain Aggregate Stock Market Volatility?" *Journal of Finance*, 68, 987–1035.