THE CONTINUUM COLLABORATION

THE ARCHITECTURE OF SPACE-TIME SEMINARS

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SEMINAR-V: GEOMETRY AND THE INITIAL VALUE PROBLEM

In the previous seminar, an interesting point was that of the idea of the predictability of M using the data on a "slice" Σ on M. Today, we will further develop this idea.

We will talk about the idea of hypersurfaces, and then we will get the idea of an initial data set.

HYPERSURFACES

- A hypersurface is the image of a three-manifold under an emedding.
- This is to say that the image of a three-manifold Ω with an embedding $\Phi: \Omega \to M$, or $\Sigma = \Phi(\Omega)$
- The nature of the homeomorphic character of Φ ensures that no two slices Σ intersect, a condition on our definition of every time-like curve being intersected only once.

HYPERSURFACES

- We want to lay a condition on Σ that some scalar field t would be constant.
- For some $p \in M, p \in \Sigma$, we wish to set $t(p) = 0$.

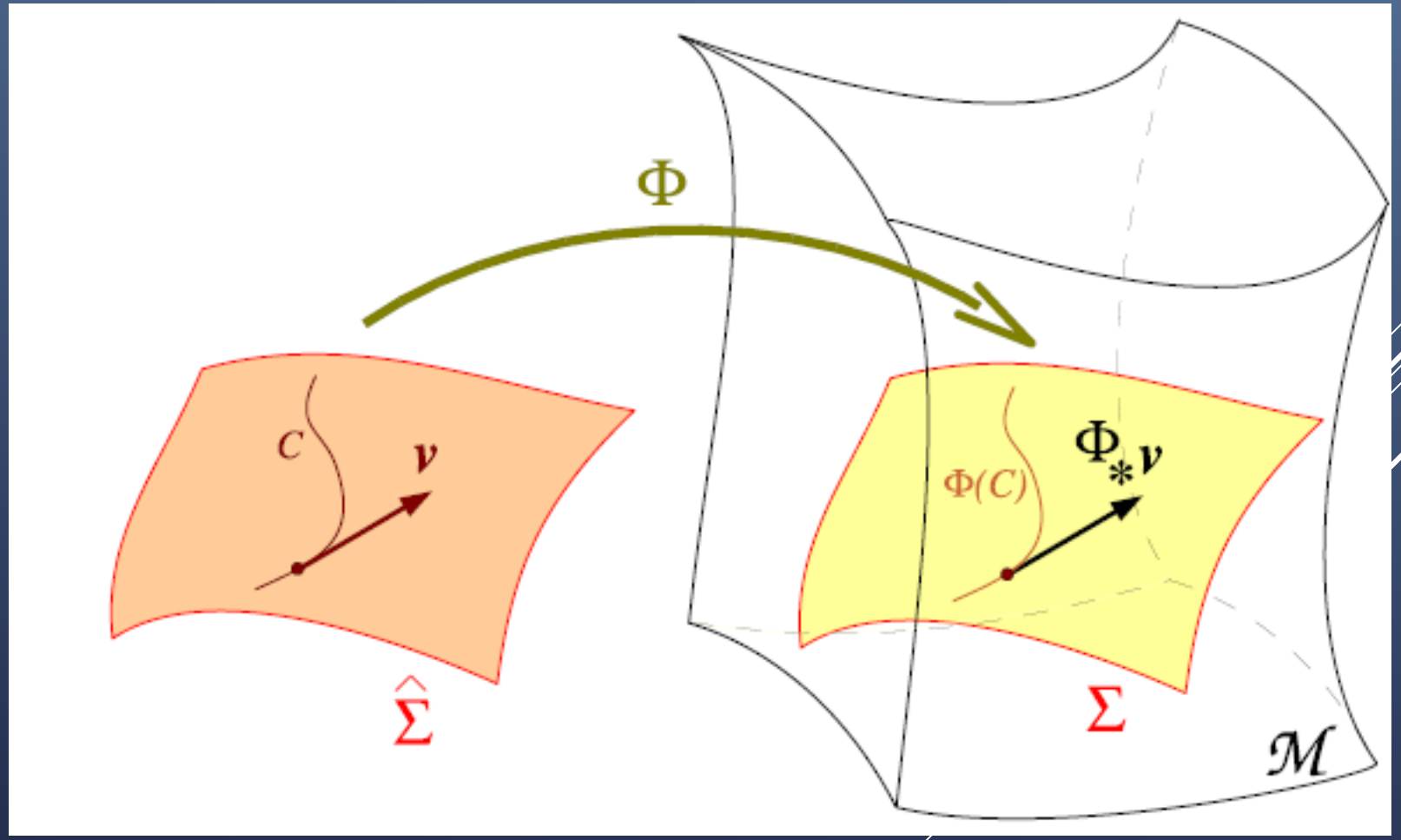
- We can define a coordinate system $x^{\partial} = (x, y, z)$ such that we have

$$\Phi: \Omega(x^{\partial}) \to M(x^{\mu})$$

- Where $x^{\partial} = (x, y, z)$ and $x^{\mu} = (0, x, y, z)$. This is our coordinate condition on Ω.

- An embedding of Ω (represented here as $\bar{\Sigma}$) in M:

- arXiv:0703035



HYPERSURFACES

- Since the mapping takes curves on Ω and maps them into M, vectors are also mapped. Thus we define the Push-forward mapping as the mapping between vectors in $T_p\Omega$ and T_pM.

$$\Phi_*: T_p\Omega \to T_pM$$

- Suppose we have a vector $V^\partial = (V^x, V^y, V^z)$. This can be related into x^μ in T_pM as

$$\Phi_*: V^\partial \to \Phi_*\left(V^\partial\right) = V^\mu \equiv (0, V^x, V^y, V^z)$$

- We define an induced metric (or the first fundamental form) as the pull-back of the metric:

$$\gamma = \Phi^* g$$

HYPERSURFACES

- The hypersurface where the form γ is induced can be distinguished into three classes:
- 1. If γ is positive-definite, then we say that the hypersurface is space-like,
- 2. If γ is Lorentzian, then we say that the hypersurface is time-like,
- 3. If γ is degenerate, then we say that the hypersurface is null.

TOWARDS THE "CAUCHY"

- A Cauchy surface is a space-like Σ such that every non-space-like curve (also called as causal curves) is intersected by Σ only once.
- A manifold M with CTCs does not contain Cauchy surfaces, as a hypersurface can then intersect two points along the curve twice.
- (M, g) that have a Cauchy surface are called globally hyperbolic Space-times. The topology of such a manifold would be $\Sigma \times \mathbb{R}$.

A NICE EXAMPLE OF AN INITIAL VALUE PROBLEM

- Consider (M, g) with $ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t)[d\sigma^2]$, where $d\sigma^2$ is a Riemannian metric of the spatial components of our model.
- We start by physical conditions on the model – the first condition is the homogeneous and isotropic nature of M, and the solution being of the form of dust, which greatly simplifies our model.

- Solving the field equations would give us three expressions, G_{00}, G_{0i} and G_{ij}.
- How would you solve for, say ρ?

- A nice way to solve the resulting field equations would be to take some time t, and choose a set of values for a and ρ so that the equation $3H^2 - \rho = 0$ would be solved, where $H = \frac{\dot{a}}{a}$.

- Assume that this holds for the other equations, and understand the evolution of the model.

- What about a more definite idea of initial data?

THE STATIC UNIVERSE PROBLEM

- The first model of the Universe was the Einstein static Universe, which had zero acceleration. The field equations were

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = kT_{\mu\nu}$$

- A problem is that Λ is "fine-tuned" – if the value of this term was even slightly changed, the model would no longer be static – it would have non-zero acceleration.

- This was the beginning of the initial data problem in GR – to understand the nature of the evolution of M based on an initial data set.

PROPERTIES OF THE INITIAL DATA

- The static Universe is an *unstable* model, where even minor perturbations in the initial data set would cause the nature of the evolution of the model to change drastically.

- There is also the idea of singularities, as seen in the previous seminar, which arise in certain models. In those models that are highly symmetric, the question of the nature of singularities is a very reasonable one.

GEOMETRIC IDEA OF INITIAL DATA

- The initial data problem can be written as a more formal question, namely, *how do we understand the evolution of M from the "initial data" on a hypersurface?*

- The initial value problem can be summarized as follows: consider a three-surface Σ, and an induced metric h_{ab}. We require to find such an "initial data set" that describe the evolution of M, along with a symmetric rank two tensor $K_{\mu\nu}$ (which we would consider the second fundamental form).

GEOMETRIC IDEA OF INITIAL DATA

- Suppose that we lay a physical condition on the Space-time, such as the vacuum constraint, $R_{\mu\nu} = 0$. The Choquet-Bruhat theorem states that

if an initial data set $(\Sigma, h_{ab}, K_{\mu\nu})$ satisfy the vacuum constraint $R_{\mu\nu} = 0$, with Σ a space-like hypersurface, h_{ab} an induced metric and $K_{\mu\nu}$ a symmetric 2 –tensor, then $K_{\mu\nu}$ is the second fundamental form satisfying the Gauss-Codazzi equations.

GEOMETRIC IDEA OF INITIAL DATA

- The Gauss and Codazzi equations are constraint equations relating the Riemann tensor and the nature of hypersurface.
- The second fundamental form satisfies this set of equations. We will talk about this set of equations soon.
- We will talk about some important remarks on the initial value problem in GR.

SOME REMARKS

- A part of understanding Choquet-Bruhat's result is to consider the idea of harmonic slicing. The de Donder harmonic condition is that considering each coordinate in x^μ as a scalar, we ask for $\square_g x^\mu = 0$. Harmonic slicing is to set some coordinate (in our case x^0) to satisfy

$$\square_g x^0 = 0$$

Which leaves x^∂ for our hypersurface. This will be discussed in later seminars.

- The initial value problem also has implications on the geometric aspects of M.

- The following is a set of steps on understanding the initial value problem on M.

SOME REMARKS

- 1. Choose a hypersurface Σ,
- 2. h and K are the induced metric and the two-tensor,
- 3. Choose coordinates on Σ which would be the local coordinates on $\Sigma \times \mathbb{R}$,
- 4. Solve the constraints for a "slice" of time for the initial data,
- 5. This would be the form of initial data (Σ, h, K).

- A set of lecture notes on these seminars will be provided soon.

THANK YOU

- An extension of this seminar series is planned from 19th of June. We will inform you soon.

- References:

- [1] The Large scale structure of Space-time, Hawking and Ellis
- [2] The Cauchy problem in General Relativity, Hans Ringstrom
- [3] Constraint equations and the well-posedness of Einstein equations, Vladmir Sicca