Exploiting rivals' strengths

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Online Appendix

In this appendix, we provide a characterization of the solution to the Ramsey-Boiteux problem with non-linear prices and adverse selection presented in Section 4 of the paper.

As noted in the paper, the Ramsey-Boiteux problem can be formulated as a problem of optimal control:

$$\begin{aligned} &\max_{q_1(\theta)\geq 0, q_2(\theta)\geq 0} \int_{\theta_{\min}}^{\theta_{\max}} \left[u\left(q_1(\theta), q_2(\theta), \theta\right) - c_2 q_2(\theta) - v(\theta)\right] f(\theta) d\theta \\ &\text{s.t.} \quad \tfrac{dv(\theta)}{d\theta} = u_\theta(q_1(\theta), q_2(\theta), \theta) \\ &v(\theta) \geq v^R(\theta), \end{aligned} \tag{A.1}$$

where $q_1(\theta)$ and $q_2(\theta)$ are the control variables and $v(\theta)$, which denotes the surplus that the buyer obtains in equilibrium, is the state variable. The type-dependent reservation utility $v^R(\theta)$ is

$$v^R(\theta) = u(0, q^E(\theta), \theta) - c_2 q^E(\theta)$$

where

$$q_2^E(\theta) = \arg\max_{q_2\geq 0}[u(0, q_2, \theta) - c_2 q_2].$$

By the sorting condition, $q^E(\theta)$ is increasing in θ. Therefore, there exists $\theta^E \geq \theta_{\min}$ such that $q_2^E(\theta^E) = 0$ and $q_2^E(\theta) > 0$ for any $\theta > \theta^E$.

Let us consider first the relaxed problem in which the participation constraint is simply $v^R(\theta) = 0$. In this case, the virtual surplus function is:

$$s(q_1, q_2, \theta) = u(q_1, q_2, \theta) - c_2 q_2 - \frac{1 - F(\theta)}{f(\theta)} u_\theta(q_1, q_2, \theta).$$

For simplicity, we make the following standard regularity conditions:

Condition 1 *The virtual surplus function $s(q_1, q_2, \theta)$ is strictly concave in q_1 and q_2.*

Condition 2 *The inverse hazard rate $\frac{1-F(\theta)}{f(\theta)}$ is non-increasing in θ.*

Condition 3 $u_{\theta\theta q_i}(q_1, q_2, \theta) \geq 0$.

Under these conditions, the solution to the relaxed problem $\{\tilde{q}_1(\theta), \tilde{q}_2(\theta)\}$ can be obtained by pointwise maximization of $s(q_1, q_2, \theta)$ and is monotonically increasing in θ. Necessary and sufficient conditions for a maximum then are:

$$\begin{array}{ll} u_{q_1} - \frac{1-F(\theta)}{f(\theta)} u_{\theta q_1} \leq 0 & \left[u_{q_1} - \frac{1-F(\theta)}{f(\theta)} u_{\theta q_1}\right] q_1 = 0 \\ u_{q_2} - c_2 - \frac{1-F(\theta)}{f(\theta)} u_{\theta q_2} \leq 0 & \left[u_{q_2} - c_2 - \frac{1-F(\theta)}{f(\theta)} u_{\theta q_2}\right] q_2 = 0. \end{array} \tag{A.2}$$

We must distinguish between two cases, depending on whether $q_2 = 0$ or $q_2 > 0$. Denote the solution to the relaxed problem when $q_2 = 0$ by $q_1^M(\theta) = \arg\max_{q_1 \geq 0} s(q_1, 0, \theta)$. To avoid proliferation of cases, let us assume that $\theta_{\min}$ is sufficiently low that:

$$q_1^M(\theta_{\min}) = 0.$$

By monotonicity of $q_1^M(\theta)$, there exists $\theta^M > \theta_{\min}$ such that $q_1^M(\theta^M) = 0$ and $q_1^M(\theta) > 0$ for any $\theta > \theta^M$.

To simplify the exposition, we make one more regularity assumption:

Condition 4 *The curves $q_1^M(\theta)$ and $q_2^E(\theta)$ intersect at most once.*

It is then easy to show that if the curves intersect at all, then $q_2^E(\theta)$ cuts $q_1^M(\theta)$ from above.[1] Denote by θ^{EM} the intersection point.

Next, denote by $\{q_1^*(\theta), q_2^*(\theta)\} = \arg\max_{q_1>0, q_2>0} s(q_1, q_2, \theta)$ the solution to (A.2) when $q_2 > 0$, and let $\hat{\theta}$ be implicitly defined by the condition $q_1^M(\theta) = q_1^*(\theta)$.

The solution $\{\tilde{q}_1(\theta), \tilde{q}_2(\theta)\}$ to the relaxed problem then is:

$$\tilde{q}_1(\theta) = \begin{cases} 0 & \text{for } \theta \leq \theta^M \\ q_1^M(\theta) & \text{for } \theta^M \leq \theta \leq \hat{\theta} \\ q_1^*(\theta) & \text{for } \hat{\theta} \leq \theta \leq \theta_{\max} \end{cases} \qquad \tilde{q}_2(\theta) = \begin{cases} 0 & \text{for } \theta \leq \hat{\theta} \\ q_2^*(\theta) & \text{for } \hat{\theta} \leq \theta \leq \theta_{\max}. \end{cases}$$

If c_2 is sufficiently high, the solution to the relaxed problem satisfies the type-dependent participation constraint, $v(\theta) \geq v^R(\theta)$ for all types and thus $\{\tilde{q}_1(\theta), \tilde{q}_2(\theta)\}$ is the solution to the original problem (A1). In particular, this occurs when $c_2 \geq c^M$, where c^M is implicitly defined by the condition:

$$\theta^E(c_2) = \theta^M.$$

In words, at $c_2 = c^M$ the curves $q_1^M(\theta)$ and $q_1^E(\theta)$ intersect at $\theta = \theta^M$, where they both vanish. This implies that $q_1^M(\theta) > q_1^E(\theta)$ whenever the two quantities are positive.

To show that when $c_2 \geq c^M$ the participation constraint is automatically satisfied, we must consider three cases. First, when $\theta \leq \theta^M$, the result is obvious as $v(\theta) = 0 = v^R(\theta)$. Second, for $\theta^M \leq \theta \leq \hat{\theta}$ we have:

$$v(\theta) = \int_{\theta_{\min}}^{\theta} u_\theta(q_1^M(s), 0, s) ds \geq v^R(\theta) = \int_{\theta_{\min}}^{\theta} u_\theta(0, q^E(s), s) ds,$$

[1] This follows from the fact that $q_1^M(\theta)$ satisfies the no-distortion-at-the-top property, which implies that $q_1^M(\theta_{\max}) > q_2^E(\theta_{\max})$.

where the inequality follows from $u_{\theta q_i} \geq 0$ and $q_1^M(\theta^M) \geq q^E(\theta^M)$. Finally, for $\theta \geq \hat{\theta}$ we have:

$$v(\theta) = v(\hat{\theta}) + \int_{\hat{\theta}}^{\theta} u_\theta(q_1^*(s), q_2^*(s), s)ds \geq v^R(\theta) = v^R(\hat{\theta}) + \int_{\hat{\theta}}^{\theta} u_\theta(0, q^E(s), s)ds.$$

The inequality follows from the same arguments as above, observing that $q_1^*(s) \geq q_1^M(\theta) \geq q^E(\theta)$ and $q_2^*(s) \geq 0$.

Consider now the case $c_2 < c^M$. For low-demand states, the participation constraint now binds. To characterize the solution, we apply Proposition 5.5 in Jullien (2000). To this end, we must first show that program (A.1) satisfies the conditions of Full Participation, Homogeneity, Potential Separation and Weak Convexity.

Weak Convexity requires that $v(\theta)$ in the relaxed problem is more strongly convex than $v^R(\theta)$. But this is what we have just shown above. To show that Potential Separation holds, define

$$(\ell_1^E(g,\theta), \ell_2^E(g,\theta)) = \arg\max_{q_1,q_2}\{u(q_1,q_2,\theta) - \frac{g - F(\theta)}{f(\theta)} u_\theta(q_1,q_2,\theta)\}$$

for $g \in [0,1]$, Potential Separation requires that $\ell_i^E(g,\theta)$ is non-decreasing in θ, which is true under the same regularity conditions that guarantee monotonicity of quantities. Homogeneity is obvious, as it requires that $v^R(\theta)$ can be implemented by continuous and non decreasing quantity: by construction, this quantity is $q_2^E(\theta)$. Finally, Full Participation requires that aggregate quantities are strictly positive for types $\theta > \theta^E$, which must be the case since for these types $v^R(\theta) > 0$.

Since the assumptions of Proposition 5.5 in Jullien (2000) hold true in our problem, we can conclude that the solution partitions the set of types into three sets:

- buyers who are excluded;
- buyers who obtain exactly their reservation utility $v^R(\theta)$;
- buyers whose net payoff is strictly greater than $v^R(\theta)$.

The buyers who are excluded are those for which $\theta \leq \theta^E$. For the buyers who obtain exactly their reservation utilities, equilibrium quantities must solve the following constrained maximization of the virtual surplus,

$$\max_{q_1(\theta),q_2(\theta)} s(q_1(\theta), q_2(\theta), \theta)$$

$$\text{s.t. } u_\theta(q_1(\theta), q_2(\theta), \theta) = u_\theta(0, q_2^E(\theta), \theta).$$

The constraint on the rate of change of the buyer's surplus with respect to θ $u_\theta(q_1(\theta), q_2(\theta), \theta) = u_\theta(0, q_2^E(\theta), \theta)$ is equivalent to the original constraint in terms of the surplus level, $u(q_1(\theta), q_2(\theta), \theta) = u(0, q_2^E(\theta), \theta)$.

The solution to this constrained maximization problem may be interior, in which case it is denoted by $\bar{q}_1(\theta) > 0$ and $\bar{q}_2(\theta) > 0$, or it may be a corner solution with $q_2 = 0$ and $q_1 = q_2^E$. Let $\bar{\theta}$ denote the threshold above which the solution is interior. Like $\hat{\theta}$, it depends on c_2. Since $\bar{\theta} = 0$ and $\hat{\theta} > 0$ for $c_2 = 0$ and $\bar{\theta} \geq \hat{\theta}$ for c_2 sufficiently high, there exists a $\hat{c}$ such that $\bar{\theta} = \hat{\theta}$.

When $c_2 \leq \hat{c}$, the solution to the Ramsey-Boiteux problem therefore is:

$$q_1^{RB}(\theta) = \begin{cases} 0 & \text{for } \theta \leq \theta^E \\ q_2^E(\theta) & \text{for } \theta^E \leq \theta \leq \bar{\theta} \\ \bar{q}_1(\theta) & \text{for } \bar{\theta} \leq \theta \leq \hat{\theta} \\ q_1^*(\theta) & \text{for } \theta > \hat{\theta} \end{cases} \qquad q_2^{RB}(\theta) = \begin{cases} 0 & \text{for } \theta \leq \bar{\theta} \\ \bar{q}_2(\theta) & \text{for } \bar{\theta} \leq \theta \leq \hat{\theta} \\ q_2^*(\theta) & \text{for } \theta > \hat{\theta} \end{cases}$$

When instead $c^M > c_2 > \hat{c}$, the solution to the Ramsey-Boiteux problem is:

$$q_1(\theta) = \begin{cases} 0 & \text{for } \theta \leq \theta^E \\ q^E(\theta) & \text{for } \theta^E \leq \theta \leq \theta^{EM} \\ q_1^M(\theta) & \text{for } \theta^{EM} \leq \theta \leq \hat{\theta} \\ q_1^*(\theta) & \text{for } \theta > \hat{\theta} \end{cases} \qquad q_2(\theta) = \begin{cases} 0 & \text{for } \theta \leq \hat{\theta} \\ q_2^*(\theta) & \text{for } \theta > \hat{\theta}. \end{cases}$$

To illustrate, consider the quadratic-uniform case where the payoff function is quadratic:

$$u(q_1, q_2, \theta) = \theta(q_1 + q_2) - \frac{1}{2}\left(q_1^2 + q_2^2\right) - \gamma q_1 q_2,$$

and θ is uniformly distributed in the interval $[0, 1]$. The parameter $\gamma \in [0, 1]$ represents the degree of product substitutability. In this above, we have,

$$\begin{aligned} q_2^E(\theta) &= \theta - c_2 \\ q_1^M(\theta) &= 2\theta - 1 \\ q_1^*(\theta) &= (2\theta - 1)\frac{1-\gamma}{1-\gamma^2} + c_2\frac{\gamma}{1-\gamma^2} \\ q_2^*(\theta) &= (2\theta - 1)\frac{1-\gamma}{1-\gamma^2} - c_2\frac{1}{1-\gamma^2} \\ \bar{q}_1(\theta) &= \frac{1}{2}\left(\theta + c_2\frac{\gamma}{1-\gamma}\right) \\ \bar{q}_2(\theta) &= \theta - c_2 - \frac{1}{2}\left(\theta + c_2\frac{\gamma}{1-\gamma}\right). \end{aligned}$$

The critical thresholds for θ are:

$$\theta^M = 1/2,$$

$$\theta^E = c_2$$
$$\theta^{EM} = 1 - c_2,$$
$$\hat{\theta} = \frac{1 + c_2 - \gamma}{2(1-\gamma)},$$
$$\bar{\theta} = c_2 \frac{2-\gamma}{1-\gamma},$$

and the critical thresholds for c_2 are

$$c^M = \frac{1}{2}$$

and

$$\bar{c} = \frac{1-\gamma}{3-2\gamma}.$$

References

[1] Jullien, Bruno. 2000. "Participation Constraints in Adverse Selection Models." *Journal of Economic Theory* 93 (1): 1–47.