Homogenization and PCE method : Application in tokamak plasma (*)

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Abstract

Homogenization methods for dielectric mixtures have existed for over two decades, but their limitations regarding the wavelength of incoming beam did not allow them to be used extensively in tokamak plasmas. We present a new method which does not have the same limitations, with application to a dielectric plasma mixture with embedded filamentary structures of different density than the background plasma.

The Polynomial chaos expansion (**PCE**) method determines, in a computationally efficient way, the evolution of uncertainty in a dynamical system due to the probabilistic uncertainty in the system parameters.

In this work, by use of the PCE in conjunction with the homogenization method we calculate the statistical properties of the output (reflection-transmission) of a slab-scattering system for uncertain parameters regarding tokamak plasma and blobs that approximate the plasma-blob dielectric mixture. This scattering configuration models the propagation of radio frequency (RF) waves through turbulent tokamak plasma, with significant applications in RF heating, current drive and diagnostics.

Homogenization Method

The method of homogenization addresses the subject of composite dielectric media, producing an equivalent homogeneous medium which can be studied in place of the real medium, with respect to its electromagnetic properties. Different formalisms have been developed and used in many cases [1],[2]. The thought of using existing formalisms for plasma physics applications comes naturally. An area of special interest is the SOL region of tokamak plasmas, where filamentary structures are formed, due to turbulent flows. These formations are called *blobs*.

However, with respect to the above problem, existing methods fall short because of their limitations regarding the ratio of incoming beam wavelength to blob size. A new method has been recently developed [3] which utilizes Fourier transformed quantities of the electromagnetic fields, transferring approximations to another part of the problem (namely, the numerical integration part) rather than the start, where Rayleigh approximations were used in previous applications.

The Polynomial Chaos Expansion Method

Real world systems in science and engineering often depend on uncertain inputs. Uncertainty Quantification (UQ) quantifies the impact of the system's input to the system's output quantities of interest (QoI). The Polynomial Chaos Expansion (PCE) method is among the most popular probabilistic UQ methods.

In PCE it is assumed that the system's input uncertainty is described by a d-dimensional random vector $\Xi = (\Xi_1, \Xi_2, \ldots, \Xi_d)$ with Ξ_i, $i = 1, \ldots, d$ independent identically distributed random variables (i.i.d) and realizations $\xi = (\xi_1, \xi_2, \ldots, \xi_d)$. Then the joint probability density function (pdf) is $f(\xi) = \prod_{k=1}^{d} f(\xi_k)$, where $f(\xi_k)$ is the pdf of Ξ_k. If the QoI, $u(\Xi)$ is a scalar with finite variance, $u(\Xi)$ can be expanded (PCE method) as:

$$u(\Xi) = \sum_{j=1}^{\infty} c_j \psi_j(\Xi), \tag{1}$$

where $\psi_j(\Xi)$ are multivariate polynomials that are orthogonal to $f(\xi)$, and c_j are the PCE deterministic coefficients that need to be estimated.

In reality the summation in eq. 1 is truncated to the first P terms:

$$u(\Xi) = \sum_{j=1}^{P} c_j \psi_j(\Xi) + \varepsilon(\Xi), \tag{2}$$

where $\varepsilon(\Xi)$ is the truncation error of the PCE. The construction of the multi-dimensional polynomial basis function $\psi_j(\Xi)$ is based on the Askey ([4]) scheme which is suitable for general non-Gaussian random inputs. In particular if $\psi_{j_k}(\Xi_k)$ specify a complete set of univariate polynomials of order j_k, where $j_k \in \mathbb{N} \cup \{0\}$ that are orthonormal with respect to the pdf $f(\xi_k)$, the multi-dimensional basis functions are:

$$\psi_j(\Xi) = \prod_{k=1}^{d} \psi_{j_k}(\Xi_k), \tag{3}$$

where $j = (j_1, \ldots, j_d)$ specifies the order of $\psi_j(\Xi)$. If the maximum total order of $\psi_j(\Xi)$ is p_{max}, ($\sum_{k=1}^{d} j_k \leq p_{max}$), the number P of basis functions in the truncated PCE expansion, eq. 2, is:

$$P = \frac{(p_{max} + d)!}{d! p_{max}!}. \tag{4}$$

If u has finite variance, the PCE in eq. 2 converges in the mean-square sense as p_{max} or P tend to infinity. In addition if the univariate polynomial basis is chosen according to the Askey scheme, the error term in eq. 2 tends to zero exponentially fast. The PCE coefficients c_j are computed by projecting the QoI into the corresponding PCE basis:

$$c_j = E\left[u\psi_j\right]. \tag{5}$$

In reality eq. 5 is inefficient to apply, and other methods have been developed. These fall into two categories, intrusive and non-intrusive methods. In intrusive methods the deterministic solvers that relate the system output and input is modified, which can be problematic. In non-intrusive methods the deterministic solvers are considered as a black box that provide a QoI sampling from a particular system input sampling. In this work the PCE coefficients are determined using a non-intrusive method based on the Least Squares (LS) approximation. In the PCE-LS method we sample the d-dimensional input random vectors $\{\xi_i\}_{i=1}^N$ according to the $f(\xi)$ pdf (Monte Carlo sampling), and then by using the deterministic solvers we obtain the QoI $\{u(\xi_i)\}_{i=1}^N$. Then eq. 2 can be written in the form:

$$u = \Psi c + \varepsilon, \tag{6}$$

where $u = (u(\xi_1), u(\xi_2), \ldots, u(\xi_N))^T$, $\Psi_{i,j} = \psi_j(\xi_i) \in \mathbb{R}^{N \times P}$, the error vector $\varepsilon \in \mathbb{R}^N$ and $c = (c_1, c_3, \ldots, c_P)^T$. It is oberved that eq. 6 is a regression task and the PCE coefficient vector can be estimated in the LS sense according to:

$$c = \left(\Psi^T \Psi\right)^{-1} \Psi^T u. \tag{7}$$

The LS solution 7 is stable as long as $N > P$. As a rule of thump, for PCE applications, $N = 4P$ is a good choice. From the PCE coefficients, c, the mean value and variance of the QoI are directly calculated [4].

Results

The scattering of (RF) plane waves at 170Ghz, by three anisotropic dielectric layers is considered. The first and third layers are a plasma semi-infinite layer described by a relative permittivity tensor $\varepsilon_p = [4.83\ i0.86\ 0; -i0.86\ 4.69\ 0; 0\ 0\ 5.76]$ and density $n_p = 5 \cdot 10^{19} m^{-3}$. The middle layer of thickness $d = 5cm$, consists of a mixture of plasma and blobs with relative permittivity tensors ε_p and $\varepsilon_b = [10.66\ i11.52\ 0; -i11.52\ 10.66\ 0; 0\ 0\ 0.72]$ respectively and $n_b = 10^{20} m^{-3}$. Using the homogenization theory this mixed layer is considered equivalent to a layer of anisotropic permittivity $\varepsilon_{eqv.}$. The homogenization method uses as inputs the percentage of blob in the middle layer p_b and the major R_b and minor r_b axes of the blob, assuming it is of elliptical shape and provides $\varepsilon_{eqv.}$. The system to be analyzed by the PCE method consists of the homogenization method code [3] in series with the multilayer anisotropic scattering solver [5]-[6]. The system's input is the three dimensional random vector $\xi = (p_b, R_b, r_b)$ and its QoI is the reflection R (or transmission T) of the scattering solver. The QoI statistics are calculated by the PCE method for various incident angles of the RF wave. The input parameters are uniformly

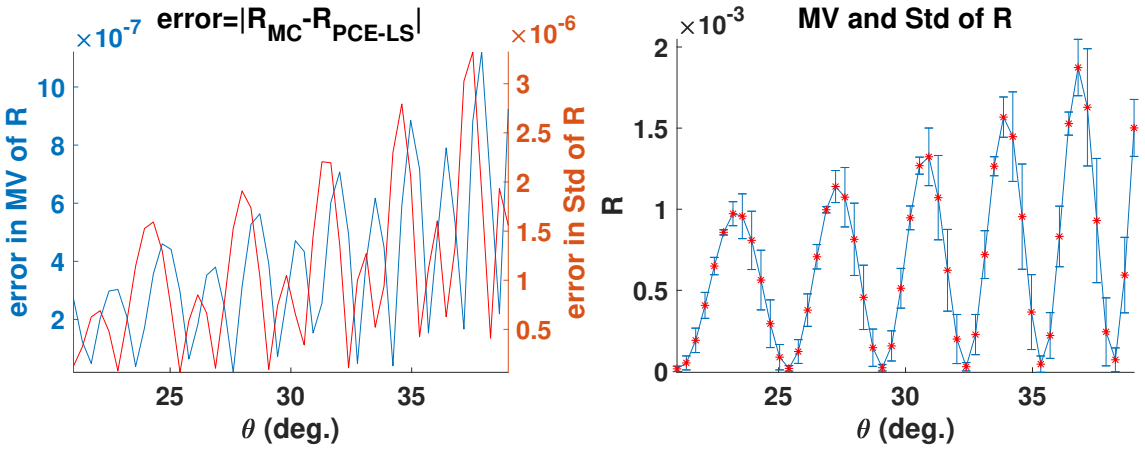

Figure 1: Mean value (a) and standard deviation (b) of reflection (R) using the Monte Carlo (MC) and Polynomial Chaos Expansion-Least Squares (PCE-LS) method as a function of the plane wave angle of incidence (θ)

distributed (5% around $\xi = (0.2, 0.5, 0.5)$) and the univariate basis are the Legendre polynomials (so as to achieve exponential convergence [4] in eq. 2). In Fig. 1, the QoI (R) statistics are shown as a function of θ. The MC method directly calculates R statistics as reference. It is seen that PCE-LS and MC results are in excellent agreement, even though PCE-LS uses only 140 calls to the scattering solver.

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