ON THE GEOMETRIC FEATURES OF THE CAUSAL DEFINITIONS OF SPACE-TIME

VAIBHAV R. KALVAKOTA[1]

INTRODUCTION:
General Relativity puts forward the definitions of Causal Structures that are much needed to understand the entire structure of Space-time. These definitions arise due to the *Pseudo-Riemannian* nature of Space-time, and these are very important to write down a clear set of definitions that govern the Causal behaviour of Space-time. We first discuss the relation between two events – a set of Causal definitions that give us an idea of the relation between two intervals that are elements in the Pseudo-Riemannian manifold, labelled as *events* (all the points $p \in M$ are defined as events). We will discuss extensively of the geometric nature of thee definitions, which describe the nature of points on M . We talk of the light-cone structures, which we will use to understand and depict these Causal structures. We discuss in a geometric detail these definitions, after which we discuss about energy conditions. From there, we model Singularities, and how they can be predicted using the satisfaction of certain conditions by the Space-time. We then discuss the Raychaudari theorem, given by sir Amal Raychaudari, which was the first Singularity theorem that identified the modelling of Singularities. We then discuss the Pattern, Penrose and the Hawking Singularity theorem, and their structure, or the basis on which they are defined. We will then talk briefly of the Cauchy initial data problem, and we will talk of the implications of it. We will discuss the working of an initial data set, and then discuss how they can be mathematically understood. We will focus on an analysis of the Cauchy problem, and we will look through the various points that describe an initial data set on a manifold.

PACS numbers: 04.20.Gz, 04.20.Dw, 98.80.Jk

MSC classification: 83C05, 83C75

Key words: Causal Structure, Singularity theorems, Cauchy initial value problem

1 CAUSAL DEFINITIONS

If we consider a set S^n in n-dimensions that describes a Vector space, and let $v \in S^n$, then we will define the following definition that will define the nature of v [1][2][4]:

Definition 1.1: *The vector v is defined to be light-like, space-like, or time-like if it respectively satisfies $g(v,\, v) = 0$, $g(v,v) < 0$ or $g(v,v) > 0$. The vector is otherwise said to be "Causal" if $v \neq 0$, and it is either time-like or light-like.*

We assume that the space S^n is *Minkowskian*, or that the metric has a signature $(+ - - -)$. In general, if we consider the structure of the Light cone diagram, it would be a collection of bounded time-like, light-like and space-like vectors, with light-like vectors forming a surface that separates the areas of the time-like and space-like vectors, and is inclined at an angle of 45° with the spatial and time axes.

The mathematical structure of Causality can be based on the nature of the vector. Let the vector v separate two events p and q . Then, we can define a *horismos* relation as the following [2]:

Definition 1.2: *A horismos relation exists between two events and q if they can be joined via a future light-like curve. This is depicted as $p \to q$. Further, we can define Chronological precedence and Causal precedence. Chronological precedence is when p and q can be joined via a Time-like future curve, depicted as $p \ll q$. Causal precedence is when the can be joined via a Causal future directed curve.*

In general, if a set $\sum$ exists on M such that two events p, q are related via a time-like curve, then we say that p and q are chronologically related, and $\Sigma(p,q)$ is a set of causal curves that are future directed (in another definition, it is the collection of all curves not space-like from p to q), with (past) end at p and (future) end at q . If Σ(p, q) is not empty, then there is said to be a causal

[1]ORCID: 0000-0002-7537-351X
Email: Vkalvakota@vikasconcept.com, Vaibhavkalvakota@gmail.com

[2]The document by Alfonso Parrado and Jose Senovilla Causal structures and causal boundaries gives a very analytic and intuitive introduction to the causal definitions discussed here.

relation between p and q.

In terms of a parameterization, if we define a $I \subseteq R$ and an everywhere differentiable curve γ mapping from I to M , then the respective tangent vectors at each point $p \in \gamma$ describe whether the curve γ is time/space/light like, if the corresponding tangent vector is time/space/light like. The same analogy applies if the curve γ is causal – the corresponding tangent vector should be future or past-directed. We will define $I^{\pm}$ and $J^{\pm}$ for which:

I^{+}, I^{-} *depicts the chronological future and past respectively, and*

J^{+}, J^{-} *depicts the causal future and past respectively.*

In a mathematical presentation, we can define

$$I^{+}(p) = (x \in M : p \ll x)$$

$$J^{+}(p) = (x \in M : p < x)$$

The *horismos* analogue of these is simply defined as

$$E^{+}(p) = J^{+}(p) - I^{+}(p)$$

A *boundary* ∂ is defined to be such that it is *achronal*, if it is defined on a set S which is such that no two points $p, q \in S$ can be joined via a future-directed time-like curve. A boundary ∂S is then said to be *achronal*. In general, if a *Hypersurface* is *achronal*, then it refers to that no two points p, q $\in H$ can be joined via a time-like curve. A space-like *Hypersurface* is one whose normal vectors are entirely time-like.

The *Causal axiom* states that no points $p, q\,(p \neq q)$ can follow $p < q < p$. The *Causal set* of a Lorentzian manifold now can be written as the following definition:

Definition 1.3: *A Causal set R is defined as a set the obeys the Causal axiom, with a defined relation that is such, that no two elements p, $q \in R$ are such that $p < q < p$ with neither p, q equal to each other.*

If we were to construct two events p, q $\in R$ such that $I^{+}(q) \subset I^{+}(r)$ for $r \in I^{-}(p)$, and $I^{+}(p) \subset I^{+}(r)$ if $r \in I^{-}(q)$, then we state that $x = y$. A *Causal space* is a collection of such R and causal operators (such as $\rightarrow, <$ etc.).

An overview of some important features of the Causal structure of a Lorentzian manifold M can be defined by considering the above mapped definitions, as:

Definition 1.4: *If we consider a Lorentzian manifold M , then we can define M to be:*

- *Chronological if $x \notin I^{+}$ for all $x \in M$*
- *Causal if $J^{+} \cap J^{-} = \{x\}$ for all $x \in M$*
- *Distinguishing if $I^{+}(x) = I^{+}(y)$ only if $x = y$*
- *Globally Hyperbolic, if there exists a Cauchy Hypersurface in M*

M is said to be *Causally simple* if M is both distinguishing, and is equipped with the feature that for all $p \in M$, $J(p)$ (respectively past and future for $J^{-}(p)$ and $J^{+}(p)$) is a closed set.

In order to define a *Cauchy Hypersurface*, we will need to define a *Domain of Dependence*. Simply written, we can define the following [7]:

Definition 1.5: *A future (respectively past) Domain of Dependence $D^{\pm}(g)$ is the collection of all points $p \in g$ (for an achronal g) such that all past (respectively future) inextendible causal curve γ that passes through p intersects g , where by "inextendible" we mean that the curve γ has no endpoints, or no proper superset containing γ .*

For a (future) $D^{+}(g)$, we can define a $H^{+}(g)$ that forms the *boundary* of $D^{+}(g)$. Here, we call $H^{+}(g)$ as the *future Cauchy Horizon*. The following is a depiction of the (future) *Cauchy Horizon* of V:

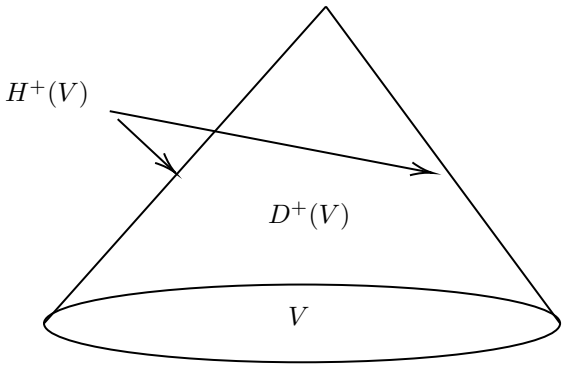

Fig-1: An example of the relation between D and H . The arrows indicate the light-like boundaries of the Surface $D^{+}(V)$ inside.

If a closed *achronal Hypersurface* M is defined such that $D(M) = M$ (where M is the entire manifold), then M is said to be a *Cauchy Hypersurface*. If we have F that is such that $D(F)$ is not the entire manifold M , then we call F *Partially Cauchy Hypersurface*.

In Definition 1.4 we defined that if a *Cauchy Hypersurface* exists, then M is said to be *Globally Hyperbolic*. This point is very important in our discussion on Singularity models and their Causal structure, as we will be discussing eventually. The Leray definition [38] of a *globally hyperbolic* space is as follows:

Definition 1.6: *M is globally hyperbolic if for all $p \in M$, $J^{+}(p) \cap J^{-}(p)$ is compact.*

An important property of a *globally hyperbolic* M is that if is *diffeomorphic* to a *smooth Cauchy Hypersurface*. Or, a *globally hyperbolic* M with a smooth *Cauchy Hypersurface* I follows that it is diffeomorphic as $R \times I$.

We will now discuss *Causality* and *Energy conditions*, which are very important for understanding the structure

of Singularity theorems.

2 ENERGY CONDITIONS

Energy conditions lay out the constraints necessary to determine the nature of matter fields. These are also important in determining the possibility of Singularities in a model, since the matter content determines this crucial point. We will now define some important definitions that will guide us through understanding the structure of Singularity theorems.

It becomes necessary to define a set of conditions that ensure that a given model can hold a *Singularity*. For this, we lay a set of *Energy conditions* that can be considered to show whether a Space-time houses Singularities. These conditions are crucial because they lay the foundation of the notion of a Singularity, that is the diverging effect [2][5][13][18][20].
If we consider a perfect fluid, the Energy-Momentum tensor takes the form

$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + p g^{\mu\nu}$$

The *weak Energy condition* is satisfied if the value of $T^{\mu\nu}$ is never negative, i.e. it satisfies $T^{\mu\nu} \geq 0$. Therefore, we can write down the value of the expression completely as

$$T^{\mu\nu} u^\mu u^\nu \geq 0$$

The *strong Energy condition* has to do with the *Raychaudari equation*, which can be derived from the *Riemann tensor* in terms of the *Covariant derivative*. Consider a u^μ vector field. For this, the Riemann tensor is defined as

$$\nabla_\mu \nabla_\nu u^\alpha - \nabla_\nu \nabla_\mu u^\alpha = R^\alpha_{\gamma\mu\nu} u^\gamma$$

By *contracting* the terms α *and* γ , and then by considering that the norm of the vector is either light-like or time-like (i.e. $u^\mu u_\mu = 0$ or -1 for respectively light-like or time-like), we can derive the expression,

$$-u^\nu \nabla_\nu \nabla_\mu u^\mu = S_{\mu\nu} S^{\mu\nu} + R_{\delta\nu} u^\delta u^\nu$$

Here, S is the collection of the symmetric components in the entire expansion after contraction by $\alpha = \gamma$. (Sir Raychaudari elaborated on this geometric aspect of the Riemann tensor to write down the Raychaudari equation. This would later be one of the most important results in General relativity that would allow an elaborate understanding of Singularity theorems.) We know write the divergence expansion as

$$\theta \equiv \nabla_\mu u^\mu$$

Which on expansion results in a collection of the *shear* tensor and the *Raychaudari scalar*. The above expression splits into S^μ_μ . Since the Riemann tensor value determines the condition, we now write the *strong Energy condition* as

$$\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right) V^\mu V^\nu \geq 0$$

One reason why this is important is that it ensures that for any model satisfying the Strong condition, there is a form of a *convergence condition*, which is to say that neighbouring time-like geodesics focus, or *converge*. This can also be written as $R_{\mu\nu} u^\mu u^{\mu\nu} \geq 0$. The *null Energy condition* is a fundamental condition that lays out that

$$T_{\mu\nu} u^\mu u^\nu \geq 0$$

If the above are both satisfied along with the chronology condition, then we say that the *causality condition* is held true for M.

A *Singularity* is a region where:

1. Either the coordinate system can no longer map the geometry of that region, or

2. *No* coordinate system can map that region.

(1) is simply a point where the coordinate system cannot define the neighbourhood of. For example, the *Event Horizon* of a Black hole is such that it arises as a singularity *solely* for the usual coordinates. That singularity at $R = R_{Sch}$ (called a *coordinate singularity*) can be covered by extending the coordinate system (here the *Eddington Finkelstein coordinates*). On the other hand, (2) is a point (respectively a collection of points) on M that are such that even a coordinate-independent system of curvature definitions cannot be used to describe the point. An example is the central singularity of a Black hole. Since the very definition of the singularity at $R = 0$ (called a *curvature singularity*) is that the Curvature tensor breaks down, it is a *real* singularity in a sense. Our discussion will be on these Singularities, and how they can be understood more deeply [32].

A *Singularity theorem* requires that there exist an incompatible set of energy and causal conditions. For instance, as we will see, a model satisfying the *generic Energy condition* can be shown to house *geodesic Incompleteness* by the *Raychaudari theorem*. Further, a continuation of the *Energy conditions* can be used to show (under the *Pattern Singularity theorem*) that under the *Energy conditions* and a reasonable *initial value condition*, a model can be shown to incorporate some form of *geodesic Incompleteness*. Then, a combination of the *Energy conditions* and the concept of a *Closed Trapped Surface* can bring a new conceptualisation of the notion of *Singularity theorems*, under the Penrose and Hawking theorems [2][5][13][14[18][19][20].

We will now discuss Singularity theorems, and their Causal structure.

3 SINGULARITY THEOREMS

From the above mentioned θ expansion, we can write out the Raychaudari theorem as:

Theorem 3.1: *If the θ expansion is positive, and the strong Energy condition is satisfied, then the energy density diverges at an instant in the past (or future if the expansion is negative) of the fluid model.*

This is the Raychaudari theorem, which was given by sir Amal Raychaudari[3]. This was a most important formulation, since the divergence of a fluid meant that a Singularity could be modelled. An extension is provided by considering some additional points, under the necessity of these *Energy conditions*. We can develop the *Pattern Singularity theorem* as:

Theorem 3.2: *If the following conditions are met, then the model defined contains geodesic Incompleteness:*

1. *The strong Energy condition is satisfied,*
2. *The Causality conditions are held true,*
3. *Suitable sufficiency in differentiability of (M, g)*
4. *A corresponding set of initial conditions (i.e. conditions on the focusing of causal geodesics).*

(1) has to do with the *focusing* effect of causal geodesics as predicted by the model.

(4) states the importance of the focusing effects. This point is necessary to form a condition that provides the deciding point, where M is *singular* or not. For instance, we have models that are *globally hyperbolic* (that is to say that it is a *Cauchy Hypersurface*), and *also* satisfy the generic condition, but are non-singular, i.e. no form of singularities exist. Therefore, (4) is a necessary condition to define those models which *appear* to satisfy all the necessary conditions of singularity theorems against those which *truly* are singular [13][27].

We will now define the notion of a *Closed Trapped Surface*[2][34].

Consider a sphere that from all points on its boundaries emits light rays inward and outward. If the Space-time was expanding, the light rays sphere "S" will form two spheres of light, one larger than S and the other smaller. On the other hand, if the Space-time was contracting, then the spheres formed would *both* be smaller than S. A mathematical picture is to assume a set $\wp$ whose nature is *achronal*. If this set has a compact $E^{+}(\wp)$, then we call this as a *future-trapped surface*.

Penrose's theorem uses this very notion to write down a Singularity theorem that further describes the conditions for a Space-time M to be singular [2][13][18][20]:

Theorem 3.3: *If for all null vectors the strong Energy condition is satisfied, and if a non-compact Γ Cauchy Hypersurface exists for a Space-time with a Closed trapped surface G , then there is a null geodesic Incompleteness on (M, g) .*

The description is based on that (M, g) can be said to be *geodesically Incomplete* if

(1) The *strong Energy condition* is satisfied, which guarantees that gravity is *always* attractive,

(2) *Global Hyperbolicity* exists on (M, g) .

(3) The surface is such that there is a trapped surface formed by any two collection of light rays in M (correspondingly *future* or *past* of M .

We have set G to be non-compact, which is a major point in showing that M is truly null geodesically Incomplete. The proof considers that in order to say that M is geodesically complete, it should hold that Γ is compact, and by considering a series of considerations, we reach a contradiction that show that M is then not geodesically complete.

If we had a point p whose light cone was such, that the past light-cone extended from the future light-cone of p, then we say that the light cone of p *re-converges* [2][18][20].

The *generic Energy condition* can be also written for a time-like 4 -vector as

$$R_{\alpha\beta\gamma\delta}u^{\beta}u^{\gamma} \neq 0$$

We can now write the Hawking-Penrose theorem as [2][5][13][14][18][19][20]

Theorem 3.4: *If a Space-time (M, g) is such that:*

(1) The strong Energy condition holds for every light-like/time-like vector,

(2) The generic Energy condition holds,

(3) The Chronology axiom holds true, i.e. no p, $q \in M$ exist so that $p < q < p$ for $p \neq q$,

(4) A set I (respectively future and past trapped for a compact $E^{+}(I)$ or $E^{-}(I)$ exists on M .

Since *(1)* and *(2)* imply that there exists a form of *conjugate pair* for every light-like/time-like geodesic that cannot be extended, many texts write a more simpler form of theorem 3.4, as

(1) Every inextendible light-like/time-like geodesic has a conjugate pair,

(2) The Chronology principle is true,

[3]The original paper by sir Raychaudari is *A Raychaudhuri, Phys. Rev. 89, 417 (1953).*

(3) A set I as previously defined.

The following of this theorem was actually from the original 1967 Hawking theorem [2][13][19], which provided the backing of understanding Singularities. The Hawking-Penrose theorem has a lot of significance in Cosmology. Namely, it can be used to directly show if there was an initial singularity. For instance, by considering the Cosmic Microwave Background, we can show that there is a form of incompleteness that arises where re-converging past non-Space-like geodesics are found. This can be used to show that the Universe satisfies the necessary conditions required for the Hawking-Penrose theorem to hold.

If we were to consider a set H such that we want to understand the evolution of this set on a manifold M , where the set is a combination of a *Hypersurface* in M , we would require to think about *how* the nature of M , the "constraints" and the conditions that M places on a *Hypersurface* in M affects the set H and its existence in itself. We will now turn our attention to the Cauchy initial data problem [12][24][25][28][33]. The Cauchy problem lays this very problem, and many important results, such as the stability and

4 THE CAUCHY INITIAL VALUE PROBLEM

If we consider a manifold M which has a given set of initial data $(H, g_{\mu\nu}, L)$ where H is a Riemannian manifold that acts as a *Spacelike Hypersurface* on M . An example would be to consider the Friedmann-Robertson model, and solve the initial conditions. Consider the pair (M, g) to be in accordance with the *Cosmological Principle* (that is to say that M is, in general, homogeneous and isotropic). If we take the metric for a Euclidean surface as $\overline{g}$, for $M = I \times R^3$, where I is an open interval on R^3 , then the metric for M can be written as $g = -dt^2 + a^2(t)\,\overline{g}$.

a lies in the interval $(0, \infty)$. We can then solve the Einstein Field Equations, which individually will give equations for each component. These can be broken into the *constraint* and *evolution* [25][33] equations.

If we define a coordinate system x^α which satisfies the following wave equation, [25][27]

$$\Box_g x^\alpha = 0$$

We call such a coordinate system to be "harmonic". The Cauchy problem focuses on an initial data set $H = (H, g_{\mu\nu}, L)$ such that the manifold M evolves under H . If we consider the vacuum Einstein equations, which are a set of PDEs that are quasi-linear second order. The reduction can be taken in terms of *Gauge* functions. Considering the vacuum Field equations, if we wrote the Ricci tensor in terms of a term $\Gamma_\alpha \equiv g^{\mu\nu}\Gamma_{\mu\alpha\nu}$ (called as the *wave coordinates condition*), we can change it with a new quantity f_α , which we will define to be dependent on the metric, but not dependent on the derivatives of the metric in itself[4]. We will write down

$$d_\alpha = f_\alpha - \Gamma_\alpha$$

We define a new Ricci tensor using the above setting[5], where we will add a certain quantity that is defined in terms of the above quantity d ; in other words, we are writing the changed Ricci tensor in terms of the system that has been defined using the above conditions.

$$r_{\mu\nu} = R_{\mu\nu} + \Pi_{\mu\nu}$$

Where the additional term $\Pi_{\mu\nu}$ is defined in terms of the Covariant derivatives of d_μ ,

$$\Pi_{\mu\nu} = \nabla_{(\mu} d_{\nu)} \equiv \frac{1}{2}\left(\nabla_\mu d_\nu + \nabla_\nu d_\mu\right)$$

Since we are imposing that the entire process is satisfying the vacuum constraint, we can reduce the above definition of the changed Ricci tensor back in terms of the original Ricci tensor and the addition Π . We end up with the original Ricci tensor, which from above can be written as

$$R_{\mu\nu} = -\Pi_{\mu\nu}$$

From this, we can derive the curvature, by bringing the left side into the Einstein tensor, which is defined for a given metric g as a collection of Riemann tensor and the Ricci scalar (which is defined by operating the Ricci tensor with the inverse metric $g^{\mu\nu}$, or $R = g^{\mu\nu}R_{\mu\nu}$):

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

In order to do so, consider that due to the Bianchi identities, the divergence of $G_{\mu\nu}$ (here the Einstein tensor) will *have* to be zero. If we take the trace of the relation between R and Π , we notice that it reduces to

$$G_{\mu\nu} = -\Pi_{\mu\nu} + \frac{1}{2}\nabla^\alpha d_\alpha g_{\mu\nu}$$

[4] Yvonne Choquet-Bruhat's 2014 paper, Beginnings of the Cauchy problem discusses analytically the Cauchy problem in detail.

[5] What we are doing here is considering the gauge function approach. Since we have defined a new Ricci tensor dependent on the term d, we are in a way defining the original Ricci tensor in terms of an extra term, which, as we shall see in the next few sentences, is defined in terms of the covariant derivatives of d.

We can now take the divergence of the above to a wave equation for d , which will be a composite expression in the Covariant derivatives of d and the Ricci tensor. This is labelled as *Gauge function* for a vacuum solution.

In wave coordinates, we can consider an expression of the metric that will give us our derivatives for all the components of g :

$$\Box_g g_{\mu\nu} = I_{\mu\nu}\left(g, \partial g\right)$$

Here, we have considered the wave-operator, the components of g and the derivatives of the components of g . The constraint equations can be written to read

$$\nabla^{\ \alpha} N_{\alpha\beta} - \nabla^{\beta} trN = 0$$

$$R - N^{\alpha\beta} N_{\alpha\beta} + \left(trN\right)^2 = 0$$

Here, the terms R and N represent the curvature scalar and the second fundamental form respectively. These constraint equations arise from the *Gauss-Codazzi* equations. The evolution equations of the problem can be written in terms of the *Lapse* (the 00 components of g) and the *Shift* (the $0i$ components of g).

Yvonne Choquet Bruhat, in 1952 showed that for an initial set H satisfying the vacuum constraint equations, the following theorem holds:

Theorem 4.1: *If an initial data set* $H\left(H, g_{\mu\nu}, L\right)$ *is such that it satisfies the vacuum constraint equations, then there is a pair* $\left(M^{n+1}, g\right)$ *that* $H \subset M$ *is space-like, for a given Second Fundamental form* $L = K\left(X, Y\right)$ *and a Hypersurface* H *, considering that* g *is a Riemann metric.*

The definition of L is such that we are considering various geometric properties of H . We define L to be defined as

$$L \equiv K\left(\mathrm{X},\ \mathrm{Y}\right) = g\left(D_X N, Y\right)$$

Where D_X denotes the affine Connection, and N the normal vector field to H . The respective equations that we will relate to the *Gauss* and *Codazzi* equations are given by[6]:

$$R^M\left(\mathrm{A},\ \mathrm{B},\ \mathrm{C},\ \mathrm{D}\right) = R^H\left(\mathrm{A},\ \mathrm{B},\ \mathrm{C},\ \mathrm{D}\right) + \Sigma$$

$$\Sigma = K\left(\mathrm{A},\ \mathrm{D}\right) K\left(\mathrm{B},\ \mathrm{C}\right) - K\left(\mathrm{A},\ \mathrm{C}\right) K\left(B, D\right)$$

$$R^M\left(\mathrm{A},\ \mathrm{B},\ \mathrm{N},\ \mathrm{C}\right) = \nabla_A K\left(\mathrm{B},\ \mathrm{C}\right) - \nabla_B K\left(A, C\right)$$

Where respectively $A, B, C, D \in TH$, and N is the normal vector field as seen previously. The terms R^M and R^H respectively are the Riemann tensors, which run from indices $X = \mu = \left(1, 2 \ldots . n\right)$.

Proving theorem 4.1 has involves some important results that have to do with the wave system. We take that the metric g holds the Einstein vacuum solution true, i.e. it follows that $R_{\mu\nu} = 0$. Next, we will define a function f that is such that it *only* depends on g – it does not depend on the derivatives of g at all. From this, we will construct an arbitrary vector field F that is a function of g and f . Since we already know that G is equipped with following $\nabla \cdot G = 0$, we can further write down a form of *uniqueness*, by adding another polynomial function $p\left(F, F^1\right)$, where F^1 denotes the first-order derivatives of F .

The initial data set H is such that we define a set as seen before, with g being a metric of signature $(-+++)$, a Riemann metric g , and a *Hypersurface* H . If this can be the initial set for a pair (M, g) is given by the constraint equations imposed. The constraint equations as seen above show the set of conditions that regulate the initial data H to exist plausibly. Solving the aspects that are regulated by the vacuum constraint is in steps that involve allowing a given Hypersurface $H \subset M$ to be equipped with forms g and g that are such that they solve the vacuum constraint, and then by choosing the Gauge functions that behave like f .

The Cauchy problem considers some aspects that are of particularly great interest [2][10][17][21][22][23], and are very important in understanding the results from General Relativity. Some of them are:

1. Stability,
2. Singularities,
3. Predictability.
4. Gravitational waves.

Stability has to do with the high degree of homogeneity [10][12]. For example, consider a model M that follows from the *Cosmological Principle* that it is perfectly homogeneous and isotropic. It then is a very straightforward question to ask whether a very minor disturbance in the initial data set resolving to M is such that it gives a *stable* future for M. Gravitational radiation [35][36][37] also arises from the Cauchy initial value problem, where we consider the vacuum equations to be satisfied by a manifold where "ripples" of the Space-time are generated by the accelerated motion of compact and dense objects. These propagate at the speed of light, and travel along null *Hypersurfaces* to M There are several other aspects of the Cauchy problem that are very important to understand, such as Singularities and Predictability as mentioned above.

[6] Parametric Manifolds by Stuart Boersma and Tevian Dray gives an extensive introduction to the Gauss-Codazzi equations, and a detailed description of

5 CONCLUSION

We have discussed the important of Causal definitions in modelling Space-time in General Relativity. We have discussed the geometric features associated with these Causal definitions, and we have defined the basic properties of the Causal structure of Lorentzian manifolds. We then have written the energy conditions required to formulate Singularity theorems. We discussed the nature of Singularities, and how they can be mathematically formulated for a given Space-time model. We then mathematically discussed the Cauchy problem, and the implications of an initial data set on a manifold.

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