VALID INEQUALITIES AND FACETS FOR MULTI-MODULE (SURVIVABLE) CAPACITATED NETWORK DESIGN PROBLEM

A Dissertation

by

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ABSTRACT

In this dissertation, we develop new methodologies and algorithms to solve the multi-module (survivable) network design problem. Many real-world decision-making problems can be modeled as network design problems, especially on networks with capacity requirements on arcs or edges. In most cases, network design problems of this type that have been studied involve different types of capacity sizes (modules), and we call them the multi-module capacitated network design (MMND) problem. MMND problems arise in various industrial applications, such as transportation, telecommunication, power grid, data centers, and oil production, among many others.

In the first part of the dissertation, we study the polyhedral structure of the MMND problem. We summarize current literature on polyhedral study of MMND, which generates the family of the so-called cutset inequalities based on the traditional mixed integer rounding (MIR). We then introduce a new family of inequalities for MMND based on the so-called n-step MIR, and show that various classes of cutset inequalities in the literature are special cases of these inequalities. We do so by studying a mixed integer set, the cutset polyhedron, which is closely related to MMND. We We also study the strength of this family of inequalities by providing some facet-defining conditions. These inequalities are then tested on MMND instances, and our computational results show that these classes of inequalities are very effective for solving MMND problems. Generalizations of these inequalities for some variants of MMND are also discussed.

Network design problems have many generalizations depending on the application. In the second part of the dissertation, we study a highly applicable form of SND, referred to as multi-module SND (MM-SND), in which transmission capacities on edges can be sum of integer multiples of differently sized capacity modules. For the first time, we formulate MM-SND as a mixed integer program (MIP) using preconfigured-cycles (p-cycles) to reroute flow on failed edges. We derive several classes of valid inequalities for this MIP, and show that the valid inequalities previously developed in the literature for single-module SND are special cases of our inequalities. Furthermore, we show that our valid inequalities are facet-defining for MM-SND in many cases. Our computa-

tional results, using a heuristic separation algorithm, show that these inequalities are very effective in solving MM-SND. In particular they are more effective than compared to using single-module inequalities alone.

Lastly, we generalize the inequalities for MMND for other mixed integer sets relaxed from MMND and the cutset polyhedron. These inequalities also generalize several valid inequalities in the literature. We conclude the dissertation by summarizing the work and pointing out potential directions for future research.

DEDICATION

To my parents

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1. INTRODUCTION

More and more decision-making problems in today's world have been modeled as network design problems, ranging from distribution networks of online retailers to data networks of cloud providers. In such problems, networks are constructed by connecting the nodes and decisions are made on using limited resources to deliver products between the nodes. In particular, in the *capacitated* network design problem, decisions are made on installation of flow transfer capacities and routing of flow from a set of source nodes to a set of sink nodes in a network. The objective is to minimize the cost of installing capacities and transferring flows over the links.

Telecommunication cable network is a prime example of capacitated networks, and the design of telecommunication networks has been widely studied in the literature due to rapid growth of the Internet and the importance of ensuring its quality of service. In telecommunication networks, the nodes can be clients/servers or any device that broadcasts, receives, or distributes signals or data. The end-users of the Internet have certain demand for speed which is measured by bit-rates, which is growing exponentially with the development of new telecommunication technologies. In order to satisfy such demand to ensure the quality of service, Internet providers install transmission facilities like fiber-optic cables or any device that transmits these signals or data to build links between the nodes of the network. Such transmission facilities each have a certain capacity on the maximum bit-rates it is able to transmit, as known as it's bandwidth. Higher transmission capacity between two nodes can be achieved by installing multiple transmission facilities. Usually, there are different types of transmission facilities available on the market with different capacities and cost structures.

Construction of such a telecommunication network requires huge capital investments from the telecommunication service providers. Therefore, the telecommunication service providers aim to design the network by making decisions on routing of the demands and installation of the transmission facilities at the minimum cost. This network design problem is also re-optimized on a regular basis due to change in the demand to adjust the routing of flow or expand the capacities if

necessary.

Similar analogies for flow and capacities can be found in many applications other than telecommunication networks, such as transportation networks and power grid networks. Design of networks in these fields can be modeled and solved using similar methods. Therefore, network design problems are fundamental in industrial applications where many scenarios can be abstracted as networks.

In this dissertation, we focus our study on two major types of network design problems. In the first part of the dissertation, We study the multi-module capacitated network design problem which arise in many of the aforementioned industrial applications. In the second part of the dissertation, we study the multi-module survivable network design problem, in which extra capacities are installed to protect edge failures. This problem are most applicable to networks where edge failures are of high risk and costly impact, such as telecommunication networks and power grid networks.

1.1 Multi-Module Capacitated Network Design

Network design problems have been widely studied in the literature. While many studies limit their problem in the setting of a single type of capacity, in many network applications, the flow transfer capacities are available for purchase in the form of modules of different sizes, each associated with a fixed-charge cost, where the unit cost of capacity for each module is different (typically smaller for larger module sizes). The total flow transfer capacity on each link can be constructed by installing any combination of any integer multiple of these capacity modules.

In such situations, the decision-making about installing capacities becomes more complicated than the case where we have only a single-sized capacity module, as we need to determine the composition of capacity modules of different sizes to be installed on each link. In this dissertation, we are interested in this type of problem, referred to as the *multi-module capacitated network design* (MMND) problem. MMND can be defined on directed or undirected networks and formulated as mixed integer programs (MIPs).

Telecommunication is one of the major application areas where MMND problems arise. In this context, different capacity modules correspond to transmission facilities such as fiber-optic cables

of different bandwidths. Typically, a set of cable types with different bandwidths are available for purchase [1], and the cost of purchasing cables constitutes the majority of the total cost for network design. Consequently, the telecommunication applications have had a significant role in motivating study of MMND and its variants in the MIP literature from different perspectives; see for example [2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. These studies differ from each other in many aspects, including the link model used to formulate the MIP. For telecommunication networks, three types of link model are common in the literature: the directed, the undirected, and the bidirected link model. We will name the MIP problem formulated using each of these link models the directed MMND, the undirected MMND, and the bidirected MMND, respectively. These formulations will be illustrated in Section 3.1. Note that in this literature MMND has also been referred to as network loading [7], capacity installation [3], or multifacility capacitated network design [12] problem, among other names.

A considerable number of studies have addressed the MIP formulations of MMND from the cutting plane perspective [2, 3, 5, 7, 8, 9, 10]. In these studies, the so-called cutset inequalities are among the most effective classes of inequalities for network design problems. This class of inequalities is derived for the convex hull of a certain relaxation of the directed/undirected/bidrected MMND, called the *cutset polyhedron* for the respective problem (see Section 3.3 for more details).

Upon closer examination, it can be easily shown that almost all aforementioned cutset inequalities can be derived by applying the traditional mixed integer rounding (MIR), referred to as the 1-step MIR, on the base inequalities formed by certain aggregation and relaxation of the defining constraints of the corresponding cutset polyhedron (see [13] and [14] for more details on 1-step MIR inequalities). Even though capacity modules play a central role in developing these cutset inequalities, almost all these inequalities are developed using the information of only one of the capacity modules as noted in Table 1.1. The only exception to this is the cutset facet for the 3-module problem in [7] involving only capacity variables, which is a 2-step MIR inequality as we will show later. In the case of other multi-module problems, in particular the multi-module capacitated lot-sizing problem, it has been shown that using information of all modules to develop

cuts results in much more effective cuts compared to cuts that only use the information of a single module [15, 16]. Motivated by this observation, in this dissertation, we develop cutset inequalities for MMND using the information of all the modules for any number of modules. We show their theoretical strength, and demonstrate they are computationally very effective in solving MMND, especially compared to cutset inequalities derived based on the information of a single module. In developing these inequalities, we employ the n-step MIR theory.

Table 1.1: Summary of major relevant studies on network design problems

Reference	Problem name (link model*)	M*	n*	Inequalities	
				Name in reference	Name in this dissertation
[7]	network loading (u)	≤ 3, div	≤ 2	cutset	simple cutset
[2]	capacity expansion (b)	2, div	1	flow-cut-set	flow cutset
[3]	network design (d)	≤ 2, div	1	cut-set	general cutset
[12]	network design (d)	any	1	multifacility cut-set	general cutset
[10]	network design (d,u,b)	any	1	flow cutset	general cutset
This dissertation	MMND (d,u,b)	any	any	-	n-step cutset

*: "d", "u", and "b" denote directed, undirected, and bidirected, respectively. "div" means only divisible module sizes were studied. M denotes the number of modules in the problem and n is the number of modules used to derive the inequality.

Kianfar and Fathi [17] presented the n-step MIR inequalities for the mixed integer knapsack set (see Section 2.2.2 for more details). The (1-step) MIR inequalities [13, 14] and the 2-step MIR inequalities [18] are special cases of the n-step MIR inequalities for $n = 1$ and 2, respectively. Generalizations of the n-step MIR inequalities have been shown to be facet-defining for several generalizations of the mixed integer knapsack set [19, 15, 16, 20, 21]. Inequalities derived based on the n-step MIR theory and its generalizations have been previously proven to be very effective cuts for other multi-module problems such as the single node capacitated network design problem [22] and the multi-module lot-sizing (MMLS) problem [23, 15, 21]. In this dissertation, we utilize n-step MIR to derive valid and facet-defining inequalities for MMND.

We propose a new family of valid inequalities for MMND on directed networks, referred to as the *n-step cutset inequalities*, for any integer $n \leq M$. The n-step general cutset inequality uses

the information of n modules. As a result, our inequalities can use the information of any desired number of modules, and particularly, all the modules when $n = M$. We show that the "cut-set inequality" in [3] and the "multifacility cut-set inequality" in [12, 10] are special cases of the n-step general cutset inequality. We then introduce the *n-step simple cutset inequality* and the *n-step flow cutset inequality* as special cases of the n-step general cutset inequality. We show that the n-step general cutset inequalities, the n-step flow cutset inequalities, and the n-step simple cutset inequalities are facet-defining under certain conditions for the directed cutset polyhedron. Based on a result in [10], these inequalities are also facet-defining for the convex hull of the directed MMND.

We show that the n-step simple, flow, and general cutset inequalities, applied using a polynomial time separation algorithm, are computationally very effective in solving the directed MMND test instances. For our 2-module test instances, the average total solution time (including cut generation) with our 2-step cuts added was 0.35 times that of CPLEX 12.7 in its default setting. This time was also 0.59 times the solution time when only 1-step cuts (which only use the information of a single module) are added. With the 2-step cuts, the number of branch-and-bound nodes was also 0.23 times the number of nodes with default CPLEX and 0.38 times that of the 1-step cuts. For the 3-module problems, the average total solution time with the 3-step cuts added was 0.45 times that of CPLEX in its default setting, 0.45 times the solution time when only 1-step cuts are added, and 0.56 times the solution time when the 2-step cuts are added. With the 3-step cuts, the number of branch-and-bound nodes was 0.32 times the number of nodes with default CPLEX, 0.42 times that of the 1-step cuts, and 0.55 times that of the 2-step cuts.

We generalize the n-step cutset inequalities for other link models of MMND studied in the literature, namely the undirected MMND and the bidirected MMND. The generalized n-step cutset inequalities can be shown to be facet-defining for the convex hulls of the undirected and bidirected MMND, as well as their respective cutset polyhedron. We show that the "cutset inequality" in [7] and the "flow-cut-set inequality" in [2] are special cases of the n-step general cutset inequality for the undirected MMND and the bidirected MMND, respectively. We also generalize the n-step

cutset inequalities for all link models of MMND with multicommodity.

1.2 Multi-Module Survivable Capacitated Network Design

Most networks in real-world applications are vulnerable to edge failure. Telecommunication [24] and power transmission [25] networks are prime examples of networks that are subject to frequent edge failures, and hence, service interruptions. As a result, the capacitated *Survivable Network Design* (SND) problem, i.e. the problem of designing a network with the minimum flow-routing plus capacity-installation cost such that it can still function when an edge failure happens, is a crucial problem in network science and engineering.

In this dissertation, we study a highly applicable form of the SND problem where the flow capacities on edges are created by adding integer multiples of differently sized *capacity modules*. We refer to this problem as the *multi-module SND problem* (MM-SND). Telecommunication networks are a prime example of where capacity is multi-modular [2, 8, 10], e.g., fiber-optic cables or wireless routers each are available in different bandwidths and any number of them can be combined in connecting two adjacent nodes. Despite ubiquity of multi-modular edge capacities, most polyhedral studies on the SND problem have been focused on the single-module SND (SM-SND), i.e., where the capacity on an edge can only be a binary or integer multiple of a single-sized module. We refer to Soriano et al. [24] for an introduction and review on telecommunication network design and survivability. We refer to Grötschel et al. [26] and Kerivin and Mahjoub [27] for comprehensive reviews on formulation, algorithms, and polyhedral results of the SND. A few forms of network design or capacity expansion problems involving multiple differently sized capacity modules have been addressed from polyhedral perspective before [28, 29], but the definition of the problem and the manner in which survivability is addressed in these studies are totally different and unrelated to our definition of the MM-SND.

In this dissertation, we propose two models for MM-SND. One of them determines the initial network capacity installation decisions and the extra capacity installation decisions separately in two MIPs, and is more time-efficient; the other integrates the two decisions in a single MIP, which achieves lowest cost possible. Our results show that the time-efficient (hierarchical) model is $\sim$12

times faster than the cost-efficient (integrated) model, while the cost-efficient model achieves $\sim 8\%$ less cost than the time-efficient model. We focus our subsequent results on the integrated model.

We develop several families of valid inequalities for the MM-SND, namely, the n-step flow cutset inequalities, the n-step p-cycle flow partition inequalities, and the n-step survivable subset-Q inequalities (Section 4.4). We show that the partition inequalities of Rajan and Atamtürk for the SM-SND [30, 31] are special cases of our proposed inequalities.

We show that some special cases of the n-step flow cutset inequalities and the n-step p-cycle flow partition inequalities, which we refer to as the n-step simple cutset inequalities and the n-step survivable partition inequalities, respectively, are facet-defining for the convex hull of MM-SND.

We propose a method to generate primary p-cycles to initialize the p-cycle construction method presented in [31], which was based on column generation. We performed computational experiments to evaluate the effectiveness of our proposed inequalities using a heuristic separation algorithm. Our results show that our cuts are very effective. On average, both the time and the number of branch-and-cut nodes taken to solve our test instances by CPLEX 12.7 with our cuts added was 0.45 times that by CPLEX in its default settings (Section 4.6).

1.3 Dissertation Structure

The dissertation is organized as follows: In Chapter 2, we present a brief review on mixed integer programming, its solution techniques, and polyhedral results to the extent required for the results in this dissertation. We present our research results on the multi-module capacitated network design (MMND) problem in Chapter 3, on the multi-module survivable network design (MM-SND) problem in Chapter 4, and some generalizations of MMND and the cutset polyhedron in Chapter 5. We conclude the dissertation in Chapter 6 along with some future research plans.

2. NECESSARY BACKGROUND

In this chapter, we review theoretical and algorithmic aspects of linear and mixed integer programming to the level that is necessary to present our results. In particular, we review basics of linear and mixed integer programming in Section 2.1, especially algorithms used to solve mixed integer programs and basic polyhedral theory. In section 2.2, we review concepts and generalizations of the so-called mixed integer rounding (MIR), which forms the foundation of methodologies developed in this dissertation. In section 2.3, we introduce basic terminologies in networks and the notations used to present our models and results in this dissertation.

2.1 Linear and Mixed Integer Programming

Linear programming is an advanced method to solve various problems in science and engineering, such as production planning, facility location, scheduling, transportation and telecommunication network design, and many others. The two key sets of identifiers for linear programs (LPs) are decision variables that represents the business or operation decisions that needs to be made, and constraints that describes certain conditions that decision variables need to satisfy. The goal of an LP is to minimize or maximize the objective function by moving the values of the decision variables.

Mixed integer programs (MIPs), sometimes referred to as mixed integer linear programs (MILPs), are a type of linear programs (LPs) where all or part of the decision variables are discrete. Within the scope of this dissertation, we assume that the objective function and all constraints are linear.

A mixed integer program can be written as

$$\begin{aligned} \min \quad & cv + hy \\ s.t. \quad & Av + Gy \leq b \\ & y \in \mathbb{Z}^n; v \in \mathbb{R}^p \end{aligned}$$

where A is an m by n matrix, G is an m by p matrix, c and h are row-vectors of dimensions n and p, respectively, and v and y are the decision variables. The linear problem obtained by dropping the integrality restrictions on decision variables of a MIP is called the linear relaxation of the MIP.

2.1.1 Branch-And-Bound

Branch-and-bound is a fundamental algorithm to solve MIPs. The basic idea behind branch-and-bound is to divide a MIP into smaller subproblems and solve for their linear relaxations which can be solved efficiently using linear programming algorithms such as simplex, and merge the solutions of the subproblems. For a maximization problem, the steps of branch-and-bound is as follows: the algorithm starts at the root node of a branch-and-bound tree, which corresponds to the linear relaxation of the original MIP. Each node in the tree corresponds to a subset of the feasible region of the root node linear program. The solution of each node can be either a feasible solution to the original MIP, in which case its objective is a lower bound on the optimal objective of the original MIP, or a linear relaxation solution whose variables have fractional values, in which case its objective is an upper bound on the optimal objective of the original MIP. Each node can be either branched or pruned. It is branched if the linear program of the node has at least one fractional variable $y_i = y_i^*$. In this case, two child nodes can be created by adding the constraint $y_i < \lfloor y_i^* \rfloor$ for the first node and $y_i \geq \lceil y_i^* \rceil$ for the second node. On the other hand, it is pruned in one of the three cases:

- If the linear relaxation objective on the node is smaller than the lower bound or larger than the upper bound.
- If the linear relaxation has a feasible solution to the original MIP.
- If the linear relaxation problem is infeasible.

The branch-and-bound algorithm terminates when all nodes are pruned. There are different strategies for choosing the branching scheme and node selection scheme. Generally, trade-offs exist between searching for better bounds and searching for better feasible solutions. The performance

of branch-and-bound depends strongly on the problem and the schemes selected. More details on branch-and-bound can be found in [13, 14].

2.1.2 Cutting Plane Algorithm

Cutting plane algorithm is another method to solve MIPs. Cutting planes are additional constraints added to the linear relaxation of the original MIP to cut off areas of the feasible region that do not contain integer solutions. In the ideal case, cutting planes can cut off all areas outside the minimal region that contains the set of integer solutions (the so-called convex hull), and linear relaxation will lead to optimal integer solutions. For most problems, however, we cannnot find all such cutting planes, and in this case, cutting planes can usually lead to better linear relaxation objective value, and, hence, can be integrated within a branch-and-bound process to provide better bounds. The integrated algorithm is called branch-and-cut.

Typically, there are exponentially many cutting planes for a MIP. Only a subset of them will be effective in providing better bounds, and this set of cutting planes are added to the formulation by a cutting plane algorithm. The steps in a cutting plane algorithm is as follows: Given a MIP, we solve its linear relaxation, and generate a cut that separates the optimal solution of the linear relaxation and the set of feasible integer solutions of the original MIP. Such procedure to generate a cut is called a separation procedure. The cut is added to the linear relaxation and the process is repeated again, until some stopping criterion is satisfied. A well-designed cutting plane algorithm can dramatically boost the efficiency of branch-and-bound. More details on cutting planes and branch-and-cut can be found in [13, 14, 32, 33, 34].

2.1.3 Polyhedral Theory

In this section, we review some definitions and theoretical results in polyhedral theory to the extent required to present our research results. More details can be found in [13, 14].

Definition 1. *The feasible region of a MIP (denoted by $P_{MIP} \subseteq \mathbb{Z}^n \times \mathbb{R}^p$) is the set of points $(y, v) \in \mathbb{Z}^n \times \mathbb{R}^p$ which satisfy its constraints, i.e., $P_{MIP} := \{(y, v) \in \mathbb{Z}^n \times \mathbb{R}^p : Av + Gy \geq b\}$.*

Definition 2. *A* polyhedron *is a subset of $\mathbb{R}^p$ described by a finite set of linear constraints $P =$*

$\{v \in \mathbb{R}^p : Av \geq b\}$.

Definition 3. *Given a set $X \subseteq \mathbb{R}^n$, the convex hull of X, denoted $conv(X)$, is defined as: $conv(X) = \{x : x = \sum_{i=1}^{t} \lambda_i x^i, \sum_{i=1}^{t} \lambda_i = 1, \lambda_i \geq 0$ for $i = 1, \ldots, t$ over all finite subsets $\{x^1, \ldots, x^t\}$ of $X\}$.*

Theorem 1 ([13])**.** *$conv(P_{MIP})$ is a polyhedron, if the data A, G, b is rational.*

Definition 4. *An inequality $\pi x \leq \pi_0$ is a* valid inequality *for $X \subseteq \mathbb{R}^n$ if $\pi x \leq \pi_0$ for all $x \in X$.*

Theorem 2 ([13])**.** *If $\pi x \leq \pi_0$ is valid for $X \subseteq \mathbb{R}^n$, it is also valid for $conv(X)$.*

Definition 5. *The points $x^1, \ldots, x^k \in \mathbb{R}^n$ are* affinely independent *if the $k1$ directions $x^2 - x^1, \ldots, x^k - x^1$ are linearly independent, or alternatively the k vectors $(x^1, 1), \ldots, (x^k, 1) \in \mathbb{R}^{n+1}$ are linearly independent.*

Definition 6. *The* dimension *of P, denoted $dim(P)$, is one less than the maximum number of affinely independent points in P.*

Definition 7. *F defines a* face *of the polyhedron P if $F = \{x \in P : \pi x = \pi_0\}$ for some valid inequality $\pi x \geq \pi_0$ of P.*

Definition 8. *F is a* facet *of P if F is a face of P and $dim(F) = dim(P)1$.*

Definition 9. *If F is a face of P with $F = \{x \in P : \pi x = \pi_0\}$, the valid inequality $\pi x \geq \pi_0$ is said to* represent *or* define *the face.*

2.1.4 Column Generation

Column generation algorithm is one of the practical methods to solve large scale linear programs. The basic idea behind column generation is that in a basic feasible solution of a linear program, only basic variables are non-zero. Therefore one could start with only those variables in the formulation and add the other variables to the formulation. The steps of column generation is as follows: the algorithm start with a subset of all variables, called the reduced master. The linear

relaxation of the reduced master is solved, and a pricing problem based on the linear relaxation solution is solved for the variables that are not in the reduced master to evaluate their improvements to the objective function. Those variables with bigger improvements are added to the reduced master, and the process is repeated again until no improvement to the objective function can be made. More details on column generation can be found in [35].

2.2 Generalizations of Mixed Integer Rounding

2.2.1 Mixed Integer Rounding

Mixed integer rounding (MIR) is a fundamental cut-generating procedure to develop cutting planes [13, 14]. For a single-constraint mixed integer base set,

$$Q := \{(y, s) \in \mathbb{Z} \times \mathbb{R}_+ : \alpha_1 y + s \geq \beta\}$$

where $\alpha_1 > 0$ and $\beta \in \mathbb{R}$, the inequality

$$y_1 + \frac{v}{\beta - \alpha_1 \lfloor \beta/\alpha_1 \rfloor} \geq \left\lceil \frac{\beta}{\alpha_1} \right\rceil \tag{2.1}$$

is valid and facet-defining for $conv(Q)$.

MIR procedure is widely applied in optimization solvers, because many types of inequalities can be aggregated and relaxed to obtain a set of the form of Q [36].

2.2.2 n-Step Mixed Integer Rounding

We briefly review the n-step MIR inequalities. Following MIR inequalities, Kianfar and Fathi [17] developed the n-step MIR inequalities for the mixed integer knapsack set

$$K := \left\{(z, s) \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+ : \sum_{i \in I} C_i z_i + s \geq b\right\}$$

where $C_i, b \in \mathbb{R}$. Given $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} > 0$, $n \in \{1, \ldots, |I|\}$, for any $u \in \mathbb{R}$ define the recursive reminders

$$u^{(k)} := u^{(k-1)} - \alpha_k \left\lfloor \frac{u^{(k-1)}}{\alpha_k} \right\rfloor, k = 1, \ldots, n, \tag{2.2}$$

and $u^{(0)} := u$. Define $\sum_k^l(.) = 0$ and $\prod_k^l(.) = 1$ if $k > l$. The n-step MIR inequality for K is

$$\sum_{i \in I} \mu^n_{\alpha,b}(C_i) z_i + s \geq \mu^n_\alpha(b), \tag{2.3}$$

where for any $u \in \mathbb{R}$ the n-step MIR function is defined as

$$\mu^n_{\alpha,b}(u) = \begin{cases} b^{(n)} \sum_{k=1}^{m} \prod_{l=k+1}^{n} \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{u^{(k-1)}}{\alpha_k} \right\rfloor + b^{(n)} \prod_{l=m+2}^{n} \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{u^{(m)}}{\alpha_{m+1}} \right\rceil \\ \qquad\qquad \text{if } u \in \mathcal{L}^n_m; m = 0, 1, \ldots, n-1 \\ b^{(n)} \sum_{k=1}^{n} \prod_{l=k+1}^{n} \left\lceil \frac{b^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{u^{(k-1)}}{\alpha_k} \right\rfloor + u^{(n)} \text{ if } u \in \mathcal{L}^n_n, \end{cases} \tag{2.4}$$

with the partitioning of $\mathbb{R}$ by

$$\begin{aligned} \mathcal{L}^n_m &= \{u \in \mathbb{R} : u^{(k)} < b^{(k)}, k = 1, \ldots, m, u^{(m+1)} \geq b^{(m+1)}\} \\ &\qquad\qquad \text{for } m = 0, \ldots, n-1; \\ \mathcal{L}^n_n &= \{u \in \mathbb{R} : u^{(k)} < b^{(k)}, k = 1, \ldots, n\}. \end{aligned} \tag{2.5}$$

Kianfar and Fathi [17] showed that (2.3) is valid for K if the *n-step MIR conditions*

$$\alpha_k \left\lceil \frac{b^{(k-1)}}{\alpha_k} \right\rceil \leq \alpha_{k-1}, k = 2, \ldots, n \tag{2.6}$$

hold, and is facet-defining for the convex hull of K under several additional conditions (see Theorem 10 of [22] and Corollary 1 of [19]).

2.2.3 Valid Inequalities for Multi-Module Mixed Integer Programming Problems

n-step MIR theory was proved to be successful filling the research gaps on providing strong valid facet-defining inequalities for multi-module problems. The n-step MIR facets for mixed integer knapsack set [17], and the n-step mingling facets for the mixed integer knapsack set with bounded variables [19] were developed using n-step MIR. In [22], the partition inequalities [37] were proved to be special cases of the n-step MIR inequalities, and new facets were derived for variations of the single node capacitated flow set. n-step MIR was also used to develop the mixed n-step MIR inequalities for the n-mixing set [21], the n-step cycle inequalities for the continuous multi-mixing set [15, 16], and the n-step conic MIR inequalities for the mixed integer polyhedral conic set [38]. Especially, the mixed n-step MIR inequalities [21] and the n-step cycle inequalities [15] are also strong cuts for the multi-module lot-sizing (MMLS) problem [21, 15, 23] and the multi-module facility location (MMFL) problem [21], which are in fact special cases of MMND.

n-step MIR theory was applied to variations of the single node flow set to generate facet-defining inequalities [22]. It was proved in [22] that the partition inequalities in [37] are a subclass of the inequalities derived by n-step MIR for when the capacities are divisible; n-step MIR give facets for these sets with arbitrary coefficients that either dominate or are not obtainable by the partition inequalities. n-step MIR has also been used to develop facets for generalizations of the mixed MIR set [21] and continuous mixing set [15]. It has also been used to develop strong cuts for MMLS and MMFL. In this dissertation, we use techniques based on n-step MIR to derive strong cuts for MMND.

Sanjeevi and Kianfar [21] developed the n-step counterpart of the mixing procedure by Günlük and Pochet [39] to mix the n-step MIR inequalities. They showed that the n-step mixed MIR inequalities are valid for the mixing set and are facet-defining for the convex hull of the mixing set under certain conditions. They also introduced MMLS and MMFL, and showed how to generate valid inequalities for these problems using the mixed n-step MIR inequalities.

Bansal and Kianfar [15] generalized the cycle inequalities from [40] using n-step MIR theory. They showed that the n-step cycle inequalities are valid for the continuous multi-mixing polyhe-

dron and are facet-defining for the convex hull of the polyhedron under certain conditions. They showed how to generate valid cuts for MMLS with or without backlogging, and computationally tested that the cycle cuts are very effective in solving MMLS problems.

2.3 Notations

Let $G = (V, A)$ be a directed graph with V and A being the set of nodes and arcs of G, respectively. For any two non-empty set of nodes $U, W \subset V$, let $\delta(U, W)$ denote the set of arcs from the nodes in U to the nodes in W. For any $v \in V$, let $\delta^+(v) = \delta(v, V \setminus \{v\})$ be the set of arcs that have v as their tail, and let $\delta^-(v) = \delta(V \setminus \{v\}, v)$ be the set of arcs that have v as their head. Assume we have M differently sized capacity modules, indexed by $1, \ldots, M$. Let $C_1, \ldots, C_M$ be the sizes of these capacity modules. Without loss of generality, we assume that $C_1 > C_2 > \ldots > C_M > 0$. For each $a \in A$, the unit cost of flow along arc $a \in A$ is denoted by h_a, and the pre-installed capacity on arc $a \in A$ is denoted by g^a.

Let $H = (V, E)$ be an undirected graph, where E is the set of (undirected) edges. For each edge $e \in E$, we introduce a pair of anti-parallel (directed) arcs e^+ and e^-. Let A be the set of all such arcs, i.e., $A = \{e^+ = ij, e^- = ji : e = (i, j) \in E\}$. For each arc $a \in A$, let $\bar{a}$ be the arc in the opposite direction of a, i.e., $\bar{a} = \{ij : a = ji\}$. For any $v \in V$, $\delta^+(v)$ and $\delta^-(v)$ can be defined similarly to those in the directed graph. Let h_a denote unit flow cost on arc $a \in A$, f_t^e denote the fixed-charge cost of capacity module t installed on edge $e \in E$, and g^e denote the pre-installed capacity on edge $e \in E$ for undirected networks.

We assume there is a single type of commodity with multiple sources and multiple sinks over the network. A demand d_v is associated with each node such that $\sum_{v \in V} d_v = 0$. For source nodes we have $d_v > 0$ and for all other nodes we have $d_v \leq 0$.

Let $U \subset V$ be a nonempty subset of V and $\overline{U} = V \setminus U$. Also, let $A_U^+ = \delta(U, \overline{U})$ be the set of arcs from nodes in U to nodes in $\overline{U}$, $A_U^- = \delta(\overline{U}, U)$ be the set of arcs from nodes in $\overline{U}$ to nodes in U, and $A_U = A_U^+ \cup A_U^-$ be the set of all arcs crossing the partition with respect to $(U, \overline{U})$.

Let a directed p-cycle be a directed cycle with at least three arcs (we consider the case of undirected p-cycles in Section 4.5). Let $\mathcal{R}$ be the set of all directed p-cycles in G.

3. MULTI-MODULE CAPACITATED NETWORK DESIGN PROBLEM

We present our results for the multi-module capacitated network design problem (MMND) in this chapter. In particular, we review the mixed integer programming models for MMND in section 3.1, current literature on MMND and the family of cutset inequalities in sections 3.2 and 3.3. In section 3.4, we present the n-step cutset inequalities, a new class of inequalities developed for MMND on directed networks in this dissertation. We study the strength of these inequalities in section 3.5. We show the effectiveness of these inequalities in section 3.6. In section 3.7, we generalize the n-step cutset inequalities for MMND on other network models.

3.1 Models

On a directed network, MMND is formulated as the following MIP.

$$\min \sum_{a \in A} (h_a x_a + \sum_{t=1}^{M} f_t^a y_t^a) \tag{3.1}$$

$$\sum_{a \in \delta^+(v)} x_a - \sum_{a \in \delta^-(v)} x_a = d_v, v \in V \tag{3.2}$$

$$x_a \leq \sum_{t=1}^{M} C_t y_t^a + g^a, a \in A \tag{3.3}$$

$$(x, y) \in \mathbb{R}_+^{|A|} \times \mathbb{Z}_+^{M|A|}, \tag{3.4}$$

In the above formulation, the flow variable x_a is the flow to be transferred along arc $a \in A$, and the capacity variable y_t^a is the number of capacity module t, $t = 1, \ldots, M$, to be installed on arc $a \in A$. We refer to the problem defined by (3.1)-(3.4) as the directed MMND.

For undirected networks, two types of link models have been studied for MMND, namely the undirected link model and the bidirected link model. In both models, the capacity variable y_t^e is defined for the edge $e \in E$ and each module t, while the flow variables x_{e^+}, x_{e^-} are defined for the arcs e^+ and e^- with respect to e. In the undirected link model, the summation of flows in both directed arcs corresponding to an edge is bounded by the edge capacity. Therefore, this problem,

referred to as the undirected MMND in this dissertation, is formulated as follows:

$$\min \sum_{a \in A} h_a x_a + \sum_{e \in E} \sum_{t=1}^{M} f_t^e y_t^e \tag{3.5}$$

$$\sum_{a \in \delta^+(v)} x_a - \sum_{a \in \delta^-(v)} x_a = d_v, v \in V \tag{3.6}$$

$$x_{e^+} + x_{e^-} \leq \sum_{t=1}^{M} C_t y_t^e + g^e, e \in E \tag{3.7}$$

$$(x, y) \in \mathbb{R}_+^{2|E|} \times \mathbb{Z}_+^{M|E|}. \tag{3.8}$$

In the bidirected link model, the flow in each of the two arcs corresponding to an edge is bounded by the edge capacity. Therefore, this problem, referred to as the biderected MMND in this dissertation, is formulated by (3.5), (3.6), (3.8) and the following constraint

$$\max\{x_{e^+}, x_{e^-}\} \leq \sum_{t=1}^{M} C_t y_t^e + g^e, e \in E \tag{3.9}$$

instead of (3.7). The bidirected MMND can be transformed to a directed MMND where the capacities on the forward and backward arcs between a pair of nodes are the same. See [9, 10] for more details on these link models.

3.2 Literature Review

Magnanti and Mirchandani [7] studied the cutset inequalities that only include the capacity variables for the undirected MMND with $M \leq 3$, where the module sizes are *divisible*, i.e., $C_2|C_1$ and $C_3|C_2$. Bienstock and Günlük [2] proposed the flow-cut-set inequalities to include flow variables for the bidirected MMND with $M = 2$ divisible capacity modules. Chopra et al. [3] presented the cut-set inequalities for the directed MMND with $M \leq 2$ divisible capacity modules. Atamtürk [12] generalized the cut-set inequalities in [3] to the so-called multifacility cut-set inequalities for the directed MMND with any M (not necessarily divisible) capacity modules. Raack et al. [9, 10] generalized the result in [12] for the undirected and the bidrected MMND. Table 1.1

summarizes some pertinent features of the aforementioned studies.

Many similar problems have also been broadly studied in the literature. Bienstock and Muratore [41] and Atamturk and Rajan [30] proposed survivable/partition inequalities based on cutset inequalities for different forms of single-module network design problems with survivability constraints.

Different forms of the single node flow set, which arises either from network design models [42] or from generalizations of a row of a mixed-integer program [43, 37], are also related to the MMND problems and cutset polyhedron. It can be shown that the partition inequalities in [37] can be obtained by MIR [22]. The arc residual capacity inequalities in [42] for a simple version of the single node flow set, which were later generalized for 2MND, can in fact also be obtained using MIR. Later, Atamturk and Rajan [44] proposed a separation algorithm for residual capacity inequalities.

Other than cutset inequalities, which are based on a 2-partition of the network, three-partition inequalities were introduced by Magnanti et al. [42] and further discussed in [8, 2, 45, 46].

Brockmüller et al. [47] introduced c-strong inequality. It was further studied by Atamturk and Rajan [44] and van Hoesel et al. [48].

Besides the types of inequalities we mentioned above, Bienstock et al. [45], Günlük [5] and Avella et al. [49] mentioned metric inequalities for network flow problems as a general structure of generating inequalities. Louveaux and Wolsey [43] generalized the flow-cover inequalities to the general integer case. Brockmüller et al. [47], van Hoesel et al. [48], Louveaux and Wolsey [43], and Avella et al. [49] used lifting to strengthen their inequalities.

3.3 The Cutset Polyhedra And The Cutset Inequalities in Literature

In this section, we first briefly review the cutset inequalities previously introduced in the literature.

Let X^d, X^u, and X^b be the convex hulls of the set of feasible solutions to the directed, undi-

rected, and bidirected MMND, respectively, i.e.,

$$X^d := conv\{(x, y) : (x, y) \text{ satisfies (3.2)(3.3)(3.4)}\},$$
$$X^u := conv\{(x, y) : (x, y) \text{ satisfies (3.6)(3.7)(3.8)}\},$$
$$X^b := conv\{(x, y) : (x, y) \text{ satisfies (3.6)(3.8)(3.9)}\}.$$

As mentioned in Section 1.1, cutset inequalities for the directed, undirected, and bidirected MMND are in fact valid inequalities for certain relaxations of X^d, X^u, and X^b, respectively, which are referred to as the cutset polyhedron for the respective problem [12]. For the directed MMND, the cutset polyhedron is defined as follows. Let $U \subset V$ be a nonempty subset of V and $\overline{U} = V \setminus U$. Also, let $A_U^+ = \delta(U, \overline{U})$ be the set of arcs from nodes in U to nodes in $\overline{U}$, $A_U^- = \delta(\overline{U}, U)$ be the set of arcs from nodes in $\overline{U}$ to nodes in U, and $A_U = A_U^+ \cup A_U^-$ be the set of all arcs crossing the partition.

Let $d = \sum_{v \in U} d_v$. We can assume without loss of generality that $d \geq 0$, since if $d < 0$, then from $\sum_{v \in V} d_v = 0$ we know that $\sum_{v \in \overline{U}} > 0$, in which case we can switch U and $\overline{U}$.

The cutset polyhedron corresponding to the partition $(U, \overline{U})$ for the directed MMND is defined as

$$P^d := conv\Bigg\{(x, y) \in \mathbb{R}_+^{|A_U|} \times \mathbb{Z}_+^{M|A_U|} : \tag{3.10}$$
$$\sum_{a \in A_U^+} x_a - \sum_{a \in A_U^-} x_a = d, \tag{3.11}$$
$$x_a \leq \sum_{t=1}^{M} C_t y_t^a + g^a, a \in A_U\Bigg\}, \tag{3.12}$$

where (3.11) is obtained by aggregating (3.2) for $v \in U$. Notice that P^d is a relaxation of X^d since for every point $p = (x, y)$ in X^d, the subvector of p restricted to only variables corresponding to A_U is in P^d. Therefore any inequality that is valid for P^d is also valid for X^d.

Similarly, for the undirected MMND, the corresponding cutset polyhedron can be defined over a partition $(U, \overline{U})$. Let E_U be the set of edges crossing the partition. Each edge $e \in E$ is represented

by its two antiparallel arcs e^+ and e^-. Let A_U be the set of all such arcs, i.e., $A_U = \{e^+ = ij, e^- = ji : e = (i,j) \in E_U\}$. Let A_U^+, A_U^-, and d be defined the same as those in P^d. Then the cutset polyhedron for the undirected MMND is

$$P^u := conv\Bigg\{(x,y) \in \mathbb{R}_+^{|A_U|} \times \mathbb{Z}_+^{M|E_U|} : \tag{3.13}$$

$$\sum_{a \in A_U^+} x_a - \sum_{a \in A_U^-} x_a = d, \tag{3.14}$$

$$x_{e^+} + x_{e^-} \leq \sum_{t=1}^{M} C_t y_t^e + g^e, e \in E_U\Bigg\}, \tag{3.15}$$

and for the bidirected MMND, the corresponding cutset polyhedron P^b is the same as P^u except that

$$\max\{x_{e^+}, x_{e^-}\} \leq \sum_{t=1}^{M} C_t y_t^e + g^e, e \in E_U \tag{3.16}$$

is in place of (3.15).

Magnanti and Mirchandani [7] considered the undirected MMND with a single source $s \in V$ and a single sink $t \in V$. Let $d_s > 0$ be the supply of s, $d_t = -d_s$ be the demand of t, and $d_v = 0$ for $v \in V \setminus \{s,t\}$. For any $U \subset V$ such that $s \in U, t \notin U$, by definition of d in the cutset polyhedron, we have $d = d_s$.

The undirected MMND problems they consider have $M \leq 3$ divisible modules, and no pre-installed capacities. For $M = 3$, the capacity modules are $(C_1, C_2, C_3) = (\lambda C, C, 1)$, where C and λ are constant integers. For a given partition $(U, \overline{U})$ such that $s \in U$, $t \in \overline{U}$, let $d = d_s$, $r = d - C\lfloor d/C \rfloor$, $q = (d - \lambda C \lfloor d/\lambda C \rfloor - r)/C$, and $p = (d - qC - r)/\lambda C$. The following inequalities are valid and facet-defining for the convex hull of their 3-module undirected MMND:

$$\sum_{e \in E_U} \left(y_3^e + r y_2^e + \lambda r y_1^e \right) \geq r \left\lceil \frac{d}{C} \right\rceil, \tag{3.17}$$

$$\sum_{e \in E_U} \left(y_3^e + \min(qC + r, C) y_2^e + (qC + r) y_1^e \right) \geq (qC + r) \left\lceil \frac{d}{\lambda C} \right\rceil, \tag{3.18}$$

$$\sum_{e\in E_U}\left(y_3^e + ry_2^e + r(q+1)y_1^e\right) \geq r(q+1)\left\lceil \frac{d}{\lambda C}\right\rceil. \tag{3.19}$$

These inequalities are referred to as the cutset inequalities. They showed that for certain cost vectors, adding these inequalities to the linear programming relaxation of the 3-module undirected MMND yield integer optimal solutions.

Bienstock and Günlük [2] studied the bidirected MMND with two divisible capacity modules $(C_1, C_2) = (C, 1)$, where C is a constant integer. Given a partition $(U, \overline{U})$ of V and the corresponding cutset polyhedron, let $E_\subset$ be a subset of E_U. Each edge $e \in E_\subset$ can be represented by its two antiparallel arcs e^+ and e^-. Let $A_\subset$ be the set of all such arcs, i.e., $A_\subset = \{e^+ = ij, e^- = ji : e = (i,j) \in E_\subset\}$. Let $A_\subset^+ \subseteq A_\subset$ be the set of arcs that have tails in U and heads in $\overline{U}$, $A_\subset^- = A_\subset \setminus A_\subset^+$, and $r = d - \sum_{e\in E_\subset} g^e - C\left\lfloor (d - \sum_{e\in E_\subset} g^e)/C \right\rfloor$. They introduced the flow-cut-set inequality of the form

$$\sum_{a\in A_U^+\setminus A_\subset^+} x_a + \sum_{e\in E_\subset}(y_2^e + ry_1^e) \geq r\left\lceil \frac{d}{C}\right\rceil. \tag{3.20}$$

They showed that the flow-cut-set inequalities define facets of the convex hull of the 2-module bidirected MMND under certain conditions.

Chopra et al. [3] studied directed MMND problems with the same single-source and single-sink assumption as that in [7]. These problems have $M \leq 2$ modules and no pre-installed capacities. For $M = 2$, the modules $(C_1, C_2) = (C, 1)$, where C is a constant integer. They showed that the 1-module directed MMND problem is NP-hard, and the 2-module directed MMND problem is NP-hard even with zero flow costs. For a given partition $(U, \overline{U})$ of V such that the source $s \in U$ and the sink $t \in \overline{U}$, let d, r be defined the same as those in [7], $A_\subset^+ \subseteq A_U^+$, and $A_\subset^- \subseteq A_U^-$. They showed the following inequality is valid for the 2-module directed MMND:

$$\sum_{a\in A_U^+\setminus A_\subset^+} x_a - \sum_{a\in A_\subset^-} x_a + \sum_{a\in A_\subset^+}\left(y_2^a + ry_1^a\right) + \sum_{a\in A_\subset^-}\left((C-r)y_1^a + y_2^a\right) \geq r\left\lceil \frac{d}{C}\right\rceil. \tag{3.21}$$

(3.21) is referred to as the cut-set inequality in [3]. We refer to (3.21) as the general cutset inequal-

ity.

Atamtürk [12] studied P^d directly (without pre-installed capacities). The general cutset inequality (3.21) was generalized to the multifacility cut-set inequality for P^d with any fixed number of modules. For a given partition $(U, \overline{U})$ of V and the corresponding cutset polyhedron, let $A^+_{\subset} \subseteq A^+_U$, and $A^-_{\subset} \subseteq A^-_U$. The multifacility cut-set inequality has the form

$$\sum_{t=1}^{M} \phi^+_s(C_t) \sum_{a \in A^+_{\subset}} y^a_t + \sum_{t=1}^{M} \phi^-_s(C_t) \sum_{a \in A^-_{\subset}} y^a_t + \sum_{a \in A^+_U \setminus A^+_{\subset}} x_a - \sum_{a \in A^-_{\subset}} x_a \geq r_s \left\lceil \frac{d}{C_s} \right\rceil, \tag{3.22}$$

where for some $s \in \{1, \ldots, M\}$, $r_s = d - C_s \lfloor d/C_s \rfloor$,

$$\phi^+_s(C_t) := \min \left\{ C_t - \left\lfloor \frac{C_t}{C_s} \right\rfloor \left(C_s - (d - C_s \left\lfloor \frac{d}{C_s} \right\rfloor) \right), \left\lceil \frac{C_t}{C_s} \right\rceil \left(d - C_s \left\lfloor \frac{d}{C_s} \right\rfloor \right) \right\},$$
$$\phi^-_s(C_t) := \min \left\{ C_t - \left\lfloor \frac{C_t}{C_s} \right\rfloor \left(d - C_s \left\lfloor \frac{d}{C_s} \right\rfloor \right), \left\lceil \frac{C_t}{C_s} \right\rceil \left(C_s - (d - C_s \left\lfloor \frac{d}{C_s} \right\rfloor) \right) \right\}.$$

Atamtürk [12] showed that the multifacility cut-set inequalities define facets of P^d under certain conditions. Raack et al. [9, 10] generalized these inequalities for the undirected and the bidirected MMND. They also provided conditions under which facet-defining inequalities of P^d, P^u and P^b are also facet-defining for X^d, X^u and X^b.

3.4 n-Step Cutset Inequalities

In this section, we introduce a new class of valid inequalities for X^d and P^d. This class of inequalities is derived for each $n \in \{1, \ldots, M\}$, and we refer to them as the *n-step cutset inequalities*. The most general inequality from this class is the *n-step general cutset inequalities* in Theorem 3.

Theorem 3. *Given a partition $(U, \overline{U})$ of V and the corresponding cutset polyhedron P^d, let $A^+_{\subset} \subseteq A^+_U$, $A^-_{\subset} \subseteq A^-_U$, and $D = d - \sum_{a \in A^+_{\subset}} g^a + \sum_{a \in A^-_{\subset}} g^a$. Given $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} > 0$ and $n \in \{1, \ldots, M\}$, if the n-step MIR conditions* (2.6) *are satisfied, i.e., $\alpha_k \left\lceil D^{(k-1)}/\alpha_k \right\rceil \leq \alpha_{k-1}, k =$*

$2, \ldots, n$, *the* n*-step general cutset inequality*

$$\sum_{t=1}^{M} \mu^n_{\alpha,D}(C_t) \sum_{a \in A^+_{\subset}} y^a_t + \sum_{t=1}^{M} \Big(C_t + \mu^n_{\alpha,D}(-C_t) \Big) \sum_{a \in A^-_{\subset}} y^a_t + \sum_{a \in A^+_U \setminus A^+_{\subset}} x_a - \sum_{a \in A^-_{\subset}} x_a \geq \mu^n_{\alpha,D}(D) - \sum_{a \in A^-_{\subset}} g^a \tag{3.23}$$

is valid for P^d.

Proof. Rewrite the flow conservation constraint (3.11) as

$$\sum_{a \in A^+_{\subset}} x_a + \sum_{a \in A^+_U \setminus A^+_{\subset}} x_a - \sum_{a \in A^-_{\subset}} x_a - \sum_{a \in A^-_U \setminus A^-_{\subset}} x_a = d.$$

Relaxing x_a, $a \in A^+_{\subset}$ using the capacity constraints (3.12), and x_a, $a \in A^-_U \setminus A^-_{\subset}$ using the nonnegativity constraints $x_a \geq 0$, we have

$$\sum_{t=1}^{M} C_t \sum_{a \in A^+_{\subset}} y^a_t + \sum_{a \in A^+_U \setminus A^+_{\subset}} x_a - \sum_{a \in A^-_{\subset}} x_a \geq d - \sum_{a \in A^+_{\subset}} g^a. \tag{3.24}$$

Adding and subtracting the capacity constraints (3.12) for $a \in A^-_{\subset}$, (3.24) can be rewritten as

$$\sum_{t=1}^{M} C_t \sum_{a \in A^+_{\subset}} y^a_t + \sum_{t=1}^{M} (-C_t) \sum_{a \in A^-_{\subset}} y^a_t + \sum_{a \in A^+_U \setminus A^+_{\subset}} x_a + \sum_{a \in A^-_{\subset}} (\sum_{t=1}^{M} C_t y^a_t + g^a - x_a) \geq D. \tag{3.25}$$

We can treat each $\sum_{a \in A^+_{\subset}} y^a_t$ and $\sum_{a \in A^-_{\subset}} y^a_t$ as z_i in K and apply the n-step MIR function $\mu^n_{\alpha,D}(\cdot)$ on coefficients associated with each $\sum_{a \in A^+_{\subset}} y^a_t$ and $\sum_{a \in A^-_{\subset}} y^a_t$. Also, we treat $\sum_{a \in A^+_U \setminus A^+_{\subset}} x_a + \sum_{a \in A^-_{\subset}} (\sum_{t=1}^{M} C_t y^a_t + g^a - x_a)$ as s, and D as b in K. Applying the n-step MIR inequality, we get exactly (3.23). □

Remark 1. *If* $\alpha_1, \ldots, \alpha_n$ *are divisible, then the* n*-step MIR conditions* (2.6) *are automatically satisfied regardless of the value of* D.

Remark 2. *The n-step cutset inequality is also valid for a variant of P^d with variable capacities, where the constraints $x_a \le C_a y^a + g^a, a = 1, \dots, M$ are in place of* (3.12)*. This is because these capacity constraints can be aggregated over $a = 1, \dots, M$ to form capacity constraints in the form of* (3.12) *without pre-installed capacities. A special case of such variant with $A_U^- = \emptyset$ is discussed in [19, 50].*

Special Cases:

- **Cut-set inequality.** The cut-set inequality (3.21) is obtained by setting $n = 1, \alpha = C_1$ in (3.23).

- **Multifacility cut-set inequality.** For the multifacility cut-set inequality (3.22), we note that although the functions $\phi_s^+(\cdot)$ and $\phi_s^+(\cdot)$ depend on the values of all the capacity modules $C_1, \dots, C_M$, they can be derived using 1-step MIR, a single-parameter theory. Given $s \in \{1, \dots, M\}$, this inequality can be obtained by setting $n = 1$ and $\alpha = C_s$ in (3.23). To see this, we substitute $n = 1$ and $\alpha = C_s$ into the MIR function (2.4). Then, by the criteria we partition $\mathbb{R}$ in (2.5), we have that for any $u \in \mathbb{R}_+$,

$$\mu^1_{C_s,D}(u) = \begin{cases} D^{(1)} \left\lceil \frac{u}{C_s} \right\rceil = \left\lceil \frac{u}{C_s} \right\rceil \left(D - C_s \left\lfloor \frac{D}{C_s} \right\rfloor \right) & \text{if } u^{(1)} \ge D^{(1)}, \\ D^{(1)} \left\lfloor \frac{u}{C_s} \right\rfloor + u^{(1)} = u - \left\lfloor \frac{u}{C_s} \right\rfloor \left(C_s - (D - C_s \left\lfloor \frac{D}{C_s} \right\rfloor) \right) & \\ & \text{if } u^{(1)} < D^{(1)}, \end{cases}$$

and

$$\mu^1_{C_s,D}(-u) = \begin{cases} D^{(1)} \left\lceil \frac{-u}{C_s} \right\rceil = -\left(D - C_s \left\lfloor \frac{D}{C_s} \right\rfloor \right) \left\lfloor \frac{u}{C_s} \right\rfloor & \text{if } (-u)^{(1)} \ge D^{(1)}, \\ D^{(1)} \left\lfloor \frac{-u}{C_s} \right\rfloor + (-u)^{(1)} = C_s \left\lceil \frac{u}{C_s} \right\rceil - u - \left(D - C_s \left\lfloor \frac{D}{C_s} \right\rfloor \right) \left\lceil \frac{u}{C_s} \right\rceil & \\ & \text{if } (-u)^{(1)} < D^{(1)}. \end{cases}$$

Substituting the above 1-step MIR function values into (3.23) gives exactly (3.22).

- **n-Step flow cutset inequality.** By setting $A^-_\subset = \emptyset$ in (3.23), we get

$$\sum_{t=1}^{M} \mu^n_{\alpha,D}(C_t) \sum_{a \in A^+_\subset} y^a_t + \sum_{a \in A^+_U \setminus A^+_\subset} x_a \geq \mu^n_{\alpha,D}(D). \tag{3.26}$$

  We refer to (3.26) as the *n-step flow cutset* inequality.

- **n-Step simple cutset inequality.** By setting $A^+_\subset = A^+_U$, $A^-_\subset = \emptyset$ in (3.23), we get

$$\sum_{t=1}^{M} \mu^n_{\alpha,D}(C_t) \sum_{a \in A^+_U} y^a_t \geq \mu^n_{\alpha,D}(D). \tag{3.27}$$

  We refer to (3.27) as the *n-step simple cutset* inequality.

Example 1. *Consider a directed cutset polyhedron P^d with 3 capacity modules, where*

$$P^d = conv\Big\{(x,y) \in \mathbb{R}^6_+ \times \mathbb{Z}^{18}_+ : \\ x_1 + x_2 + x_3 - x_4 - x_5 - x_6 = 31, \\ x_a \leq 20y^a_1 + 15y^a_2 + 7y^a_3, a = 1, \ldots, 6\Big\}$$

with $A^+_U = \{1,2,3\}$, $A^-_U = \{4,5,6\}$, $d = 31$, and $g^a = 0$ for all $a \in A_U$. Let $A^+_\subset = \{1,2\}$ and $A^-_\subset = \{4,5\}$. We illustrate some of the special cases of the n-step general cutset inequality (3.23)*.*

1-step cutset inequalities. *The 1-step simple cutset inequality can be written as $\mu^1_{20,31}(20)(y^1_1 + y^2_1 + y^3_1) + \mu^1_{20,31}(15)(y^1_2 + y^2_2 + y^3_2) + \mu^1_{20,31}(7)(y^1_3 + y^2_3 + y^3_3) \geq \mu^1_{20,31}(31)$, or*

$$11(y^1_1 + y^2_1 + y^3_1) + 11(y^1_2 + y^2_2 + y^3_2) + 7(y^1_3 + y^2_3 + y^3_3) \geq 22. \tag{3.28}$$

The 1-step flow cutset inequality is

$$11(y_1^1 + y_1^2) + 11(y_2^1 + y_2^2) + 7(y_3^1 + y_3^2) + x_3 \geq 22. \tag{3.29}$$

The multifacility cut-set inequalities (3.22) *in [12] are*

$$\begin{aligned}
&11(y_1^1 + y_1^2) + 11(y_2^1 + y_2^2) + 7(y_3^1 + y_3^2) \\
&\qquad +9(y_1^4 + y_1^5) + 9(y_2^4 + y_2^5) + 7(y_3^4 + y_3^5) + x_3 - x_4 - x_5 \geq 22, \\
&2(y_1^1 + y_1^2) + 1(y_2^1 + y_2^2) + 1(y_3^1 + y_3^2) \\
&\qquad +19(y_1^4 + y_1^5) + 14(y_2^4 + y_2^5) + 7(y_3^4 + y_3^5) + x_3 - x_4 - x_5 \geq 3, \\
&9(y_1^4 + y_1^5) + 7(y_2^4 + y_2^5) + 3(y_3^4 + y_3^5) \\
&\qquad +12(y_1^4 + y_1^5) + 9(y_2^4 + y_2^5) + 4(y_3^4 + y_3^5) + x_3 - x_4 - x_5 \geq 15,
\end{aligned} \tag{3.30}$$

for $s = 1, 2$*, and* 3*, respectively. Notice that* (3.30) *is also the cut-set inequality in [3].*

2-step cutset inequalities. *The 2-step simple cutset inequality can be written as* $\mu^2_{(20,15),31}(20)(y_1^1 + y_1^2 + y_1^3) + \mu^2_{(20,15),31}(15)(y_2^1 + y_2^2 + y_2^3) + \mu^2_{(20,15),31}(7)(y_3^1 + y_3^2 + y_3^3) \geq \mu^2_{(20,15),31}(31)$*, which is*

$$11(y_1^1 + y_1^2 + y_1^3) + 11(y_2^1 + y_2^2 + y_2^3) + 7(y_3^1 + y_3^2 + y_3^3) \geq 22.$$

Notice that for this particular example, the 2-step simple cutset inequality is the same as the 1-step simple cutset inequality (3.28)*. Similarly, the 2-step flow cutset inequality is the same as* (3.29)*, and the 2-step general cutset inequality is the same as* (3.30)*.*

3-step cutset inequalities. *The 3-step simple cutset inequality is*

$$8(y_1^1 + y_1^2 + y_1^3) + 8(y_2^1 + y_2^2 + y_2^3) + 4(y_3^1 + y_3^2 + y_3^3) \geq 16.$$

The 3-step flow cutset inequality is

$$8(y_1^1 + y_1^2) + 8(y_2^1 + y_2^2) + 4(y_3^1 + y_3^2) + x_3 \geq 16.$$

The 3-step general cutset inequality is

$$\begin{aligned} &8(y_1^1 + y_1^2) + 8(y_2^1 + y_2^2) + 4(y_3^1 + y_3^2) + 12(y_1^4 + y_1^5) \\ &+11(y_2^4 + y_2^5) + 7(y_3^4 + y_3^5) + x_3 - x_4 - x_5 \geq 16. \end{aligned}$$

3.5 Facet-Defining n-Step Cutset Inequalities

In this section we study the facet-defining properties of the n-step cutset inequalities. Specifically, we give sufficient conditions for the n-step general cutset inequality (3.23), the n-step flow cutset inequality (3.26), and the n-step simple cutset inequality (3.27) to be facet-defining for P^d and X^d. Given a directed cutset polyhedron P^d, let $A^+_\subset \subseteq A^+_U$, $A^-_\subset \subseteq A^-_U$, and $D = d - \sum_{a\in A^+_\subset} g^a + \sum_{a\in A^-_\subset} g^a$. In order to prove the results, we define the following points and directions. Notice that for all directions and points we illustrate below, only nonzero values are mentioned.

Definition 10. *Let i, j, δ, ω be indices of arcs. Define the following points :*

(a) For any $i \in A^+_\subset, j \in A^+_U \setminus A^+_\subset$, the points $\mathcal{A}^{i,j}_l, l = 1, \ldots, n$:

$$y^i_t = \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor, t = 1, \ldots, l, x_i = \sum_{t=1}^{l} \alpha_t \left\lfloor \frac{d^{(t-1)}}{\alpha_t} \right\rfloor + g^i,$$
$$x_{i'} = g^{i'}, i' \in A^+_\subset \setminus \{i\} \cup A^-_\subset, y^j_l = 1, x_j = D^{(l)},$$

and the points $\mathcal{A}_l^{i,j}, l = n+1, \ldots, M$*:*

$$y_t^i = \begin{cases} \left\lfloor \frac{d^{(t-1)}}{\alpha_t} \right\rfloor, \ t = 1, \ldots, n, \\ 1, \qquad\quad t = l, \end{cases} \quad x_i = \sum_{t=1}^{n} \alpha_t \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor + \min\{C_l, D^{(n)}\} + g^i,$$

$$x_{i'} = g^{i'}, i' \in A_{\subset}^{+} \setminus \{i\} \cup A_{\subset}^{-}, y_n^j = 1, x_j = \max\{0, D^{(n)} - C_l\}.$$

(b) For any $i \in A_{\subset}^{+}$*, the points* $\mathcal{B}_l^i, l = 1, \ldots, n$*:*

$$y_t^i = \begin{cases} \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor, \quad t = 1, \ldots, l-1, \\ \left\lceil \frac{D^{(t-1)}}{\alpha_t} \right\rceil, \quad t = l, \end{cases} \quad x_i = D + g^i, x_{i'} = g^{i'}, i' \in A_{\subset}^{+} \setminus \{i\} \cup A_{\subset}^{-},$$

and the points $\mathcal{B}_l^i, l = n+1, \ldots, M$*:*

$$y_t^i = \begin{cases} \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor, \quad t = 1, \ldots, n, \\ 1, \qquad\quad t = l, \end{cases} \quad x_i = D + g^i, x_{i'} = g^{i'}, i' \in A_{\subset}^{+} \setminus \{i\} \cup A_{\subset}^{-}.$$

(c) For any $i \in A_{\subset}^{+}, \delta \in A_{\subset}^{-}, \omega \in A_U^{-} \setminus A_{\subset}^{-}$*, the points* $\mathcal{C}_l^{i,\delta,\omega}, l = 2, \ldots, n$*:*

$$y_t^i = \begin{cases} \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor + 1, \ t = 1, \\ \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor, \ t = 2, \ldots, n+1-l, \\ \left\lceil \frac{D^{(t-1)}}{\alpha_t} \right\rceil - 1, \ t = n+2-l, \\ 0, \ t = n+3-l, \ldots, n, \end{cases} \quad x_i = \sum_{t=1}^{n+2-l} \alpha_t \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor + \alpha_1 + g^i,$$

$$x_{i'} = g^{i'}, i' \in A_{\subset}^{+} \setminus \{i\}, y_l^{\delta} = 1, x_{\delta} = \alpha_l + g^{\delta}, x_{\delta'} = g^{\delta'}, \delta' \in A_{\subset}^{-} \setminus \{\delta\},$$

$$y_{n+2-l}^{\omega} = 1, x_{\omega} = \alpha_1 - \alpha_l - D^{(n+2-l)}.$$

(d) For any $i \in A_{\subset}^{+}, \omega \in A_U^{-} \setminus A_{\subset}^{-}$, the point $\mathcal{F}^{i,\omega}$:

$$y_1^i = \left\lceil \frac{D}{\alpha_1} \right\rceil, x_i = \alpha_1 \left\lceil \frac{D}{\alpha_1} \right\rceil + g^i, x_{i'} = g^{i'}, i' \in A_{\subset}^{+} \setminus \{i\},$$
$$y_1^\omega = 1, x_\omega = \alpha_1 \left\lceil \frac{D}{\alpha_1} \right\rceil - D, x_\delta = g^\delta, \delta \in A_{\subset}^{-}.$$

(e) For any $i \in A_{\subset}^{+}, j \in A_U^{+} \setminus A_{\subset}^{+}, \delta \in A_{\subset}^{-}$, the point $\mathcal{G}^{i,j,\delta}$:

$$y_t^i = \begin{cases} \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor + 1, \ t = 1, \\ \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor, \ t = 2, \ldots, n, \end{cases} \quad x_i = \sum_{t=1}^{n} \alpha_t \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor + \alpha_1 + g^i,$$
$$x_{i'} = g^{i'}, i' \in A_{\subset}^{+} \setminus \{i\}, y_n^j = 1, x_j = D^{(n)}, y_1^\delta = 1, x_\delta = \alpha_1 + g^\delta,$$
$$x_{\delta'} = g^{\delta'}, \delta' \in A_{\subset}^{-} \setminus \{\delta\}.$$

(f) For any $i \in A_{\subset}^{+}, \delta \in A_{\subset}^{-}$, the point $\mathcal{H}^{i,\delta}$:

$$y_t^i = \begin{cases} \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor + 1, & t = 1, \\ \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor, & t = 2, \ldots, n, \end{cases} \quad x_i = \sum_{t=1}^{n} \alpha_t \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor + \alpha_1 + g^i,$$
$$x_{i'} = g^{i'}, i' \in A_{\subset}^{+} \setminus \{i\}, y_1^\delta = 1, x_\delta = \alpha_1 - D^{(n)} + g^\delta,$$
$$x_{\delta'} = g^{\delta'}, \delta' \in A_{\subset}^{-} \setminus \{\delta\}.$$

(g) For $i \in A_U, t = 1, \ldots, n$, the direction $\mathcal{E}_t^i$ where $y_t^i = 1$, $y_{t'}^i = 0$ for $t' \neq t$, and $x_i = 0$ for $i \in A_U$.

Observation 1 is helpful for checking if a point is in P^d.

Observation 1. *Given $n \in \{1, \ldots, M\}$ and $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, for any $t = 1, \ldots, n$, $D = \sum_{k=1}^{t} \alpha_k \left\lfloor D^{(k-1)}/\alpha_k \right\rfloor + D^{(t)} \leq \sum_{k=1}^{t} \alpha_k \left\lfloor D^{(k-1)}/\alpha_k \right\rfloor + \alpha_{t+1} \left\lceil D^{(t)}/\alpha_{t+1} \right\rceil$.*

Next, we provide some lemmas that will be used to prove the main result of this section.

Lemma 1. *Define* $\sum_{t=1}^{0}(\cdot) := 0$ *and* $\prod_{t=1}^{0}(\cdot) := 1$. *Given* $n \in \{1, \ldots, M\}$ *and* $\alpha = \{\alpha_1, \ldots, \alpha_n\}$,

(a) $\sum_{t=1}^{l-1} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor + \prod_{k=l}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil = \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil, l = 1, \ldots, n,$

(b) $\sum_{t=1}^{n} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \left\lfloor \frac{D^{(t-1)}}{\alpha_t} \right\rfloor = \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil - 1.$

Proof. These can be proved similarly to Lemma 6 of [21]. □

Lemma 2. *Given* $n \in \{1, \ldots, M\}$, $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, *and* $D \in \mathbb{R}$ *such that* $D^{(n)} > 0$, *for any* $u \in \mathbb{R}$ *such that* $0 < u \leq \alpha_1$,

(a) If $D^{(s)} \leq u \leq \alpha_s, s \in \{1, \ldots, n\}$, *then* $\mu^n_{\alpha,D}(u) = D^{(n)} \prod_{k=s+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil$.

(b) If $0 < u \leq D^{(n)}$, *then* $\mu^n_{\alpha,D}(u) = u$.

(c) If $u = \alpha_1$, *then* $\mu^n_{\alpha,D}(-u) = -D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil$.

(d) If $D^{(s)} \leq \alpha_1 - u \leq \alpha_s$ *for some* $s \in \{2, \ldots, n\}$, *then* $\mu^n_{\alpha,D}(-u) = D^{(n)} \prod_{k=s+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil - D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil$.

Proof. (a) This is from Lemma 1 of [22].

(b) This is from Lemma 1 of [22].

(c) In this case we have $(-u)^{(1)} = \ldots = (-u)^{(n)} = 0$ and $(-u) \in \mathcal{L}^n_n$, so $\mu^n_{\alpha,D}(-u) = D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \left\lfloor \frac{-u}{\alpha_1} \right\rfloor = -D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \left\lceil \frac{u}{\alpha_1} \right\rceil = -D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil$.

(d) This can be proved similarly to Lemma 1 of [22]. Since $D^{(s)} \leq \alpha_1 - u \leq \alpha_s$, then $(-u)^{(1)} = \ldots = (-u)^{(s-1)} = \alpha_1 - u$. Let q be the smallest integer such that $(-u)^{(q+1)} \geq D^{(q+1)}$ holds, and let $q = n$ otherwise. Thus $(-u) \in \mathcal{L}^n_q$. By definition $D^{(1)} \geq \ldots \geq D^{(n)}$, if $\alpha_1 - u \geq D^{(s)}$, then $q+1 \leq s$. Thus we have $(-u)^{(1)} = \ldots = (-u)^{(q)} = \ldots = (-u)^{(s-1)} = \alpha_1 - u \leq \alpha_s < \ldots < \alpha_q < \ldots < \alpha_1$. Then $\mu^n_{\alpha,D}(-u) = D^{(n)} \sum_{k=1}^{q} \prod_{l=k+1}^{n} \left\lceil \frac{D^{(l-1)}}{\alpha_l} \right\rceil \left\lfloor \frac{(-u)^{(k-1)}}{\alpha_k} \right\rfloor + D^{(n)} \prod_{l=q+2}^{n} \left\lceil \frac{D^{(l-1)}}{\alpha_l} \right\rceil \left\lceil \frac{(-u)^{(q)}}{\alpha_{q+1}} \right\rceil = D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \left\lfloor \frac{-u}{\alpha_1} \right\rfloor + D^{(n)} \prod_{l=q+2}^{n} \left\lceil \frac{D^{(l-1)}}{\alpha_l} \right\rceil$
$= -D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \left\lceil \frac{u}{\alpha_1} \right\rceil + D^{(n)} \prod_{l=q+2}^{n} \left\lceil \frac{D^{(l-1)}}{\alpha_l} \right\rceil = -D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil + D^{(n)} \prod_{l=s+1}^{n} \left\lceil \frac{D^{(l-1)}}{\alpha_l} \right\rceil$.

The last equality holds because if $q \leq s-2$, then $D^{(s-1)} \leq \ldots \leq D^{(q+1)} \leq \alpha_1 - u < \alpha_s < \ldots < \alpha_{q+2}$, then $\left\lceil D^{(l-1)}/\alpha_l \right\rceil = 1, l = q+2, \ldots, s$.

□

Now we are ready to present the main results of this section.

Theorem 4. *Given a directed cutset polyhedron P^d, $n \in \{1, \ldots, M\}$, and $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, let $A^+_\subset \subseteq A^+_U$, and $A^-_\subset \subseteq A^-_U$. The n-step general cutset inequality* (3.23) *is facet-defining for P^d if*

(a) $n = M, \alpha = (C_1, \ldots, C_M)$,

(b) $D^{(n+2-t)} < \alpha_1 - \alpha_t \leq \alpha_{n+2-t}$ *for* $t = 2, \ldots, n$, $D^{(n)} > 0$,

(c) $\frac{D^{(t-1)}}{\alpha_t} < \left\lceil \frac{D^{(t-1)}}{\alpha_t} \right\rceil \leq \frac{\alpha_{t-1}}{\alpha_t}, t = 2, \ldots, n$,

(d) $A^+_\subset \neq \emptyset, A^+_U \setminus A^+_\subset \neq \emptyset, A^-_\subset \neq \emptyset, A^-_U \setminus A^-_\subset \neq \emptyset$.

Proof. Define $\prod_{k=n+2}^{n}(\cdot) := 0$ and $\prod_{k=n+1}^{n}(\cdot) := 1$. Under conditions (a) and (b), substituting the n-step MIR function (2.4) corresponding to the ones of Lemma 2 into (3.23) yields

$$\sum_{t=1}^{n} D^{(n)} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \sum_{a \in A^+_\subset} y^a_t + \sum_{t=1}^{n} \Big(\alpha_t + D^{(n)} \prod_{k=n+3-t}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil$$
$$- D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \Big) \sum_{a \in A^-_\subset} y^a_t + \sum_{a \in A^+_U \setminus A^+_\subset} x_a - \sum_{a \in A^-_\subset} x_a \geq D^{(n)} \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil - \sum_{a \in A^-_\subset} g^a. \tag{3.31}$$

The equality corresponding to (3.31) is

$$\sum_{t=1}^{n} D^{(n)} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \sum_{a \in A^+_\subset} y^a_t + \sum_{t=1}^{n} \Big(\alpha_t + D^{(n)} \prod_{k=n+3-t}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil$$
$$- D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \Big) \sum_{a \in A^-_\subset} y^a_t + \sum_{a \in A^+_U \setminus A^+_\subset} x_a - \sum_{a \in A^-_\subset} x_a = D^{(n)} \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil - \sum_{a \in A^-_\subset} g^a. \tag{3.32}$$

Let

$$\sum_{t=1}^{n} \sum_{a \in A_U} \beta_t^a y_t^a + \sum_{a \in A_U} \pi_a x_a = \theta \tag{3.33}$$

be a hyperplane passing through the face defined by (3.32). We prove (3.33) must be a scalar multiple of (3.32) plus the flow balance equality (3.11).

For $i \in (A_U^+ \setminus A_\subset^+) \cup (A_U^- \setminus A_\subset^-)$, $t = 1, \ldots, n$, consider the direction $\mathcal{E}_t^i$. $\mathcal{E}_t^i$ is an unbounded direction for both P^d and (3.32), and hence a direction for the face defined by (3.33). This implies that $\beta_t^i = 0$ for all $i \in i \in (A_U^+ \setminus A_\subset^+) \cup (A_U^- \setminus A_\subset^-)$, $t = 1, \ldots, n$.

Now, for any $i \in A_\subset^+$, $l = 1, \ldots, n$, and $\omega \in A_U^- \setminus A_\subset^-$, consider the points $\mathcal{B}_l^i$ and $\mathcal{F}^{i,\omega}$. It is easy to check that $\mathcal{B}_l^i$ and $\mathcal{F}^{i,\omega}$ are in P^d by Observation 1, and by (a) of Lemma 1, $\mathcal{B}_l^i$ and $\mathcal{F}^{i,\omega}$ satisfy (3.32). Then $\mathcal{B}_l^i$ and $\mathcal{F}^{i,\omega}$ must satisfy (3.33). Now, for any $i \in A_\subset^+$ and $\omega \in A_U^- \setminus A_\subset^-$, substituting $\mathcal{B}_1^i$ and $\mathcal{F}^{i,\omega}$ into (3.33) and subtracting one equality from the other, we have $\pi_i(\alpha_1 \lceil D/\alpha_1 \rceil - D) + \pi_\omega(\alpha_1 \lceil D/\alpha_1 \rceil - D) = 0$, which implies that $\pi_i = -\pi_\omega$ for $i \in A_\subset^+$, and $\omega \in A_U^- \setminus A_\subset^-$. Now, since all points of P^d satisfy the flow balance equality (3.11), we may add multiples of the flow balance equality to facet-defining inequalities without changing them. Therefore without loss of generality we assume that $\pi_\gamma = 0$ for some $\gamma \in A_\subset^+$. This implies that

$$\pi_i = 0, i \in A_\subset^+ \cup (A_U^- \setminus A_\subset^-). \tag{3.34}$$

Next, for any $i \in A_\subset^+$, $j \in A_U^+ \setminus A_\subset^+$, and $\delta \in A_\subset^-$, consider the points $\mathcal{G}^{i,j,\delta}$ and $\mathcal{H}^{i,\delta}$. It is easy to check that $\mathcal{G}^{i,j,\delta}$ and $\mathcal{H}^{i,\delta}$ are in P^d by Observation 1, and by (b) of Lemma 1, $\mathcal{G}^{i,j,\delta}$ and $\mathcal{H}^{i,\delta}$ satisfy (3.32). Then $\mathcal{G}^{i,j,\delta}$ and $\mathcal{H}^{i,\delta}$ must satisfy (3.33). Now, for any $i \in A_\subset^+$, $j \in A_U^+ \setminus A_\subset^+$, and $\delta \in A_\subset^-$, substituting $\mathcal{G}^{i,j,\delta}$ and $\mathcal{H}^{i,\delta}$ into (3.33) and subtracting one equality from the other, we have $(\pi_\delta + \pi_j)D^{(n)} = 0$, which implies that $\pi_\delta = -\pi_j$ for $\delta \in A_\subset^-$ and $j \in A_U^+ \setminus A_\subset^+$. Thus, there exists $\tau \in A_U^+ \setminus A_\subset^+$ such that

$$\pi_j = \pi_\tau, j \in A_U^+ \setminus A_\subset^+, \pi_\delta = -\pi_\tau, \delta \in A_\subset^-. \tag{3.35}$$

Now, for any $i \in A_{\subset}^{+}$, and $j \in A_U^{+} \setminus A_{\subset}^{+}$, consider the point $\mathcal{A}_n^{i,j}$. It is easy to check that $\mathcal{A}_n^{i,j}$ is in P^d by Observation 1, and by (b) of Lemma 1, $\mathcal{A}_n^{i,j}$ satisfies (3.32). Then $\mathcal{A}_n^{i,j}$ must satisfy (3.33). For any $i \in A_{\subset}^{+}$, and $j \in A_U^{+} \setminus A_{\subset}^{+}$, substituting $\mathcal{A}_n^{i,j}$ and $\mathcal{B}_n^{i}$ into (3.33) and subtracting one equality from the other, we obtain

$$\beta_n^i = \pi_\tau D^{(n)}, i \in A_{\subset}^{+}. \tag{3.36}$$

Now, if we substitute $\mathcal{B}_n^i$, $\mathcal{B}_{n-1}^i$, ..., $\mathcal{B}_1^i$ one after another into (3.33) and subtract one equality from another, we obtain $\beta_{l-1}^i = \beta_l^i \left\lceil D^{(l-1)}/\alpha_l \right\rceil$ for $l = n, n-1, \ldots, 2$, which implies

$$\beta_t^i = \pi_\tau D^{(n)} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil, i \in A_{\subset}^{+}, t = 1, \ldots, n. \tag{3.37}$$

Next, for any $i \in A_{\subset}^{+}$, $j \in A_U^{+} \setminus A_{\subset}^{+}$, and $\delta \in A_{\subset}^{-}$, consider the points $\mathcal{A}_n^{i,j}$ and $\mathcal{H}^{i,\delta}$. Substituting $\mathcal{A}_n^{i,j}$ and $\mathcal{H}^{i,\delta}$ into (3.33), and subtracting one equality from the other, we have

$$\beta_1^\delta = \pi_\tau \big(\alpha_1 - D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \big), \delta \in A_{\subset}^{-}. \tag{3.38}$$

Next, for $i \in A_{\subset}^{+}$, $\delta \in A_{\subset}^{-}$, and $\omega \in A_U^{-} \setminus A_{\subset}^{-}$, consider the points $\mathcal{C}_l^{i,\delta,\omega}$, $l = 2, \ldots, n$. It is easy to check that $\mathcal{C}_l^{i,\delta,\omega}$ is in P^d by Observation 1, and by (a) of Lemma 1, $\mathcal{C}_l^{i,\delta,\omega}$ satisfy (3.32). Then $\mathcal{C}_l^{i,\delta,\omega}$ must satisfy (3.33). Now for $i \in A_{\subset}^{+}$, $\delta \in A_{\subset}^{-}$, and $\omega \in A_U^{-} \setminus A_{\subset}^{-}$, if we substitute $\mathcal{H}^{i,\delta}$ and $\mathcal{C}_2^{i,\delta,\omega}$ into (3.33) and subtract one equality from the other, we have $\beta_2^\delta = \beta_1^\delta + \pi_\tau \big(\alpha_2 - (\alpha_1 - D^{(n)})\big)$, which implies that

$$\beta_2^\delta = \pi_\tau \big(\alpha_2 + D^{(n)} - D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \big), \delta \in A_{\subset}^{-}. \tag{3.39}$$

If we substitute $\mathcal{H}^{i,\delta}$ and $\mathcal{C}_3^{i,\delta,\omega}$ into (3.33) and subtract one equality from the other, we have $\beta_3^\delta = \beta_1^\delta + \pi_\tau \big(D^{(n)} \left\lfloor D^{(n-1)}/\alpha_n \right\rfloor - (\alpha_1 - D^{(n)} - \alpha_3)\big)$, which implies

$$\beta_3^\delta = \pi_\tau \big(\alpha_3 + D^{(n)} \left\lceil \frac{D^{(n-1)}}{\alpha_n} \right\rceil - D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \big), \delta \in A_{\subset}^{-}. \tag{3.40}$$

Substituting $\mathcal{C}_3^{i,\delta,\omega}$, $\mathcal{C}_4^{i,\delta,\omega}$, …, $\mathcal{C}_n^{i,\delta,\omega}$ one after another into (3.33) and subtracting one equality from another, we have $\beta_{l+1}^{\delta} = \beta_l^{\delta} + \pi_\tau\big(D^{(n)} \prod_{k=n+3-l}^{n} \lceil D^{(k-1)}/\alpha_k \rceil \lfloor D^{(n+1-l)}/\alpha_{n+2-l} \rfloor - \alpha_l + \alpha_{l+1}\big)$, $l = 3, \ldots, n-1$, which implies

$$\beta_l^{\delta} = \pi_\tau\big(\alpha_l + D^{(n)} \prod_{k=n+3-l}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil - D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil\big), i \in A_{\subset}^{-}, l = 3, \ldots, n. \tag{3.41}$$

By (3.34)(3.35)(3.36)(3.37)(3.38)(3.39) and (3.41), (3.33) is reduced to

$$\begin{aligned} &\sum_{t=1}^{n} \pi_\tau D^{(n)} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \sum_{a \in A_{\subset}^{+}} y_t^a + \sum_{t=1}^{n} \pi_\tau\big(\alpha_t + D^{(n)} \prod_{k=n+3-t}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \\ &- D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil\big) \sum_{a \in A_{\subset}^{-}} y_t^a + \sum_{a \in A_U^{+} \setminus A_{\subset}^{+}} \pi_\tau x_a - \sum_{a \in A_{\subset}^{-}} \pi_\tau x_a = \theta. \end{aligned} \tag{3.42}$$

Finally, substituting $\mathcal{B}_n^i, i \in A_{\subset}^{+}$ into (3.42), then by (a) of Lemma 1, we have

$$\theta = \pi_\tau\big(D^{(n)} \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil - \sum_{a \in A_{\subset}^{-}} g^a\big), \tag{3.43}$$

which reduces (3.42) to

$$\begin{aligned} &\sum_{t=1}^{n} \pi_\tau D^{(n)} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \sum_{a \in A_{\subset}^{+}} y_t^a \\ &+ \sum_{t=1}^{n} \pi_\tau\big(\alpha_t + D^{(n)} \prod_{k=n+3-t}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil - D^{(n)} \prod_{k=2}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil\big) \sum_{a \in A_{\subset}^{-}} y_t^a \\ &+ \sum_{a \in A_U^{+} \setminus A_{\subset}^{+}} \pi_\tau x_a - \sum_{a \in A_{\subset}^{-}} \pi_\tau x_a = \pi_\tau\big(D^{(n)} \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil - \sum_{a \in A_{\subset}^{-}} g^a\big). \end{aligned} \tag{3.44}$$

(3.44) is a scalar multiple of (3.32) (the scalar is π_τ). This completes the proof. □

Next, we give sufficient conditions for the n-step flow cutset inequality (3.26) to be facet-defining for P^d. Note that although the n-step flow cutset inequality (3.26) is a special case of the n-step general cutset inequality (3.23), facet-defining n-step flow cutset inequalities and facet-

defining n-step general cutset inequalities under Theorem 4 and Theorem 5 will be two separate classes, because the conditions for them to be facet-defining are exclusive.

Theorem 5. *Given a directed cutset polyhedron P^d, $n \in \{1, \ldots, M\}$, and $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, let $A^+_{\subset} \subseteq A^+_U$. The n-step flow cutset inequality* (3.26) *is facet-defining for P^d if*

(a) $\alpha = (C_1, \ldots, C_n)$,

(b) $D^{(n)} > 0$,

(c) $\frac{D^{(t-1)}}{\alpha_t} < \left\lceil \frac{D^{(t-1)}}{\alpha_t} \right\rceil \leq \frac{\alpha_{t-1}}{\alpha_t}, t = 2, \ldots, n$,

(d) $A^+_{\subset} \neq \emptyset, A^+_U \setminus A^+_{\subset} \neq \emptyset$.

Proof. Let s be the index of the last capacity module whose size is larger than $D^{(n)}$, i.e., $s = max\{t \in \{1, \ldots, n\} : \alpha_t > D^{(n)}\}$. Then, we have $\alpha_t > D^{(n)}$, $t = n+1, \ldots, s$ and $\alpha_t \leq D^{(n)}$, $t = s+1, \ldots, M$. Under condition (a), substituting the n-step MIR function (2.4) corresponding to the ones of Lemma 2 into (3.26) yields

$$\begin{aligned} \sum_{t=1}^{n} D^{(n)} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \sum_{a \in A^+_{\subset}} y^a_t + \sum_{t=n+1}^{s} D^{(n)} \sum_{a \in A^+_{\subset}} y^a_t \\ + \sum_{t=s+1}^{M} C_t \sum_{a \in A^+_{\subset}} y^a_t + \sum_{a \in A^+_U \setminus A^+_{\subset}} x_a \geq D^{(n)} \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil . \end{aligned} \tag{3.45}$$

The equality corresponding to (3.45) is

$$\begin{aligned} \sum_{t=1}^{n} D^{(n)} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \sum_{a \in A^+_{\subset}} y^a_t + \sum_{t=n+1}^{s} D^{(n)} \sum_{a \in A^+_{\subset}} y^a_t \\ + \sum_{t=s+1}^{M} C_t \sum_{a \in A^+_{\subset}} y^a_t + \sum_{a \in A^+_U \setminus A^+_{\subset}} x_a = D^{(n)} \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil . \end{aligned} \tag{3.46}$$

Let

$$\sum_{t=1}^{M} \sum_{a \in A_U} \beta^a_t y^a_t + \sum_{a \in A_U} \pi_a x_a = \theta \tag{3.47}$$

be a hyperplane passing through the face defined by (3.46). We prove that (3.47) is a scalar multiple of (3.46).

For $i \in (A_U^+ \setminus A_\subset^+) \cup A_U^-$, $t = 1, \ldots, M$, consider the direction $\mathcal{E}_t^i$. $\mathcal{E}_t^i$ is an unbounded direction for both P^d and (3.46), and hence a direction for the face defined by (3.47). This implies that $\beta_t^i = 0$, $i \in (A_U^+ \setminus A_\subset^+) \cup A_U^-$, $t = 1, \ldots, M$.

Next, for any $i \in A_\subset^+$ and $\omega \in A_U^-$, consider the points $\mathcal{B}_1^i$ and $\mathcal{F}^{i,\omega}$. It is easy to check that $\mathcal{B}_1^i$ and $\mathcal{F}^{i,\omega}$ satisfy (3.46) by (a) of Lemma 1, and by Observation 1, $\mathcal{B}_1^i$ and $\mathcal{F}^{i,\omega}$ are in P^d. Then $\mathcal{B}_1^i$ and $\mathcal{F}^{i,\omega}$ must satisfy (3.47). Substituting $\mathcal{B}_1^i$ and $\mathcal{F}^{i,\omega}$ into (3.47) and subtracting one equality from the other, we have $(\alpha_1 \lceil D/\alpha_1 \rceil - D)(\pi_i + \pi_j) = 0$, which implies that $\pi_i = -\pi_j$ for any $i \in A_\subset^+, \omega \in A_U^-$.

Now, since all points of P^d satisfy the flow balance equality (3.11), we may add multiples of the flow balance equality to facet-defining inequalities without changing them. Therefore without loss of generality we assume that $\pi_\gamma = 0$ for some $\gamma \in as^+$. This implies that

$$\pi_i = 0, i \in A_\subset^+ \cup A_U^-. \tag{3.48}$$

Next, for any $i \in A_\subset^+, j \in A_U^+ \setminus A_\subset^+$, consider the points $\mathcal{B}_n^i$ and $\mathcal{A}_n^{i,j}$. It is easy to check that they are in P^d by Observation 1, and by $(a)(b)$ of Lemma 1, they all satisfy (3.46). Then they must satisfy (3.47). Substituting them into (3.47), and subtracting one equality from the other, we have $\beta_n^i = D^{(n)}\pi_j$. Since this is true for any $j \in A_U^+ \setminus A_\subset^+$, this implies that there exists $\tau \in A_U^+ \setminus A_\subset^+$ such that

$$\pi_j = \pi_\tau, j \in A_U^+ \setminus A_\subset^+, \beta_n^i = D^{(n)}\pi_\tau, i \in A_\subset^+. \tag{3.49}$$

Now, for any $i \in A_\subset^+$, $j \in A_U^+ \setminus A_\subset^+$, consider the points $\mathcal{A}_n^{i,j}$ and $\mathcal{A}_l^{i,j}$, $l \in \{n+1, \ldots, s\}$. It is easy to check they are in P^d by Observation 1, and by $(a)(b)$ of Lemma 1, they all satisfy (3.46). Then they must satisfy (3.47). Substituting them into (3.47), and subtracting one equality from the other, we have

$$\beta_l^i = D^{(n)}\pi_\tau, i \in A_\subset^+, l = n+1, \ldots, s. \tag{3.50}$$

Next, for any $i \in A_{\subset}^{+}$, $j \in A_U^{+} \setminus A_{\subset}^{+}$, consider the points $\mathcal{A}_n^{i,j}$ and $\mathcal{A}_l^{i,j}, l \in \{s+1, \ldots, M\}$. It is easy to check they are in P^d by Observation 1, and by (b) of Lemma 1, they all satisfy (3.46). Then they must satisfy (3.47). Substituting them into (3.47), and subtracting one equality from the other, we have

$$\beta_l^i = C_l \pi_\tau, i \in A_{\subset}^{+}, l = s+1, \ldots, M. \tag{3.51}$$

Next, for any $i \in A_{\subset}^{+}$, consider the points $\mathcal{B}_l^i$, $\mathcal{B}_{l-1}^i$, $l = 2, \ldots, n$. It is easy to check they are in P^d by Observation 1, and by (a) of Lemma 1, they all satisfy (3.46). Then they must satisfy (3.47). Substituting them one after another into (3.47), and subtracting one equality from the other, we have $\mathcal{B}_{l-1}^i = \mathcal{B}_l^i \left\lceil D^{(l-1)}/\alpha_l \right\rceil$, $l = 2, \ldots, n$. Since $\beta_n^i = D^{(n)}\pi_\tau$, this implies that

$$\beta_l^i = D^{(n)} \prod_{t=l+1}^{n} \left\lceil \frac{D^{(t-1)}}{\alpha_t} \right\rceil \pi_\tau, i \in A_{\subset}^{+}, l = 1, \ldots, n. \tag{3.52}$$

So far, (3.47) has been reduced to

$$\begin{aligned} \sum_{t=1}^{n} D^{(n)}\pi_\tau \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \sum_{a \in A_{\subset}^{+}} y_t^a + \sum_{t=n+1}^{s} D^{(n)}\pi_\tau \sum_{a \in A_{\subset}^{+}} y_t^a \\ + \sum_{t=s+1}^{M} C_t \pi_\tau \sum_{a \in A_{\subset}^{+}} y_t^a + \sum_{a \in A_U^{+} \setminus A_{\subset}^{+}} \pi_\tau x_a = \theta. \end{aligned} \tag{3.53}$$

Finally, substituting $\mathcal{B}_1^i$ for some $i \in A_{\subset}^{+}$ into (3.53), we have $\theta = D^{(n)}\pi_\tau \prod_{k=1}^{n} \left\lceil D^{(k-1)}/\alpha_k \right\rceil$, which reduces (3.53) to

$$\begin{aligned} \sum_{t=1}^{n} D^{(n)}\pi_\tau \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \sum_{a \in A_{\subset}^{+}} y_t^a + \sum_{t=n+1}^{s} D^{(n)}\pi_\tau \sum_{a \in A_{\subset}^{+}} y_t^a \\ + \sum_{t=s+1}^{M} C_t \pi_\tau \sum_{a \in A_{\subset}^{+}} y_t^a + \sum_{a \in A_U^{+} \setminus A_{\subset}^{+}} \pi_\tau x_a = D^{(n)}\pi_\tau \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil. \end{aligned} \tag{3.54}$$

(3.54) is a scalar multiple of (3.46) (the scalar is π_τ). This completes the proof. □

Finally, we give sufficient conditions for the n-step simple cutset inequalities (3.27) to be facet-

defining for P^d.

Theorem 6. *Given a directed cutset polyhedron P^d, $n \in \{1, \ldots, M\}$, and $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, the n-step simple cutset inequality* (3.27) *is facet-defining for P^d if*

(a) $\alpha = (C_1, \ldots, C_n)$,

(b) $0 < D^{(n)} \leq \alpha_t, t = 1, \ldots, M$,

(c) $\frac{D^{(t-1)}}{\alpha_t} < \left\lceil \frac{D^{(t-1)}}{\alpha_t} \right\rceil \leq \frac{\alpha_{t-1}}{\alpha_t}, t = 2, \ldots, n$.

Proof. Under conditions (a) and (b), substituting the n-step MIR function (2.4) corresponding to the ones of Lemma 2 into (3.27) yields

$$\sum_{t=1}^{n} D^{(n)} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \sum_{a \in A_U^+} y_t^a + \sum_{t=n+1}^{M} D^{(n)} \sum_{a \in A_U^+} y_t^a \geq D^{(n)} \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil. \tag{3.55}$$

The equality corresponding to (3.55) is

$$\sum_{t=1}^{n} D^{(n)} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \sum_{a \in A_U^+} y_t^a + \sum_{t=n+1}^{M} D^{(n)} \sum_{a \in A_U^+} y_t^a = D^{(n)} \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil. \tag{3.56}$$

Let

$$\sum_{t=1}^{M} \sum_{a \in A_U} \beta_t^a y_t^a + \sum_{a \in A_U} \pi_a x_a = \theta \tag{3.57}$$

be a hyperplane passing through the face defined by (3.56). We prove that (3.57) is a scalar multiple of (3.56).

For $i \in A_U^-$, $t = 1, \ldots, M$, consider the direction $\mathcal{E}_t^i$. $\mathcal{E}_t^i$ is an unbounded direction for both P^d and (3.56), and hence a direction for the face defined by (3.57). This imples that $\beta_t^i = 0$, $i \in A_U^-$, $t = 1, \ldots, M$.

Next, for any $i \in A_U^+$ and $\omega \in A_U^-$, consider the points $\mathcal{B}_1^i$ and $\mathcal{F}^{i,\omega}$. By similar argument to the proof of Theorem 5, we have $\pi_i = -\pi_j$ for any $i \in A_U^+, \omega \in A_U^-$. Now since we may

add multiples of the flow balance equality to facet-defining inequalities without changing them, by similar argument to the proof of Theorem 5, we have $\pi_i = 0$, $i \in A_U$.

Next, for any $i \in A_U^+$, consider the points $\mathcal{B}_l^i$, $l = 1, \ldots, n$. By similar argument to the proof of Theorem 5, if we substitute $\mathcal{B}_n^i$ and $\mathcal{B}_{n-1}^i$ into (3.57) and subtract one equality from the other, we have $\beta_{n-1}^i = \left\lceil d^{(n-1)}/\alpha_n \right\rceil \beta_n^i$. If we substitute $\mathcal{B}_n^i$, $\mathcal{B}_{n-1}^i$, ..., $\mathcal{B}_1^i$ one after another into (3.57) and subtract one equality from another, we have $\beta_{l-1}^i = \left\lceil D^{(l-1)}/\alpha_l \right\rceil \beta_l^i$, $l = 2, \ldots, n$, which implies $\beta_l^i = \prod_{k=l+1}^n \left\lceil D^{(k-1)}/\alpha_k \right\rceil \beta_n^i$, $i \in A_U^+$, $l = 1, \ldots, n$.

Next, for any $i \in A_U^+$, consider the points $\mathcal{B}_l^i$, $l = n+1, \ldots, M$. By similar arguments to the proof of Theorem 5, $\mathcal{B}_l^i$ satisfy (3.57). If we substitute $\mathcal{B}_n^i$ and $\mathcal{B}_l^i$ into (3.57), and subtract one equality from the other, we have $\beta_l^i = \beta_n^i, i \in A_U^+, l = n+1, \ldots, M$.

Now, if $|A_U^+| = 1$, (3.57) is reduced to

$$\sum_{t=1}^{n} \beta_n^i \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil y_t^i + \sum_{t=n+1}^{M} \beta_n^i y_t^i = \theta \tag{3.58}$$

for some $i \in A_U^+$. Otherwise if $|A_U^+| > 1$, then for any $i, j \in A_U^+$, if we substitute the points $\mathcal{B}_1^i$ and $\mathcal{B}_1^j$ into (3.57) and subtract one equality from the other, we have $\beta_1^i = \beta_1^j$. Therefore $\beta_l^i = \beta_l^j$, $l = 1, \ldots, M$. Since our choices of i and j are arbitrary, there exists some $\tau \in A_U^+$ such that $\beta_l^i = \beta_l^\tau$, $l = 1, \ldots, M$ for any $i \in A_U^+$. Then (3.57) is reduced to

$$\sum_{t=1}^{n} \beta_n^\tau \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \sum_{a \in A_U^+} y_t^a + \sum_{t=n+1}^{M} \beta_n^\tau \sum_{a \in A_U^+} y_t^a = \theta, \tag{3.59}$$

which is the same as (3.58).

Finally, if we substitute $\mathcal{B}_1^\tau$ into (3.59), we have $\theta = \prod_{k=1}^{n} \left\lceil D^{(k-1)}/\alpha_k \right\rceil \beta_n^\tau$, which reduces (3.59) to

$$\sum_{t=1}^{n} \beta_n^\tau \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil \sum_{a \in A_U^+} y_t^a + \sum_{t=n+1}^{M} \beta_n^\tau \sum_{a \in A_U^+} y_t^a = \beta_n^\tau \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{\alpha_k} \right\rceil. \tag{3.60}$$

(3.60) is a scalar multiple of (3.56) (the scalar is $\beta_n^\tau / D^{(n)}$). This completes the proof. □

Example 1 (continued)**.** *Among the inequalities mentioned in **Example 1**, the 3-step general cut-set inequality is facet-defining for P^d by Theorem 4. The 1- and 3-step flow cutset inequalities are facet-defining for P^d by Theorem 5. The 1- and 3-step simple cutset inequalities are facet-defining for P^d by Theorem 6. The 1-step general cutset inequalities, i.e. the multifacility cut-set inequalities are facet-defining for P^d by the results in [12].*

The n-step cutset inequalities are not only facet-defining for P^d, but also for X^d when proper conditions are satisfied. This is straightforward by the result of Raack et al. in [9]. Note that this result was presented for MMND assuming no pre-installed capacities on arcs, i.e., $g^a = 0$ for all $a \in A$, but it still holds without changing the proof for MMND assuming pre-installed capacities are present.

Lemma 3 ([9])**.** *Let $\emptyset \subset U \subset V$, and for any $V' \subset V$, define $G[V'] = (V', A_{V'})$ where $A_{V'} = \{a = ij \in A : i, j \in V'\}$. Let*

$$\sum_{t=1}^{M} \sum_{a \in A} \beta_t^a y_t^a + \sum_{a \in A} \pi_a x_a = \theta$$

be a facet-defining inequality of P^d. Then it is also facet-defining for X^d if both $G[U]$ and $G[\overline{U}]$ are strongly connected.

By this lemma, together with Theorem 4, 5, and 6, we have the following:

Corollary 1. *The n-step general cutset (resp. flow cutset, simple cutset) inequality is facet-defining for X^d if in addition to the conditions in Theorem 4 (resp. 5, 6), $G[U]$ and $G[\overline{U}]$ are strongly connected.*

3.6 Computational Results

In this section, we test the effectiveness of the n-step cutset inequalities on our randomly generated test instances. We illustrate the random graph generation procedure in Section 3.6.1, the separation heuristic in Section 3.6.2, and the experimental setup and our results in Section 3.6.3. We note that in our computations we assume no pre-installed capacities on arcs, i.e., $g^a = 0, a \in A$.

3.6.1 Graph Generation

We first generate random graphs for MMND. Our idea of generating random graphs is similar to the ones in [8, 11]. We generate graphs with 50 nodes, where the coordinates of the nodes are uniformly distributed on a 100×100 region in the Euclidean plane. 5 of the nodes are randomly chosen to be sources, and 30 of the rest of the nodes are randomly chosen to be sinks. Based on [11], real-life graphs should have low arc degree, and arcs with large length should be avoided. Therefore, we randomly choose an out degree for each node to be equal to 2 or 5 with a probability of 0.2, and 3 or 4 with a probability of 0.3. For source nodes, we then add its out degree by 10. The maximum length of an arc is set to be 50. For each node, we add a directed arc from it to the node closest to it, until its degree requirement is satisfied, or there are no more nodes within range of length 50. Then, for each source node, we check if every other node of the graph can be reached from this source node. If not, then we reject this graph and generate a new one. We add parameters and create an instance of MMND only if a valid graph is generated.

3.6.2 Separation

Given an LP relaxation optimal solution $(\hat{x}, \hat{y})$ of an MMND instance, the number of n-step general cutset inequalities (3.23) is exponential with respect to A_U, A_C^+, A_C^-, n, and α. Finding the most violated inequality with respect to A_U, A_C^+, A_C^-, n, and α simultaneously is an NP-hard problem even for the special case where $M = 1$ with a single source and a single sink [12]. In our computations, we assume α and n are given, and we use a simple heuristic to determine A_U (see Section 3.6.3). Given A_U, n and α, finding the most violated n-step general cutset inequality can be done in linear time by setting A_C^+ and A_C^- as follows:

$$A_C^+ = \Big\{ a \in A_U^+ : \sum_{t=1}^{M} \mu_{\alpha,d}^n(C_t)\hat{y_t^a} \leq \hat{x_a} \Big\},$$

$$A_C^- = \Big\{ a \in A_U^- : \sum_{t=1}^{M} (C_t + \mu_{\alpha,d}^n(-C_t))\hat{y_t^a} < \hat{x_a} \Big\}.$$

However, previous computational efforts [51, 2, 10] and our tests on cutset inequalities indicate that the following aspects should be considered when adding cutset inequalities:

- Simple cutset inequalities contribute the most on reducing time and integrality gap for network design problems.
- The most violated cutset inequalities do not necessarily perform the best in computational tests.
- Adding too many general cutset inequalities to the formulation can lead to unacceptable CPU time to solve the problem.

In this dissertation, we design a new separation heuristic that focuses on the following:

- It prefers generating violated n-step simple cutset inequalities rather than n-step flow and general cutset inequalities.
- It prefers generating violated n-step general cutset inequalities with the least number of flow variables rather than the most violated n-step general cutset inequality.
- It limits the number of general cutset inequalities added to the formulation.

In our separation, we consider the n-step simple cutset inequalities and the n-step general cutset inequalities hierarchically. For each given A_U, n, and α, we first check if the corresponding n-step simple cutset inequality is violated by $(\hat{x}, \hat{y})$. If so, we add the n-step simple cutset inequality to the formulation. Notice that the n-step simple cutset inequality is a special case of the n-step general cutset inequality where $A^+_\subset = A^+_U$, $A^+_U \setminus A^+_\subset = \emptyset$, $A^-_\subset = \emptyset$, and $A^-_U \setminus A^-_\subset = A^-_U$. Therefore, if the n-step simple cutset inequality is not violated, we want to construct a violated n-step general cutset inequality by adaptively moving the arcs from $A^+_\subset$ to $A^+_U \setminus A^+_\subset$, and from $A^-_U \setminus A^-_\subset$ to $A^-_\subset$. We choose such arcs based on the following criteria. Let s_N be the slack of the n-step simple cutset inequality, which is calculated by $s_N = \sum_{t=1}^{M} \mu^n_{\alpha,D}(C_t) \sum_{a \in A^+_U} \hat{y}^a_t - \mu^n_{\alpha,D}(D)$. Let w_a be the slack

for arc $a \in A_U$, which is calculated by

$$w_a = \begin{cases} \hat{x_a} - \sum_{t=1}^{M} \mu_{\alpha,b}^n(C_t)\hat{y_t^a}, a \in A_U^+ \\ \sum_{t=1}^{M}(C_t + \mu_{\alpha,b}^n(-C_t))\hat{y_t^a} - \hat{x_a}, a \in A_U^-. \end{cases} \tag{3.61}$$

We sort w_a values in ascending order. Starting from the smallest w_a, we do the following: first we check if $w_a < 0$. If $w_a \geq 0$ and $s_N \geq 0$, then we conclude that no violated inequality can be obtained for the current set of A_U, n, and α. If $w_a < 0$, then we move the corresponding arc a from A_C^+ to $A_U^+ \setminus A_C^+$ if $a \in A_C^+$, or from $A_U^- \setminus A_C^-$ to A_C^- if $a \in A_U^- \setminus A_C^-$. By doing so, the slack s_N of the resulting inequality is decreased by $-w_a$, so we set $s_N = s_N + w_a$. If we still have $s_N \geq 0$, then we repeat the above process with the next smallest w_a, until $s_N < 0$, at which point we have a violated n-step general cutset inequality. Notice if $A_C^- = \emptyset$ in the resulting inequality, then we have a violated n-step flow cutset inequality.

In our computations, we noticed that violated n-step general cutset inequalities can be found for most combinations of A_U, n, and α. Adding all of them to the formulation lead to unacceptable CPU time to solve the instances. We use a technique similar to that in [52] to select a small number of inequalities to add to the formulation. It is based on calculating the orthogonality of the newly found cuts with respect to previously added cuts. The goal is to select a nearly orthogonal subset of cutting planes, which cut as deep as possible into the current LP relaxation polyhedron. Let N be the coefficient vector of the newly found violated n-step general cutset inequality, and $\mathcal{R}$ be the set of coefficient vectors of all previously added cuts to the formulation. The orthogonality of N with respect to any $R \in \mathcal{R}$ is calculated by $o_R = |R^T * N|/||R|| \cdot ||N||$, and the orthogonality of N with respect to the set of all previously added cuts $\mathcal{R}$ is defined as $o_N = \max_{R \in \mathcal{R}} o_R$. We only add the newly generated cut to the formulation if o_N is less than or equal to a fixed threshold. In our computations the threshold is tuned to be 0.3.

The above cut generating procedure is summarized in Algorithm 1.

Algorithm 1 Cut Separation and Selection

Input: Current LP relaxation solution $(\hat{x}, \hat{y})$, a set of previously added cuts $\mathcal{R}$, and A_U, α, n
Output: Coefficients $N = (\pi, \beta) \in \mathbb{R}^{|A|} \times \mathbb{R}^{M|A|}$ of a cut

1. Let $(\pi, \beta) = 0$
2. Let $\beta_t^a = \mu_{\alpha,D}^n(C_t), a \in A_U^+, t = 1, \ldots, M$
3. If $\sum_{t=1}^{M} \beta_t^a \sum_{a \in A_U^+} \hat{y}_t^a < \mu_{\alpha,D}^n(D)$

   Stop and output (π, β)

   Else

   Let $s_N = \sum_{t=1}^{M} \mu_{\alpha,D}^n(C_t) \sum_{a \in A_U^+} \hat{y}_t^a - \mu_{\alpha,D}^n(D)$ and go to 4
4. Let w_a be calculated as in (3.61) for $a \in A_U$
5. While $s_N \geq 0$

   If $\min_{a \in A_U}\{w_a\} \geq 0$

   Stop; no cutset cut can be generated

   Else

   Let $\bar{a} = argmin_{a \in A_U}\{w_a\}$
   If $\bar{a} \in A_U^+$ then let $\beta_t^{\bar{a}} = 0, t = 1, \ldots, M$, $\pi_{\bar{a}} = 1$
   If $\bar{a} \in A_U^-$ then let $\beta_t^{\bar{a}} = C_t + \mu_{\alpha,D}^n(-C_t), t = 1, \ldots, M$, $\pi_{\bar{a}} = -1$
   Let $s_N = s_N + w_{\bar{a}}$
   Let $w_{\bar{a}} = \infty$
6. For each $R \in \mathcal{R}$

   Let $o_R = |R^T * N|/||R|| \cdot ||N||$
   If $o_R > \mathbf{threshold}$

   Stop; no cutset cut is added
7. Stop and output (π, β)

3.6.3 Experimental Setup And Results

We first generated random graphs following the steps of Section 3.6.1. Once a valid graph was generated, we added the parameters to the problem. The demand of each sink node $d_v, v \in T$ was chosen from *uniform*[-190,-10]. The negative of the aggregated demand over all sinks was then randomly split among the sources. The unit flow cost h_a for each arc $a \in A$ was equal to its length, rounding down to the nearest integer. For each 2-module MMND instance, we assigned to it one of the 3 sets of capacity modules: (130,50), (170,70), and (200,80). We also assigned to it one of the 2 sets of costs associated with these capacity modules: (10000,5000) and (18000,9000) (we assumed the module cost to be the same for every arc, i.e., $f_t^a = f_t$, $a \in A$). For each 3-module MMND instance, we assigned to it one of the 3 sets of capacity module types: (130,50,20), (170,70,30), and (200,80,30). We also assigned to it one of the 3 sets of costs associated with these capacity modules: (10000,5000,2500), (18000,9000,5000), and (25000,13000,9000). The summary of the instances is listed in Table 3.1.

Table 3.1: Summary of MMND instances with 2 modules

Instance	Module Sizes	Module Costs
2_1_1	(130, 50)	(10000, 5000)
2_1_2	(130, 50)	(18000, 9000)
2_2_1	(170, 70)	(10000, 5000)
2_2_2	(170, 70)	(18000, 9000)
2_3_1	(200, 80)	(10000, 5000)
2_3_2	(200, 80)	(18000, 9000)
3_1_1	(130, 50, 20)	(10000, 5000, 2500)
3_1_2	(130, 50, 20)	(18000, 9000, 5000)
3_1_2	(130, 50, 20)	(25000, 13000, 9000)
3_1_1	(170, 70, 30)	(10000, 5000, 2500)
3_1_2	(170, 70, 30)	(18000, 9000, 5000)
3_1_2	(170, 70, 30)	(25000, 13000, 9000)
3_1_1	(200, 80, 30)	(10000, 5000, 2500)
3_1_2	(200, 80, 30)	(18000, 9000, 5000)
3_1_2	(200, 80, 30)	(25000, 13000, 9000)

We tested the n-step general cutset inequalities on 2-module and 3-module MMND instances. For each 2-module and 3-module MMND instance, we performed several runs. In the first run, we

solved it using CPLEX 12.7 in its default settings. The corresponding results are under label DEF in Table 3.2 and 3.3. For other runs, we added the n-step cutset inequalities *a priori* to the formulation and solved the instance with the added cuts using CPLEX in its default settings. We added different special cases of the n-step general cutset inequalities in separate runs and compared the performance of these cuts. For 2-module MMND instances, we considered two sets of parameters, which are $\{n = 1, \alpha = C_1\}$ and $\{n = 2, \alpha = (C_1, C_2)\}$. The two sets correspond to the 1-step general cutset inequalities and the 2-step general cutset inequalities, respectively. The corresponding results are under labels 1CUT and 2CUT in Table 3.2 and 3.3. For 3-module MMND instances, we considered an additional set of parameters, which is $\{n = 3, \alpha = (C_1, C_2, C_3)\}$. This corresponds to 3-step general cutset inequalities, and the corresponding results are under label 3CUT in Table 3.3.

In terms of A_U, we considered all partitions $(U, \overline{U})$ of V where $1 \leq |U| \leq 4$. This was based on the fact in [12] and our computations that most violated inequalities are generated from uneven partitions. Given n, α, and the LP relaxation optimal solution $(\hat{x}, \hat{y})$, for each A_U, the separation procedure in Section 3.6.2 was called and at most one violated inequality was generated. The LP relaxation problem was re-optimized when a new cut was added to the formulation. Then the next choice of A_U was considered with the updated LP relaxation optimal solution, and the separation was called again. This process was repeated for all of our choices of A_U. After adding the cuts, the cuts that are inactive at the final LP relaxation optimal solution were removed, and the instance was solved using CPLEX in its default settings.

We implemented the cutting plane algorithm in C++ with CPLEX 12.7. All the experiments were run on a PC with Intel Core i7 2.50GHz processor with 4 cores and 16 GB of RAM. The time limit for CPLEX was set to be 2 hours. The results are listed in Table 3.2 and 3.3.

Table 3.2 summarizes the computational results on the 2-module MMND instances. Each row reports the average results for 10 instances of the corresponding instance category.

We report the following statistics if applicable: under DEF, the time (in seconds) to solve the instance (T); the number of branch-and-bound nodes by CPLEX (*Nodes*); the initial integrality

Table 3.2: Results of computational experiments on 2MND instances

Instance	DEF			1CUT						2CUT					
	T_{Opt}	*Nodes*	G_0	*Cuts*	T_{Cut}	T_{Opt}	T	*Nodes*	*CG%*	*Cuts*	T_{Cut}	T_{Opt}	T	*Nodes*	*CG%*
2_1_1	458	1713521	17	45	14	106	120	502724	46	83	35	122	154	486370	77
2_1_2	672	2490527	18	46	14	313	327	1520851	46	83	38	390	417	1280358	77
2_2_1	569	2200308	22	44	15	126	142	575316	47	76	40	110	150	472718	79
2_2_2	554	3363122	24	45	15	374	390	1920436	47	77	42	194	236	778173	79
2_3_1	279	1405855	22	50	18	391	408	1375045	51	92	44	56	101	214235	81
2_3_2	762	3425010	24	48	17	536	553	2937331	51	90	44	36	81	131356	82

Table 3.3: Results of computational experiments on 3MND instances

Instance	DEF			1CUT						2CUT						3CUT					
	T_{Opt}	*Nodes*	G_0	*Cuts*	T_{Cut}	T_{Opt}	T	*Nodes*	*CG%*	*Cuts*	T_{Cut}	T_{Opt}	T	*Nodes*	*CG%*	*Cuts*	T_{Cut}	T_{Opt}	T	*Nodes*	*CG%*
3_1_1	953	3184686	15	62	26	1493	1519	3859727	54	82	45	771	816	2539234	57	85	43	223	266	643022	72
3_1_2	804	1121292	17	63	28	1471	1499	3440327	54	91	55	955	1010	1858099	62	89	49	478	527	1209506	73
3_1_3	827	1751619	20	62	26	598	624	1292236	56	94	55	414	469	926358	72	94	47	165	212	356813	75
3_2_1	1105	6626769	14	61	20	476	496	1968267	49	74	36	551	587	2860707	52	84	42	208	250	904037	73
3_2_2	1056	4360067	16	62	22	428	450	1457589	50	82	42	457	499	1655077	59	90	45	344	389	1170541	74
3_2_3	826	1438662	19	64	24	866	889	2164659	53	92	61	783	844	1074849	70	89	44	624	667	1636625	74
3_3_1	31	122339	19	58	20	36	56	128792	52	96	56	41	97	161283	60	88	38	18	57	61685	77
3_3_2	66	306930	21	65	27	58	85	201923	53	99	63	38	102	88630	65	93	40	29	69	70178	77
3_3_3	75	257924	25	69	29	82	112	248610	57	101	67	65	133	149331	74	97	46	79	125	147462	77

gap, calculated by $G_0 = 100 \times (zmip - zlp)/zmip$, where zlp, and $zmip$ are the optimal objective values of the LP relaxation and MIP, respectively. For each type of cut, we report the number of active cuts added at the root node (*Cuts*); the number of branch-and-bound nodes reported by CPLEX (*Nodes*); the percentage of the integrality gap closed by our cuts, i.e., $CG\% = 100 \times (zcut - zlp)/(zmip - zlp)$, where zlp, $zcut$, and $zmip$ are the optimal objective values of the LP relaxation without the cuts, LP relaxation with the cuts, and MIP, respectively; the time (in seconds) to generate the customized cuts (T_{Cut});the time (in seconds) to solve the instance excluding the cut generation time (T_{Opt}); and the total solution time including the cut generation time (T). In DEF, $T = T_{Opt}$.

For 2-module MMND instances, we noticed significant improvement in the time and nodes required to solve the instances by adding the 2-step cutset inequalities. On average, the gap closed by the 2-step cutset inequalities is 79.2%. The average total solution time (including cut generation) with our 2-step cuts was 0.35 times that of CPLEX 12.7 in default settings, and the number of branch-and-bound nodes was 0.23 times that of the default CPLEX. The best performance was on the category with capacity modules (200,80) and module costs (18000,9000), where the average total solution time with the 2-step cuts was 0.11 times that of CPLEX 12.7 in its default settings, and the number of branch-and-bound nodes was 0.04 times that of the default CPLEX.

Furthermore, in 4 of 6 categories, the 2-step cuts outperformed the 1-step cuts in terms of the solution time without cut-generation time T_{Opt}, and in 3 of them the 2-step cuts also had advantages in terms of the total solution time T. For all categories, the instances with the 2-step cuts require less number of nodes to solve than the instances with the 1-step cuts. On average, the total solution time (including cut generation) with our 2-step cuts was 0.59 times that with the 1-step cutset inequalities, and the number of branch-and-bound nodes was 0.38 times that with the 1-step cutset inequalities. The integrality gap closed by our 2-step cuts was 1.6 times that closed by the 1-step cuts.

Table 3.3 summarizes the results on the 3-module MMND instances. For instances with the capacity modules (200,80,30) which were easier to solve, the average total solution time by adding

the 3-step cuts was slightly worse than that with CPLEX in its default settings because of relatively long cut generation time. For harder instances, however, the improvement by adding the 3-step cuts was significant over CPLEX in its default settings. On average, the total solution time (including cut generation) with our 3-step cuts was 0.45 times that with CPLEX in default settings, 0.45 times that with only 1-step cuts added, and 0.56 times that with only 2-step cuts added. The number of branch-and-bound nodes with our 3-step cuts was 0.32 times that with default CPLEX, 0.42 times that with only 1-step cuts, and 0.55 times that with only 2-step cuts. The gap closed by the 3-step cuts was 74.8%, which was 1.4 times that with the 1-step cuts, and 1.2 times that with the 2-step cuts.

Therefore, we conclude that the 2-step cutset inequalities are very effective in solving 2-module MMND instances, and 3-step cutset inequalities are very effective in solving 3-module MMND instances. Moreover, they are more effective than the n-step general cutset inequalities that use information of less modules. We expect the n-step cutset inequalities to be effective also on MMND instances with more capacity modules.

3.7 n-Step Cutset Inequalities for Undirected And Bidirected MMND

Our results for the directed MMND can be easily generalized for the undirected and the bidirected MMND. Given a nonempty partition $(U, \overline{U})$ of V and the corresponding cutset polyhedra P^u or P^b defined in Section 3.3, let $S_1, S_2 \subseteq E_U$. Each edge $e \in E_U$ is represented by its two antiparallel arcs e^+ and e^-. Let A_1 be the set of such arcs, $A_1^+ \subseteq A_1$ be the set of arcs in A_1 who have tails in U and heads in $\overline{U}$, and $A_1^- = A_1 \setminus A_1^+$ (and define A_2, A_2^+ and A_2^- similarly for S_2). We have the following.

Theorem 7. *Given a nonempty partition $(U, \overline{U})$ of V, let $S_1, S_2 \subseteq E_U$. Define $D = \sum_{v \in U} d_v - \sum_{e \in S_1} g^e + \sum_{e \in S_2} g^e$. Given $n \in \{1, \ldots, M\}$, $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} > 0$, if the n-step MIR conditions* (2.6) *are satisfied, i.e., $\alpha_k \left\lceil D^{(k-1)}/\alpha_k \right\rceil \leq \alpha_{k-1}, k = 2, \ldots, n$, the n-step general cutset*

inequality

$$\sum_{t=1}^{M} \mu^n_{\alpha,D}(C_t) \sum_{e\in S_1} y^e_t + \sum_{t=1}^{M} \Big(C_t + \mu^n_{\alpha,D}(-C_t)\Big) \sum_{e\in S_2} y^e_t + \sum_{a\in A_U^+\setminus A_1^+} x_a - \sum_{a\in A_2^-} x_{ij} \geq \mu^n_{\alpha,D}(D) - \sum_{e\in S_2} g^e \tag{3.62}$$

is valid for X^u and X^b.

Proof. We show it for X^u and X^b respectively. For X^u, if we aggregate (3.6) for $v \in U$, we have

$$\sum_{a\in A_U^+} x_a - \sum_{a\in A_U^-} x_a = \sum_{v\in U} d_v. \tag{3.63}$$

Adding the inequalities $x_a \geq 0, a \in A_U^+ \setminus A_2^+$ to (3.63), (3.63) can be written as

$$\sum_{a\in A_1^+} x_a + \sum_{a\in A_U^+\setminus A_1^+} x_a - \sum_{a\in A_2^-} x_a \geq \sum_{v\in U} d_v. \tag{3.64}$$

The rest of the proof is similar to that of Theorem 3.

For X^b, as mentioned in Section 3.3, X^b is a special case of X^d where the arcs sharing the same edge have the same capacity. Therefore the n-step general cutset inequality (3.23) is valid for X^b. Let $A_1^+ = A_\subset^+$, $A_2^- = A_\subset^-$, $S_1 = \{e : e^+ \in A_1^+ \text{ or } e^- \in A_1^+\}$, and $S_2 = \{e : e^+ \in A_2^+ \text{ or } e^- \in A_2^+\}$. Then (3.23) becomes exactly (3.62). □

Remark 3. *The n-step flow cutset inequality for X^u and X^b is obtained by setting $S_2 = \emptyset$ in* (3.62)*, and the n-step simple cutset inequality is obtained by setting $S_1 = E_U, S_2 = \emptyset$ in* (3.62)*.*

Special Cases:

- **Cut-set inequality.** The cut-set inequality (3.17) can be obtained by setting $n = 1, \alpha = C_2, S_1 = E_U, S_2 = \emptyset$ in (3.62). The cut-set inequality (3.18) can be obtained by setting $n = 1, \alpha = C_1, S_1 = E_U, S_2 = \emptyset$ in (3.62). The cut-set inequality (3.19) is in fact a 2-step

MIR inequality [18]. This inequality can be rewritten as

$$D^{(2)} \left\lceil \frac{D^{(1)}}{C_2} \right\rceil \sum_{e \in E_U} y_1^e + D^{(2)} \sum_{e \in E_U} y_2^e + \sum_{e \in E_U} y_3^e \geq D^{(2)} \left\lceil \frac{D^{(1)}}{C_2} \right\rceil \left\lceil \frac{D}{C_1} \right\rceil. \tag{3.65}$$

(3.65) can be obtained by setting $n = 2, \alpha = (C_1, C_2), S_1 = E_U, S_2 = \emptyset$ in (3.62). This is the only class of inequalities of our knowledge in the literature that needs the information of more than one module to be derived. It is also mentioned in [7] that for MMND with any general number of divisible capacity modules and $C_M = 1$, (3.65) can be generalized to

$$\sum_{t=1}^{M} D^{(M-1)} \prod_{k=t+1}^{M-1} \left\lceil \frac{D^{(k-1)}}{C_k} \right\rceil \sum_{e \in E_U} y_t^e + \sum_{e \in E_U} y_M^e \geq D^{(M-1)} \prod_{k=t+1}^{M-1} \left\lceil \frac{D^{(k-1)}}{C_t} \right\rceil, \tag{3.66}$$

which can be obtained by setting $n = M-1, \alpha = (C_1, \ldots, C_M), S_1 = E_U, S_2 = \emptyset$ in (3.62).

- **Flow-cut-set inequality.** The flow-cut-set inequality (3.20) can be obtained by setting $n = 1, \alpha = C_1, S_2 = \emptyset$ in (3.62).

Our next theorem shows that the n-step cutset inequalities are facet-defining for P^u and P^d.

Theorem 8. *The n-step general cutset (resp. flow cutset, simple cutset) inequality is facet-defining for P^u and P^b under conditions similar to Theorem 4 (resp. 5, 6).*

Based on a result in [10] that is similar to Lemma 3, we have the following corollary that shows the n-step cutset inequalities are also facet-defining for X^u and X^b.

Corollary 2. *The n-step general cutset (resp. flow cutset, simple cutset) inequality is facet-defining for X^u (resp. X^b) if it is facet-defining for P^u (resp. P^b), and the graphs induced by U and $\overline{U}$ are connected.*

4. MULTI-MODULE SURVIVABLE NETWORK DESIGN PROBLEM

In this chapter, we present our results for the multi-module survivable network design problem (MM-SND). We introduce different models for MM-SND using directed p-cycles in section 4.1. In section 4.2, we review existing studies on modeling and solving MM-SND. We develop valid inequalities and show they are facet-defining for the convex hull of MM-SND in section 4.4. We generalize these inequalities for MM-SND using undirected p-cycles in section 4.5. We present our computational experiment results in section 4.6.

4.1 Models

To formulate MM-SND in this dissertation, we consider a bidirected link network [2, 10]. Each edge can have flows in the two opposite directions, and flow in each direction is bounded by the capacity installed on the edge. It is assumed that each edge is a member of at least one undirected cycle. As assumed in [30, 31, 24], we assume that only one edge fails at a time, i.e., the network has to survive only one edge failure until it is repaired.

We adapt most of our notations from Section 2.3 For each p-cycle $R \in \mathcal{R}$, let z_R be the amount of slack reserved on p-cycle $R \in \mathcal{R}$. This slack can be used to accommodate the rerouted flow after failure of an edge. Fig. 4.1 shows an example of the protection of a flow on an arc using the slacks reserved on two p-cycles. In Fig. 4.1 (a), the flow along arc uv (filled arrow) is protected by slacks reserved on p-cycles R_1 and R_2, denoted by z_{R_1} and z_{R_2}, where $z_{R_1} + z_{R_2} \geq x_{uv}$. The arcs in p-cycles are shown by hollow arrows. In Fig. 4.1 (b), under failure of edge (u, v), the flow from u to v is rerouted using reserved slacks on other arcs of the two p-cycles, i.e., on $u \to w \to v$ and $u \to a \to b \to v$ routes.

Our MIP formulation for the MM-SND is as follows:

$$\min \quad \sum_{a \in A} f_a x_a + \sum_{t=1}^{M} \sum_{e \in E} h_t^e y_t^e \tag{4.1}$$

$$s.t. \quad \sum_{a \in \delta^+(v)} x_a - \sum_{a \in \delta^-(v)} x_a = d_v, v \in V \tag{4.2}$$

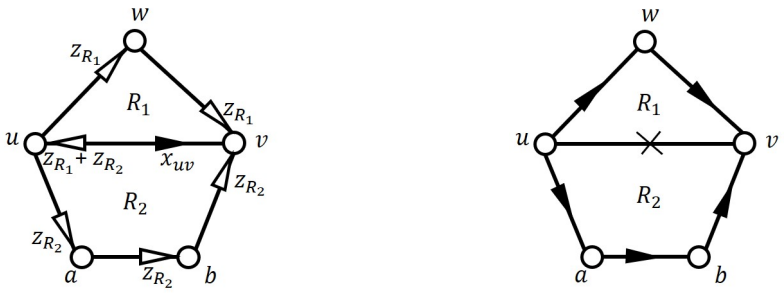
$$x_a - \sum_{R \in \mathcal{R}: \bar{a} \in R} z_R \leq 0, a \in A \tag{4.3}$$

$$x_{e^+} + \sum_{R \in \mathcal{R}: e^+ \in R} z_R \leq \sum_{t=1}^{M} C_t y_t^e, e \in E \tag{4.4}$$

$$x_{e^-} + \sum_{R \in \mathcal{R}: e^- \in R} z_R \leq \sum_{t=1}^{M} C_t y_t^e, e \in E \tag{4.5}$$

$$(x, y, z) \in \mathbb{R}_+^{2|E|} \times \mathbb{Z}_+^{M|E|} \times \mathbb{R}^{|\mathcal{R}|} \tag{4.6}$$

The objective function (4.1) minimizes the total cost of routing the flows and installing of the capacity modules over all edges. Constraints (4.2) are the flow balance constraints of the network. Constraints (4.3) ensure that the flow on an arc is no more than the sum of all reserved slacks on the p-cycles that protect that arc (see Fig. 4.1). Constraints (4.4) and (4.5) are the multi-module capacity constraints.



(a) (b)

Figure 4.1: Illustration of flow protection using two p-cycles.

An alternative model for SM-SND was proposed in [44] where the capacity installation decisions for the flow on the arcs and the reserved slacks on the p-cycles are made hierarchically. In the first stage, the network design problem without survivability requirements is solved, and the solutions are fed into the second stage problem where the reserved slacks and extra capacities are

determined.

For MM-SND, the hierarchical model is formulated as follows. In the first stage, we solve the following multi-module capacitated network design problem (MMND):

$$\min \quad \sum_{a\in A} f_a x_a + \sum_{t=1}^{M}\sum_{e\in E} h_t^e y_t^e \tag{4.7}$$

$$s.t. \quad \sum_{a\in\delta^+(v)} x_a - \sum_{a\in\delta^-(v)} x_a = d_v, v \in V \tag{4.8}$$

$$\max\{x_{e^+}, x_{e^-}\} \le \sum_{t=1}^{M} C_t y_t^e, e \in E \tag{4.9}$$

$$(x, y) \in \mathbb{R}_+^{2|E|} \times \mathbb{Z}_+^{M|E|}. \tag{4.10}$$

We refer to [53] for a comprehensive study on models and polyhedral results of MMND. Let $(\hat{x}, \hat{y})$ be the optimal solution of the MMND in stage 1. In the second stage, we solve the Spare Capacity Installation (SCI) problem:

$$\min \quad \sum_{t=1}^{M}\sum_{e\in E} h_t^e y_t^e \tag{4.11}$$

$$s.t. \quad \hat{x}_a - \sum_{R\in\mathcal{R}:\bar{a}\in R} z_R \le 0, a \in A \tag{4.12}$$

$$\hat{x}_{e^+} + \sum_{R\in\mathcal{R}:e^+\in R} z_R \le \sum_{t=1}^{M} C_t(\hat{y}_t^e + y_t^e), e \in E \tag{4.13}$$

$$\hat{x}_{e^-} + \sum_{R\in\mathcal{R}:e^-\in R} z_R \le \sum_{t=1}^{M} C_t(\hat{y}_t^e + y_t^e), e \in E \tag{4.14}$$

$$(y, z) \in \mathbb{Z}_+^{M|E|} \times \mathbb{R}^{|\mathcal{R}|}, \tag{4.15}$$

where the decision variables y_t^e in the second stage are extra capacities that need to be installed for reserved slacks on the p-cycles.

Based on our computational results, the hierarchical model is more efficient to solve than the integrated model. However, it cannot yield the overall minimum cost for flow routing as well

as capacity installation of the integrated model. For telecommunication providers, the integrated model could be the more favorable choice at the design stage of a telecommunication network, since the MM-SND problem only needs to be solved once, where the goal is to minimize the overall cost. Therefore, we dedicate most of our polyhedral results in this dissertation to the integrated model for the MM-SND.

4.2 Literature Review

In the SND literature, the disrupted flow can be restored by one of the following two schemes. *Global rerouting* methods, or path restoration, consider all demand pairs interrupted by the failed edge, and reassign flow paths for these demand pairs. Extra capacities have to be installed on the new paths and the cost is minimized. Stoer and Dahl [29] studied a multi-module SND problem with binary capacity variables using global rerouting. The authors proposed a cutting plane algorithm for this problem later in [54]. Alevras et al. [28] studied a multifacility SND problem using global rerouting with additional restrictions and proposed cutting plane heuristics for the problem. Iraschko et al. [55] formulated the SND with global rerouting and compared the formulations where the spare capacity assignment problem is solved separately or jointly with the original network design problem without survivability requirement. Kennington and Lewis [56] proposed a branch and bound algorithm for the SND with global rerouting. It is possible to achieve optimal capacity efficiency using global rerouting, i.e., to use the least possible number of extra capacities. However, the SND problems using global rerouting are computationally challenging to solve due to large sizes of its formulations.

Instead, *local rerouting* tries to only reroute the interrupted flow on the failed edge by other edges of the network. This is usually done by reserving sufficient capacities on certain predetermined structures formed by edges. Ring structure is one of the popular choices, where certain sets of nodes of the network are chosen and new edges are constructed between the nodes to form a ring. Altinkemer [57] proposed algorithms to solve SND using an enhanced ring network where local rings are connected with a backbone ring. Goldschmidt et al. [58] and Luss et al. [59] studied a similar model with interconnected rings, and proposed exact and heuristic algorithms. Slevinsky

et al. [60] designed an algorithm to deploy ring structures to achieve better capacity efficiency.

The so-called *preconfigured-cycle (p-cycle)* structure has been shown to be more cost-effective than the popular ring-based methods [61, 62]. Extra capacities are reserved on existing edges and no new edge need to be constructed. Kiaei et al. [63] conducted a comprehensive review on the research topics of the p-cycle protection method in survivable networks. Grover et al. [64] studied the formulation, heuristics, and variants for SND using p-cycles. To our best knowledge, the first formulation of the SND problem using p-cycles was proposed by Grover and Stamatelakis [62]. They compared the undirected p-cycle methods with the ring methods and concluded that p-cycle method achieves better capacity efficiency for SND. In [65], they provided theoretical insights behind these results. Rajan and Atamtürk [66, 31] proposed a formulation of the SND with directed p-cycles and introduced valid inequalities for this problem. In [30], a more complicated model was studied by them such that the interrupted flow on a chord of a p-cycle can also be rerouted by the p-cycles.

Among the literature of the SND, the SM-SND studies by Rajan and Atamtürk [30, 31] are the most closely related ones to ours. In [31] the SM-SND has been formulated by two different approaches. The first model is a hierarchical two-stage model (in which the problem is solved without flow protection in the first stage and a spare capacity installation problem is solved in the second stage). The second model is an integrated model where both stages are integrated into a single formulation. The integrated model achieves $\sim$10% more cost-efficiency compared to the two-stage model and does not take longer than the two-stage model to solve for SM-SND, as reported in [31]. In this dissertation, we also formulate the hierarchical model and the integrated model for the MM-SND and compare the cost and time efficiency of these two models. Our results show that for the MM-SND, there is clearly a trade-off between the time and the cost: the cost of the integrated model is 8% lower than that of the hierarchical model, while the integrated model takes much longer to solve.

The MM-SND problem is in general NP-hard. This is not difficult to notice since the MMND problem, the first stage problem of the hierarchical model, is NP-hard [3]. For the survivable

network design problems in general, some studies focus on solving them using algorithmic approaches. For example, Grover et al. [67] developed a heuristic to solve the spare capacity installation problem where the failed edge is protected by alternate shortest paths between its two endpoints. Balakrishnan et al. [68] considered several heuristic methods to solve the MM-SND where failed edges are protected by all other remaining edges of the network. Kennington and Lewis [56] developed a branch-and-bound algorithm to solve a similar problem.

Many studies have addressed the survivable network design formulations from the polyhedral perspective. Sakauchi et al. [69] used a cutting plane algorithm to add cutset based inequalities for the SM-SND. Grötschel et al. [26] gave a comprehensive review of polyhedral results for several models of the SND. Stoer and Dahl [29] studied a MM-SND formulation to protect node or edge failures where capacity variables are split into sums of binary variables and proposed inequalities based on projection of the formulation onto the space of the capacity variables. They developed a cutting plane algorithm for these inequalities in [54]. Bienstock and Muratore [41] considered a polyhedron with respect to a cut in a survivable network and proposed strong valid inequalities for the polyhedron. Rajan and Atamtürk [31] considered a special case of our MM-SND formulation where $M = 1$, and proposed cutset-based inequalities. Atamtürk and Rajan [30] studied a generalization of the model in [31] where cord edges of a p-cycle can also be protected by the p-cycle, and developed several other classes of cutset-based inequalities. The inequalities we developed in this dissertation are generalizations of the ones in [31] and [30].

4.3 Cutset Inequalities for SM-SND

In this section, we first review the inequalities previously introduced for SM-SND in the literature. Let X be the convex hull of the set of solutions that satisfy (4.2)-(4.6). For any set of nodes $V' \subseteq V$, let $G[V']$ be the graph induced by nodes in V', i.e., $G[V'] = (V', E[V'])$ where $E[V'] = \{e = (i,j) \in E : i,j \in V'\}$. For any set of edges $E' \subseteq E$, let $A_{E'}$ be the set of arcs whose corresponding edge is in E', i.e., $A_{E'} = \{e^+ = ij, e^- = ji : e = (i,j) \in E'\}$. For any set of arcs $A' \in A$, let $[A']$ be the set of edges such that it has a corresponding arc in A', i.e., $[A'] = \{e \in E : e^+ \in A' \text{ or } e^- \in A'\}$.

All inequalities in discussion of this dissertation are derived based on a given partition of V. Let $U \subset V$, and $\bar{U} = V \setminus U$. Let $d = \sum_{v \in U} d_v$. We can assume without loss of generality that $d \geq 0$, since otherwise we have $\sum_{v \in \bar{U}} > 0$, in which case we can switch U and $\bar{U}$. Let E_U be the set of edges crossing the partition. For any subset $E_\subset \subseteq E_U$, each edge $e \in E_\subset$ is represented by its two antiparallel arcs e^+ and e^-. Let $A_\subset$ be the set of all such arcs, i.e., $A_\subset = \{e^+ = ij, e^- = ji : e = (i,j) \in E_\subset\}$. Let $A_\subset^+$ be the set of arcs whose tails are in U, and $A_\subset^-$ be the set of arcs whose tails are in $\bar{U}$.

For any arc $a \in A$, let $\mathcal{R}^a$ be the set of p-cycles R such that $a \in R$. Let $z_a = \sum_{R \in \mathcal{R}^a} z_R$. By our definition, for any $a \in A$, $\mathcal{R}^{\bar{a}}$ will be the set of p-cycles that *protect* a, since x_a will be rerouted using the arcs of the p-cycles that are in $\mathcal{R}^{\bar{a}}$ in case edge $[a]$ fails (see Fig. 4.1). For any arc $a \in A$ and a set of arcs crossing the partition $A_\subset \subseteq A_U \setminus \{a\}$, let $\mathcal{R}^a_{A_\subset} \subseteq \mathcal{R}^a$ be the set of p-cycles that cross the partition using only a and an arc in $A_\subset$, i.e., $\mathcal{R}^a_{A_\subset} = \{R \in \mathcal{R}^a : R \cap A_\subset \neq \emptyset\}$.

Rajan and Atamtürk [30, 66] considered the following inequalities for the single-module survivable capacitated network design problem. Let X_S be the convex hull of the set defined by the constraints of SM-SND. Given a partition $(U, \bar{U})$ of V, let $d = \sum_{v \in U} d_v$ and $r = 2d - \lfloor 2d \rfloor$. Given $E_\subset \subseteq E_U$, the 2-partition inequality

$$r \sum_{e \in E_\subset} y_1^e + \sum_{a \in A_U^+ \setminus A_\subset^+} (x_a + z_a - x_{\bar{a}}) \geq r \lceil 2d \rceil \tag{4.16}$$

is valid for X_S. Rajan and Atamtürk [66] showed that the 2-partition inequality is facet-defining for a relaxation set of X_S, and a special case of the 2-partition inequality when $E_\subset = E_U$ is facet-defining for X_S.

In [30], Atamtürk and Rajan proposed two classes of inequalities for the SM-SND problem using directed p-cycles where the p-cycles can also be used to recover flow of their chord arcs. Let X_S^C be the convex hull of the set defined by the constraints of the SM-SND problem in [30]. For any $a \in A$ and $A_\subset \in A_U \setminus \{a\}$, let $\mathcal{C}^a_{A_\subset}$ be the set of p-cycles that protect a and cross the partition using some arc in $A_\subset$. Note that in this problem, $\bar{a}$ may not be in all p-cycles $R \in \mathcal{C}^a_{A_\subset}$ because a

may be a chord arc of some of the p-cycles. Now, given an arc $a_0 \in A_U^+$ and $I \subseteq A_U^+ \setminus \{a_0\}$, let $r' = d - \lfloor d \rfloor$. The p-cycle flow partition inequality

$$\sum_{a \in I} x_a + \sum_{R \in \mathcal{C}_I^{a_0}} z_R + r' \sum_{e \in [A_U^+ \setminus I \setminus a_0]} y_1^e \geq r' \lceil d \rceil . \tag{4.17}$$

is valid and facet-defining for X_S^C under certain conditions.

We note that inequalities (4.16) and (4.17) can be derived using mixed integer rounding (MIR). We develop inequalities specifically for the MM-SND in later sections that are not obtainable by MIR, and show that (4.16) and (4.17) are special cases of these inequalities.

Another type of inequality presented in [30] is the p-cycle flow subset-Q inequality. For any $Q \subseteq A_U^+$, let $q = |Q| \geq 2, \eta = \lceil d_A \rceil , \eta_q = \lceil q\eta/(q-1) \rceil$, and $r_q = q\eta - (q-1) \lfloor q\eta/(q-1) \rfloor$. The p-cycle flow subset-Q inequality

$$\frac{q}{r'} \sum_{a \in I} x_a + \frac{1}{r} \sum_{a \in I} \sum_{R \in \mathcal{C}_I^a} z_R + r_q \sum_{e \in [Q]} y_e + (r_q + 1) \sum_{e \in [A_U^+ \setminus Q \setminus I]} y_e \geq r_q \eta_q \tag{4.18}$$

is valid and facet-defining for X_S^C under certain conditions.

4.4 Valid Inequalities and Facets for MM-SND

In this section we illustrate the valid inequalities and facets for MM-SND. We first define a mixed integer set closely related to X. This set comes from aggregating and relaxing the constrains of the MM-SND formulation with respect to edges crossing a given partition $(U, \bar{U})$. This set is defined as follows:

$$P = conv\Bigg\{ \sum_{a \in A_U^+} x_a - \sum_{a \in A_U^-} x_a = d \tag{4.19}$$

$$x_a \leq z_{\bar{a}}, a \in A_U \tag{4.20}$$

$$x_{e^+} + z_{e^+} \leq \sum_{t=1}^{M} C_t y_t^e, e \in E_U \tag{4.21}$$

$$x_{e^-} + z_{e^-} \leq \sum_{t=1}^{M} C_t y_t^e, e \in E_U \tag{4.22}$$

$$\sum_{a \in A_U^+} z_a = \sum_{a \in A_U^-} z_a \tag{4.23}$$

$$(x, y, z) \in \mathbb{R}_+^{2|E_U|} \times \mathbb{Z}_+^{M|E_U|} \times \mathbb{R}^{2|E_U|} \Big\} \tag{4.24}$$

The equality (4.19) is obtained by aggregating the flow balance equality (4.2) for $v \in U$. The equality (4.23) is based on the fact that any directed p-cycle that crosses the partition using some arc $a \in A_U^+$ must come back across the partition using some arc $a' \in A_U^-$, as illustrated in Fig. 4.2. Therefore if we sum the reserved slacks over all arcs that cross the partition for each direction, we obtain equality (4.23). It is easy to see that any valid inequality for P is also valid for X.

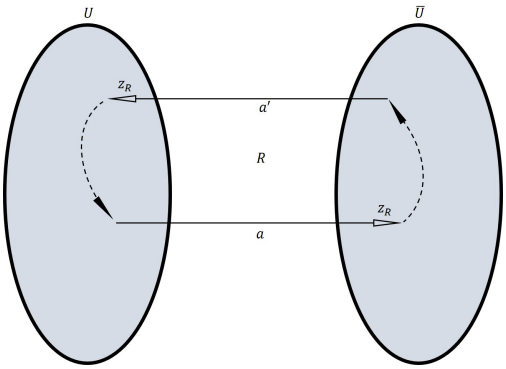


Figure 4.2: A p-cycle that crosses the partition.

For simplicity of the notations, we define the following functions: given $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ where $\alpha_1 > \ldots > \alpha_n > 0$, for any $u \in \mathbb{R}$, define

$$\mu_\alpha^l(u) = \begin{cases} u^{(n)} \prod_{k=l+1}^{n} \left\lceil \frac{u^{(k-1)}}{C_k} \right\rceil, l = 0, 1, \ldots, n, \\ min(u^{(n)}, C_l), l = n+1, \ldots, M. \end{cases}$$

Next we present a Lemma that is used in the proof of the theorems to verify that certain points

belong to X.

Lemma 4. *For non-negative integers a and c, define $\sum_a^c(.) = 0$ and $\prod_a^c(.) = 1$ if $a > c$. Given $\{\alpha_1, \ldots, \alpha_n\}$ where $\alpha_1 > \ldots > \alpha_n > 0$, for any $u \in \mathbb{R}$,*

(a) $u = \sum_{k=1}^{t} \alpha_k \left\lfloor \frac{u^{(k-1)}}{\alpha_k} \right\rfloor + u^{(t)} \leq \sum_{k=1}^{t} \alpha_k \left\lfloor \frac{u^{(k-1)}}{\alpha_k} \right\rfloor + \alpha_{t+1} \left\lceil \frac{u^{(t)}}{\alpha_{t+1}} \right\rceil, t = 0, \ldots, n,$

(b) $\sum_{t=1}^{l-1} \mu_\alpha^t(u) \left\lfloor \frac{u^{(t-1)}}{\alpha_t} \right\rfloor + \mu_\alpha^{l-1}(u) = \mu_\alpha^0(u), l = 1, \ldots, n,$

(c) $\sum_{t=1}^{n} \mu_\alpha^t(u) \left\lfloor \frac{u^{(t-1)}}{\alpha_t} \right\rfloor = \mu_\alpha^0(u) - 1.$

Proof. (a) is straightforward by definition of the recursive reminder. (b) and (c) can be proved similarly as those in [21]. □

4.4.1 n-Step Flow Cutset Inequalities

Our first class of inequalities are generalizations of the n-step flow cutset inequalities for the MMND problem [53] and the 2-partition inequalities for SM-SND [31].

Theorem 9. *Let $D = 2d$. Given $E_\subset \subseteq E_U$, if the n-step MIR conditions, i.e., $\left\lceil D^{(t-1)}/C_t \right\rceil \leq C_{t-1}/C_t, t = 2, \ldots, n$, hold, the n-step flow cutset inequality*

$$\sum_{t=1}^{M} \mu_C^t(D) \sum_{e \in E_\subset} y_t^e + \sum_{a \in A_U^+ \setminus A_\subset^+} (x_a + z_a - x_{\bar{a}}) \geq \mu_C^0(D) \tag{4.25}$$

is valid for X.

Proof. The flow balance constraints of P (4.19) can be rewritten as

$$\sum_{a \in A_\subset^+} x_a + \sum_{a \in A_U^+ \setminus A_\subset^+} x_a - \sum_{a \in A_\subset^-} x_a - \sum_{a \in A_U^- \setminus A_\subset^-} x_a = d. \tag{4.26}$$

Adding $x_a \geq 0, a \in A_U^-$ to (4.26), we have

$$\sum_{a \in A_\subset^+} x_a + \sum_{a \in A_U^+ \setminus A_\subset^+} x_a \geq d. \tag{4.27}$$

Using $x_a \leq z_{\bar{a}}, a \in A_U^+$, (4.27) is relaxed to

$$\sum_{a \in A_\subset^+} z_{\bar{a}} + \sum_{a \in A_U^+ \setminus A_\subset^+} z_{\bar{a}} \geq d. \tag{4.28}$$

Using (4.23), (4.28) can be written as

$$\sum_{a \in A_\subset^+} z_a + \sum_{a \in A_U^+ \setminus A_\subset^+} z_a \geq d. \tag{4.29}$$

Adding (4.29) to (4.26), we have

$$\sum_{a \in A_\subset^+} (x_a + z_a) + \sum_{a \in A_U^+ \setminus A_\subset^+} (x_a + z_a) - \sum_{a \in A_\subset^-} x_a - \sum_{a \in A_U^- \setminus A_\subset^-} x_a \geq D. \tag{4.30}$$

Using (4.21) and (4.22) for $a \in A_\subset^+$ and $x_a \geq 0$ for $a \in A_\subset^-$, (4.30) is relaxed to

$$\sum_{t=1}^{M} \sum_{e \in E_\subset} C_t y_t^e + \sum_{a \in A_U^+ \setminus A_\subset^+} x_a + \sum_{a \in A_U^+ \setminus A_\subset^+} z_a - \sum_{a \in A_U^- \setminus A_\subset^-} x_a \geq D. \tag{4.31}$$

Notice that $\sum_{a \in A_U^+ \setminus A_\subset^+} z_a - \sum_{a \in A_U^- \setminus A_\subset^-} x_a = \sum_{a \in A_U^+ \setminus A_\subset^+} (z_a - x_{\bar{a}}) \geq 0$. Now, if $C_M < C_{M-1} < \ldots < C_{n+1} < D^{(n)}$, then we can treat $\sum_{e \in E_\subset} y_t^e$ as γ_i, $i = 1, \ldots, n$, $\sum_{t=n+1}^{M} \sum_{e \in E_\subset} C_t y_t^e + \sum_{a \in A_U^+ \setminus A_\subset^+} x_a + \sum_{a \in A_U^+ \setminus A_\subset^+} z_a - \sum_{a \in A_U^- \setminus A_\subset^-} x_a$ as w, and D as b in K. By applying n-step MIR on (4.31), we get exactly (4.25).

Otherwise, let s be the index of the smallest module size that is greater than $D^{(n)}$, so that $C_{n+1} > \ldots > C_s > D^{(n)} > C_{s+1} > \ldots > C_M$. Then we have $D^{(n+1)} = \ldots = D^{(s)} = D^{(n)}$ and $\left\lceil D^{(n)}/C_{n+1} \right\rceil = \ldots = \left\lceil D^{(s-1)}/C_s \right\rceil = 1$. We can apply s-step MIR on (4.31), and it is straightforward to check the the resulting inequality is exactly (4.25). □

Special Cases:

- **2-Partition inequality.** The 2-partition inequality (4.16) is a special case of (4.25) with $M = n = 1$ and $C_1 = 1$.

- **n-Step simple cutset inequality.** If we let $E_{\subset} = E_U$, (4.25) becomes

$$\sum_{t=1}^{M} \mu_C^t(D) \sum_{e \in E_U} y_t^e \geq \mu_C^0(D). \tag{4.32}$$

We refer to (4.32) as the n-step simple cutset inequality. We prove in our next theorem that (4.32) is facet-defining for X under certain conditions.

Theorem 10. *The n-step simple cutset inequality* (4.32) *is facet-defining for X if*

(a) $|E_U| \geq 3, \frac{D^{(t-1)}}{C_t} < \left\lceil \frac{D^{(t-1)}}{C_t} \right\rceil \leq \frac{C_{t-1}}{C_t}, t = 2, \ldots, n, D > C_n, C_M > D^{(n)}$,

(b) $2d^{(1)} > C_1, \left\lceil \frac{d^{(t-1)}}{C_t} \right\rceil = \left\lceil \frac{D^{(t-1)}}{C_t} \right\rceil, t = 2, \ldots, n$,

(c) $G[U], G[\bar{U}]$ *are 2-connected.*

Proof. Under condition (a), (4.32) can be written as

$$\sum_{t=1}^{n} D^{(n)} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{C_k} \right\rceil \sum_{e \in E_U} y_t^e + \sum_{t=n+1}^{M} D^{(n)} \sum_{e \in E_U} y_t^e \geq D^{(n)} \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{C_k} \right\rceil. \tag{4.33}$$

The equality corresponding to (4.32) is

$$\sum_{t=1}^{n} D^{(n)} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{C_k} \right\rceil \sum_{e \in E_U} y_t^e + \sum_{t=n+1}^{M} D^{(n)} \sum_{e \in E_U} y_t^e = D^{(n)} \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{C_k} \right\rceil. \tag{4.34}$$

Let

$$\sum_{t=1}^{M} \sum_{e \in E} \beta_t^e y_t^e + \sum_{a \in A} \pi_a x_a + \sum_{R \in \mathcal{R}} \delta_R z_R = \theta \tag{4.35}$$

be a hyperplane passing through the face defined by (4.34). We prove (4.35) is a scalar multiple of (4.34) plus a linear combination of the flow balance equalities (4.2).

For $e \in E \setminus E_U, t = 1, \ldots, M$, consider the directions $\mathcal{E}_t^e$. $\mathcal{E}_t^e$ are directions for both X and (4.34), so this implies that in (4.35), $\beta_t^e = 0, e \in E \setminus E_U, t = 1, \ldots, M$.

Now, consider a spanning arborescence $T \subset A$ of G rooted in $r \in U$ using arcs in $A_{E[U]}$ and $A_{E[\bar{U}]}$ and some arc $a_0 \in A_U^+$. Such spanning arborescence exists because $G, G[U]$, and $G[\bar{U}]$

are connected. Since we can add flow balance equalities (4.2) to facet-defining equalities without changing them and $|T| = |V| - 1$, we can assume without loss of generality that $\pi_a = 0, a \in T$.

Let $R^0 \in \mathcal{R}^{\bar{a_0}}$. Let $a_1 = R^0 \cap A_U^+$. Let $\mathcal{M}$ be a large enough constant, and ϵ be a small enough constant. We construct a feasible solution $p = (x, y, z)$ to X such that

$$y_t^e = \begin{cases} \mathcal{M}, e \in E \setminus E_U, t = 1, \dots, n, \\ \left\lceil \frac{d}{C_1} \right\rceil, e = [a_0], [a_1], t = 1, \\ 0, e \in E_U \setminus \{[a_0], [a_1]\}, \end{cases} \quad z_R = \begin{cases} \mathcal{M}, R \cap A_U = \emptyset, \\ d + \epsilon, R = R^0, \\ 0, otherwise, \end{cases} \quad x_a = \begin{cases} \epsilon, a \in T \setminus \{a_0\}, \\ d, a = a_0, \\ 0, a \in A_U \setminus \{a_0\}, \end{cases}$$

and $x_a, a \in A \setminus (A_U \cup T)$ be chosen such that the flow balance constraints (4.2) are satisfied (see Fig. 4.3). Note that a solution with $x_a > 0$ for $a \in T \setminus \{a_0\}$ exists, because if $x_a = 0$ for some $a \in T \setminus \{a_0\}$, then we can construct another point where $x_a > 0$ as follows. If $a = ij \in A_{E[U]}$, since $G[U]$ is connected, there is a path from j to i using arcs in $A_{E[U]}$ that forms a directed cycle with ij. Because neither the capacities installed on the edges of this cycle nor the slacks of the p-cycles passing through the arcs of this cycle are saturated, we can send a circulation along this cycle of amount ϵ such that $x_{ij} > 0$. Similarly, a point can be constructed where $x_a > 0$ if $a \in A_{E[\bar{U}]}$. Note that by condition (a) we have $2d^{(1)} > C_1$, then $2\lceil d/C_1 \rceil = \lceil D/C_1 \rceil$. By Lemma 4 it is easy to check that p satisfies (4.34).

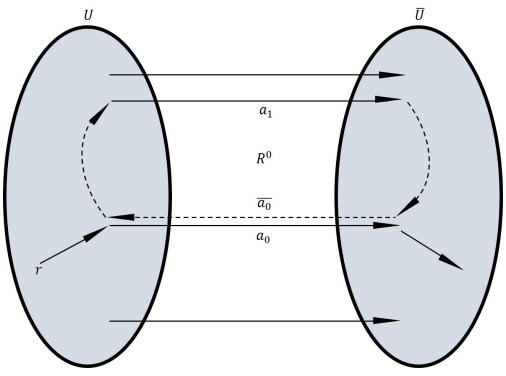


Figure 4.3: A feasible solution p.

Let $\mathcal{R}_U$ be the set of p-cycles that cross the partition, i.e., $\mathcal{R}_U = \{R \in \mathcal{R} : R \cap A_U \neq \emptyset\}$. For any $R \in \mathcal{R} \setminus \mathcal{R}_U$, since the edge capacities are not saturated, we can construct a new point $\hat{p}$ from p by increasing z_R by ϵ without leaving X. $\hat{p}$ satisfies (4.34), so it satisfies (4.35). Therefore, by comparing $\hat{p}$ and p we have $\delta_R = 0, R \in \mathcal{R} \setminus \mathcal{R}_U$.

Now consider any arc $a_2 \in A \setminus (T \cup A_U)$. Since T is a spanning arborescence, $[T] \subset E$ is a spanning tree of G, and the edge $[a_2]$ and a subset of edges $E_{a_2} \subset [T]$ form an undirected cycle. Let $A_{a_2} = A_{E_{a_2}} \cap T$. The scenarios of the directions of a_2 and the arcs in A_{a_2} can be one of the following cases, as illustrated in Fig. 4.4:

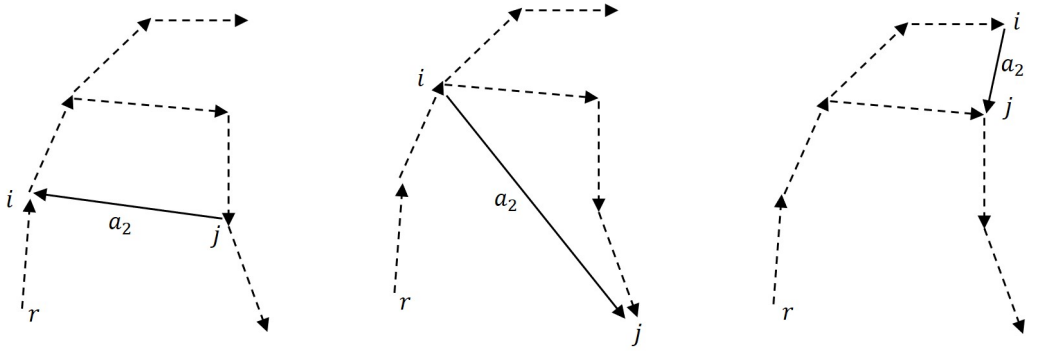
Case A. a_2 and arcs in A_{a_2} form a directed cycle. We can construct a new feasible solution from p by sending a circulation of amount ϵ along the cycle without leaving X.

Case B. $a_2 = ij$, and the arcs in A_{a_2} form a path from i to j. We can construct a new feasible solution from p by decreasing ϵ amount of flow along the path and increasing the same amount along a_2 without leaving X.

Case C. There are two paths in T from r to i and j, namely P_{ri}, P_{rj}, respectively. We can construct a new feasible solution from p without leaving X by decreasing ϵ amount of flow along arcs in P_{rj} and increasing the same amount along arc a_2 and the arcs in P_{ri}.

In all of the above cases, we get a new point $\hat{p}$ that satisfies (4.2)-(4.6) and (4.34). Therefore, it also satisfies (4.35), and since $\pi_a = 0$ for all $a \in T$, by comparing $\hat{p}$ and p, we have that $\pi_a = 0, a \in A \setminus (T \cup A_U)$.

Now for R^0, we can increase z_{R^0} by ϵ without leaving X. Therefore $\delta_{R^0} = 0$. Let R^1 be the p-cycle where all arcs of R^0 are reversed, i.e., $R^1 = \{a = ij : ji \in R^0\}$. By similar argument we have $\delta_{R^1} = 0$. For arc a_1, consider a new feasible solution $\hat{p}$ to X from p by decreasing x_{a_0} by ϵ and increasing x_{a_1} and z_{R^1} by ϵ (and set appropriate flows for $a \in A \setminus A_U$ if necessary). By comparing $\hat{p}$ and p we have $\pi_{a_1} = \pi_{a_0} = 0$. For arc $\bar{a_1}$, since capacity on edge $[a_1]$ is not saturated, we can get a new feasible solution $\hat{p}$ to X from p by sending a circulation along the directed cycle $(a_1, \bar{a_1})$. By comparing $\hat{p}$ and p we have $\pi_{\bar{a_1}} = 0$. Now since $G[U]$ and $G[\bar{U}]$ are connected, for every $e \in E_U \setminus [a_0]$, there exists a p-cycle $R^0 \in \mathcal{R}^{\bar{a_0}}$ such that $R^0 \cap A_e^+ \neq \emptyset$, which means our choices of



(a) case A. (b) case B. (c) case C.

Figure 4.4: Possible directions of a_2 and A_{a_2}.

R^0 and a_1 are arbitrary. Therefore, we can repeat the argument for each $e \in E_U \setminus [a_0]$ and $R \in \mathcal{R}^{\bar{a_0}}$, and we have $\pi_a = 0, a \in A_U$ and $\delta_R = 0, R \in \mathcal{R}^{\bar{a_0}}$. For $R \in \mathcal{R}_U \setminus \mathcal{R}^{\bar{a_0}}$, we can repeat the above argument by replacing $[a_0]$ with $[a_3] = [R] \cap E_U$, and conclude that $\delta_R = 0, R \in \mathcal{R}_U \setminus \mathcal{R}^{\bar{a_0}}$. Eventually, we have $\pi_a = 0, a \in A$ and $\delta_R = 0, R \in \mathcal{R}$. Also, since we can use any $e \in E_U$ to replace either $[a_1]$ or $[a_0]$, we have $\beta_1^e = \beta_1^{e_\tau}, e \in E_U$ for some fixed $e_\tau \in E_U$.

Next, consider the points $p_t, t = 2, \ldots, n$ which are the same as p except that $y_l^{[a_0]} = \left\lfloor d^{(l-1)}/C_l \right\rfloor$, $l = 1, \ldots, t-1$ and $y_t^{[a_0]} = \left\lceil d^{(t-1)}/C_t \right\rceil$. By condition (b), it is easy to check that p_t satisfy (4.2)-(4.6) and (4.34), and hence they satisfy (4.35). Comparing them one by one, we have $\beta_{l-1}^{[a_0]} = \left\lceil D^{(l-1)}/C_l \right\rceil \beta_l^{[a_0]}, l = 2, \ldots, n$, so $\beta_l^{[a_0]} = \prod_{t=l+1}^{n} \left\lceil D^{(t-1)}/C_t \right\rceil \beta_n^{[a_0]}, l = 1, \ldots, n-1$. Combining with the result above, we have $\beta_l^e = \prod_{t=l+1}^{n} \left\lceil D^{(t-1)}/C_t \right\rceil \beta_n^e = \prod_{t=l+1}^{n} \left\lceil D^{(t-1)}/C_t \right\rceil \beta_n^{e_\tau}, l = 1, \ldots, n-1, e \in E_U$.

Next, consider the points $p_t, t = n+1, \ldots, M$ which are the same as p_n except that $y_n^{[a_0]} = \left\lfloor \frac{d^{(n-1)}}{C_n} \right\rfloor$ and $y_t^{[a_0]} = 1$. By conditions (a) and (b), it is easy to check that $p_t, t = n+1, \ldots, M$ satisfy both (4.2)-(4.6) and (4.34). Therefore, they also satisfy (4.35). Comparing them one by one, we have $\beta_t^{[a_0]} = \beta_n^{[a_0]}, t = n+1, \ldots, M$. Combining this with the result above, we have $\beta_t^e = \beta_n^e = \beta_n^{e_\tau}, t = n+1, \ldots, M, e \in E_U$.

Now (4.35) is reduced to

$$\sum_{t=1}^{n} \beta_n^{e_\tau} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{C_k} \right\rceil \beta_n^{e_\tau} \sum_{e \in E_U} y_t^e + \sum_{t=n+1}^{M} \beta_n^{e_\tau} \sum_{e \in E_U} y_t^e = \theta. \tag{4.36}$$

Finally, substituting p into (4.36), we have $\theta = \beta_n^{e_\tau} \prod_{k=1}^{n} \left\lceil D^{(k-1)}/C_k \right\rceil$, which reduces (4.36) to

$$\sum_{t=1}^{n} \beta_n^{e_\tau} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{C_k} \right\rceil \sum_{e \in E_U} y_t^e + \sum_{t=n+1}^{M} \beta_n^{e_\tau} \sum_{e \in E_U} y_t^e = \beta_n^{e_\tau} \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{C_k} \right\rceil. \tag{4.37}$$

(4.37) is a scalar multiple of (4.34) (the scalar is $\beta_n^{e_\tau}/D^{(n)}$). This completes the proof. □

The n-step flow cutset inequality (4.25) is not always facet-defining for X. However, it is indeed facet-defining for P under conditions provided by the following theorem.

Theorem 11. *The n-step flow cutset inequality* (4.25) *is facet-defining for P if*

(a) $\frac{D^{(t-1)}}{C_t} < \left\lceil \frac{D^{(t-1)}}{C_t} \right\rceil \leq \frac{C_{t-1}}{C_t}, D > C_n$,

(b) $E_{\subset} \neq \emptyset$.

Proof. We first define the following points. Note that only the nonzero elements are mentioned.

Definition 11. *Define the following points:*

(a) For any $a \in A_{\subset}^{+}, l = 1, \ldots, n$, the points $\mathcal{A}_l^a$:

$$y_t^{[a]} = \begin{cases} \left\lfloor \frac{D^{(t-1)}}{C_t} \right\rfloor, & t = 1, \ldots, l-1, \\ \left\lceil \frac{D^{(t-1)}}{C_t} \right\rceil, & t = l, \qquad x_a = z_a = z_{\bar{a}} = d. \\ 0, & t = l+1, \ldots, n, \end{cases}$$

(b) For any $a \in A^+_{\subset}, l = n+1, \ldots, M$ and some $\tau \in A^+_U \setminus A^+_{\subset}$, the points $\mathcal{A}^{a\tau}_l$:

$$y_t^{[a]} = \begin{cases} \left\lfloor \frac{D^{(t-1)}}{C_t} \right\rfloor, & t = 1, \ldots, n, \\ 1, & t = l, \end{cases} \quad x_a = z_a = z_{\bar{a}} = \Bigg(\sum_{t=1}^{n} C_t \left\lfloor \frac{D^{(t-1)}}{C_t} \right\rfloor + \min\{C_t, D^{(n)}\} \Bigg) / 2, y_n^{\tau} = 1, x_\tau = z_\tau = z_{\bar{\tau}} = \max\{0, (D^{(n)} - C_l)/2\}.$$

(c) For any $a \in A^+_{\subset}$, the points $\mathcal{B}^a$:

$$y_1^{[a]} = \left\lceil \frac{D}{C_1} \right\rceil, x_a = d, z_a = z_{\bar{a}} = C_1 \left\lceil \frac{D}{C_1} \right\rceil /2,$$

and the points $\mathcal{B}^a_1$:

$$y_1^{[a]} = \left\lceil \frac{D}{C_1} \right\rceil, x_a = z_a = z_{\bar{a}} = C_1 \left\lceil \frac{D}{C_1} \right\rceil /2, x_{\bar{a}} = C_1 \left\lceil \frac{D}{C_1} \right\rceil /2 - d.$$

(d) For any $a_1, a_2 \in A^+_{\subset}$, the point $\mathcal{C}^{a_1 a_2}$:

$$y_t^{[a_1]} = \left\lfloor \frac{D}{C_t} \right\rfloor, t = 1, \ldots, n, x_{a_1} = z_{a_1} = z_{\bar{a_1}} = d - D^{(n)}/2,$$
$$y_n^{[a_2]} = 1, x_{a_2} = z_{a_2} = z_{\bar{a_2}} = D^{(n)}/2,$$

the points $\mathcal{C}^{a_1 a_2}_1$:

$$y_t^{[a_1]} = \left\lfloor \frac{D}{C_t} \right\rfloor, t = 1, \ldots, n, x_{a_1} = z_{a_1} = d - D^{(n)}/2, z_{\bar{a_1}} = d - D^{(n)}/2 + \epsilon,$$
$$y_n^{[a_2]} = 1, x_{a_2} = z_{\bar{a_2}} = D^{(n)}/2, z_{a_2} = D^{(n)}/2 + \epsilon,$$

and the points $\mathcal{C}^{a_1 a_2}_2$:

$$y_t^{[a_1]} = \left\lfloor \frac{D}{C_t} \right\rfloor, t = 1, \ldots, n, x_{a_1} = z_{a_1} = z_{\bar{a_1}} = d - C_n/2,$$
$$y_n^{[a_2]} = 1, x_{a_2} = z_{\bar{a_2}} = z_{a_2} = C_n/2.$$

(e) For any $a_1 \in A_{\subset}^{+}, a_2 \in A_U^{-} \setminus A_{\subset}^{-}$, the point $\mathcal{D}^{a_1 a_2}$:

$$y_1^{[a_1]} = \left\lceil \frac{D}{C_1} \right\rceil, x_{a_1} = z_{a_1} = z_{\bar{a_1}} = C_1 \left\lceil \frac{D}{C_1} \right\rceil /2, y_1^{[a_2]} = 1, x_{a_2} = z_{a_2} = z_{\bar{a_2}} = D^{(n)}/2,$$

the point $\mathcal{D}_1^{a_1 a_2}$:

$$y_1^{[a_1]} = \left\lceil \frac{D}{C_1} \right\rceil, x_{a_1} = z_{\bar{a_1}} = C_1 \left\lceil \frac{D}{C_1} \right\rceil /2 - \epsilon, z_{a_1} = C_1 \left\lceil \frac{D}{C_1} \right\rceil /2,$$
$$y_1^{[a_2]} = 1, x_{a_2} = z_{\bar{a_2}} = D^{(n)}/2 - \epsilon, z_{a_2} = D^{(n)}/2,$$

and the point $\mathcal{D}_2^{a_1 a_2}$:

$$y_1^{[a_1]} = \left\lceil \frac{D}{C_1} \right\rceil, x_{a_1} = z_{\bar{a_1}} = C_1 \left\lceil \frac{D}{C_1} \right\rceil /2 - \epsilon, z_{a_1} = C_1 \left\lceil \frac{D}{C_1} \right\rceil /2 + \epsilon,$$
$$y_1^{[a_2]} = 1, x_{a_2} = z_{\bar{a_2}} = D^{(n)}/2 - \epsilon, z_{a_2} = D^{(n)}/2 + \epsilon.$$

(f) For any $a_1 \in A_{\subset}^{+}, a_2 \in A_U^{+} \setminus A_{\subset}^{+}$, the point $\mathcal{F}^{a_1 a_2}$:

$$y_t^{[a_1]} = \left\lfloor \frac{D}{C_t} \right\rfloor, t = 1, \ldots, n, x_{a_1} = z_{a_1} = z_{\bar{a_1}} = d - D^{(n)}/2,$$
$$y_n^{[a_2]} = 1, x_{a_2} = z_{a_2} = z_{\bar{a_2}} = D^{(n)}/2,$$

and the point $\mathcal{F}_1^{a_1 a_2}$:

$$y_t^{[a_2]} = \left\lfloor \frac{D}{C_t} \right\rfloor, t = 1, \ldots, n, x_{a_1} = z_{\bar{a_1}} = d - D^{(n)}/2 - \epsilon, z_{a_1} = d - D^{(n)}/2 + \epsilon,$$
$$y_n^{[a_2]} = 1, x_{a_2} = z_{\bar{a_2}} = D^{(n)}/2 + \epsilon, z_{a_2} = D^{(n)}/2 - \epsilon.$$

(g) For $e \in E_U, t = 1, \ldots, n$, the direction $\mathcal{E}_t^e$ where $y_t^e = 1$.

Under condition (a), (4.25) can be written as

$$\sum_{t=1}^{n} D^{(n)} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{C_k} \right\rceil \sum_{e \in E_{\subset}} y_t^e + \sum_{t=n+1}^{M} min(D^{(n)}, C_t) \sum_{e \in E_{\subset}} y_t^e + \sum_{a \in A_U^+ \setminus A_{\subset}^+} (x_a + z_a - x_{\bar{a}}) \geq D^{(n)} \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{C_k} \right\rceil. \tag{4.38}$$

The equality corresponding to (4.38) is

$$\sum_{t=1}^{n} D^{(n)} \prod_{k=t+1}^{n} \left\lceil \frac{D^{(k-1)}}{C_k} \right\rceil \sum_{e \in E_{\subset}} y_t^e + \sum_{t=n+1}^{M} min(D^{(n)}, C_t) \sum_{e \in E_{\subset}} y_t^e + \sum_{a \in A_U^+ \setminus A_{\subset}^+} (x_a + z_a - x_{\bar{a}}) = D^{(n)} \prod_{k=1}^{n} \left\lceil \frac{D^{(k-1)}}{C_k} \right\rceil. \tag{4.39}$$

Let

$$\sum_{t=1}^{M} \sum_{e \in E_U} \beta_t^e y_t^e + \sum_{a \in A_U} \pi_a x_a + \sum_{a \in A_U} \delta_a z_a = \theta \tag{4.40}$$

be a hyperplane passing through the face defined by (4.39). We prove that (4.40) must be a scalar multiple of (4.39) plus a linear combination of equalities (4.19) and (4.23).

For $e \in E_U \setminus E_{\subset}, t = 1, \ldots, n$, consider the directions $\mathcal{E}_t^e$. $\mathcal{E}_t^e$ are directions for both P and (4.39), so this implies that in (4.40), $\beta_t^e = 0, e \in E_U \setminus E_{\subset}, t = 1, \ldots, n$.

Next, for $a \in A_{\subset}^+$, consider the points $\mathcal{A}_1^a$ and $\mathcal{B}^a$. By Lemma 4 they satisfy (4.39), and since they are in P, they must satisfy (4.40). Substituting $\mathcal{A}_1^a$ and $\mathcal{B}^a$ into (4.40) and subtracting one equality from the other, we have $\delta_{\bar{a}} = -\delta_a$.

Now, since all points of P satisfy the equality (4.23), we may add multiples of (4.23) to facet-defining inequalities without changing them. Therefore we can without loss of generality assume that $\exists \tau_1 \in A_{\subset}^+$ such that $\delta_{\tau_1} = 0$. If $|A_{\subset}^+| = 1$, then we have $\delta_a = 0, a \in A_{\subset}$. Otherwise, for $a_1, a_2 \in A_{\subset}^+$, consider the points $\mathcal{C}^{a_1 a_2}$ and $\mathcal{C}_1^{a_1 a_2}$. By Lemma 4 they satisfy (4.39), and since they are in P, they must satisfy (4.40). Substituting $\mathcal{C}^{a_1 a_2}$ and $\mathcal{C}_1^{a_1 a_2}$ into (4.40) and subtracting one equality from the other, we have $\delta_{\bar{a_1}} = -\delta_{a_2}$. Since both our choices of a_1 and a_2 are arbitrary, and

$\delta_{\tau_1} = 0$ for some $\tau_1 \in A_{\subset}^+$, this implies that $\delta_a = 0, a \in A_{\subset}$.

Next, for $a \in A_{\subset}^+$, consider the points $\mathcal{B}^a$ and $\mathcal{B}_1^a$. By Lemma 4 they satisfy (4.39), and since they are in P, they must satisfy (4.40). Substituting $\mathcal{B}^a$ and $\mathcal{B}_1^a$ into (4.40) and subtracting one equality from another, we have $\pi_{\bar{a}} = -\pi_a$.

Now, since all points of P satisfy the equality (4.19), we may add multiples of (4.19) to facet-defining inequalities without changing them. Therefore we can without loss of generality assume that $\exists \tau_2 \in A_{\subset}^+$ such that $\pi_{\tau_2} = 0$. If $|A_{\subset}^+| = 1$, then we have $\pi_a = 0, a \in A_{\subset}$. Otherwise, for $a_1, a_2 \in A_{\subset}^+$, consider the points $\mathcal{C}^{a_1 a_2}$ and $\mathcal{C}_2^{a_1 a_2}$. By Lemma 4 they satisfy (4.39), and since they are in P, they must satisfy (4.40). Substituting $\mathcal{C}^{a_1 a_2}$ and $\mathcal{C}_2^{a_1 a_2}$ into (4.40) and subtracting one equality from the other, we have $\pi_{a_1} = \pi_{a_2}$. Since both our choices of a_1 and a_2 are arbitrary, and $\pi_{\tau_2} = 0$ for some $\tau_2 \in A_{\subset}^+$, this implies that $\pi_a = 0, a \in A_{\subset}$.

Next, for any $a_1 \in A_{\subset}^+$ and $a_2 \in A_U^- \setminus A_{\subset}^-$, consider the points $\mathcal{D}_1^{a_1 a_2}$ and $\mathcal{D}_2^{a_1 a_2}$. By Lemma 4 they satisfy (4.39), and since they are in P, they must satisfy (4.40). Substituting $\mathcal{D}_1^{a_1 a_2}$ and $\mathcal{D}_2^{a_1 a_2}$ into (4.40) and subtracting one equality from another, we have $\delta_{a_2} = -\delta_{a_1}$. Since our choice of a_2 is arbitrary and $\delta_a = 0$ for any $a \in A_{\subset}^+$, this implies that $\delta_a = 0, a \in A_U^- \setminus A_{\subset}^-$.

Next, for any $a_1 \in A_{\subset}^+$ and $a_2 \in A_U^- \setminus A_{\subset}^-$, consider the points $\mathcal{D}^{a_1 a_2}$ and $\mathcal{D}_1^{a_1 a_2}$. By Lemma 4 they satisfy (4.39), and since they are in P, they must satisfy (4.40). Substituting $\mathcal{D}^{a_1 a_2}$ and $\mathcal{D}_1^{a_1 a_2}$ into (4.40) and subtracting one equality from another, we have $\pi_{a_2} = \delta_{\bar{a_2}}$, which implies that $\pi_a = \delta_{\bar{a}}, a \in A_U^- \setminus A_{\subset}^-$.

Next, for $a_1 \in A_{\subset}^+$ and $a_2 \in A_U^+ \setminus A_{\subset}^+$, consider the points $\mathcal{F}^{a_1 a_2}$ and $\mathcal{F}_1^{a_1 a_2}$. By Lemma 4 they satisfy (4.39), and since they are in P, they must satisfy (4.40). Substituting $\mathcal{F}^{a_1 a_2}$ and $\mathcal{F}_1^{a_1 a_2}$ into (4.40) and subtracting one equality from another, we have $\pi_{a_2} = \delta_{a_2}$. Since our choice of a_2 is arbitrary and $\pi_a = -\delta_{ba}$ for all $a \in A_U^- \setminus A_{\subset}^-$, we have $\pi_a = \delta_a = -\pi_{\bar{a}}, a \in A_U^+ \setminus A_{\subset}^+$.

Next, for $a_1 \in A_{\subset}^+$ and $a_2 \in A_U^+ \setminus A_{\subset}^+$, consider the points $\mathcal{A}_n^{a_1}$ and $\mathcal{F}^{a_1 a_2}$. Substituting $\mathcal{A}_n^{a_1}$ and $\mathcal{F}^{a_1 a_2}$ into (4.40) and subtracting one equality from another, we have $\beta_n^{[a_1]} = D^{(n)}/2(\pi_{a_2} + \delta_{a_2}) = D^{(n)}\pi_{a_2}$. Since both our choices of a_1 and a_2 are arbitrary, we can replace either a_1 with any other arc in $A_{\subset}^+$, or a_2 with any other arc in $A_U^+ \setminus A_{\subset}^+$. Therefore, we have $\exists \gamma \in E_{\subset}$ such that

$\beta_n^e = \beta_n^\gamma, e \in E_\subset$ and $\exists \tau_3 \in A_\subset^+$ such that $\pi_a = \pi_{\tau_3}, a \in A_\subset^+$. Also, we have $\beta_n^\gamma = D^{(n)}\pi_{\tau_3}$.

So far, (4.40) has been reduced to

$$\sum_{e\in E_\subset}\sum_{t=1}^{M}\beta_t^e y_t^e + \sum_{a\in A_\subset^+}\pi_{\tau_3}(x_a + z_a - x_{\bar{a}}) = \theta. \tag{4.41}$$

Next, for $a \in A_\subset^+$, consider the points $\mathcal{A}_n^a, \mathcal{A}_{n-1}^a, \ldots, \mathcal{A}_1^a$. By Lemma 4 they satisfy (4.39), and since they are in P, they must satisfy (4.40). Substituting these points one after the other into (4.40), and subtracting one equality after another, we have $\beta_{t-1}^{[a]} = \beta_t^{[a]}\lceil D^{(t-1)}/C_t\rceil, t = n, \ldots, 2$. Since $\beta_n^{[a]} = D^{(n)}\pi_{\tau_3}$, this implies that $\beta_t^e = \prod_{k=t+1}^n \left\lceil \frac{D^{(k-1)}}{C_k} \right\rceil D^{(n)}\pi_{\tau_3}, t = 1, \ldots, n$.

Now, let s be the index of the last capacity module whose size is larger than $D^{(n)}$, i.e., $s = \max\{t \in \{1, \ldots, n\} : C_t > D^{(n)}\}$. For any $a \in A_\subset^+$, consider the points $\mathcal{A}_{n+1}^{a\tau_3}, \ldots, \mathcal{A}_M^{a\tau_3}$. By Lemma 4 they satisfy (4.39), and since they are in P, they must satisfy (4.40). Substituting the points $\mathcal{A}_n^a$ and $\mathcal{A}_l^{a\tau_3}, l \in \{n+1, \ldots, s\}$ into (4.40), and subtracting one equality after the other, we have $\beta_l^{[a]} = \beta_n^{[a]} = D^{(n)}\pi_{\tau_3}$. Substituting the points $\mathcal{A}_n^a$ and $\mathcal{A}_l^{a\tau_3}, l \in \{s+1, \ldots, M\}$ into (4.40), and subtracting one equality after another, we have $\beta_l^{[a]} = \beta_n^{[a]} - (D^{(n)} - C_l)\pi_{\tau_3} = C_l\pi_{\tau_3}$.

Now (4.40) has been reduced to

$$\sum_{t=1}^{n} D^{(n)} \sum_{e\in E_\subset}\prod_{k=t+1}^{n}\left\lceil \frac{D^{(k-1)}}{C_k}\right\rceil \pi_{\tau_3} y_t^e + \sum_{t=n+1}^{M}\sum_{e\in E_\subset}\pi_{\tau_3}\min\{D^{(n)}, C_t\}y_t^e + \sum_{a\in A_U^+\setminus A_\subset^+}\pi_{\tau_3}(x_a - x_{\bar{a}} + z_a) = \theta. \tag{4.42}$$

Finally, substituting $\mathcal{A}_1^a, a \in A_\subset^+$ into (4.42), we have $\theta = \pi_{\tau_3} D^{(n)} \prod_{k=1}^n \left\lceil \frac{D^{(k-1)}}{C_k} \right\rceil$. Then (4.42) is reduced to

$$\begin{aligned}\sum_{t=1}^{n} D^{(n)} \sum_{e\in E_\subset}\prod_{k=t+1}^{n}\left\lceil \frac{D^{(k-1)}}{C_k}\right\rceil \pi_{\tau_3} y_t^e + \sum_{t=n+1}^{M}\sum_{e\in E_\subset}\pi_{\tau_3}\min\{D^{(n)}, C_t\}y_t^e \\ + \sum_{a\in A_U^+\setminus A_\subset^+}\pi_{\tau_3}(x_a - x_{\bar{a}} + z_a) = \pi_{\tau_3}\prod_{k=1}^{n}\left\lceil \frac{D^{(k-1)}}{C_k}\right\rceil D^{(n)}.\end{aligned} \tag{4.43}$$

(4.43) is a scalar multiple of (4.42) (this scalar is π_{τ_3}). This completes our proof. □

4.4.2 n-Step p-Cycle Flow Partition Inequalities

Our next class of inequalities generalize the p-cycle flow partition inequalities for SM-SND [30].

Theorem 12. *Given an arc $a_0 \in A_U^+$ and $I \subseteq A_U^+ \setminus a_0$, if the n-step MIR conditions are satisfied, i.e., $\left\lceil d^{(t-1)}/C_t \right\rceil \leq C_{t-1}/C_t, t = 2, \ldots, n$, the n-step p-cycle flow partition inequality*

$$\sum_{t=1}^{M} \mu_C^t(d) \sum_{e \in [A_U^+ \setminus I \setminus a_0]} y_t^e + \sum_{a \in I} x_a + \sum_{R \in \mathcal{R}_I^{\bar{a_0}}} z_R \geq \mu_C^0(d) \tag{4.44}$$

is valid for X.

Proof. We start from the flow balance equality (4.19) of X. Adding the inequalities $x_a \geq 0, a \in A_U^-$ to it, we have

$$\sum_{a \in A_U^+} x_a \geq d. \tag{4.45}$$

This can be written as

$$\sum_{a \in A_U^+ \setminus a_0} x_a + x_{a_0} \geq d. \tag{4.46}$$

Using constraint (4.3) for a_0, (4.46) can be relaxed to

$$\sum_{a \in A_U^+ \setminus a_0} x_a + \sum_{R \in \mathcal{R}^{\bar{a_0}}} z_R \geq d. \tag{4.47}$$

By definition $\mathcal{R}^{\bar{a_0}} = \mathcal{R}^{\bar{a_0}}_{A_U^+ \setminus a_0} = \mathcal{R}^{\bar{a_0}}_{A_U^+ \setminus I \setminus a_0} \cup \mathcal{R}_I^{\bar{a_0}}$, so (4.47) can be rewritten as

$$\sum_{a \in A_U^+ \setminus I \setminus a_0} x_a + \sum_{a \in I} x_a + \sum_{R \in \mathcal{R}^{\bar{a_0}}_{A_U^+ \setminus I \setminus a_0}} z_R + \sum_{R \in \mathcal{R}_I^{\bar{a_0}}} z_R \geq d. \tag{4.48}$$

By definition $\mathcal{R}^{\bar{a_0}}_{A_U^+ \setminus I \setminus a_0} = \mathcal{R}^{A_U^+ \setminus I \setminus a_0} \cap \mathcal{R}^{\bar{a_0}}$, then we have $\mathcal{R}^{\bar{a_0}}_{A_U^+ \setminus I \setminus a_0} \subseteq \mathcal{R}^{A_U^+ \setminus I \setminus a_0} = \cup_{a \in A_U^+ \setminus I \setminus a_0} \mathcal{R}^a$.

Thus $\sum_{R\in\mathcal{R}^{\bar{a_0}}_{A_U^+\setminus I\setminus a_0}} z_R$ can be further relaxed to $\sum_{a\in A_U^+\setminus I\setminus a_0} z_a$. Therefore (4.48) can be relaxed to

$$\sum_{a\in I} x_a + \sum_{a\in A_U^+\setminus I\setminus a_0} (x_a + z_a) + \sum_{R\in\mathcal{R}_I^{\bar{a_0}}} z_R \geq d. \tag{4.49}$$

Using the capacity constraints (4.4) and (4.5), (4.49) is relaxed to

$$\sum_{t=1}^{M} \sum_{e\in[A_U^+\setminus I\setminus a_0]} C_t y_t^e + \sum_{a\in I} x_a + \sum_{R\in\mathcal{R}_I^{\bar{a_0}}} z_R \geq d. \tag{4.50}$$

The rest of the proof is similar to that of Theorem 9 and is omitted. □

Special Cases:

- **p-Cycle flow partition inequality.** The p-cycle flow partition inequality (4.17) is a special case of (4.44) with $M = 1$ and $C_1 = 1$.

- **n-Step p-cycle survivable partition inequality.** When $I = \emptyset$, (4.44) becomes

$$\sum_{t=1}^{M} \mu_C^t(d) \sum_{e\in[A_U^+\setminus a_0]} y_t^e \geq \mu_C^0(d). \tag{4.51}$$

We refer to (4.51) as the n-step p-cycle survivable partition inequality. We show in our next theorem that (4.51) is facet-defining for X under certain conditions.

Theorem 13. *The n-step p-cycle survivable partition inequality* (4.51) *is facet-defining for X if*

(a) $|E_U| \geq 3, \frac{d^{(t-1)}}{C_t} < \left\lceil \frac{d^{(t-1)}}{C_t} \right\rceil \leq \frac{C_{t-1}}{C_t}, t = 2, \ldots, n, d > C_n, C_M > d^{(n)}$,

(b) $G[U], G[\bar{U}]$ *are 2-connected.*

Proof. The proof is similar to that of Theorem 10 and is omitted. □

4.4.3 n-Step Survivable Subset-Q Inequalities

In our computational experiments, the following class of inequalities helps tighten the formulation. Let $Q \subseteq E_U$ such that $3 \leq |Q| := q \leq |E_U|$. Define $\overline{C_t} := \mu_C^t(d) = d^{(n)} \prod_{l=t+1}^{n} \left\lceil d^{(l-1)}/C_l \right\rceil$, $t = 1, \ldots, n$, $\overline{C} = \{\overline{C_1}, \ldots, \overline{C_M}\}$, and $\overline{d} := q\mu_C^0(d)/(q-1) = q \prod_{l=1}^{n} d^{(n)} \left\lceil d^{(l-1)}/C_l \right\rceil /(q-1)$.

Theorem 14. *Given $n \in \{1, \ldots, M\}$, if the n-step MIR conditions are satisfied, i.e., $\left\lceil \overline{d}^{(t-1)}/\overline{C_t} \right\rceil \leq \overline{C_{t-1}}/\overline{C_t}, t = 2, \ldots, n$, the n-step survivable subset-Q inequality*

$$\sum_{t=1}^{M} \mu_{\overline{C}}^t(\overline{d}) \sum_{e \in E_U} y_t^e + \sum_{t=1}^{M} \frac{\overline{C_t}}{q-1} \sum_{e \in E_U \setminus Q} y_t^e \geq \mu_{\overline{C}}^0(\overline{d}) \tag{4.52}$$

is valid for X.

Proof. Let $Q^+ = \{ij \in A_U^+ : (i,j) \in Q\}$. For each $a \in Q^+$, consider the survivable partition inequality (4.51) corresponding to a:

$$\sum_{t=1}^{M} \prod_{k=t+1}^{M} \mu_C^t(d) \sum_{e \in [A_U^+ \setminus a]} y_t^e \geq \mu_C^0(d). \tag{4.53}$$

If we sum (4.53) over each $a \in Q^+$, and divide the aggregate inequality by $q-1$, the resulting inequality is

$$\sum_{t=1}^{M} \overline{C_t} \sum_{e \in E_U} y_t^e + \sum_{t=1}^{M} \frac{\overline{C_t}}{q-1} \sum_{e \in E_U \setminus Q} y_t^e \geq \overline{d}. \tag{4.54}$$

Applying n-step MIR on (4.54), we get exactly (4.52). □

In our computational experiments, the n-step survivable subset-Q inequality (4.52) helps tighten the formulation and thus also reduces the integrality gap. Whether it is facet-defining for X is an open question.

4.4.4 n-Step Three Partition Inequalities

The three-partition inequality is another class of inequalities that is often studied in the network design problem literature (see for example [2, 8, 31, 42, 70].) To construct such an inequality, the

set of nodes of the network are partitioned into three partitions instead of two (see Fig. 4.5). The inequality considers the relationship among the edges crossing the cut with respect to each pair of partitions.

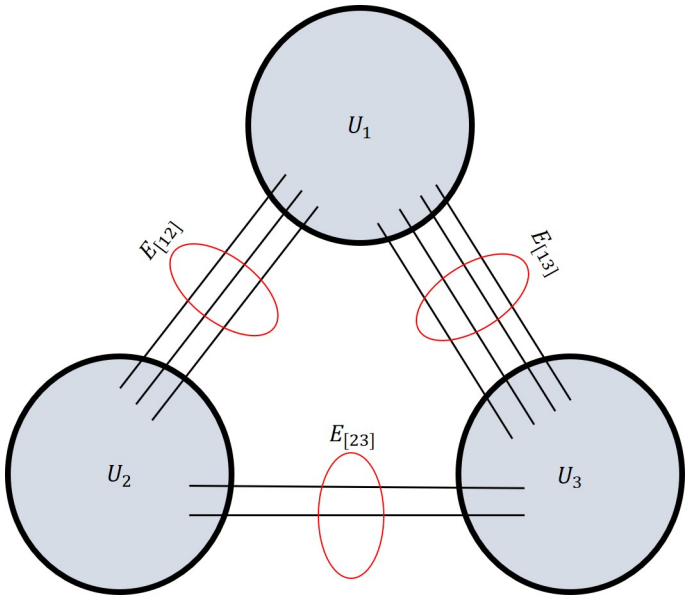


Figure 4.5: A three-partition of the network.

Formally, let (U_1, U_2, U_3) be a partition of V. Let $E_{[12]}$ be the set of edges crossing the cut with respect to U_1 and U_2. Define $E_{[13]}$ and $E[23]$ similarly. Let A_{12} be the set of arcs that have heads in U_1 and tails in U_2. Define $A_{21}, A_{13}, A_{31}, A_{23}$, and A_{32} similarly. Let $D_1 = \sum_{v \in U_1} d_v$. Define D_2 and D_3 similarly. Let $\theta = 2(\max\{D_1, D_2 + D_3\} + \max\{D_2, D_1 + D_3\} + \max\{D_3, D_1 + D_2\})$, and $\overline{C} = \{\overline{C_1}, \ldots, \overline{C_M}\} = \{2C_1, \ldots, 2C_M\}$. We have the following inequality.

Theorem 15. *Given $n \in \{1, \ldots, M\}$, if the n-step MIR conditions are satisfied, i.e., $\lceil \theta^{(t-1)}/\overline{C_t} \rceil \leq \overline{C_{t-1}}/\overline{C_t}, t = 2, \ldots, n$, the n-step three-partition inequality*

$$\sum_{t=1}^{M} \sum_{e \in E_{[12]} \cup E_{[13]} \cup E_{[23]}} \mu^n_{\overline{C},\theta}(\overline{C_t}) y^e_t \geq \mu^n_{\overline{C},\theta}(\theta) \tag{4.55}$$

is valid for X.

Proof. By aggregating the flow balance constraints (4.2) for $v \in U_1$, we have

$$\sum_{a \in A_{21} \cup A_{31}} x_a - \sum_{a \in A_{12} \cup A_{13}} x_a = D_1. \tag{4.56}$$

(4.56) can be relaxed to

$$\sum_{a \in A_{21} \cup A_{31}} x_a \geq D_1. \tag{4.57}$$

Using constraints (4.3) for $a \in A_{21} \cup A_{31}$, (4.57) can be relaxed to

$$\sum_{a \in A_{21} \cup A_{31}} z_{\bar{a}} \geq D_1. \tag{4.58}$$

Now since any directed p-cycle that crosses U_1 and U_2 using some arc $a \in A_{21}$ must come back across the partition using some arc $a' \in A_{12}$ (and similarly for p-cycles crossing U_1 and U_3), as illustrated in Fig. 4.2, (4.58) can be written as

$$\sum_{a \in A_{21} \cup A_{31}} z_a \geq D_1. \tag{4.59}$$

Adding (4.59) to (4.57), we have

$$\sum_{a \in A_{21} \cup A_{31}} (x_a + z_a) \geq 2D_1. \tag{4.60}$$

Using the capacity constraints (4.4) and (4.5), (4.60) is relaxed to

$$\sum_{a \in E_{[12]} \cup E_{[13]}} \sum_{t=1}^{M} C_t y_t^e \geq 2D_1. \tag{4.61}$$

On the other hand, by aggregating the flow balance constraints (4.2) for $v \in U_2 \cup U_3$, we have

$$\sum_{a \in A_{12} \cup A_{13}} x_a - \sum_{a \in A_{21} \cup A_{31}} x_a = D_2 + D_3. \tag{4.62}$$

By similar arguments to the above, (4.62) can be relaxed to

$$\sum_{a\in E_{[12]}\cup E_{[13]}} \sum_{t=1}^{M} C_t y_t^e \geq 2(D_2 + D_3). \tag{4.63}$$

(4.61) and (4.63) can be written together as

$$\sum_{a\in E_{[12]}\cup E_{[13]}} \sum_{t=1}^{M} C_t y_t^e \geq 2\max\{D_1, D_2 + D_3\}. \tag{4.64}$$

By similar arguments we can obtain the inequalities

$$\sum_{a\in E_{[12]}\cup E_{[23]}} \sum_{t=1}^{M} C_t y_t^e \geq 2\max\{D_2, D_1 + D_3\} \tag{4.65}$$

and

$$\sum_{a\in E_{[13]}\cup E_{[23]}} \sum_{t=1}^{M} C_t y_t^e \geq 2\max\{D_3, D_1 + D_2\}. \tag{4.66}$$

Adding (4.64), (4.65) and (4.66), we have

$$\sum_{a\in E_{[12]}\cup E_{[13]}\cup E_{[23]}} \sum_{t=1}^{M} \overline{C_t} y_t^e \geq \theta. \tag{4.67}$$

By applying the n-step MIR inequality on (4.67), we get exactly (4.55). □

Whether the three-partition inequality (4.55) is facet-defining for X is an open question.

Example 2. *Consider the network H illustrated in Fig. 4.6.*

In this example we have $V = \{1, 2, 3, 4, 5, 6\}$. The number beside each node is the demand or supply associated with the node. For any $i, j \in V$, we use ij to denote the directed arc from i to j, and $[ij]$ to denote the undirected edge between i and j. We represent a directed cycle by the set of

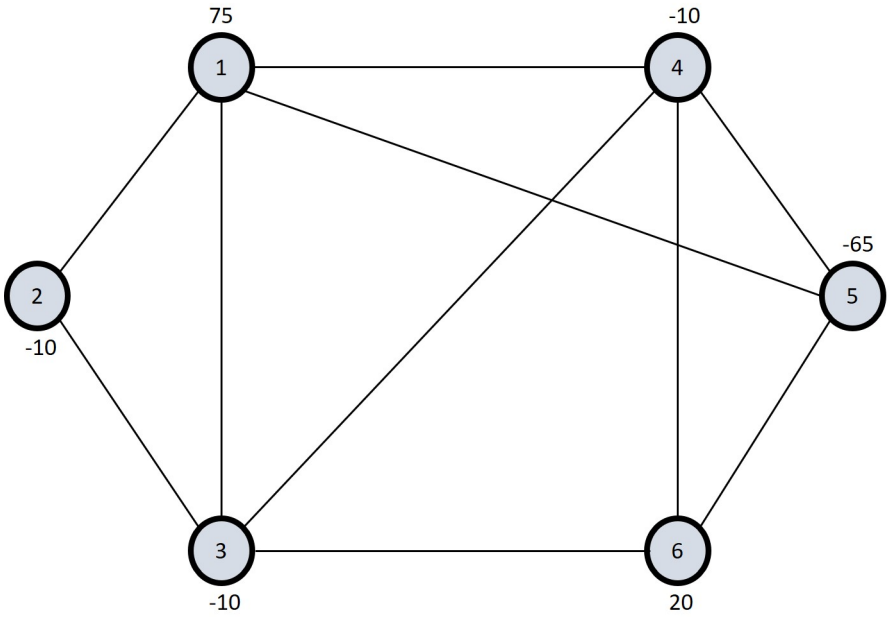


Figure 4.6: A survivable network.

its arcs. The following are some of the directed p-cycles in H*:*

$$R_1 = \{12, 23, 31\}, R_2 = \{13, 34, 41\}, R_3 = \{14, 45, 51\}, R_4 = \{34, 46, 63\},$$

$$R_5 = \{12, 23, 34, 41\}, R_6 = \{13, 36, 64, 41\}, R_7 = \{13, 36, 65, 51\},$$

$$R_8 = \{34, 45, 56, 63\}, R_9 = \{12, 23, 36, 64, 41\}, R_{10} = \{12, 23, 36, 65, 51\},$$

$$R_{11} = \{13, 36, 65, 54, 41\}, R_{12} = \{12, 23, 36, 65, 54, 41\}.$$

For any directed p-cycle R*, we use* $\overline{R}$ *to represent the p-cycle obtained by reversing the arcs in* R*. For example,* $\overline{R_1} = \{21, 13, 32\}$.

Assume there are two types of capacity modules where $(C_1, C_2) = (20, 6)$*. We illustrate the inequalities proposed in previous sections for MM-SND.*

For inequalities based on 2-partition of the network, let $U = \{1, 2, 3\}$*. Then we have* $d = 55, d^{(1)} = 15, d^{(2)} = 3, D = 110, D^{(1)} = 10, D^{(2)} = 4$.

Let $E_{\subset} = \{[14], [15], [34]\}$. The n-step flow cutset inequality is

$$8(y_1^{[14]} + y_1^{[15]} + y_1^{[34]}) + 4(y_2^{[14]} + y_2^{[15]} + y_2^{[34]}) + (x_{36} + z_{36} - x_{63}) \geq 48. \tag{4.68}$$

In the space of original variables, (4.68) *is*

$$\begin{aligned} 8(y_1^{[14]} + y_1^{[15]} + y_1^{[34]}) + 4(y_2^{[14]} + y_2^{[15]} + y_2^{[34]}) + (x_{36} \\ + z_{\bar{R}_4} + z_{R_6} + z_{R_7} + z_{\bar{R}_8} + z_{R_9} + z_{R_{10}} + z_{R_{11}} + z_{R_{12}} - x_{63}) \geq 48. \end{aligned} \tag{4.69}$$

It is facet-defining for the mixed integer set P with respect to $(U, \bar{U})$ by Theorem 9. The n-step simple cutset inequality is

$$8(y_1^{[14]} + y_1^{[15]} + y_1^{[34]} + y_1^{[36]}) + 4(y_2^{[14]} + y_2^{[15]} + y_2^{[34]} + y_1^{[36]}) \geq 48. \tag{4.70}$$

It is facet-defining for X by Theorem 10.

Let $a = 15, I = \{14, 34\}$. The n-step p-cycle flow inequality is

$$9y_1^{[36]} + 3y_2^{[36]} + x_{14} + x_{34} + z_{R_3} \geq 27. \tag{4.71}$$

The n-step p-cycle survivable partition inequality is

$$9(y_1^{[36]} + y_1^{[14]} + y_1^{[34]}) + 3(y_2^{[36]} + y_2^{[14]} + y_2^{[34]}) \geq 27. \tag{4.72}$$

It is facet-defining for X by Theorem 13.

Let $Q = \{14, 15, 34\}$. The n-step subset-Q inequality with respect to Q is

$$3(y_1^{[14]} + y_1^{[15]} + y_1^{[34]}) + 1.5(y_2^{[14]} + y_2^{[15]} + y_2^{[34]}) + 7.5y_1^{[36]} + 3y_2^{[36]} \geq 15. \tag{4.73}$$

For the three-partition inequality, we let $U_1 = \{1, 2\}, U_2 = \{3, 6\}$, and $U_3 = \{4, 5\}$. Then the

n-step three partition inequality is

$$16(y_1^{[14]}+y_1^{[15]}+y_1^{[23]}+y_1^{[13]}+y_1^{[34]}+y_1^{[46]}+y_1^{[45]})+8(y_2^{[14]}+y_2^{[15]}+y_2^{[23]}+y_2^{[13]}+y_2^{[34]}+y_2^{[46]}+y_2^{[45]}) \geq 128. \tag{4.74}$$

4.5 MM-SND with Undirected p-Cycles

In this section, we generalize the results for the MM-SND problem with directed p-cycles to the MM-SND problem with undirected p-cycles. Since an undirected cycle has at least three edges, any undirected cycle in G is also an undirected p-cycle. For SM-SND, Grover and Stamatelakis [62] considered a hierarchical model using undirected p-cycles. A major difference between their model and the models in [66] as well as in this dissertation is that in the second-stage spare capacity installation (SCI) problem, the unsaturated capacities from the first-stage solutions are not used.

4.5.1 Mathematical Formulation

Formally, let $\mathcal{R}$ be the set of all undirected cycles in H. Let z_R be the amount of slack reserved on cycle $R \in \mathcal{R}$. The MIP formulation for the MM-SND using undirected cycles is as follows:

$$min \quad \sum_{a\in A} f_a x_a + \sum_{t=1}^{M}\sum_{e\in E} h_t^e y_t^e \tag{4.75}$$

$$s.t. \quad \sum_{a\in\delta^+(v)} x_a - \sum_{a\in\delta^-(v)} x_a = d_v, v \in V \tag{4.76}$$

$$x_a - \sum_{R\in\mathcal{R}:[a]\in R} z_R \leq 0, a \in A \tag{4.77}$$

$$x_{e^+} + \sum_{R\in\mathcal{R}:e\in R} z_R \leq \sum_{t=1}^{M} C_t y_t^e, e \in E \tag{4.78}$$

$$x_{e^-} + \sum_{R\in\mathcal{R}:e\in R} z_R \leq \sum_{t=1}^{M} C_t y_t^e, e \in E \tag{4.79}$$

$$(x, y, z) \in \mathbb{R}_+^{2|E|} \times \mathbb{Z}_+^{M|E|} \times \mathbb{R}^{|\mathcal{R}|} \tag{4.80}$$

4.5.2 Valid and Facet-defining Inequalities

Let X^u be the set of solutions that satisfy (4.76)-(4.80). Given a partition $(U, \bar{U})$ of V, the inequalities presented in Section 4.4 can be generalized for X^u as follows.

Theorem 16. *Let $D = 2d$. Given $E_C \subseteq E_U$, if the n-step MIR conditions, i.e., $\lceil D^{(t-1)}/C_t \rceil \leq C_{t-1}/C_t, t = 2, \ldots, n$, hold, the n-step flow cutset inequality*

$$\sum_{t=1}^{M} \mu_C^t(D) \sum_{e \in E_C} y_t^e + \sum_{a \in A_U^+ \setminus A_C^+} (x_a + z_{[a]} - x_{\bar{a}}) \geq \mu_C^0(D) \tag{4.81}$$

is valid for X^u.

Theorem 17. *Given an arc $a_0 \in A_U^+$ and $I \subseteq A_U^+ \setminus a$, if the n-step MIR conditions are satisfied, i.e., $\lceil d^{(t-1)}/C_t \rceil \leq C_{t-1}/C_t, t = 2, \ldots, n$, the n-step p-cycle flow partition inequality*

$$\sum_{t=1}^{M} \mu_C^t(d) \sum_{e \in [A_U^+ \setminus I \setminus a]} y_t^e + \sum_{a \in I} x_a + \sum_{R \in \mathcal{R}_{[I]}^{[a_0]}} z_R \geq \mu_C^0(d) \tag{4.82}$$

is valid for X^u, where $\mathcal{R}_{[I]}^{[a_0]}$ is the set of undirected p-cycles that cross the partition using only $[a_0]$ and an edge in $[I]$.

Special cases of (4.81) and (4.82) can be similarly shown to be facet-defining for X^u. The n-step survivable subset-Q inequality (4.52) can be similarly shown to be valid for X^u.

4.6 Computational Experiments

4.6.1 Instance Generation

In order to test the effectiveness of the valid inequalities, we first generate random MM-SND instances. The random graphs we generate have 30 nodes each which are distributed uniformly on a 100 $\times$ 100 plane. The distance between any pair of nodes is no less than 5. Each node is assigned a degree of 2 or 5 with the probability of 0.2 each and 3 or 4 with the probability of 0.3 each. Then for each node, we connect its nearest neighbor within the distance of 50 that has not

been connected with it, until its degree is fulfilled or there is no other node within the distance of 50. Then we check if the generated graph is connected. If not, then we reject the graph and generate a new one. To ensure the protection of each edge, we also check if there are two edge-disjoint paths between the endpoints of each edge, i.e., the edge is in at least one cycle. If not, then we reject the graph and generate a new one.

After a valid graph is generated, we add the parameters of the problem. The demands are added in the following fashion: first, demands are generated for source-destination node pairs. For each pair of nodes that are directly connected by an edge, we generate a demand from uniform [10,190] with the probability of 0.9, while for nodes not directly connected by an edge the probability is 0.5. After the demand pairs are generated, the demand of each node v is calculated by the sum of demands for the node pairs where v is the destination less the sum of demands for the node pairs where v is the source. For each edge $e \in E$, the routing costs h_{e^+} and h_{e^-} are the same and equal the Euclidean distance between its endpoints. A set of capacity module sizes and costs is associated with each set of instances, as illustrated in Section 4.6.4.

4.6.2 Adding p-Cycle Variables

Given a network G, the number of p-cycles in G is exponential with respect to the number of edges in G, and solving the MM-SND formulation (4.1)-(4.6) with all p-cycle variables leads to unacceptable solving time when the size of network grows. Therefore, in the computational experiments, we solve the MM-SND formulation with only a subset of cycles in $\mathcal{R}$. In [30] and [31], cycle variables are priced based on reduced costs in the LP relaxation solution and are added to the formulation one by one using a column generation approach. Here we modify this heuristic to add p-cycle variables in our problem.

We first construct the *cycle basis* [71] of G, which is the set of undirected cycles in G such that all other undirected cycles can be constructed using the cycles in the cycle basis. The cycle basis is obtained by first finding a spanning tree T of G, then for every edge $e \in G \setminus T$, e and some edges in T will form a cycle in the cycle basis. The number of cycles in the cycle basis, provided that G is connected, is $|E| - |V| + 1$. Each cycle in the cycle basis can be represented by a 0-1 vector

with $|E|$ elements, where there is a 1 if the index corresponding to an edge is in the cycle. Then, (the 0-1 vector of) every other cycle in G is constructed by taking the *exclusive disjunction* of the vectors of the cycles in the cycle basis. The exclusive disjunction of two vectors results a single vector where each of its element is one only when the respective elements of the two input vectors differ. In our computations, besides the cycles in the cycle basis, we only consider cycles formed by the exclusive or operation of two and three cycles in the cycle basis. Our intuition is that large cycles (with many edges) will less likely be used to reroute disrupted flow.

Each undirected cycle in G can be represented by its two directed p-cycles of opposite directions. Let $\mathcal{R}_B$ be the set of directed p-cycles obtained by bifurcating all undirected cycles in the cycle basis. Before the start of the column generation, we add the p-cycle variables corresponding to the directed p-cycles in $\mathcal{R}_B$ to the formulation.

We follow a procedure in [66] to generate the rest of the p-cycle variables. We calculate for each of the rest of the cycle variables a weight w_R by $w_R = \sum_{a \in R}(\gamma_a - \nu_a)$. γ_a and ν_a are values of the dual variables corresponding to constraints (4.3) and either (4.4) or (4.5) of arc a. The heuristic tries to find a p-cycle with negative weight and add the corresponding p-cycle variable to the formulation. In [31], negative-weight cycle detection algorithms were applied to find a negative-weight p-cycle. We found that for the instances in our computations, enumerating the set of p-cycles to find a negative-weight p-cycle is also a viable option when we are only considering a subset of all p-cycles as mentioned before. When a negative weight p-cycle is found, the corresponding p-cycle variable is added to the formulation, and the LP relaxation is solved again to generate the next p-cycle variable. We stop adding the p-cycle variables when no negative-weight p-cycle can be found. This heuristic only takes about 1 second in our test environment for our test instances.

In the MM-SND formulation using undirected p-cycles (4.75)-(4.80), the procedure to add the p-cycle variables is similar and is omitted.

4.6.3 Cutting Plane Algorithm and Separation

After adding the p-cycle variables, a cutting plane algorithm starts to add cuts to the formulation. Given the LP relaxation optimal solution $(\hat{x}, \hat{y}, \hat{z})$, we run a separation heuristic to find violated inequalities among those proposed in Section 4.4. Since all inequalities presented are based on partitions of V, we enumerate possible partitions of V to find violated cuts. According to [30] and our computational tests, cuts are much more often violated at uneven partitions. Therefore we enumerate all partitions of V with less than 3 nodes on one side of the partition. At each iteration, we set criteria to select violated inequalities of each class proposed in Section 4.4 and add them to the formulation. We stop the cutting plane algorithm when all possible partitions are enumerated. The criteria to select each class of violated inequalities are listed as follows.

n-step survivable partition inequality. For a given partition $(U, \bar{U})$ of V and n, there are $|A_U^+|$ n-step survivable partition inequalities, depending on the choice of a. We calculate

$$L_e = \sum_{t=1}^{M} \mu_C^t(d)\hat{y}_t^e \tag{4.83}$$

for each $e \in E_U$, and let $[a_0] = argmax_{e \in E_U}\{L_e\}$, and $a_0 = \{a \in A_U^+ : [a] = [a_0]\}$. Then the most violated inequality is given by

$$\sum_{t=1}^{M} \mu_C^t(d) \sum_{e \in [A_U^+ \setminus a_0]} y_t^e \geq \mu_C^0(d). \tag{4.84}$$

n-step survivable subset-Q inequality. Here we follow a similar approach to that in [30] to find the violated n-step survivable subset-Q inequality for each fixed q. For fixed q such that $3 \leq q \leq |E_U|$, we calculate

$$L_e = \sum_{t=1}^{M} \mu_{\overline{C}}^t(\overline{d})\hat{y}_t^e + \sum_{t=1}^{M} \frac{\overline{C_t}}{q-1}\hat{y}_t^e \tag{4.85}$$

for every $e \in E_U$. The smallest right hand side of the n-step survivable subset-Q inequality (4.52)

is obtained by setting Q as the set of edges with the q largest L_e. We add the corresponding inequality to the formulation if it is violated.

n-step p-cycle flow partition inequality. In order to find the most violated n-step p-cycle flow partition inequality (4.44), we want to minimize the left hand side terms. However, notice that $\sum_{R\in\mathcal{R}_I^{\bar{a_0}}} z_R$ on the left hand side of inequality (4.44) depends on the choice of I and there are exponentially many choices of I. To simplify the separation problem, we relax (4.44) to

$$\sum_{t=1}^{M} \mu_C^t(d) \sum_{e\in[A_U^+\setminus I\setminus a_0]} y_t^e + \sum_{a\in I}(x_a + z_a) \geq \mu_C^0(d) \tag{4.86}$$

and find the most violated inequality (4.86) instead. (4.86) is a weaker inequality than (4.44) because for each $R \in \mathcal{R}_I^{\bar{a_0}}$, $\hat{z}_R$ is added more than once in terms of $\sum_{a\in I} \hat{z}_a$. Therefore, if inequality (4.86) is violated, then (4.44) is also violated. To find the most violated inequality (4.86), we set

$$I = \Big\{a \in A_U^+ : \sum_{t=1}^{M} \mu_C^t(d)\hat{y}_t^{[a]} < \hat{x}_a + \hat{z}_a\Big\}. \tag{4.87}$$

n-step simple cutset inequality. For every fixed partition, there is a single n-step simple cutset inequality. We add it to the formulation if it is violated.

n-step flow cutset inequality. To get the most violated n-step flow cutset inequality, we set

$$E_\subset = \Big\{[a] \in E_U : \sum_{t=1}^{M} \mu_C^t(D)\hat{y}_t^{[a]} < \hat{x}_a + \hat{z}_a - \hat{x}_{\bar{a}}, a \in A_U^+\Big\}. \tag{4.88}$$

Note that we find violated inequalities in the above order and do not stop the separation after a violated inequality is found. We only remove an inequality if it is not active after the cutting plane algorithm is stopped, or it is a duplicate of an existing inequality (for example, if the most violated n-step flow cutset inequality is the same as the n-step simple cutset inequality).

The cutting plane algorithm and the separation for the valid inequalities of the MM-SND using undirected p-cycles (4.75)-(4.80) are similar.

4.6.4 Results And Analysis

Using the instance generation method in Section 4.6.1, we generated 5 sets of the MM-SND instances with 5 instances in each set. Each set has two capacity modules, with module sizes being (210, 50), (430, 110), (620, 150), (810, 200) and (990, 250), respectively. For all instances, the module costs are 30000 and 10000 for the larger and smaller capacity module, respectively. The summary of the instances are listed in Table 4.1.

In our first computational experiment, we compared the cost of network design using directed p-cycles versus undirected p-cycles. Each instance was solved by CPLEX 12.7 in default settings using formulation (4.1)-(4.6) and (4.75)-(4.80) where column generation procedure was used to add p-cycle variables. The optimal objective function values of the two formulations were recorded and compared. The results are also listed in Table 4.1. Each row shows the average statistics of 5 instances. The columns under *No. p-cycles* represent the number of p-cycle variables generated by the column generation procedure. The columns under *Obj* represent the optimal objective values. The column under *Obj%* represents the difference between the objective values of the two formulations in percentage. We observe from Table 4.1 that for the same instance, the column

Table 4.1: Summary of MM-SND instances with 2 modules

Instance	Module Sizes	Module Costs	Undirected p-Cycles		Directed p-Cycles		$Obj\%$
			No. p-cycles	Obj	No. p-cycles	Obj	
210.50	(210, 50)	(30000, 10000)	283	4215580	447	4051536	3.9%
430.110	(430, 110)	(30000, 10000)	287	2275402	436	2195738	3.5%
620.150	(620, 150)	(30000, 10000)	291	1746870	446	1685498	3.5%
810.200	(810, 200)	(30000, 10000)	293	1448348	450	1409602	2.7%
990.250	(990, 250)	(30000, 10000)	283	1273594	449	1233568	3.1%

generation heuristic generated more p-cycle variables for the formulation using directed p-cycles than using undirected p-cycles. On average, the optimal objective value of the formulation using directed p-cycles was 3.34% lower than that using undirected p-cycles, which can be a significant cost reduction in telecommunication network design. Therefore in our subsequent computations,

we only considered the MM-SND formulation using directed p-cycles.

In our second computational experiment, we compared the costs of the hierarchical model (4.1)-(4.6) and the integrated model (4.7)-(4.10) and (4.11)-(4.15). Each instance was solved using the two models. For the integrated model we recorded the optimal objective value directly, while for the hierarchical model we recorded the sum of the optimal objective values of the MMND problem (4.7)-(4.10) and the SCI problem (4.11)-(4.15). The results are listed in Table 4.2. Each row shows average statistics of 5 instances. The columns under Obj_H, Obj_I, Obj_{diff} and $Obj\%$ represent the optimal objective values of the hierarchical model, the optimal objective values of the integrated model, the difference in objective values between the two models, and the difference in percentage.

Table 4.2: Cost comparison of hierarchical and integrated models

Instance	Obj_H	Obj_I	Obj_{diff}	$Obj\%$
di210.50	4333764	4051536	282228	6.5%
di430.110	2389122	2195738	193384	8.1%
di620.150	1847972	1685498	162474	8.8%
di810.200	1521474	1409602	111872	7.4%
di990.250	1343614	1233568	110046	8.2%

We observe from Table 4.2 that the optimal objective values of the integrated model were consistently smaller than that of the hierarchical model. On average the objective values of the integrated model was 7.5% lower than that of the hierarchical model. In our subsequent computations, we focused on the MM-SND instances formulated using the integrated model.

In our next computational experiment, we tested the effectiveness of the inequalities listed in Section 4.4.

For each instance, we performed three separate runs. In the first run, we solved the instance using CPLEX under default settings. In the second run, we first ran our cutting plane algorithm. At each iteration of the algorithm, inequalities similar to the ones of Section 4.3 were searched for violation and added to the formulation (abbreviated as 1CUT). Then the instance was solved

Table 4.3: Results of computational experiments on MM-SND instances with 2 modules

Instance	Cut Type	$Cuts$	T_{Cut}	T	$Nodes$	$CG\%$
di210.50	DEF			723.2	2369208.0	
	DEF+1CUT	35.8	5.0	444.5	1680013.8	26.8
	DEF+1,2CUT	53.6	8.9	340.2	1225559.6	36.8
di430.110	DEF			286.2	888681.8	
	DEF+1CUT	37.8	5.0	173.1	570521.0	25.5
	DEF+1,2CUT	59.4	8.1	135.5	322232.0	40.2
di620.150	DEF			528.8	1161922.0	
	DEF+1CUT	47.6	5.2	319.7	789927.2	32.0
	DEF+1,2CUT	65.4	9.3	160.1	334919.8	43.3
di810.200	DEF			312.8	904465.4	
	DEF+1CUT	47.8	5.2	220.3	609501.6	30.1
	DEF+1,2CUT	70.0	9.5	213.8	555654.8	44.3
di990.250	DEF			427.4	819675.0	
	DEF+1CUT	52.0	5.1	425.7	683051.2	31.0
	DEF+1,2CUT	74.8	8.7	193.5	373481.6	43.2

using CPLEX. These inequalities were modified to suit 2-module instances. Essentially, they are obtained by taking $C = C_1$ in the $\mu_C^t(\cdot)$ function in the inequalities presented in Section 4.4. We refer to them as 1-step flow (simple) cutset inequalities, 1-step p-cycle flow (survivable) partition inequalities, and 1-step survivable subset-Q inequalities. In the third run, we ran our cutting plane algorithm, and at each iteration, the 1-step inequalities as well as the 2-step inequalities, i.e., the ones obtained by taking $C = \{C_1, C_2\}$ in the $\mu_C^t(\cdot)$ function in the inequalities presented in Section 4.4, were searched for violation and added to the formulation (abbreviated as 1,2CUT). Then the instance was solved using CPLEX. In all runs, CPLEX default cuts, abbreviated as DEF, were generated. The results are shown in Table 4.3. Each row represents the average statistic of 5 instances.

The column under Cut Type indicates the types of cuts added to the formulation in three separate runs. The column under $Cuts$ shows the total number of cuts added by our cutting plane algorithm (excluding CPLEX default cuts) at the root node. The columns under T_{Cut}, T and $Nodes$ are the CPU time to generate cuts, the CPU time to solve the problem (which is the sum of the cut generation time and the time reported by CPLEX to reach optimality), and the number of branch-and-cut nodes reported by CPLEX. The column under $CG\%$ shows the percentage of the

integrality gap closed by our cuts, calculated by $CG\% = 100 \times (zcut - zlp)/(zmip - zlp)$, where $zcut$ is the optimal objective value of the LP relaxation with the cuts.

We first compare the results from the instances after adding both 1- and 2-step inequalities with those directly obtained by CPLEX default. The average total solution time (including cut generation) with both types of cuts added was 0.45 times that of CPLEX 12.7 in its default setting. The number of branch-and-cut nodes was 0.45 times that of the default CPLEX on average. The average gap closed by the cuts was 41.6%.

Next, we compare the results after adding both 1- and 2-step inequalities with those after only adding the 1-step cuts. The average total solution time (including cut generation) with both cuts added was 0.68 times that with only adding 1-step cuts. The number of branch-and-cut nodes was 0.63 times that with only adding 1-step cuts. Adding both types of cuts closed 12.5% more integrality gap than adding 1-step cuts alone.

To further test the performance of the inequalities proposed in Section 4.4, we conducted computational experiments on 3-module MM-SND instances as well. We generated 4 sets of instances with 5 instances in each set. Each set has three capacity modules, with module sizes being (210, 50, 13), (430, 110, 26), (810, 200, 39) and (990, 250, 120), respectively. For all instances, the module costs are (30000, 10000, 2800) for the three capacity modules, from largest to smallest.

We solved the integrated model of the 3-module MM-SND instances using directed p-cycles. Besides similar runs to those for the 2-module MM-SND instances, we performed an additional run for the 3-module MM-SND instances, where all 1-, 2-, and 3-step inequalities, i.e., the ones obtained by taking $C = \{C_1, C_2, C_3\}$ in the $\mu_C^t(\cdot)$ function in the inequalities presented in Section 4.4, were searched for violation and added to the formulation (abbreviated as 1,2,3CUT). All other settings were similar to those for the 2-module MM-SND instances. The results are listed in Table 4.4.

We first compare the results from the instances after adding all 1-,2-, and 3-step inequalities with those obtained by CPLEX default. The average total solution time (including cut generation) with all types of cuts added was 0.64 times that of CPLEX 12.7 in its default settings. The number

Table 4.4: Results of computational experiments on MM-SND instances with 3 modules

Instance	Cut Type	$Cuts$	T_{Cut}	T	$Nodes$	$CG\%$
di210.50.13	DEF			1683.8	7212684.8	
	DEF+1CUT	32.0	5.2	1977.8	8071693.8	33.9
	DEF+1,2CUT	55.0	10.0	1971.9	7470053.5	35.6
	DEF+1,2,3CUT	61.8	11.9	1175.3	4872422.3	37.1
di430.110.26	DEF			209.6	767485.0	
	DEF+1CUT	39.0	5.1	162.4	521849.4	28.5
	DEF+1,2CUT	65.4	8.4	157.6	508635.2	31.2
	DEF+1,2,3CUT	68.0	11.1	123.2	341990.6	33.0
di810.200.39	DEF			243.1	783594.8	
	DEF+1CUT	49.6	5.1	187.4	557301.6	32.4
	DEF+1,2CUT	78.2	9.0	214.7	732831.8	37.4
	DEF+1,2,3CUT	96.0	12.8	98.2	227058.4	40.0
di990.250.120	DEF			1439.4	4351328.0	
	DEF+1CUT	53.8	5.2	1389.0	3640756.2	31.2
	DEF+1,2CUT	81.4	8.7	947.6	2982989.6	34.7
	DEF+1,2,3CUT	107.0	12.1	894.2	2596440.6	39.1

of branch-and-cut nodes was 0.61 times that of the default CPLEX on average. The average gap closed by the cuts was 37.3%.

Next, we compare the results after adding all 1-,2-, and 3-step inequalities with those after adding only 1-step inequalities and after adding only 1- and 2-step inequalities. The average total solution time (including cut generation) with all types of cuts added was 0.61 times that with only adding 1-step cuts and 0.70 times that with both 1- and 2-step cuts. The average number of branch-and-cut nodes was 0.63 times that with only adding 1-step cuts, and 0.69 times that with both 1- and 2-step cuts. Adding all types of cuts closed 5.8% more integrality gap than adding only 1-step cuts, and 2.6% more integrality gap than adding both 1- and 2-step cuts.

Based on our results, we conclude that the n-step inequalities for MM-SND are very effective in solving MM-SND instances with 2 and 3 modules, in comparison with CPLEX 12.7 in its default settings and the inequalities proposed for SM-SND in the literature.

5. GENERALIZATIONS OF MMND AND THE CUTSET POLYHEDRA

In this chapter, we extend our polyhedral study on MMND and the related cutset polyhedra to more general problems and mixed integer sets. In section 5.1, we generalize the n-step cutset inequalities to the multi-commodity MMND problems which are common in desicion-making problems. In section 5.2, we study a generalization of the cutset polyhedron, the continuous cutset polyhedron, and build connection between the results of this dissertation and several results in the literature.

5.1 The Multi-Commodity Directed MMND Problem

Understanding the polyhedral structure of MMND motivates us to study the more general multi-commodity MMND problem (MCMMND). Multi-commodity scenarios often arise in the backbone networks of telecommunication networks [51]. In this section, we discuss how to generalize the n-step cutset inequalities for MCMMND with all three types of link models.

For MCMMND, the network structures and the capacity modules are defined similarly to those of MMND. Instead of a single commodity of demand, we now have a set of commodities Q, indexed by k. Each commodity is identified by a single-source-single-sink pair of nodes, i.e., for each $k \in Q$, there is a single source node $s \in V$ with supply $d_s^k > 0$ and a single sink node $t \in V$ with demand $d_t^k = -d_s^k$.

For the directed MCMMND, let h_a^k be the unit cost of flow along arc $a \in A$ for commodity $k \in Q$. The mixed integer programming formulation for the directed MCMMND is

$$\min \sum_{a \in A} (\sum_{k \in Q} h_a^k x_a^k + \sum_{t=1}^{M} f_t^a y_t^a) \tag{5.1}$$

$$\sum_{a \in \delta^+(v)} x_a^k - \sum_{a \in \delta^-(v)} x_a^k = d_v^k, v \in V, k \in Q \tag{5.2}$$

$$\sum_{k \in Q} x_a^k \leq \sum_{t=1}^{M} C_t y_t^a + g^a, a \in A \tag{5.3}$$

$$(x, y) \in \mathbb{R}_+^{|Q||A|} \times \mathbb{Z}_+^{M|Q||A|}, \tag{5.4}$$

where x_a^k is now the amount of flow transferred along arc a for commodity k. Let X_{MC}^d be the convex hull of the set defined by (5.2)-(5.4). The corresponding cutset polyhedron with respect to a partition $(U, \overline{U})$ can be defined as

$$P_{MC}^d := conv\Bigg\{(x, y) \in \mathbb{R}_+^{|Q||E_U|} \times \mathbb{Z}_+^{M|E_U|} : \tag{5.5}$$

$$\sum_{a \in A_U^+} x_a^k - \sum_{a \in A_U^-} x_a^k = d^k, k \in Q \tag{5.6}$$

$$\sum_{k \in Q} x_a^k \leq \sum_{t=1}^{M} C_t y_t^a + g^a, a \in A_U \quad \Bigg\} \tag{5.7}$$

where $d^k = \sum_{v \in U} d_v^k$.

The generalization of the n-step general cutset inequality for directed MCMMND is stated in our next theorem.

Theorem 18. *Given a nonempty partition $(U, \overline{U})$ and the corresponding cutset polyhedron P_{MC}^d, $n \in \{1, \ldots, M\}, \alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} > 0, K \subseteq Q$, $A_\subset^+ \subseteq A_U^+$, and $A_\subset^- \subseteq A_U^-$, let $\overline{D} = \sum_{k \in K} d^k - \sum_{a \in A_\subset^+} g^a + \sum_{a \in A_\subset^-} g^a$. If the n-step MIR conditions* (2.6) *hold, i.e., $\alpha_t \left\lceil \overline{D}^{(t-1)} / \alpha_t \right\rceil \leq \alpha_{t-1}, t = 2, \ldots, n$, the n-step general cutset inequality*

$$\begin{aligned} \sum_{t=1}^{M} \mu_{\alpha,\overline{D}}^n(C_t) \sum_{a \in A_\subset^+} y_t^a + \sum_{t=1}^{M} \Big(C_t + \mu_{\alpha,\overline{D}}^n(-C_t)\Big) \sum_{a \in A_\subset^-} y_t^a \\ + \sum_{a \in A_U^+ \setminus A_\subset^+} \sum_{k \in K} x_a^k - \sum_{a \in A_\subset^-} \sum_{k \in K} x_a^k \geq \mu_{\alpha,\overline{D}}^n(\overline{D}) - \sum_{a \in A_\subset^-} g^a \end{aligned} \tag{5.8}$$

is valid for P_{MC}^d and X_{MC}^d.

Proof. If we aggregate the flow balance constraints (5.6) for $k \in K$, then relax the capacity constraints (5.7) to $\sum_{k \in K} x_a^k \leq \sum_{t=1}^{M} C_t y_t^a + g^a$ for A_U, and make change of variables $x_a = \sum_{k \in K} x_a^k, a \in A_U$, we have constructed a directed cutset polyhedron P^d with variables x_a and y_t^a,

$a \in A_U, t = 1, \ldots, M$. The n-step general cutset inequality (3.23) is valid for P^d, and if we write the n-step general cutset inequality with $x_a = \sum_{k \in K} x_a^k, a \in A_U$, we get exactly (5.8). $\square$

The multi-commodity undirected and bidirected MMND can be defined similarly to the multi-commodity directed MMND. Let X_{MC}^u and X_{MC}^b be the convex hulls of the multi-commodity undirected MMND and the multi-commodity bidrected MMND, respectively. We give without proof in our next theorem the n-step general cutset inequality for X_{MC}^u and X_{MC}^b.

Theorem 19. *Given a nonempty partition $(U, \overline{U})$ and the corresponding cutset polyhedron, $n \in \{1, \ldots, M\}, \alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} > 0, K \subseteq Q$, and $S_1, S_2 \subseteq E_U$, let $\overline{D} = \sum_{k \in K} d^k - \sum_{e \in S_1} g^e + \sum_{e \in S_2} g^e$. If the n-step MIR conditions* (2.6) *are satisfied, i.e., $\alpha_t \left\lceil \overline{D}^{(t-1)} / \alpha_t \right\rceil \leq \alpha_{t-1}, t = 2, \ldots, n$, the n-step general cutset inequality*

$$\begin{aligned} &\sum_{t=1}^{M} \mu_{\alpha,\overline{D}}^n(C_t) \sum_{e \in S_1} y_t^e + \sum_{t=1}^{M} \left(C_t + \mu_{\alpha,\overline{D}}^n(-C_t) \right) \sum_{e \in S_2} y_t^e \\ &+ \sum_{a \in A_U^+ \setminus A_1^+} \sum_{k \in K} x_a^k - \sum_{a \in A_2^-} \sum_{k \in K} x_a^k \geq \mu_{\alpha,\overline{D}}^n(\overline{D}) - \sum_{e \in S_2} g^e \end{aligned} \tag{5.9}$$

is valid for X_{MC}^u and X_{MC}^b.

Whether the generalized n-step cutset inequalities are facet-defining for X_{MC}^d, X_{MC}^u, and X_{MC}^b under certain conditions is an open question and requires further polyhedral study in future research.

5.2 Continuous Cutset Polyhedron

In this section, we study a generalization of the directed cutset polyhedron P^d. Consider the mixed integer set

$$Y^d := \Bigg\{ (r, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+^{|A_U|} \times \mathbb{Z}_+^{M|A_U|} : \tag{5.10}$$

$$\sum_{a \in A_U^+} x_a - \sum_{a \in A_U^-} x_a - r \leq d, \tag{5.11}$$

$$x_a \leq \sum_{t=1}^{M} C_t y_t^a, a \in A_U \tag{5.12}$$

$$y_t^a \leq u_t^a, a \in A_U, t = 1, \ldots, M. \quad \Big\} \tag{5.13}$$

We call Y^d the *continuous cutset polyhedron* because of the existence of an extra continuous variable. This set often arises as mixed integer generalizations of knapsack sets, or a row of a mixed integer program [43]. We present some valid inequalities for this set and show how they are related to cutset based inequalities for P^d.

5.2.1 Symmetric n-Step Cutset Inequalities

In [19], the correspondence between valid inequalities for K and the following set

$$K_{\leq} = \left\{(z,t) \in \mathbb{Z}_+^{|I|} \times \mathbb{R}_+ : \sum_{i \in I} C_i z_i \leq b + t\right\}$$

is given as follows.

Lemma 5 (Lemma 1 of [19])**.** *The inequality $\pi z + s \geq \pi_0$ is valid for K if and only if the inequality $(a - \pi)z \leq b - \pi_0 + t$ is valid for $K_{\leq}$.*

We use this result to show that the following:

Theorem 20. *Given $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} > 0$ and $n \in \{1, \ldots, M\}$, if the n-step MIR conditions* (2.6) *are satisfied, i.e., $\alpha_k \left\lceil d^{(k-1)}/\alpha_k \right\rceil \leq \alpha_{k-1}, k = 2, \ldots, n$, the symmetric n-step general cutset inequality*

$$\begin{aligned} \sum_{t=1}^{M} \mu_{\alpha,d}^n(C_t) \sum_{a \in A_C^+} y_t^a + \sum_{t=1}^{M} \Big(C_t + \mu_{\alpha,d}^n(-C_t)\Big) \sum_{a \in A_C^-} y_t^a \\ + d - \sum_{i \in A_C^+} x_i + \sum_{a \in A_U^- \setminus A_C^-} x_i + r \geq \mu_{\alpha,D}^n(d) \end{aligned} \tag{5.14}$$

is valid for Y^d.

Proof. From the proof of Theorem 3 we know that the n-step cutset inequality

$$\sum_{t=1}^{M} \mu^n_{\alpha,d}(C_t) \sum_{a\in A^+_C} y^a_t + \sum_{t=1}^{M} \mu^n_{\alpha,d}(-C_t) \sum_{a\in A^-_C} y^a_t + \sum_{t=1}^{M} C_t \sum_{a\in A^-_C} y^a_t + \sum_{a\in A^+_U\setminus A^+_C} x_a - \sum_{a\in A^-_C} x_a \geq \mu^n_{\alpha,d}(d) \tag{5.15}$$

is valid for the set defined by

$$\Big\{(x,y)\in \mathbb{R}^{|E_U|}_+ \times \mathbb{Z}^{n\times|E_U|}_+ : \sum_{t=1}^{M} C_t \sum_{a\in A^+_C} y^a_t - \sum_{t=1}^{M} C_t \sum_{a\in A^-_C} y^a_t + \sum_{a\in A^+_U\setminus A^+_C} x_a + \sum_{a\in A^-_C} \Big(\sum_{t=1}^{M} C_t y^a_t - x_a\Big) \geq d\Big\}.$$

Treat each $\sum_{a\in A^+_C} y^a_t$ and $\sum_{a\in A^-_C} y^a_t$ as z_i, $\sum_{a\in A^+_U\setminus A^+_C} x_a + \sum_{a\in A^-_C}(\sum_{t=1}^{n} C_t y^a_t - x_a)$ as s and d as b in K.

On the other hand, consider the following relaxation of Y^d: in (5.11), relaxing $x_a, a \in A^-_U \setminus A^-_C$ to their upper bound using (5.12), and dropping the variables $x_a, a \in A^+_U \setminus A^+_C$, we have

$$-\sum_{t=1}^{M} C_t \sum_{a\in A^-_C} y^a_t - \sum_{a\in A^-_U\setminus A^-_C} x_a + \sum_{a\in A^+_C} x_a - r \leq d. \tag{5.16}$$

This can be rewritten as

$$\sum_{t=1}^{M} C_t \sum_{a\in A^+_C} y^a_t - \sum_{t=1}^{M} C_t \sum_{a\in A^-_C} y^a_t \leq \sum_{t=1}^{M} C_t \sum_{a\in A^+_C} y^a_t + \sum_{a\in A^-_U\setminus A^-_C} x_a - \sum_{a\in A^+_C} x_a + r + d. \tag{5.17}$$

Treat each $\sum_{a\in A^+_C} y^a_t$ and $\sum_{a\in A^-_C} y^a_t$ as z_i, $\sum_{a\in A^+_C}(\sum_{t=1}^{n} C_t y^a_t - x_a) + \sum_{A^-_U\setminus A^-_C} x_a + r$ as s and d as b in K. By Lemma 5, the inequality

$$\sum_{t=1}^{M} \Big(C_t - \mu^n_{\alpha,d}(C_t)\Big) \sum_{a\in A^+_C} y^a_t + \sum_{t=1}^{M} \Big(-C_t - \mu^n_{\alpha,d}(-C_t)\Big) \sum_{a\in A^-_C} y^a_t \leq d - \mu^n_{\alpha,d}(d) + \sum_{t=1}^{M} \sum_{a\in A^+_C} C_t y^a_t - \sum_{a\in A^+_C} x_a + \sum_{a\in A^-_U\setminus A^-_C} x_a + r \tag{5.18}$$

is valid for the set defined by

$$\Big\{(r, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+^{|A_U|} \times \mathbb{Z}_+^{n \times |A_U|} : (5.17) \text{ hold}\Big\}.$$

By reorganizing terms (5.18) is exactly (5.14). □

A special case of the symmetric n-step general cutset inequality can be obtained by setting $u_t^a = +\infty, A_C^+ = A_U^+, A_U^- = \emptyset$, and $r = 0$. The resulting inequality

$$\sum_{t=1}^{M} \mu_{\alpha,d}^n(C_t) \sum_{a \in A_U^+} y_t^a + d - \sum_{a \in A_U^+} x_a \geq \mu_{\alpha,d}^n(d) \tag{5.19}$$

can be seen as a generalization of the so-called residual capacity inequality [42] in the single-commodity case. Setting $M = 1$ and $\alpha = C_1$ in (5.19), we obtain the arc residual capacity inequality

$$d^{(1)} \sum_{a \in A_U^+} y_1^a + D - \sum_{a \in A_U^+} x_a \geq d^{(1)} \left\lceil \frac{d}{C_1} \right\rceil \tag{5.20}$$

in [42]. Setting $M = 2$ and $\alpha = C_2$ in (5.19), we obtain the generalized residual capacity inequality

$$\sum_{a \in A_U^+} y_1^a + d^{(1)} \sum_{a \in A_U^+} y_2^a + d - \sum_{a \in A_U^+} x_a \geq d^{(1)} \left\lceil \frac{d}{C_2} \right\rceil \tag{5.21}$$

in [8].

Let P_0^d be a special case of P^d where $g^a = 0, a \in A_U$. Then P_0^d is a special case of Y^d where $u_t^a = +\infty, r = 0$, and $\leq$ is changed to $=$. It is straightforward to show the following.

Corollary 3. *Given $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} > 0$ and $n \in \{1, \ldots, M\}$, if the n-step MIR conditions* (2.6) *are satisfied, i.e., $\alpha_k \left\lceil d^{(k-1)}/\alpha_k \right\rceil \leq \alpha_{k-1}, k = 2, \ldots, n$, the symmetric n-step general cutset*

inequality

$$\sum_{t=1}^{M} \mu^n_{\alpha,d}(C_t) \sum_{a \in A^+_C} y^a_t + \sum_{t=1}^{M} \Big(C_t + \mu^n_{\alpha,d}(-C_t) \Big) \sum_{a \in A^-_C} y^a_t + d - \sum_{i \in A^+_C} x_i + \sum_{a \in A^-_U \setminus A^-_C} x_i \geq \mu^n_{\alpha,d}(d) \tag{5.22}$$

is valid for P^d_0.

In fact, inequality (5.22) is exactly the n-step general cutset inequality (3.23) plus the flow balance equality (3.11) of P^d_0. Similarly, inequality (5.19), which can be shown to be valid for P^d_0, is exactly the n-step simple cutset inequality (3.27) plus the flow balance equality (3.11) of P^d_0.

5.2.2 MIR-Flow-Cover Inequalities

Notice that the upper bounds on the capacity variables are not used in the generalization of the symmetric n-step cutset inequality. In this section, we generalize the MIR-flow-cover inequalities in [43] to incorporate the upper bounds as well.

Definition 12. *Let* (B_1, B_2) *be an* integer flow cover *for* Y^d *if*

1. $B_1 \subseteq A^+_U, B_2 \subseteq A^-_U$
2. *there exists* $k \in B_1$ *and* $\tau \in \{1, \ldots, n\}$ *such that*

$$\sum_{t \in \{1,\ldots,M\} \setminus \{\tau\}} C_t u^k_t + \sum_{a \in C_1 \setminus k} \sum_{t=1}^{M} C_t u^a_t - \sum_{a \in C_2} \sum_{t=1}^{M} C_t u^a_t < d,$$

and there exists unique values η^k_τ *and* λ *such that*

$$C_\tau \eta^k_\tau + \sum_{t \in \{1,\ldots,n\} \setminus \{\tau\}} C_t u^k_t + \sum_{a \in C_1 \setminus k} \sum_{t=1}^{n} C_t u^a_t - \sum_{a \in C_2} \sum_{t=1}^{n} C_t u^a_t = d + \lambda \tag{5.23}$$

with $0 < \lambda < a_k, \eta^k_\tau \in \mathbb{Z}$ *and* $1 \leq \eta^k_\tau \leq u^k_\tau$.

Theorem 21. *Let (B_1, B_2) be an integer flow cover for Y^d. Given $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} > 0$ and $n \in \{1, \ldots, M\}$, if the n-step MIR conditions* (2.6) *are satisfied, i.e., $\alpha_k \left\lceil \bar{b}^{(k-1)}/\alpha_k \right\rceil \leq \alpha_{k-1}, k = 2, \ldots, n$, the n-step MIR-flow-cover inequality*

$$\begin{aligned}\sum_{t=1}^{M} \mu^n_{\alpha,\bar{b}}(C_t)\Big(\sum_{a\in B_1}(u^a_t - y^a_t) - \sum_{a\in L_1} y^a_t - \sum_{a\in B_2}(u^a_t - y^a_t) + \sum_{a\in L_2} y^a_t\Big) \\ + \sum_{B_1\cup L_1}(\sum_{t=1}^{M} a_t y^a_t - x_a) + \sum_{a\in R_2} x_a + r \geq \mu^n_{\alpha,\bar{b}}(\bar{b})\end{aligned} \tag{5.24}$$

is valid for Y^d, where $\bar{b} = a_\tau(u^k_\tau - \eta^k_\tau) + \lambda$, (B_1, L_1, R_1) is a partition of A^+_U, and (B_2, L_2, R_2) is a partition of A^-_U.

Proof. Using nonnegativity constraints of $x_a, a \in R_1$ and the upper bound constraints of $x_a, a \in L_2 \cup B_2$, we can relax the flow balance constraint (5.11) to

$$\begin{aligned}\sum_{a\in B_1}\sum_{t=1}^{M} C_t y^a_t + \sum_{a\in L_1}\sum_{t=1}^{M} C_t y^a_t - \sum_{a\in B_2}\sum_{t=1}^{M} C_t y^a_t - \sum_{a\in L_2}\sum_{t=1}^{M} C_t y^a_t + \sum_{a\in B_1\cup L_1} x_a \\ - \sum_{a\in B_1\cup L_1}\sum_{t=1}^{M} C_t y^a_t - \sum_{a\in R_2} x_a \leq d + r.\end{aligned} \tag{5.25}$$

Notice that equation (5.23) can be written as

$$a_\tau \eta^k_\tau + \sum_{a\in B_1}\sum_{t=1}^{M} C_t u^a_t - C_\tau u^k_\tau - \sum_{a\in B_2}\sum_{t=1}^{M} C_t u^a_t = d + \lambda. \tag{5.26}$$

Using (5.26), (5.25) can be written as

$$\begin{aligned}\sum_{a\in B_1}\sum_{t=1}^{M} C_t y^a_t + \sum_{a\in L_1}\sum_{t=1}^{M} C_t y^a_t - \sum_{a\in B_2}\sum_{t=1}^{M} C_t y^a_t - \sum_{a\in L_2}\sum_{t=1}^{M} C_t y^a_t \\ + \sum_{a\in B_1\cup L_1} x_a - \sum_{a\in B_1\cup L_1}\sum_{t=1}^{M} C_t y^a_t - \sum_{a\in R_2} x_a \leq a_\tau \eta^k_\tau \\ + \sum_{a\in B_1}\sum_{t=1}^{M} C_t u^a_t - C_\tau u^k_\tau - \sum_{a\in B_2}\sum_{t=1}^{M} C_t u^a_t - \lambda + r,\end{aligned} \tag{5.27}$$

which can be reorganized to

$$\begin{aligned}&\sum_{t=1}^{M} C_t\Big(\sum_{a\in B_1}(u_t^a-y_t^a)-\sum_{a\in L_1}y_t^a-\sum_{a\in B_2}(u_t^a-y_t^a)+\sum_{a\in L_2}y_t^a\Big)\\&+\sum_{a\in B_1\cup L_1}(\sum_{t=1}^{M}C_t y_t^a-x_a)+\sum_{a\in R_2}x_a+r\geq C_\tau(u_\tau^k-\eta_\tau^k)+\lambda\end{aligned}\tag{5.28}$$

Treating each $\sum_{a\in B_1}(u_t^a-y_t^a)-\sum_{a\in L_1}y_t^a-\sum_{a\in B_2}(u_t^a-y_t^a)+\sum_{a\in L_2}y_t^a$ as z_i, $\sum_{B_1\cup L_1}(\sum_{t=1}^{n}C_t y_t^a-x_a)+\sum_{a\in R_2}x_a+r$ as s, and $a_\tau(u_\tau^k-\eta_\tau^k)+\lambda$ as b in K, by applying n-step MIR inequality, we get exactly (5.24). □

Remark 4. *A similar procedure can be applied to a variation of Y^d where the capacities are variable (See Remark 2), which is discussed in [43]. This yields the inequality*

$$\begin{aligned}&\sum_{a\in A_U}\mu^n_{\alpha,C_k(u_k-\eta_k)+\lambda}(C_a)\Big(\sum_{a\in B_1}(u_a-y_a)-\sum_{a\in L_1}y_a-\sum_{a\in B_2}(u_a-y_a)+\sum_{a\in L_2}y_a\Big)\\&+\sum_{B_1\cup L_1}(C_a y_a-x_a)+\sum_{a\in R_2}x_a+r\geq\mu^n_{\alpha,C_k(u_k-\eta_k)+\lambda}[C_k(u_k-\eta_k)+\lambda].\end{aligned}\tag{5.29}$$

In fact, the strengthened MIR-flow-cover inequality in [43] which was obtained through lifting can be obtained from (5.29) by taking $n=1,\alpha=C_k$.

6. CONCLUSION AND FUTURE RESEARCH

In this dissertation, we studied the multi-module capacitated (survivable) network design problems, a set of fundamental optimization problems which arise in many real-world decision-making scenarios. In such problems, the capacities on the links of the network are constructed by installing multiples of different capacity modules with different sizes, and flows of the links are carried using the installed capacities. In this chapter, we conclude the results presented in this dissertation and provide directions for future research.

6.1 Conclusion

In this dissertation, we focused on developing the polyhedral results for several models of the multi-module capacitated network design problem (MMND) as well as the multi-module capacitated survivable network design problem (MM-SND). We proposed new families of cutting planes for both MMND and MM-SND based on a particular structure of the network, which partitions the network into two parts (i.e., a cutset). These inequalities generalize many cutset-based inequalities in the literature and significantly improve the efficiency to solve the MMND and MM-SND instances, based on our computational results.

First, we developed valid inequalities for the multi-module capacitated network design problem, referred to as the n-step cutset inequalities, from the convex hull of a mixed integer set called the cutset polyhedron. The cutset polyhedron is obtained from aggregating and relaxing the base constraints of the MMND. In developing these inequalities, we applied the n-step MIR theory [17], a powerful tool for generating strong valid inequalities for multi-module problems such as multi-module lot-sizing (MMLS) and multi-module facility location (MMFL) problems. We showed that the cutset-based inequalities in the literature [7, 2, 3, 12, 10] are special cases of the n-step cutset inequalities. We then showed that the n-step cutset inequalities are not only facet-defining for the cutset polyhedron, but also for the convex hull of the set defined by the original constraints of MMND. We designed a separation heuristic combined with a cutting plane algorithm to add

the cutset inequalities to the formulation, and our computational results showed that the cutset inequalities are very effective in solving the MMND instances.

Next, we studied a highly applicable generalization of MMND, the multi-module capacitated survivable network design problem , which is a fundamental problem in industries such as telecommunication and power. In such problems, we provided several formulations of MM-SND which trade off the time efficiency and the cost efficiency of the survivable network design. In these models, we used the so-called p-cycles (preconfigured cycles) to reserve slacks on the installed capacities on the edges. When an edge fails, the flows through the failed edge can take up the reserved slacks on other edges of the p-cycle to ensure normal functioning of the network. We focused on the integrated MM-SND model where the flows on arcs, the reserved slacks on p-cycles and the capacities on edges are determined in a single formulation. We developed several families of valid inequalities. The n-step flow cutset inequalities, the n-step p-cycle flow partition inequalities, and the n-step survivable subset-Q inequalities are based on two-partitions of the network (i.e., a cutset), while the three-partition inequalities are based on three-partitioning of the network. These inequalities generalize several cutset-based inequalities for survivable network design problems in the literature [31, 30]. We showed that the special cases of the n-step flow cutset inequalities and the n-step p-cycle flow partition inequalities are facet-defining for the convex hull of the set defined by the constraints of MM-SND. To make the size of the problem practical, we designed a column generation method to add a subset of p-cycle variables to the formulation. Our computational results showed that the proposed inequalities are effective in solving the MM-SND instances.

Finally, we also studied the extensions of MMND and the cutset polyhedron. For the multi-commodity MMND problem (MCMMND), we generalized the n-step cutset inequalities for MMND that incorporate a subset of commodities. For the continuous cutset polyhedron X_s where a continuous variable is added in the flow balance constraint, we showed that the symmetric n-step cutset inequalities are valid for X_s, and generalized the MIR-flow-cover inequality in [43].

6.2 Future Research

Several new research topics arises from the methodological developments in this dissertation. Some of the directions from the results in this dissertation are as follows:

1. Multi-Commodity Multi-Module Capacitated Network Design Problem. Telecommunication networks usually are comprised of both single-commodity networks (such as local access networks) as well as multi-commodity networks (such as backbone networks) [11]. We have generalized the n-step cutset inequalities for MCMMND in Section 5.1. We plan to investigate the strength of these inequalities, especially, if any special cases of these inequalities are facet-defining for the convex hull of the set defined by the constraints of MCMMND. We will also determine if there are other types of strong valid inequalities exclusive for MCMMND.

2. Relationships between the multi-module capacitated network design problem (MMND), the multi-module capacitated lot-sizing problem (MMLS), and the multi-module capacitated facility location problem (MMFL). MMLS and MMFL are two widely studied multi-module decision making problems which arise in industry applications other than MMND. It can be shown that MMFL and MMLS are special cases of MMND. Therefore, it is interesting to investigate the effectiveness of the n-step cutset inequalities on MMFL and MMLS. Especially, we will study whether the n-step cutset inequalities or any special cases of them are facet-defining for the convex hull of the set defined by the constraints of MMFL and MMLS. We will also computationally test if the n-step cutset inequalities will improve the efficiency of solving MMFL and MMLS instances.

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