A STUDY OF YU's EXTENSION OF THE COLUMN ANALOGY METHOD FOR CONTINUOUS FRAME AND ARCH ANALYSIS

by

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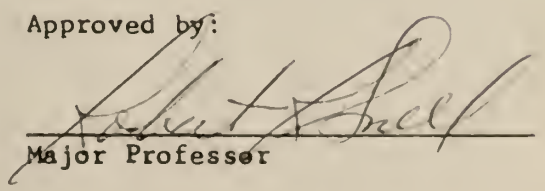

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SYNOPSIS

The basic column analogy method is applicable to beams, arches and bents which do not have more than three redundant elements. Many continuous frames and arches are outside of the limits of this restriction. The purpose of this report is to show how this restriction may be relaxed so that the column analogy method may be applied to continuous structures. An additional purpose is to show that this method is an excellent technique for use in structural analysis, especially for arches and gable frames. This method requires many calculations. However, if these calculations are written into computer programs, this method compares favorably with most other methods. The examples in this report were chosen to show the efficiency of this technique.

INTRODUCTION

The word analogy calls attention to a relation of likness between two things consisting of the resemblance not of the things themselves but of two or more circumstances or effects. In this sense, the world is filled with analogies since it requires only a slight exercise of imagination to discover such relations. Hardy Cross[1] observed an analogy between a column and a closed ring type flexural member or frame, and based on this analogy he introduced the method of column analogy in 1930. This method is familiar to most structural engineers, and is an excellent technique for solving indeterminate structures, especially, for gable frames and arches.

Hardy Cross' column analogy method[1], as originally introduced, is restricted to closed type structures with not more than three redundant elements. However, it may be applied in continuous structures with more than three redundancies if a modified concept is introduced. Yu[2] explored this extension in 1950. By introducing an ideal elastic support to the structure as shown in Fig. 1(a), the entire structure may be reduced from six to three redundancies, as shown in Fig. 1(b). In view of this reduction it is evident that the column analogy method can be applied to such structures.

The fundamentals of the extended method are shown in Fig. 1. The two-span continuous structure can be replaced by two equivalent single-span components, and the stresses in each span can be determined by the column analogy method. However, some details must be considered before the application of the method to frames such as the one shown in Fig. 1. If both spans in Fig. 1 have external loads upon them, the analysis must be divided into two steps. One step takes into consideration the external loads on

span 1 and the other considers the external loads on span 2. If span 1 is to be analyzed first, the external loads on span 2 must be taken away temporarily so that "the elastic foundation CBD" can be converted into an idealized bar, BB' . Then the method is applied to determine the stresses in each member caused by the external loads on span 1, as shown in Fig. 1(a) and (b). Similarly, the stresses in each member caused by external loads on span 2 can be determined as shown in Fig. 1(c) and (d). The final step is to combine these two analyses.

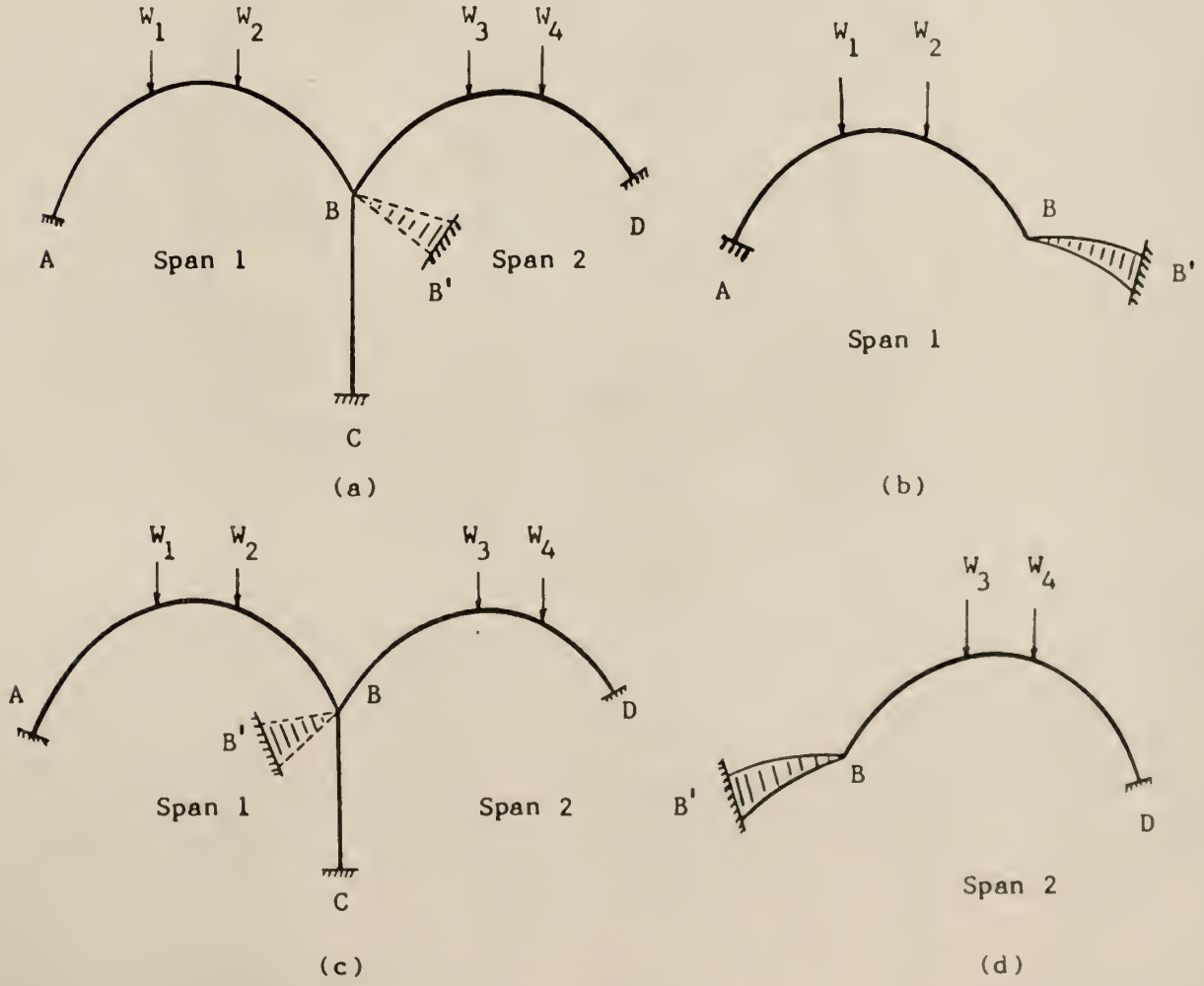


Fig. 1. Primary Features Of The Extended Method.

SIGN CONVENTIONS

Before going into the derivation of the method, it is necessary to set up a sign convention. The sign convention includes:

(1) The bending moments in the column analogy are taken as positive when they produce tension on the bottom of a beam or on the inner side of an arch or bent, as shown in Fig. 2. Positive rotations are those produced by positive moments.

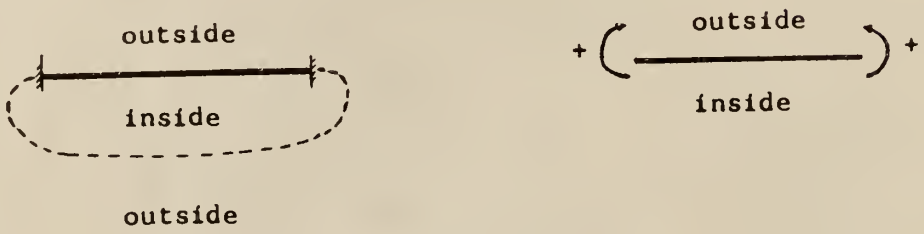


Fig. 2. Sign Conventions.

(2) Horizontal forces are positive if they pull away at the abutments (i.e. tension in the girder of a bent).

(3) Positive shear is up on the left and down on the right side of the section in question.

(4) Positive displacements correspond to positive forces. Positive bending moment is positive force times positive distance.

(5) The moment diagram for each member will be plotted on the compression side of the member.

DERIVATIONS

(I) General Derivations for Symmetrical and Unsymmetrical Sections

As shown in Fig. 3, a bent ABCD under load P is cut away at the left support while the redundant reactions X, Y, and Z are applied at this support point.

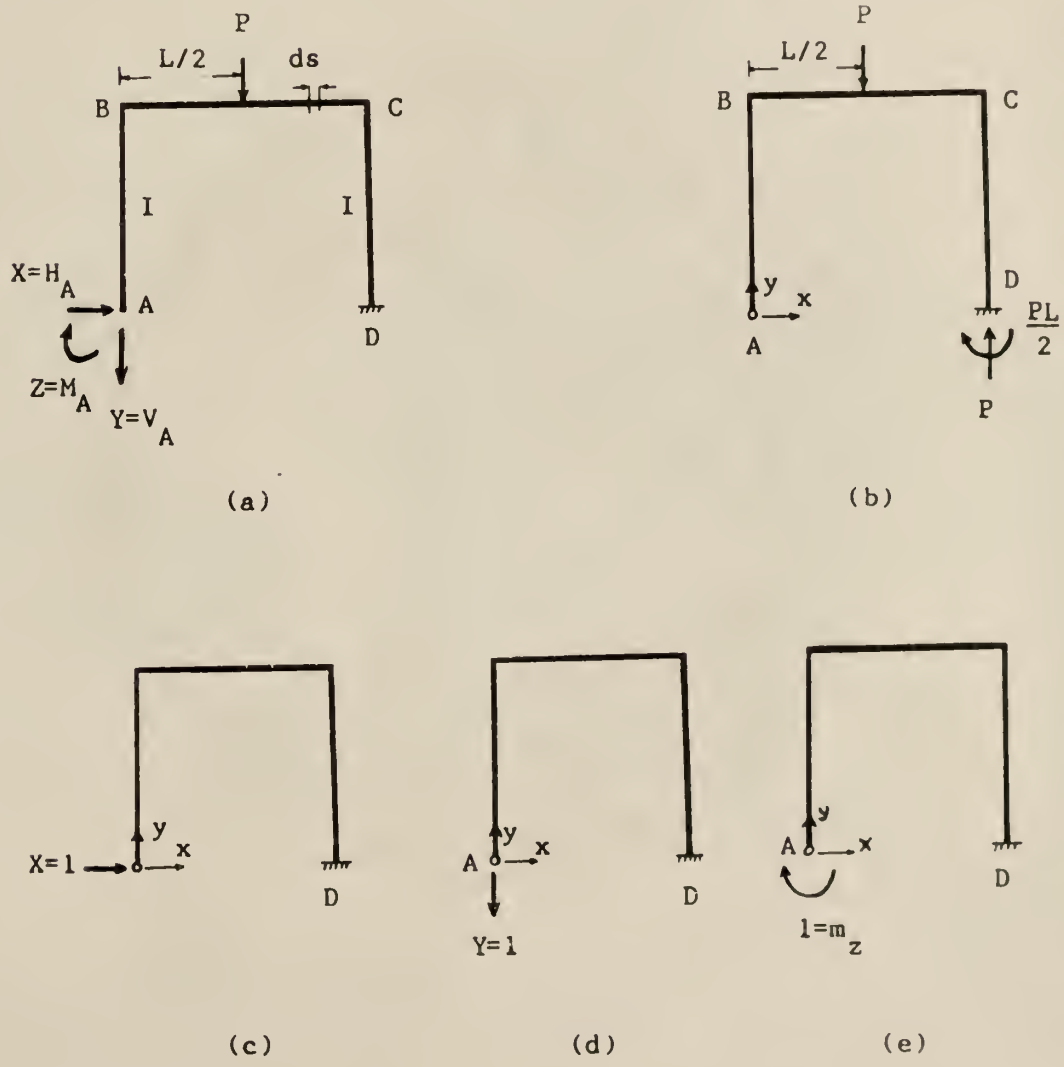


Fig. 3. Effects Of External Loads And Dummy Loads At A.

Then the equations of consistent elastic deformation can be set up as follows:

$$d_x = 0 = d'_x + Xd_{xx} + Yd_{xy} + Zd_{xz} , \tag{1a}$$

$$d_y = 0 = d'_y + Xd_{yx} + Yd_{yy} + Zd_{yz} , \tag{1b}$$

$$d_z = 0 = d'_z + Xd_{zx} + Yd_{zy} + Zd_{zz} , \tag{1c}$$

where d_x , d_y , d_z = the total displacements in the directions of X, Y, and Z respectively,

d'_x , d'_y , d'_z = the displacements in the directions of X, Y, and Z respectively caused by the applied load,

d_{xx} , d_{xy} , d_{xz} = the displacements in the directions of X, Y, and Z respectively caused by the redundant force X = 1 ,

d_{yx} , d_{yy} , d_{yz} = the displacements in the direction of X, Y, and Z respectively caused by the redundant force Y = 1 ,

d_{zx} , d_{zy} , d_{zz} = the displacements in the directions of X, Y, and Z respectively caused by the redundant force Z = 1 ,

X, Y, Z = the redundant forces, horizontal, vertical and moment, respectively.

In Eqs. (1a), (1b), and (1c), the quantities d'_x , d'_y , and d'_z can be evaluated by using the dummy unit load method as follows:

$$m_x = -y \quad ; \quad m_y = -x \quad ; \quad m_z = +1 ,$$

$$d'_x = \int \frac{M' m_x}{EI} ds = -\int \frac{M' y}{EI} ds , \tag{1a'}$$

$$d'_y = \int \frac{M' m_y}{EI} ds = -\int \frac{M' x}{EI} ds , \tag{1b'}$$

$$d'_z = \int \frac{M' m_z}{EI}\, ds = \int \frac{M'}{EI}\, ds \,, \qquad (1c')$$

where m_x , m_y , m_z = bending moments at any point in the structure due to $X = 1$, $Y = 1$, and $Z = 1$ respectively.

M' = the bending moment at any point in the structure which is caused by the applied load.

EI = flexural rigidity.

Neglecting the axial effects does not in general lead to significant errors in the analysis. Therefore, the quantities d_{xx} , d_{yy} , d_{xz} , d_{zz} , ... etc. in Eqs. (1a), (1b), and (1c) may be expressed in the following general forms:

$$d_{xx} = \int \frac{m_x^2}{EI}\, ds = \int y^2 \frac{ds}{EI} \,, \qquad (1d')$$

$$d_{yy} = \int \frac{m_y^2}{EI}\, ds = \int x^2 \frac{ds}{EI} \,, \qquad (1e')$$

$$d_{zz} = \int \frac{m_z^2}{EI}\, ds = \int \frac{ds}{EI} \,, \qquad (1f')$$

$$d_{xz} = \int \frac{m_x m_z}{EI}\, ds = -\int y \frac{ds}{EI} = d_{zx} \,, \qquad (1g')$$

$$d_{xy} = \int \frac{m_x m_y}{EI}\, ds = +\int xy \frac{ds}{EI} = d_{yx} \,, \qquad (1h')$$

$$d_{yz} = \int \frac{m_y m_z}{EI}\, ds = -\int x \frac{ds}{EI} = d_{zy} \,. \qquad (1i')$$

Thus, Eqs. (1a), (1b), and (1c) become:

$$X \int \frac{y^2\, ds}{EI} + Y \int \frac{xy\, ds}{EI} - Z \int \frac{y\, ds}{EI} - \int M' y \frac{ds}{EI} = 0 \,, \qquad (2a)$$

$$X \int \frac{xy\ ds}{EI} + Y \int \frac{x^2\ ds}{EI} - Z \int \frac{x\ ds}{EI} - \int M' x \frac{ds}{EI} = 0 , \qquad (2b)$$

$$-X \int \frac{y\ ds}{EI} - Y \int \frac{x\ ds}{EI} + Z \int \frac{ds}{EI} + \int M' \frac{ds}{EI} = 0 . \qquad (2c)$$

By solving Eqs. (1a), (1b), and (1c) by Cramer's rule, the redundant quantities, X, Y, and Z can be obtained.

$$X = \frac{\begin{vmatrix} -d'_x & d_{xy} & d_{xz} \\ -d'_y & d_{yy} & d_{yz} \\ -d'_z & d_{zy} & d_{zz} \end{vmatrix}}{J(d)} , \qquad (3a)$$

$$Y = \frac{\begin{vmatrix} d_{xx} & -d'_x & d_{xz} \\ d_{yx} & -d'_y & d_{yz} \\ d_{zx} & -d'_z & d_{zz} \end{vmatrix}}{J(d)} , \qquad (3b)$$

$$Z = \frac{\begin{vmatrix} d_{xx} & d_{xy} & -d'_x \\ d_{yz} & d_{yy} & -d'_y \\ d_{zx} & d_{zy} & -d'_z \end{vmatrix}}{J(d)} , \qquad (3c)$$

where

$$J(d) = \begin{vmatrix} d_{xx} & d_{xy} & d_{xz} \\ d_{yx} & d_{yy} & d_{yz} \\ d_{zx} & d_{zy} & d_{zz} \end{vmatrix} . \qquad (3d)$$

If the incremental area of the analogous column is defined as the quantity $\frac{ds}{EI}$ where $\frac{1}{EI}$ is treated as the width of the incremental area, then the integral terms in Eqs. (2a), (2b), and (2c) may be taken to define the quantities shown in TABLE I.

TABLE I DEFINITIONS OF AREA PROPERTIES

$\int \frac{ds}{EI} = \int dA$ = Total area of fictious strip = $A = d_{zz}$.

$\int \frac{M' ds}{EI} = \int$(intensity of load) dA

= Total load on the above area = $P = d'_z$.

$\int \frac{M' x}{EI} ds = Px$ = moment of load about y-y axis = $M_y = -d'_y$.

$\int \frac{M' y}{EI} ds = Py$ = moment of load about x-x axis = $M_x = -d'_x$.

$\int \frac{x^2 ds}{EI} = \int x^2 dA$ = moment of inertia about y-axis = $I_y = d_{yy}$.

$\int \frac{y^2 ds}{EI} = \int y^2 dA$ = moment of inertia about x-axis = $I_x = d_{xx}$.

$\int \frac{xy\ ds}{EI} = \int xy\ dA$ = product of inertia = $I_{xy} = d_{xy}$.

Since the origin of the coordinates in Fig. 3 was arbitrarily chosen, it is possible to simplify Eqs. (1a), (1b), and (1c) so that the three equations will reduce from 9 terms to 3 terms. If the quantities $d_{zx} = d_{xz} = \int \frac{y\ ds}{EI}$ and $d_{zy} = d_{yz} = \int \frac{x\ ds}{EI}$ are considered as properties of the incremental area, $\frac{ds}{EI}$, it can be seen that they will vanish if the origin of the axis is translated to the "center of gravity" of the bent. Similarly, if the axes are chosen to coincide with the principal axes of the bent, then

the integral $d_{yx} = d_{xy} = \int \frac{xy\,ds}{EI}$ which it can be seen is analogous to the product of moment of inertia of the incremental area is also equal to zero. Therefore, if the structure is symmetrical, the origin of the coordinates can be chosen on the center of gravity of the structure with an axis of symmetry. Then the quantities

$$d_{xy} = d_{yx} = d_{zy} = d_{yz} = d_{xz} = d_{zx} = 0 ,$$

and Eqs. (3a), (3b), and (3c) become

$$X = -\frac{d'_x}{d_{xx}} \quad ; \quad Y = -\frac{d'_y}{d_{yy}} \quad ; \quad Z = -\frac{d'_z}{d_{zz}} . \tag{4}$$

If the structural is unsymmetrical, the axes which were originally chosen at the center of gravity of the structure are not necessarily the principal axes, and therefore the quantities $d_{zx} = d_{xz}$ and $d_{zy} = d_{yz}$ will vanish but the quantity $d_{xy} = d_{yx}$ may still remain. Then Eqs. (3) can be expanded to become

$$X = \frac{d_{zz}\,(-d'_x\,d_{yy} + d'_y\,d_{xy})}{d_{zz}\,(d_{xx}\,d_{yy} - d_{xy}^2)} = -\frac{d'_x - d'_y\,\frac{d_{xy}}{d_{yy}}}{d_{xx} - \frac{d_{xy}^2}{d_{yy}}} , \tag{5a}$$

$$Y = \frac{d_{zz}\,(-d'_y\,d_{xx} + d'_x\,d_{yz})}{d_{zz}\,(d_{xx}\,d_{yy} - d_{xy}^2)} = -\frac{d'_y - d'_x\,\frac{d_{yz}}{d_{xx}}}{d_{yy} - \frac{d_{xy}^2}{d_{xx}}} , \tag{5b}$$

$$Z = \frac{-d'_z\,(d_{xx}\,d_{yy} - d_{xy}^2)}{d_{zz}\,(d_{xx}\,d_{yy} - d_{xy}^2)} = -\frac{d'_z}{d_{zz}} . \tag{5c}$$

The quantity $d_{xy} = d_{yx}$ in the above expressions may also be made to vanish by rotating the axes to coincide with the principal axes. By substituting the properties of TABLE I into Eq. (4), the quantities X, Y, and Z for symmetrical structures become:

$$X = -\frac{d'_x}{d_{xx}} = +\frac{M_x}{I_x} , \tag{6a}$$

$$Y = -\frac{d'_y}{d_{yy}} = +\frac{M_y}{I_y} , \tag{6b}$$

$$Z = -\frac{d'_z}{d_{zz}} = -\frac{P}{A} . \tag{6c}$$

Similarly, Eqs. (5a), (5b), and (5c) for unsymmetrical structures become:

$$X = -\frac{M_x - M_y \frac{I_{xy}}{I_y}}{I_x - \frac{I_{xy}^2}{I_y}} = +\frac{M'_x}{I'_x} , \tag{7a}$$

where $$M'_x = M_x - M_y \frac{I_{xy}}{I_y} \quad ; \quad I'_x = I_x - \frac{I_{xy}^2}{I_y} = I_x \left(1 - \frac{I_{xy}^2}{I_x I_y}\right) .$$

$$Y = -\frac{M_y - M_x \frac{I_{xy}}{I_x}}{I_y - \frac{I_{xy}^2}{I_x}} = +\frac{M'_y}{I'_y} , \tag{7b}$$

where $$M'_y = M_y - M_x \frac{I_{xy}}{I_x} \quad ; \quad I'_y = I_y - \frac{I_{xy}^2}{I_x} = I_y \left(1 - \frac{I_{xy}^2}{I_x I_y}\right) .$$

$$Z = -\frac{d'_z}{d_{zz}} = -\frac{P}{A} \quad . \tag{7c}$$

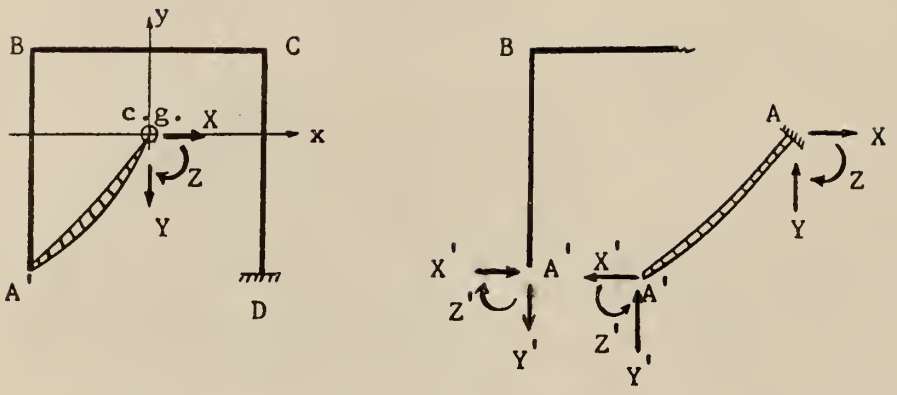


Fig. 4. Application Of Rigid Bar.

Upon examination of the superposition equations (2a), (2b), and (2c), it is found that if the axes were transferred from the left support to the elastic center as shown in Fig. 4, the reactions at the left support could be obtained by introducing a "rigid bar" to be attached from the support point to the elastic center; the reactions will then act through the elastic center. Since the "rigid bar" is undeformable, it does not change any of the coefficients of the superposition equations. Hence, by applying the statical equilibrium equations to the rigid bar as shown in Fig. (4b), the equilibrium equations will be as follows:

$$\Sigma H = 0 \quad ; \quad X' = X \quad , \tag{8a}$$

$$\Sigma V = 0 \quad ; \quad Y' = Y \quad , \tag{8b}$$

$$\Sigma M_{A'} = 0 \quad ; \quad Z' = -Z + X\,y + Y\,x \quad . \tag{8c}$$

A comparison of Eq. (8c) with the flexural stress formula for a short column,

$$f = \frac{P}{A} + \frac{M_y}{I_y}\,x + \frac{M_x}{I_x}\,y \quad ,$$

reveals that if Eqs. (6a), (6b), and (6c) are substituted into Eq. (8c) and if we let $M_i = Z'$, we obtain the following result:

$$M_i = Z' = \frac{P}{A} + \frac{M_y}{I_y}\,x + \frac{M_x}{I_x}\,y \quad . \tag{9}$$

This expression, then, yields the indeterminate moment for a symmetrical structure. Equation (9) is identical with the column flexural stress formula. If Eqs. (7a), (7b), and (7c) are substituted into Eq. (8c), the indeterminate moment for an unsymmetrical structure becomes:

$$M_i = Z' = \frac{P}{A} + \frac{M'_y}{I'_y}\,x + \frac{M'_x}{I'_x}\,y \quad , \tag{10a}$$

where

$$M'_y = M_y - M_x \frac{I_{xy}}{I_x} \quad , \tag{10b}$$

$$M'_x = M_x - M_y \frac{I_{xy}}{I_y} \quad , \tag{10c}$$

$$I'_x = I_x \left(1 - \frac{I^2_{xy}}{I_x\,I_y}\right) , \tag{10d}$$

$$I'_y = I_y \left(1 - \frac{I_{xy}^2}{I_x I_y}\right) \quad . \tag{10e}$$

The above expressions set up the column analogy method for use with both symmetrical and unsymmetrical structures. The final moment is the algebraic sum of the determinate moment, M_s , and the indeterminate moment, M_i .

$$M_T = M_s - M_i \quad . \tag{11}$$

(II) Derivation of the Extended Formulae

The fundamental innovation of this method is to introduce a "substitute member BB'" at B which takes the place of the "elastic foundation CBDE" to support the bent FAB which is to be analyzed. This concept is shown in Fig. 5. The primary requirement of the method is that the elastic displacements of the substitute member BB' and the elastic foundation CBDE must be identical. If the structure is cut at joint B, the redundant forces at this joint are as shown in Figs. 5(c) and (d). If the substitute member BB' and the elastic foundation CBDE are considered separately, the displacements of the "elastic foundation" in Fig. 5(c) must be equivalent to those of the "substitute member" in Fig. 5(d).

The elastic displacements of the elastic foundation CBDE at joint B caused by the unit redundant forces $X = 1$, $Y = 1$, and $Z = 1$ can be computed by setting up superposition equations. Since each of these forces at joint B may obviously produce three components of displacement, i.e. horizontal, vertical, and rotation, the induced elastic displacements can be expressed the same way as those in TABLE I. It is only necessary then to consider the effect of the substitute member BB' here. As shown in

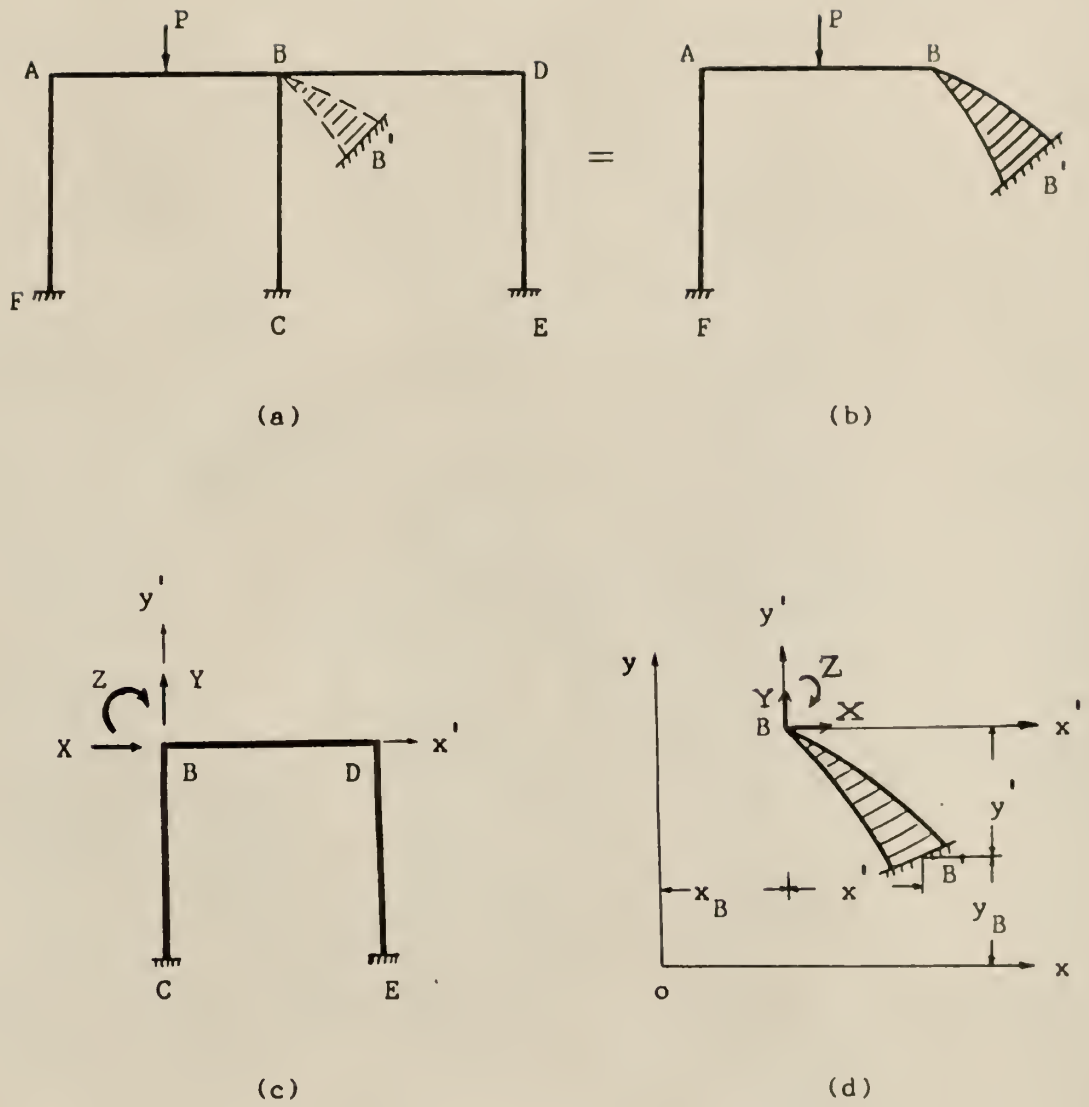


Fig. 5. Concept Of Extended Method

Fig. 5(d), if the virtual work method is to be applied, the induced displacements at B of the member BB' for a unit moment, $Z = 1$, at B become:

$$m_x = -y' \quad ; \quad m_y = +x' \quad ; \quad m_z = \text{unity} ,$$

$$d_{zx} = \int \frac{m_z\, m_x}{EI}\, ds = -\int \frac{y'\, ds}{EI} \quad ,$$

$$d_{zy} = \int \frac{m_z\, m_y}{EI}\, ds = \int \frac{x'\, ds}{EI} \quad ,$$

$$d_{zz} = \int \frac{m_z\, m_z}{EI}\, ds = \int \frac{ds}{EI} \quad .$$

Similarly, for a unit horizontal force, $X = 1$, at B:

$$d_{xx} = \int \frac{m_x^2}{EI}\, ds = \int y'^2\, \frac{ds}{EI} \quad ,$$

$$d_{xy} = \int \frac{m_x\, m_y}{EI}\, ds = -\int x'\, y'\, \frac{ds}{EI} \quad ,$$

$$d_{xz} = \int \frac{m_x\, m_y}{EI}\, ds = -\int y'\, \frac{ds}{EI} \quad ,$$

and for unit vertical force, $Y = 1$, at B:

$$d_{yx} = \int \frac{m_y\, m_x}{EI}\, ds = -\int \frac{x'\, y'}{EI}\, ds \quad ,$$

$$d_{yy} = \int \frac{m_y^2}{EI}\, ds = \int \frac{x'^2}{EI}\, ds \quad ,$$

$$d_{yz} = \int \frac{m_y\, m_z}{EI}\, ds = \int \frac{x'}{EI}\, ds \quad .$$

A comparison of these quantities with those in TABLE I shows that the expressions are identical. Therefore, the relationships between the displacements of both member BB' and bent CBDE can be set up as shown in TABLE II.

TABLE II[2] FORMULAE FOR A SUBSTITUTE MEMBER AT JOINT B

Loads at B	Line	Defl. at B	Elastic foundation (CBDE)	Substitute member (BB')
X = 1 Y = Z = 0	1	Horizontal	$d_{xx} = \int \frac{y'^2\,ds}{EI}$	$\int \frac{y'^2\,ds}{EI}$
	2	Vertical	$d_{xy} = -\int \frac{x'\,y'\,ds}{EI}$	$-\int \frac{x'\,y'}{EI}\,ds$
	3	Rotation	$d_{xz} = -\int \frac{y'\,ds}{EI}$	$-\int \frac{y'\,ds}{EI}$
Y = 1 X = Z = 0	4	Horizontal	$d_{yx} = -\int \frac{x'\,y'\,ds}{EI}$	$-\int \frac{x'\,y'\,ds}{EI}$
	5	Vertical	$d_{yy} = \int \frac{x'^2}{EI}\,ds$	$\int \frac{x'^2\,ds}{EI}$
	6	Rotation	$d_{yz} = \int \frac{x'^2}{EI}\,ds$	$\int \frac{x'\,ds}{EI}$
Z = 1 X = Y = 0	7	Horizontal	$d_{zx} = -\int \frac{y'\,ds}{EI}$	$-\int \frac{y'\,ds}{EI}$
	8	Vertical	$d_{zy} = \int \frac{x'\,ds}{EI}$	$\int \frac{x'\,ds}{EI}$
	9	Rotation	$d_{zz} = \int \frac{ds}{EI}$	$\int \frac{ds}{EI}$

The effect of the substitute member BB' must be the same as that of the bent CBDE. From lines 7, 8, and 9 in TABLE II:

$$\int \frac{ds}{EI} = A = d_{zz} \quad , \tag{12a}$$

$$\int x' \cdot \frac{ds}{EI} = d_{yz} = d_{zy} \quad , \tag{12b}$$

$$-\int y' \cdot \frac{ds}{EI} = d_{xz} = d_{zx} \quad . \tag{12c}$$

In Fig. 5(d), the coordinates of the substitute member BB' are in terms of x' and y' referred to joint B as the origin, thus, from Eqs. (12)

$$\bar{x}' = \frac{d_{yz}}{\int \frac{ds}{EI}} = \frac{d_{yz}}{d_{zz}} \quad , \tag{13a}$$

$$\bar{y}' = -\frac{d_{xz}}{\int \frac{ds}{EI}} = -\frac{d_{xz}}{d_{zz}} \quad . \tag{13b}$$

It is obvious that Eqs. (13a) and (13b) give the coordinates of the centroid of the elastic area of the substitute member BB' . Then the elastic properties referred to the elastic center of the substitute member BB' can be evaluated by translating the axes from the assumed position at B to the new position at the centroid B' . Thus the area properties at the centroid of the member BB' are evaluated to be:

$$I_{gy} = \int \frac{x'^2 ds}{EI} - A (\bar{x}')^2 = d_{yy} - \frac{d_{yz}^2}{d_{zz}} \quad , \tag{14a}$$

$$I_{gx} = \int \frac{y'^2 ds}{EI} - A (\bar{y}')^2 = d_{xx} - \frac{d_{xz}^2}{d_{zz}} \quad , \tag{14b}$$

$$I_{gxy} = -\int \frac{x' y' ds}{EI} - A(\bar{x}')(\bar{y}') = -d_{xy} + \frac{d_{xz} d_{yz}}{d_{zz}} \quad . \qquad (14c)$$

Once these properties have been evaluated, the substitute member can be provided for the next bent of the continuous frame. In this case the axes passing through B in member BB' must be translated to the new origin of the new substituted structure. The new coordinates of B' of member BB' after translating the axes to a new origin, as shown in Fig. 5(d), will be:

$$\bar{x} = \bar{x}' + x_b \quad , \qquad (15a)$$

$$\bar{y} = \bar{y}' + y_b \quad , \qquad (15b)$$

and the area properties of member BB' after translating the axes must be:

$$I_y = I_{gy} + A(\bar{x})^2 = d_{yy} - \frac{d_{yz}^2}{d_{zz}} + (\bar{x}' + x_b) d_{zz} \quad , \qquad (16a)$$

$$I_x = I_{gx} + A(\bar{y})^2 = d_{xx} - \frac{d_{xz}^2}{d_{zz}} + (\bar{y}' + y_b) d_{zz} \quad , \qquad (16b)$$

$$I_{xy} = I_{gxy} + A(\bar{x})(\bar{y}) = -d_{xy} + \frac{d_{xz} d_{yz}}{d_{zz}} + (\bar{x}' + x_b)(\bar{y}' + y_b) d_{zz} \quad . \qquad (16c)$$

Equations (13) to (16) show that it is not necessary to know the flexural rigidity EI of the substitute member BB'. Also since the "substitute member" is an ideal bar, it can be of any shape. The directions of the redundant loads as shown in Fig. 5(c) and (d) are assumed to be positive. The details of this method are shown in the examples, and the results have been checked by the slope-deflection method.

GENERAL PROCEDURES

(1) Subdivide the continuous structure into equivalent single-span's; select one span (any span) of the subdivided structure as an elastic foundation at a time.

(2) Determine the elastic properties of each subdivided span in turn.

(3) Consider all spans adjacent to the span which is to be analyzed as its "elastic foundations."

(4) Replace the "elastic foundations" by substitute member with characteristics such that a selected span of the substitute member is equivalent to a span on the elastic foundations.

(5) Determine the elastic properties of the substitute members and proceed with the application of the fundamental column analogy method.

NUMERICAL EXAMPLES

Example 1:

The middle span of the three-span symmetrical bent, as shown in Fig. 6, is loaded with a uniform load of 1 k/ft. Assuming that the bent is fixed at the bases, determine the member end moments.

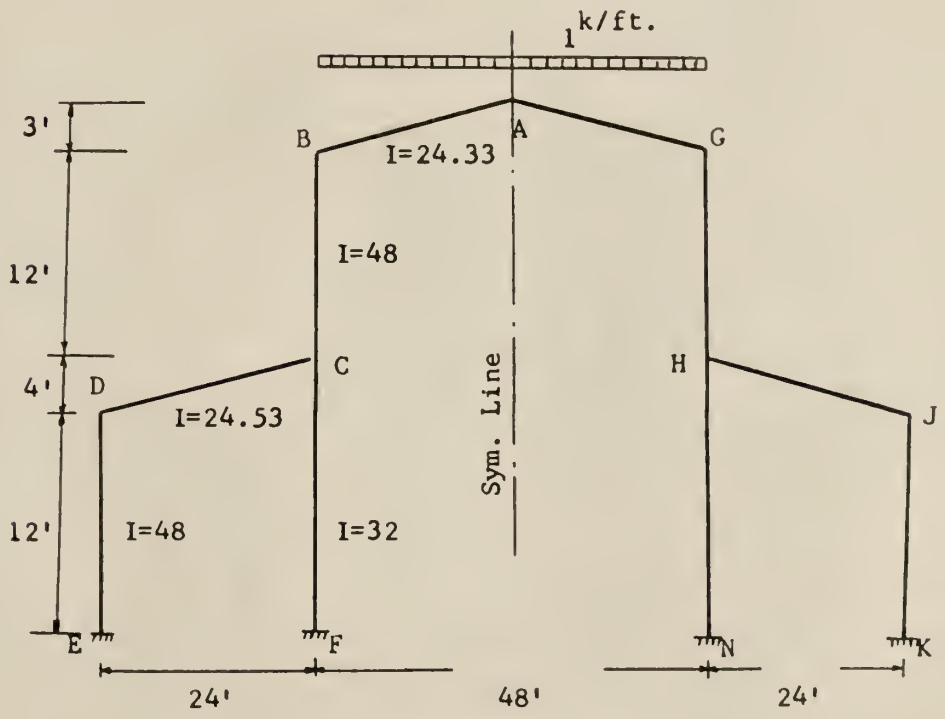
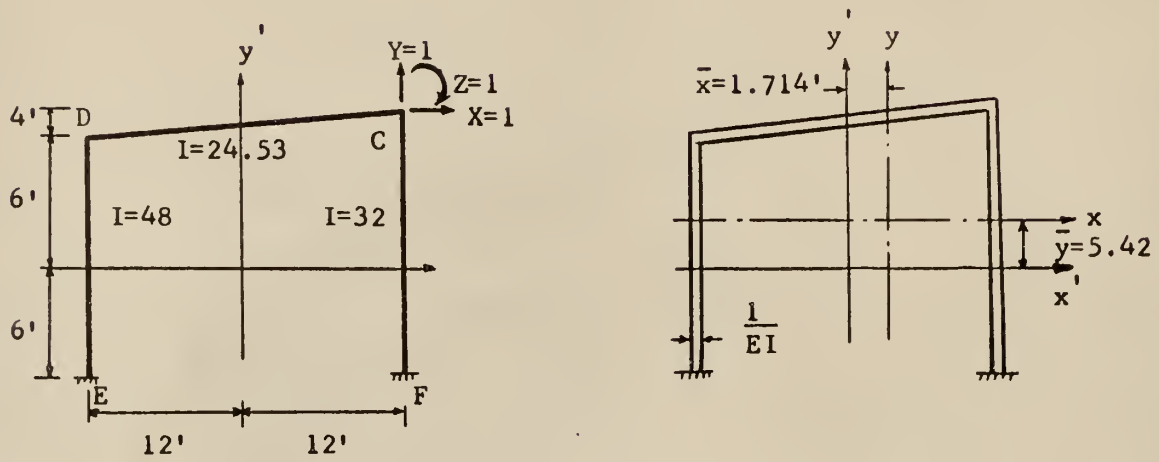


Fig. 6. Loaded Three-span Symmetrical Bent.

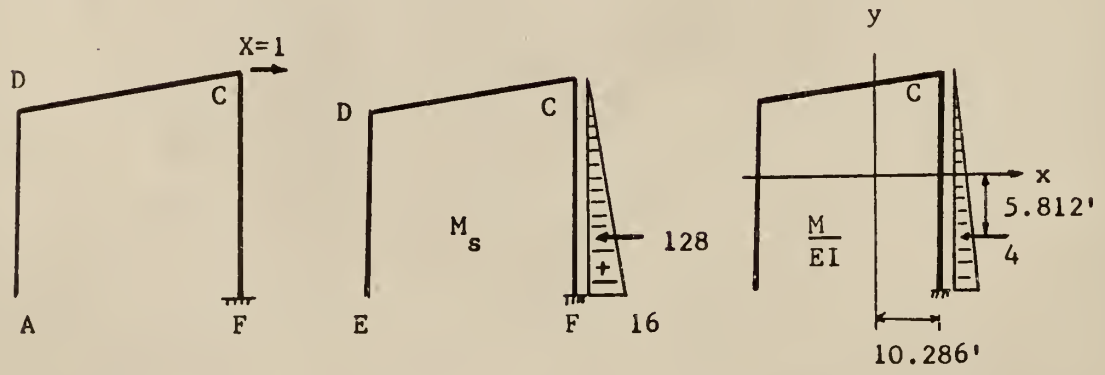
Solution by the column analogy method:

(I) The analysis of the "elastic foundation EDCF" by the column analogy method is carried out as follows: Unit redundant forces are applied at the joint C as shown in Fig. 6. The computations for the properties of the bent EDCF and the corresponding moments caused by the unit redundant forces are listed in TABLES III to IV. Since this structure is symmetrical, only

one half of the structure is analyzed, and the results for the other half can be obtained by symmetry. The analysis of bent EDCF is illustrated in Fig. 7.

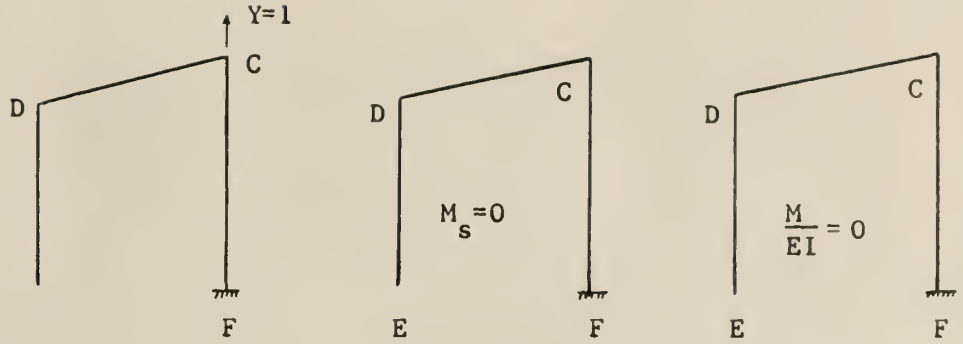


(a) Properties Of Bent EDCF.

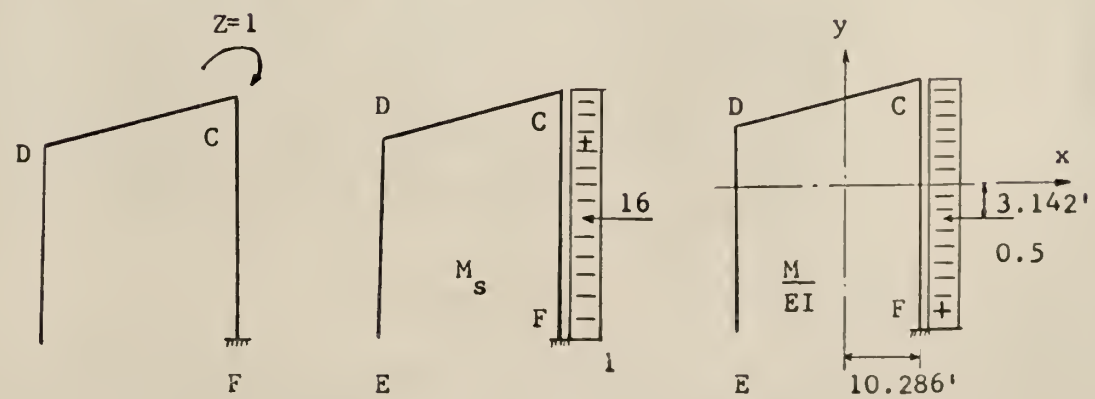


(b) Moment Diagram And M/EI Diagram Due To Redundant Force, X = 1 , Applied At Joint C.

Fig. 7. Analysis Of Bent EDCF.



(c) Moment Diagram And M/EI Diagram Due To Redundant Force, Y = 1 , Applied At Joint C.



(d) Moment Diagram And M/EI Diagram Due To Redundant Force, Z = 1 , Applied At Joint C.

Fig. 7. Analysis Of Bent EDCF. (Continued.)

TABLE III PROPERTIES OF THE BENT EDCF

GIVEN DATA					COMPUTED PROPERTIES					
BAR	L	I	X	Y	$A=\int \frac{ds}{EI}$	AX	AY	I_y	I_x	I_{xy}
DE	12.00	48.00	-12	0	0.25	-3.0	0	36.000	3.000	0
DC	24.53	24.53	0	+8	1.00	0	+8.0	48.000	65.330	+ 8.00
CF	16.00	32.00	+12	+2	0.50	+6.0	+1.0	72.000	12.670	+12.00
CORR.TO CENTROID								- 5.305	-46.305	-15.93
Σ					1.75	+3.0	+9.0	150.69	34.690	+ 4.40

REMARKS

$$\bar{X}=\frac{\Sigma Ax}{\Sigma A}=+1.714 \qquad \bar{Y}=\frac{\Sigma Ay}{\Sigma A}=+5.142$$

$$**I_y = \Sigma A\bar{X} = 5.305 \; ; \; I_x = \Sigma A\bar{Y} = 46.305 \; ; \; I_{xy} = \Sigma A\bar{X}\bar{Y} = 15.593$$

$$*I_y = AX + i_y \; ; \; I_x = AY + i_x \; ; \; I_{xy} = AXY + i_{xy}$$

TABLE IV ELASTIC LOADS AND FORMULAE FOR M_i , DUE TO REDUNDANT FORCES ON BENT EDCF

BAR	X = 1 P	X	Y	M_y	M_x	Z = 1 P	X	Y	M_y	M_x
DE	0	•	•	•	•	0	•	•	•	•
DC	0	•	•	•	•	0	•	•	•	•
CF	+4	+10.268	-5.812	+41.144	-23.248	+0.5	+10.286	-3.142	+5.142	-1.571
Σ	+4			+41.144	-23.248	+0.5			+5.143	-1.571
M_1	2.285 + 0.293X - 0.706Y					0.286 + 0.0354X - 0.0496Y				

Since the bent EDCF is unsymmetrical, the formulae for the M_i's which are listed in TABLE IV can be calculated as follows:

$$\frac{I_{xy}}{I_x} = \frac{4.4}{34.69} = 0.127 \quad ; \quad \frac{I_{xy}}{I_y} = \frac{4.4}{150.69} = 0.028$$

$$\frac{I_{xy}^2}{I_x I_y} = 0.127 \times 0.028 = 0.0035$$

$$I_y' = I_y \left(1 - \frac{I_{xy}^2}{I_x I_y}\right) = 150.24$$

$$I_x' = I_x \left(1 - \frac{I_{xy}^2}{I_x I_y}\right) = 34.55$$

If $X = 1$ is applied at C (as shown in Fig. 7 (b)), then $M_x = -23.24$, $M_y = +41.144$, $A = 1.75$, and $P = +4$, and

$$M_x' = M_x - M_y \frac{I_{xy}}{I_y} = -24.40 \ ,$$

$$M_y' = M_y - M_x \frac{I_{xy}}{I_x} = +44.1 \ ,$$

$$M_i = \frac{P}{A} + \frac{M_y'}{I_y'} x + \frac{M_x'}{I_x'} y = 2.285 + 0.293 x - 0.706 y \ .$$

If $X = 1$ is applied at C (as shown in Fig. 7(c)), then $M_x = -1.571$, $M_y = +5.143$, $A = 1.75$, and $P = +0.5$, and

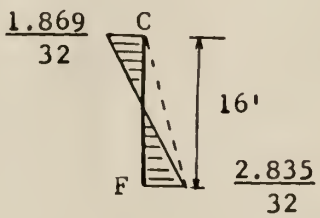
$$M_x' = M_x - M_y \frac{I_{xy}}{I_y} = -1.715 \ ,$$

$$M'_y = M_y - M_x \frac{I_{xy}}{I_x} = + 5.320 \ ,$$

$$M_i = \frac{P}{A} + \frac{M'_y}{I'_y} x + \frac{M'_x}{I'_x} y = 0.286 + 0.035 x - 0.0496 y \ .$$

In order to complete the table, the formulae for the M_i's are listed in TABLE IV. By using these formulae, the end moments of the bent EDCF are evaluated in TABLE V.

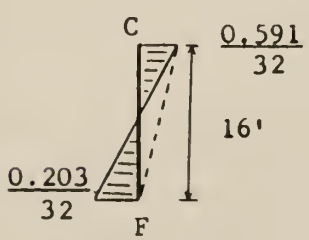
The computation of deflections at joint C is based on the values obtained in TABLE V, and is illustrated in Fig. 8. The results of the computation are listed in TABLE VI.



(a) M/EI Diagram for Computing Deflections at C.

$$d_{xx} = \frac{1}{32} (-1.869 \times \frac{16}{2} \times \frac{16}{3} + 2.835 \times \frac{16}{2} \times \frac{2 \times 16}{3}) = + 5.07$$

$$d_{xz} = \frac{1}{32} (-1.869 + 2.835) \times \frac{16}{2} = + 0.244$$



(b) M/EI Diagram for Computing Deflections at C.

$$d_{zx} = \frac{1}{32} (0.591 \times \frac{16}{2} \times \frac{16}{3} - 0.203 \times \frac{16}{2} \times \frac{2 \times 16}{3}) = + 0.244$$

$$d_{zz} = \frac{1}{32} (+0.591 - 0.203) \times \frac{16}{2} = + 0.097$$

Fig. 8. Computation Of Deflections By The Conjugate Beam Method.

TABLE V END MOMENTS DUE TO REDUNDANT FORCES

PT.	X	Y	X = 1			Z = 1		
			M_s	M_1	M	M_s	M_1	M
E	-13.7144	-11.142	0	+ 6.133	-6.133	0	+0.354	-0.354
D	-13.7144	+ 0.858	0	-2.339	+2.339	0	-0.242	+0.242
C_{CD}	+10.2866	+ 4.858	0	+1.869	-1.869	0	+0.409	-0.409
C_{CF}	+10.2866	+ 4.858	0	+1.869	-1.869	+1	+0.409	+0.591
F	+10.2866	-11.142	+16	+13.165	+2.835	+1	+1.203	-0.203

TABLE VI DEFLECTIONS OF JOINT C DUE TO THE END MOMENTS SHOWN IN TABLE V .

Horizontal	d_{xx} = + 5.070	d_{xy} = 0	d_{xz} = + 0.244
Vertical	d_{yx} = 0	d_{yy} = 0	d_{yz} = 0
Rotation	d_{zx} = + 0.244	d_{zy} = 0	d_{zz} = +0.097

The properties of the substitute member CC' can then be calculated from Eqs. (12) to Eqs. (14).

$$\frac{ds}{EI} = A = d_{zz} = + 0.097 \quad ,$$

$$\bar{x}' = \frac{d_{yz}}{d_{zz}} = 0 \quad ,$$

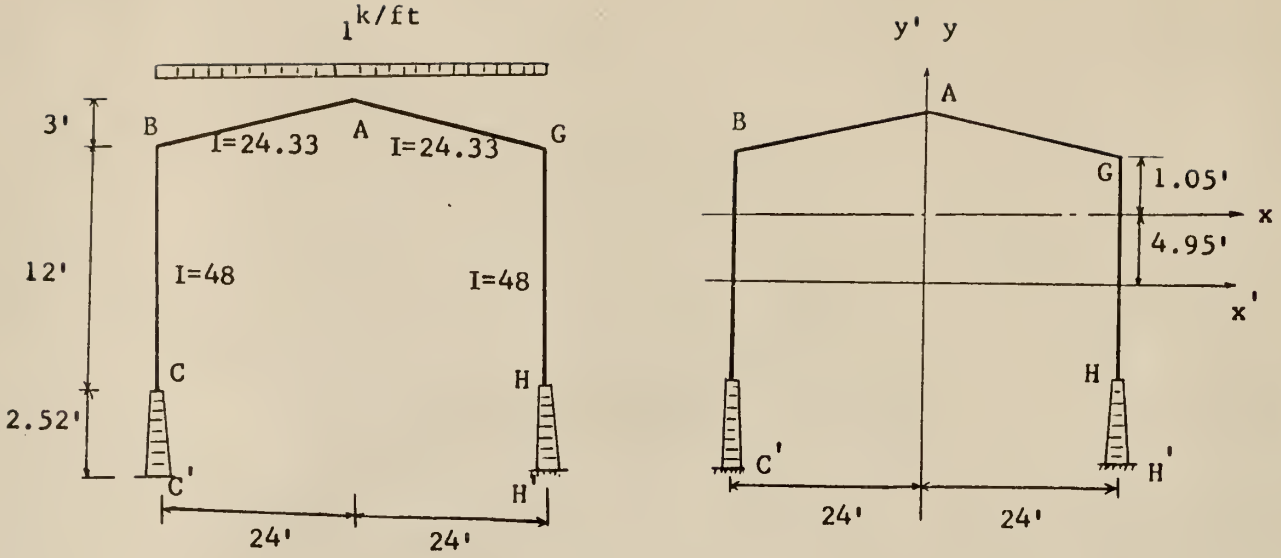
$$\bar{y}' = - \frac{d_{xz}}{d_{zz}} = - \frac{0.244}{0.097} = - 2.52 \quad ,$$

$$I_{gx} = d_{xx} - d_{zz} (\bar{y}')^2 = 5.07 - 0.097 (2.52)^2 = 4.51 \quad ,$$

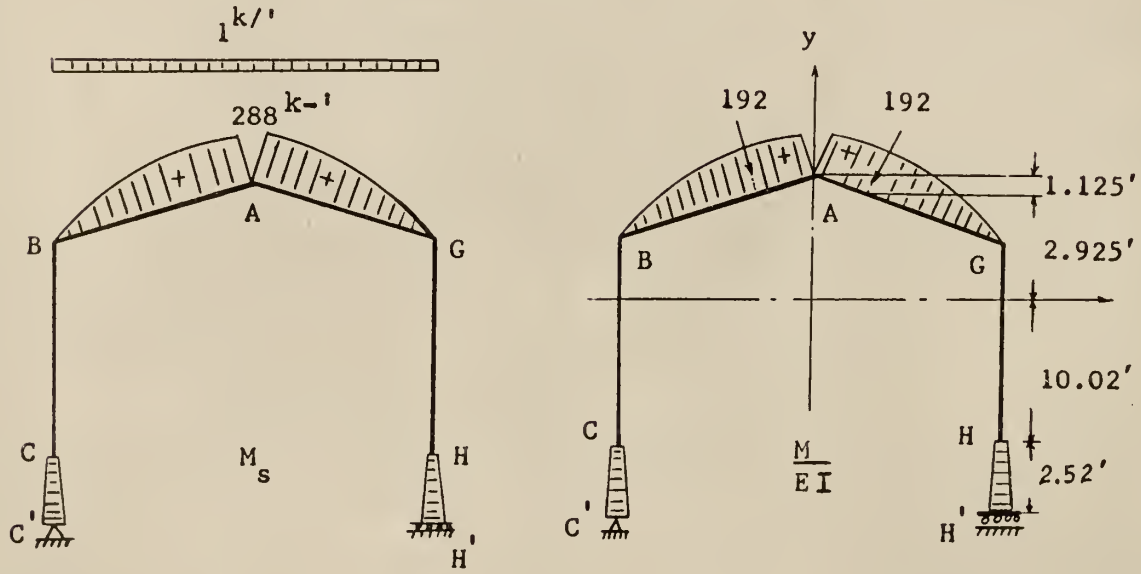
$$I_{gy} = d_{yy} - d_{zz} (\bar{x}')^2 = 0 \quad ,$$

$$I_{gxy} = - d_{xy} + \frac{d_{xz} d_{yz}}{d_{zz}} = 0 \quad .$$

(II) Since the properties of the substitute member CC' are known, the "elastic foundation EDCF" can be replaced by the substitute member CC' to support the bent CBAGH. Similarly, the substitute member HH' can also replace the bent NHJK. The computations are listed in TABLES VII to IX.



(a) Properties Of Bent C'CBAGHH'.



(b) The Determinate Moment Diagram and M/EI Diagram Due To The Applied Load.

Fig. 9. Analysis Of Bent C'CBAGHH'.

TABLE VII PROPERTIES OF BENT C'CBAGHH

BAR	DIVEN DATA				COMPUTED PROPERTIES				
	L	I	X	Y	$A=\int\frac{ds}{EI}$	AX	AY	I_y	I_x
C'C	.	.	-24	-8.52	0.097	- 2.34	-0.83	56.16	4.51
CB	12.00	48.00	-24	0	0.250	- 6.00	0	144.00	7.20
BA	24.33	24.33	-12	+7.5	1.000	-12.00	+7.50	192.00	59.60
AG	24.33	24.33	+12	+7.50	1.000	+12.00	+7.50	192.00	59.60
GH	12.00	48.00	+24	0	0.250	+ 6.0	0	144.00	7.20
HH'	.	.	+24	-8.52	0.097	+ 2.34	-0.83	56.16	4.50
CORR. TO CENTROID**								-0.00	-66.03
Σ					2.694	0	+13.34	784.32	76.59

REMARKS:

$$\bar{X}=\frac{\Sigma Ax}{\Sigma A}=0\ ,\quad \bar{Y}=\frac{\Sigma Ay}{\Sigma A}=+2.694$$

$$**I_y=\Sigma A\bar{X}^2=0\ ,\quad I_x=\Sigma A\bar{Y}^2=66.03$$

TABLE VIII ELASTIC LOADS AND FORMULA M_i DUE TO APPLIED LOAD ON BENT C'CBAGHH

BAR	P	X	Y	$A=\int\frac{ds}{EI}$	M_y	M_x
C'C	.	.	.	0.097	.	.
CB	0	.	.	0.250	0	0
BA	+192.0	-9.0	-2.925	1.000	-1728.0	+ 561.6
AG	+192.0	+9.0	+2.925	1.000	+1728.0	+ 561.6
GH	0	.	.	0.250	0	0
HH'	.	.	.	0.097	.	.
Σ	384.0			2.694	0	+1123.2

$$M_i=\frac{P}{A}+\frac{M_y}{I_y}x+\frac{M_x}{I_x}y=142.48+14.66Y$$

As shown in Fig. 10(b), the moment, $Z = 18.1^{k-ft}$, has already been obtained in TABLE IX; therefore the horizontal force, X, can then be computed by taking the free body BC and setting up the static equilibrium equation $M_B = 0$, that is:

$$X = \frac{18.1 + 157.87}{12} = 14.65^{k} .$$

Thus, the redundant forces $X = 14.65^{k}$ and $Z = 18.1^{k-ft}$ acting on bent EDCF are shown in Fig. 10(a). The end moments for the bent EDCF can then be obtained by multiplying $X = 14.65$ and $Z = 18.1$ times the corresponding coefficients in TABLE V. The results are listed in TABLE X and also plotted in Fig. 11.

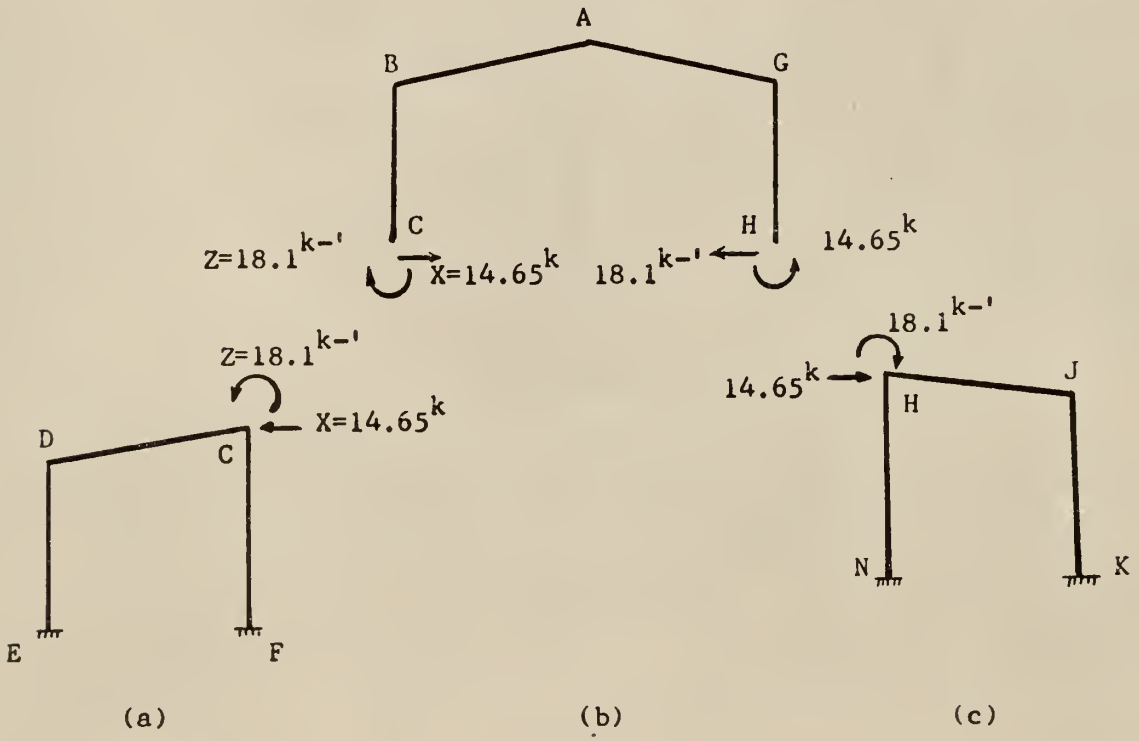


Fig. 10. Redundant Forces X and Z At Joint C.

TABLE IX END MOMENTS OF BENT C'CBAGHH

PT.	X	Y	M_s	M_i	$M=M_s-M_i$
C_{CB}	-24	-10.95	0	- 18.10	+ 18.10
B	-24	+ 1.05	0	+157.87	-157.87
A	0	+ 4.05	+288	+201.85	+ 86.15
G	+24	+ 1.05	0	+157.87	-157.87
H_{HG}	+24	-10.95	0	- 18.10	+ 18.10

TABLE X END MOMENTS OF BENT EDCF

POINT	M_1 (k-') (DUE TO X=14.65)	M_2 (k-') (DUE TO Z=18.1)	(k-') $M = M_1 + M_2$
E	+89.80	+ 6.42	+96.22
D	-34.25	- 4.38	-38.63
C_{CD}	+27.60	+ 7.41	+35.01
C_{CF}	+27.60	-10.69	+16.91
F	-41.70	+ 3.76	-37.94

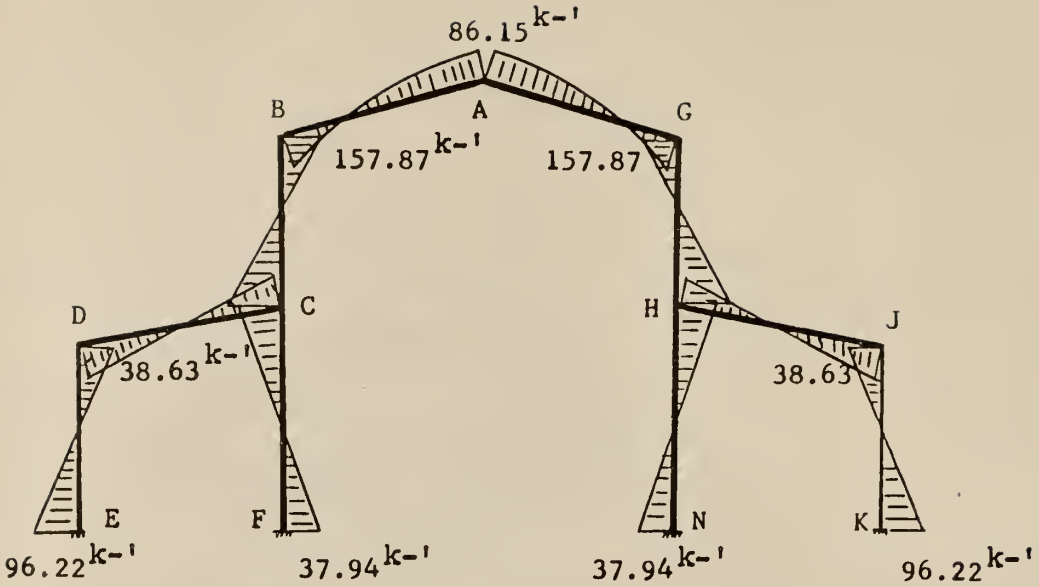


Fig. 11. Final Moment Diagram.

Example 2:

A skew frame with a curved member is loaded as shown in Fig. 12. The center line of the curved member is expressed by the equation of the parabola $u = -\frac{v^2}{125}$ referred to the u- and v-axis shown in Fig. 12. The section of the curved member is assumed to be constant. Determine the end moment.

Solution by the Column Analogy Method:

(I) Analysis of the "elastic foundation ABCD" by column analogy method: The unit redundant loads at joint C are shown in Fig. 13. The computing procedures are the same as those in Example 1, and are listed in TABLES XI to XV.

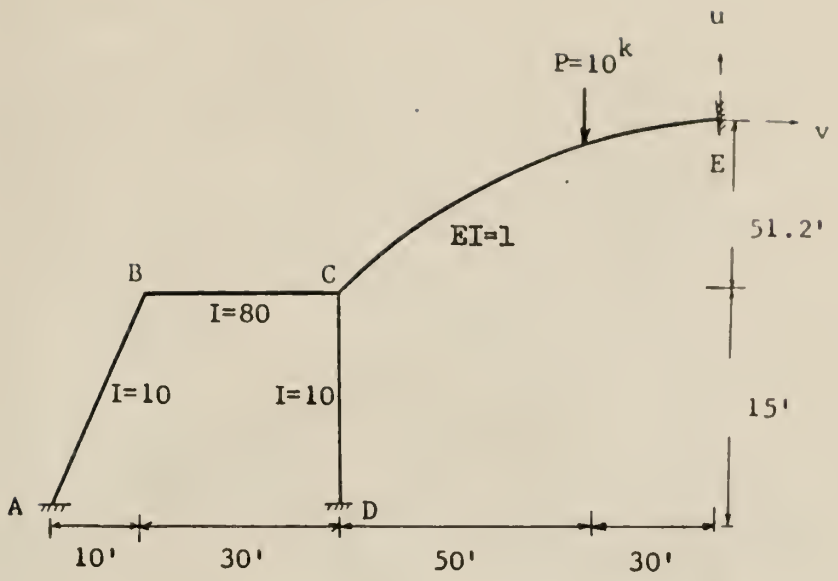
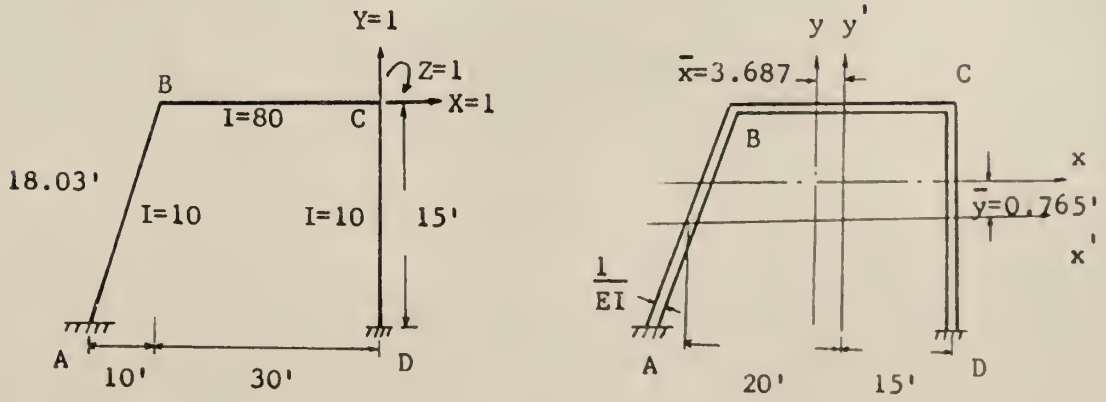
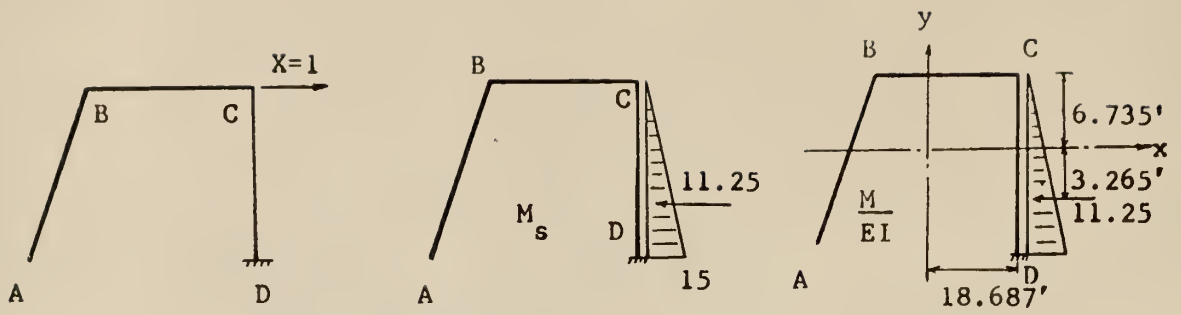


Fig. 12. Loaded Skew Frame.

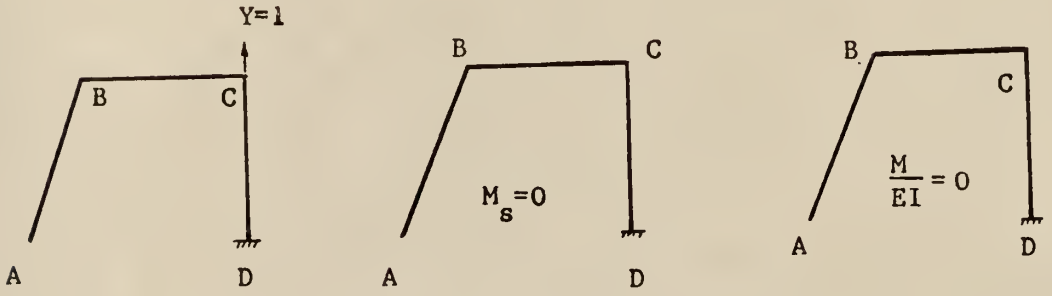


(a) Properties Of Bent ABCD.

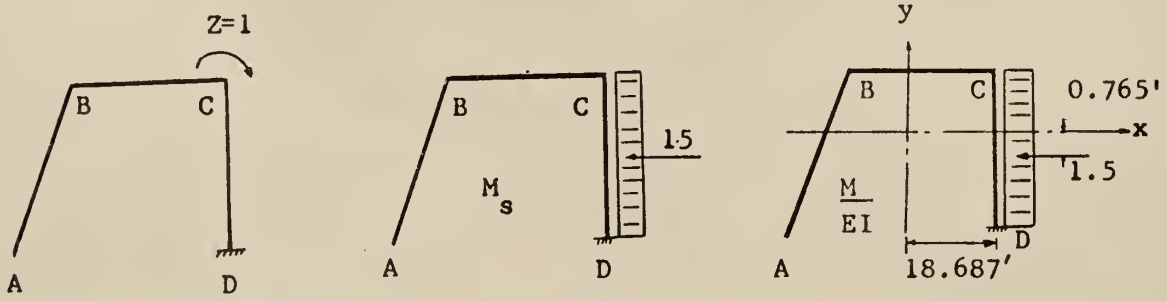
Fig. 13. Analysis Of The Bent ABCD.



(b) Moment Diagram And M/EI Diagram Due To Redundant Force X = 1 Applied At Joint C.



(c) Moment Diagram And M/EI Diagram Due To Redundant Force Y = 1 Applied At Joint C.



(d) Moment Diagram And M/EI Diagram Due To Redundant Force Z = 1 Applied At Joint C.

Fig. 13. Analysis Of Bent ABCD. (Continued.)

TABLE XI PROPERTIES OF BENT ABCD

GIVEN DATA					COMPUTED PROPERTIES					
BAR	L	I	X	Y	$A=\int \frac{ds}{EI}$	AX	AY	I_y	I_x	I_{xy}
AB	18.03	10	-20	0	1.803	-36.06	0	736.23	33.806	22.54
BC	30.00	80	0	+7.5	0.375	0	+2.813	28.13	21.100	0
CD	15.00	10	+15	0	1.500	+22.50	0	337.50	28.13	0
CORR. TO CENTROID								- 50.00	-2.151	+10.38
Σ					3.678	-13.56	+2.813	1051.86	80.885	32.92
$\bar{X}$ = -3.687			$\bar{Y}$ = +0.765		I'_x = 79.833			I'_y = 1038.186		

TABLE XII ELASTIC LOADS AND FORMULA FOR M_1 DUE TO REDUNDANT FORCES ON BENT ABCD

BAR	X = 1 P	X	Y	M_y	M_x	Z = 1 P	X	Y	M_y	M_x
AB	0	•	•	0	0	0	•	•	0	0
BC	0	•	•	0	0	0	•	•	0	0
CD	+11.25	+18.687	-3.265	+210.23	-36.73	+1.5	+18.687	-0.765	+28.03	-1.148
$\sum$	+11.25			+210.23	-36.73	+1.5			+28.03	-1.148
		$M_y' = +225.16$		$M_x' = -43.31$			$M_y' = +28.497$		$M_x' = -2.025$	
M_1	3.358 + 0.217X - 0.542Y					0.407 + 0.0274X - 0.0254Y				

TABLE XIII END MOMENTS DUE TO REDUNDANT FORCES

PT.	X	Y	X = 1			Z = 1		
			M_s	M_i	M	M_s	M_i	M
A	-21.313	-8.265	0	+2.9136	-2.9136	0	+0.03375	-0.03375
B	-11.313	+6.735	0	-3.0418	+3.0418	0	-0.07125	+0.07125
C_{CB}	+18.687	+6.735	0	+3.4580	-3.4580	0	+0.75077	-0.75077
C_{CD}	+18.687	+6.735	0	+3.4580	-3.4580	+1	+0.75077	+0.24923
D	+18.687	-8.265	+15	11.5940	+3.4064	+1	+1.12976	-0.12976

TABLE XIV DEFLECTIONS OF JOINT C DUE TO THE END MOMETNS SHOWN IN TABLE XIII .

Horizontal	d_{xx} = + 12.5805	d_{xy} = 0	d_{xz} = - 0.03865
Vertical	d_{yx} = 0	d_{yy} = 0	d_{yz} = 0
Rotation	d_{zx} = - 0.03865	d_{yz} = 0	d_{zz} = + 0.0896

By applying Eqs. (12) to (14), the properties of the substitute member CC' can be evaluated. The calculations are similar to those shown in Example 1, and the results are listed in TABLE XV.

TABLE XV PROPERTIES OF THE SUBSTITUTE MEMBER CC'

$A = \frac{ds}{EI}$	$\bar{x}'$	$\bar{y}'$	I_{gy}	I_{gx}	I_{gxy}
+0.0896	0	+0.431	0	+12.5638	0

(II) Analysis of the curved part C'CE as shown in Fig. 14. The "elastic foundation ABCD" is replaced by the substitute member CC'. The curved member CE is divided into eight equal segments so that each segment can be approximated by a straight member. The analysis of member C'CE is shown in TABLES XVI to XVIII.

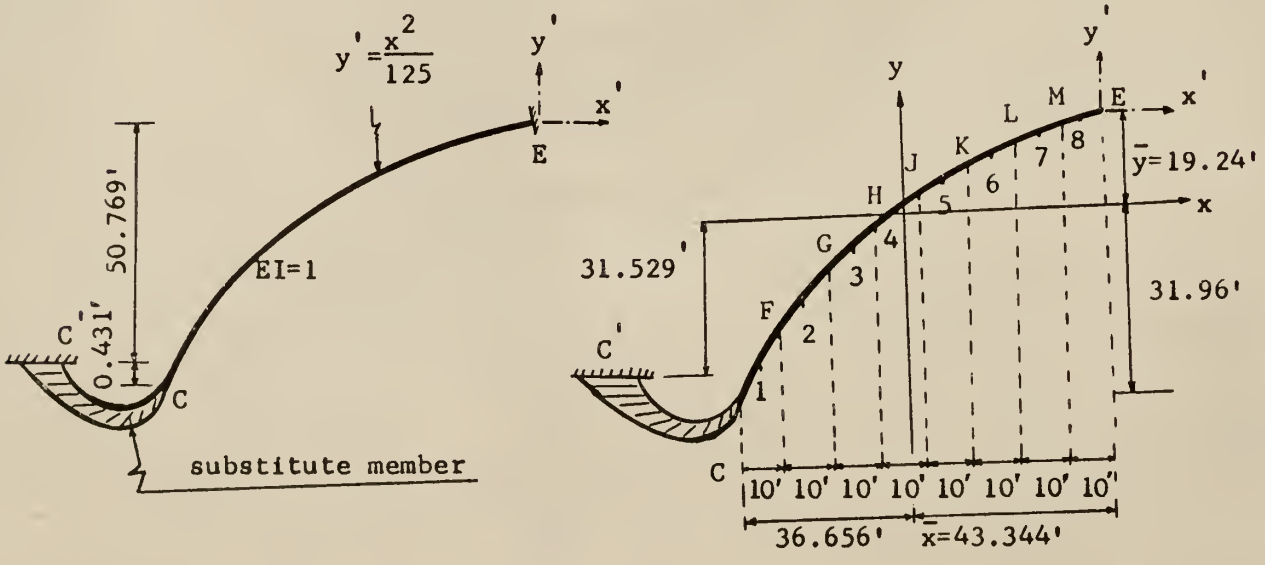


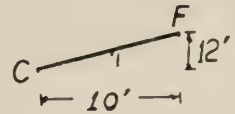
Fig. 14. Properties Of C'CE.

TABLE XVI PROPERTIES OF THE ANALOGOUS COLUMN FOR MEMBER C'CE

DIVEN DATA						COMPUTED PROPERTIES					
SEG.	PT.	L	I	X	Y	$A=\int \frac{ds}{EI}$	AX	AY	$I_y = AX^2$	$I_x = AY^2$	$I_{xy} = AXY$
C'C				-80	-50.769	0.0896	-7.1680	-4.5490	573.44	230.54	+ 363.91
CF	1	15.36	1	-75	-45.000	15.360	-1152.0	-691.20	86400.00	31104.00	+51840.00
FG	2	14.08	1	-65	-33.800	14.080	- 915.2	-475.90	59488.00	16085.42	+30933.50
GH	3	13.24	1	-55	-24.200	13.240	- 728.2	-320.41	40051.00	7753.92	+17622.40
HJ	4	12.32	1	-45	-16.200	12.320	- 554.4	-199.58	24948.00	3353.20	+ 8981.10
JK	5	11.46	1	-35	- 9.800	11.460	- 401.1	-112.31	14038.50	1100.64	+ 3930.78
KL	6	10.77	1	-25	- 5.000	10.770	- 269.3	- 53.85	6731.25	269.25	+ 1346.25
LM	7	10.28	1	-15	- 1.800	10.280	- 154.2	- 18.51	2313.00	333.07	+ 277.56
MF	8	10.03	1	- 5	- 0.200	10.030	- 50.2	- 2.01	250.75	0.40	+ 10.05
CORR. TO CENTROID**									-183383.68	-36140.67	-81409.83
Σ						97.63	-4231.7	-1878.4	51410.26	24089.77	33895.72
								***I_y' = 3850.0		I_x' = 1830.822	

REMARKS

$$\bar{X} = \frac{\Sigma Ax}{\Sigma A} = -43.344 \qquad \bar{Y} = \frac{\Sigma Ay}{\Sigma A} = -19.24$$

* 

$$Y' = -\frac{x'^2}{125} \; ; \; Y_A' = -51.2 \; ; \; Y_F' = -\frac{70^2}{125} = -39.2$$

$$Y_1' = -\frac{75^2}{125} = -45 \; ; \; Y_{CF}' = +51.2 - 39.2 = 12$$

$$L_{CF} = (10^2 + 12^2)^{\frac{1}{2}} = 15.36$$

**AX = (97.63)(-43.344) = 183383.68
AY = (97.63)(-19.24) = 36140.67
AXY = (97.63)(-43.344)(-19.24) = 81409.83

$$***I_y' = I\left(1 - \frac{I_{xy}^2}{I_x I_y}\right) = 3850.0 \; ; \; I_x = I\left(1 - \frac{I_{xy}^2}{I_x I_y}\right) = 1830.822$$

TABLE XVII ELASTIC LOADS AND FORMULA FOR M_i DUE TO APPLIED LOAD ON MEMBER C'CE

SEG.	CTR.	M_s	$A=\int\frac{ds}{EI}$	$P=\int\frac{M_s ds}{EI}$	X	Y	M_y	M_x
CC'			0.089	0	-36.656	-31.529	.	.
CF	1	0	15.360	0	-31.656	-25.760	0	0
FG	2	0	14.080	0	-21.656	-14.600	0	0
GH	3	0	13.240	0	-11.656	- 5.000	0	0
HJ	4	0	12.320	0	- 1.656	+ 3.000	0	0
JK	5	0	11.460	0	+ 8.344	+ 9.400	0	0
KL	6	- 50	10.770	- 538.5	+18.344	+14.200	- 9867.67	- 7652.1
LM	7	-150	10.280	-1542.0	+28.344	+17.400	- 43644.02	-26846.2
ME	8	-250	10.030	-2507.5	+38.344	+19.040	- 96147.58	-47742.8
Σ			97.630	-4588.0			-149659.27	-82241.1
					M'_y = -34521.73		M'_x = +16534.02	
M_i	-47.03 - 8.94X + 9.001Y							

TABLE XVIII END MOMENTS ON MEMBER C'CE

PT.	X	Y	M_s	M_i	$M=M_s - M_i$
C	-36.656	-31.96	0	- 7.33	+ 7.33
K	+13.344	+12.04	0	- 58.03	+58.03
E	+43.344	+19.24	-300	-258.57	-41.43

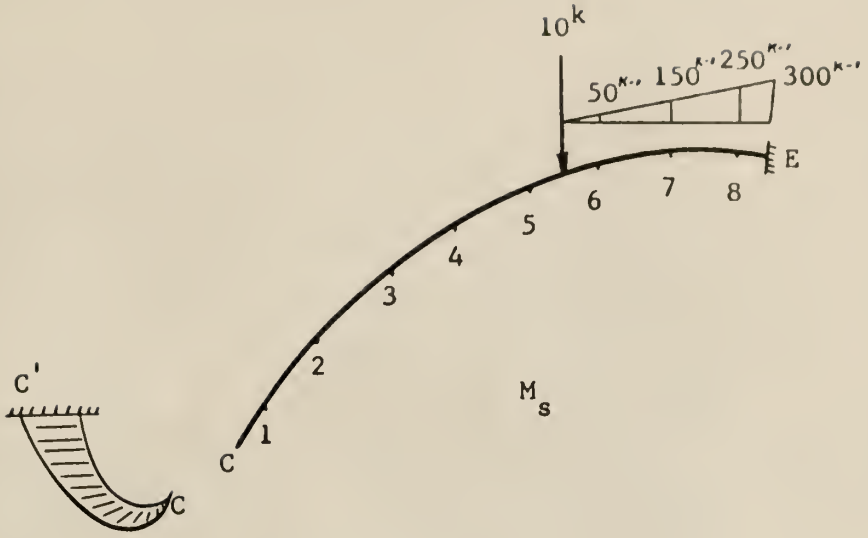


Fig. 15. Statically Determinate Moment Diagram Of Curved Member C'CE.

From TABLE XVIII and the formula for M_i, the redundant forces at joint C are computed to be $Z = 7.33^{k-ft}$ and $X = 9.001^{k}$. The combined action of $X = 9.001^{k}$ and $Z = 7.33^{k-ft}$ is shown in Fig. 16. The end moments of bent ABCD can be evaluated by multiplying $X = 9.001$ and $Z = 7.33$ to the coefficients in TABLE XIII correspondingly. The results are listed in TABLE XIX and the moment diagrams are plotted in Fig. 17.

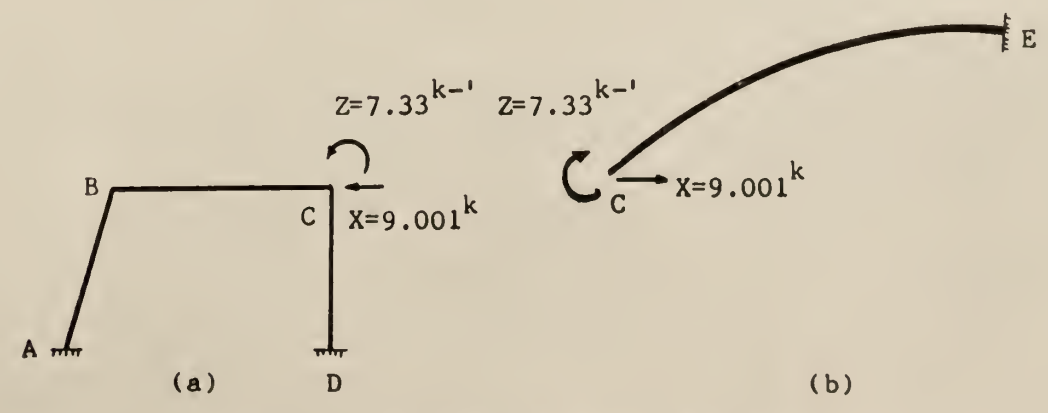


Fig. 16. Redundant Forces X and Z At Joint C.

TABLE XIX FINAL MOMENTS FOR BENT ABCD

POINT	M_1 (k-') (DUE TO X=-9.001)	M_2 (k-') (DUE TO Z=-7.33)	$M = M_1 + M_2$ (k-')
A	+26.22	+0.25	+26.47
B	-27.41	-0.52	-27.93
C	+31.10	+5.51	+36.61
C	+3..10	-1.83	+29.27
D	-30.65	+0.95	-29.70

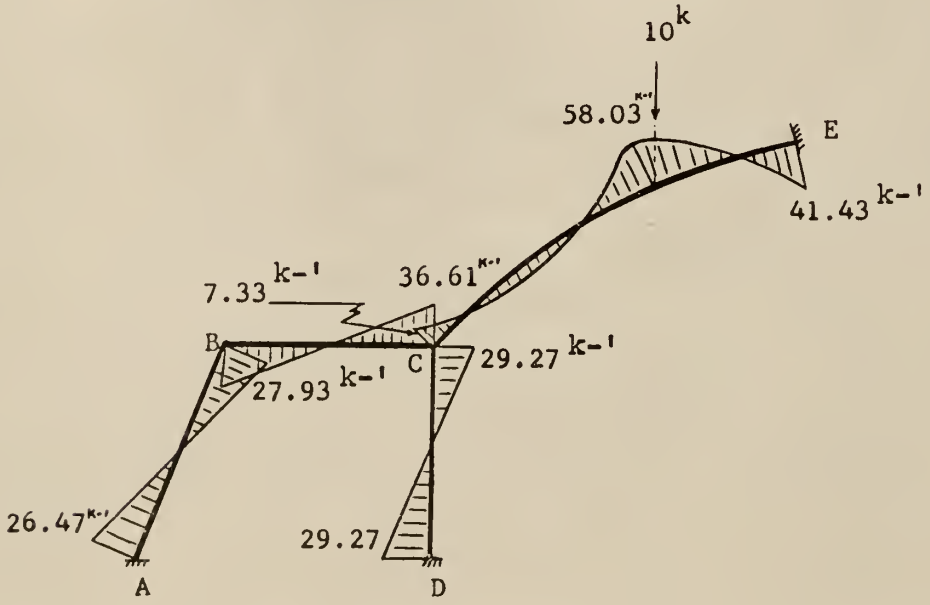


Fig. 17. Final Moments.

Example 3:

This example is the same as that in Example 2, it is solved by slope-deflection method here to check the results obtained in Example 2.

Solution by slope-deflection method:

This method is used here to check the results of the previous solution. The properties of the curved member CE are first analyzed so that the fixed end moments and stiffness factor can be determined. Having these values, "the extended slope-deflection equations"[3] can be set up. The curved member is also divided into eight equal segments as shown in Fig. 18.

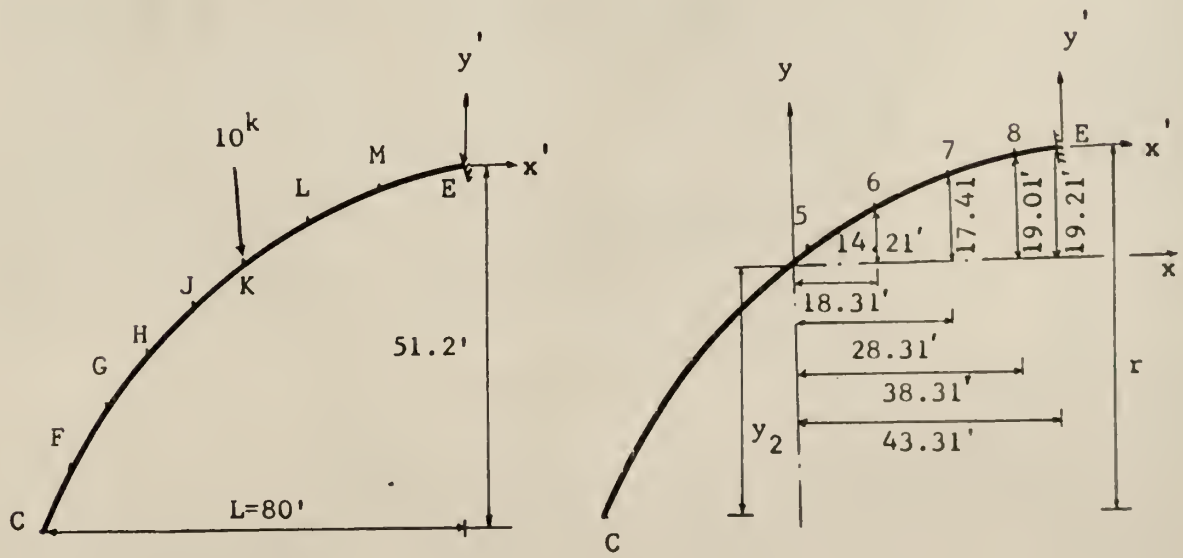


Fig. 18. Properties Of Curved Member CE.

TABLE XX PROPERTIES OF CURVED MEMBER CE

DIVEN DATA						COMPUTED PROPERTIES					
SEG	PT	L	I	X	Y	$A=\int \frac{ds}{EI}$	AX	AY	$I_y = AX^2$	$I_x=AY^2$	$I_{xy}=AXY$
CF	1	15.36	1	-75	-45.000	15.360	-1152.0	-691.20	86400.00	31104.00	+51840.00
FG	2	14.08	1	-65	-33.800	14.080	- 915.2	-475.90	59488.00	16085.42	+30933.50
GH	3	13.24	1	-55	-24.200	13.240	- 728.2	-320.41	40051.00	7753.92	+17622.40
HJ	4	12.32	1	-45	-16.200	12.320	- 554.4	-199.58	24948.00	3353.20	+ 8981.10
JK	5	11.46	1	-35	- 9.800	11.460	- 401.1	-112.31	14038.50	1100.64	+ 3930.78
KL	6	10.77	1	-25	- 5.000	10.770	- 269.3	- 53.85	6731.25	269.25	+ 1346.25
LM	7	10.28	1	-15	- 1.800	10.280	- 154.2	- 18.51	2313.00	333.07	+ 277.56
ME	8	10.03	1	- 5	- 0.200	10.030	- 50.2	- 2.01	250.75	0.40	+ 10.05
CORR. TO CENTROID									-182961.63	-35994.21	-81153.28
Σ						97.54	-4224.5	-1873.8	51258.87	24005.69	33788.36
								$I'_x = 1872.0$		$I'_y = 3900.0$	

REMARKS

$$\bar{X} = \frac{\Sigma Ax}{\Sigma A} = -43.31 \qquad \bar{Y} = \frac{\Sigma Ay}{\Sigma A} = -19.21$$

TABLE XXI ELASTIC LOADS AND FORMULA M_i

SEG	CTR.	M_s	$P=\int \frac{M_s ds}{EI}$	X	Y	M_y	M_x
KL	6	- 50	- 538.5	+18.31	+14.21	- 9857.94	- 7625.1
LM	7	-150	-1542.0	+28.31	+17.41	- 43644.02	- 26846.2
ME	8	-250	-2507.5	+38.31	+19.01	- 96062.33	- 47667.6
Σ			4588.0			- 82165.90	-149564.3
A = 97.54				M'_x = +15798.72		M'_y = -33874.71	
M_i	-47.03 - 8.69X + 8.67Y						

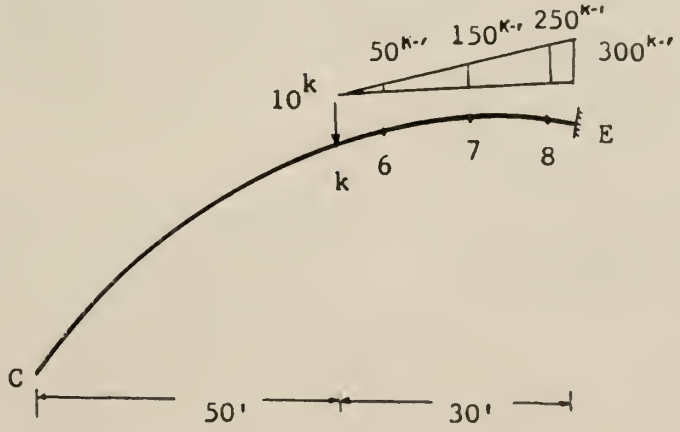


Fig. 19. Statically Determinate Moment Diagram.

From TABLE XX and the formula for M_i in TABLE XXI, the fixed end moments, fixed end thrust, and stiffness factors can be obtained as the following.

$$M_{CE}^{F} = -47.03 - 8.69\ (-36.69) + 8.67\ (-31.99) = +\ 5.53^{k-'}$$

$$M_{EC}^{F} = -\ (-47.03 - 8.69(43.31) + 8.67(19.21)) - 300 = -\ 42.37^{k-'}$$

From the formula for M_i, since $H_{CE}^{F} = +\ 8.67^{k}$

$$C_1 = \frac{L}{\int \frac{ds}{EI}} + \frac{L^3}{4\int \frac{x^2\,ds}{EI}} + \frac{y_2^{\,2}\,L}{\int \frac{y^2\,ds}{EI}} = \frac{80}{97.54} + \frac{80^3}{4(3900)} + \frac{(31.99)^2 \cdot 80}{1872}$$

$$= 77.322$$

$$C_2 = \frac{L}{\int \frac{ds}{EI}} - \frac{L^3}{4\int \frac{x^2\,ds}{EI}} + \frac{y_2^{\,2}\,L}{\int \frac{y^2\,ds}{EI}} = \frac{80}{97.54} + \frac{80^3}{4(3900)} + \frac{19.21^2 \times 80}{1872}$$

$$= 16.19$$

$$C_4^{EC} = \frac{y_2\,r\,L}{\int \frac{y^2\,ds}{EI}} = \frac{(19.21)\ (0)\ (80)}{1872} = 0$$

$$C_4^{CE} = \frac{y_2\,r\,L}{\int \frac{y^2\,ds}{EI}} = \frac{(19.21)\ (51.2)\ (80)}{1872} = 70.0$$

where C_1 = stiffness coefficient at the near end of the member CE with respect to rotation at the near end.

C_2 = stiffness coefficient at the far end of the member CE with respect to rotation at the near end.

C_4^{CE} = stiffness coefficient at end C of the member CE with respect to horizontal displacement at C.

C_4^{EC} = stiffness coefficient at end E of the member CE with respect to horizontal displacement at C.

Then the extended slope-deflection equations can be set up as follows:

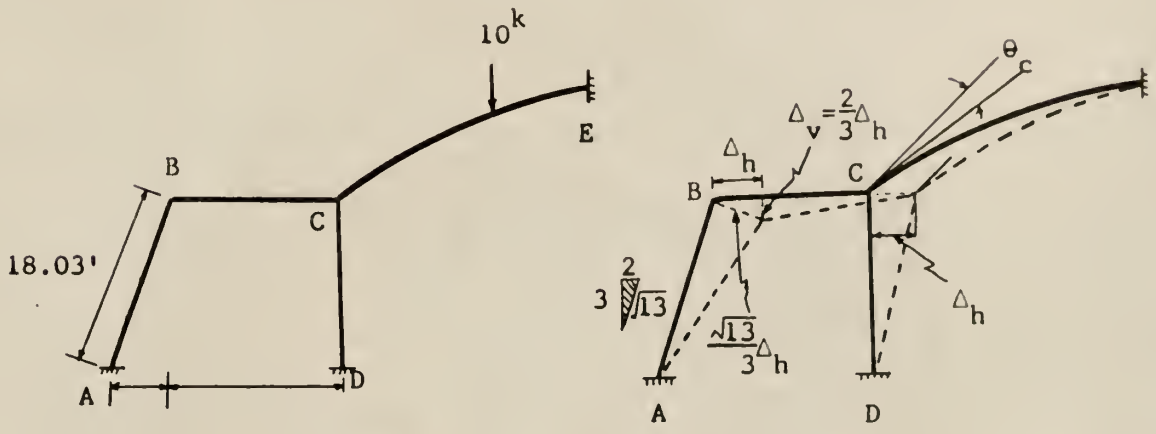


Fig. 20. Displacements Of Joints.

$$M_{AB} = 2E\frac{I}{L}(\theta_B - 3R) = 2E\frac{10}{18.03}(\theta_B - \frac{\sqrt{13}\,\Delta_h}{18.03}) = E(1.11\theta_B - 0.22\,\Delta_h)$$

$$M_{BA} = 2E\frac{I}{L}(2\theta_B - 3R) = E(2.22\theta_B - 0.22\,\Delta_h)$$

$$M_{BC} = 2E\frac{80}{30}(2\theta_B + \theta_C + 3(\frac{\Delta_v}{30})) = E(10.67\theta_B + 5.33\theta_C + 0.355\,\Delta_h)$$

$$M_{CB} = E(5.33\theta_B + 10.67\theta_C + 0.355\,\Delta_h)$$

$$M_{CD} = 2E\frac{10}{15}(2\theta_C + 3(\frac{\Delta_h}{15})) = E(2.67\theta_C - 0.267\,\Delta_h)$$

$$M_{DC} = E(1.33\theta_C - 0.267\,\Delta_h)$$

$$M_{CE} = \frac{EI}{L}\left(C_1\theta_C - C_2\theta_E - C_4^{CE}\frac{\Delta_h}{r}\right) + M_{CE}^F$$

$$= \frac{E}{80}\left(77.322\theta_C - 70\left(\frac{\Delta_h}{51.2}\right)\right) - 5.53 = E(0.966\theta_C - 0.0171\,\Delta_h) - 5.53$$

$$M_{EC} = \frac{EI}{L}\left(-C_2\theta_C + C_1\theta_E - C_4^{EC}\frac{\Delta_h}{r}\right) + M_{EC}^F$$

$$= \frac{E}{80}(-16.19\theta_C - 0) - 42.37 = E(-0.2024\theta_C) - 42.37$$

For joint moment equilibrium:

$$M_{BA} + M_{BC} = 0 \quad ; \quad \text{or}$$

$$12.89\theta_B + 5.33\theta_C + 0.135\,\Delta_h = 0 \quad , \qquad (A)$$

$$M_{CB} + M_{CD} + M_{CE} = 0 \quad ; \quad \text{or}$$

$$5.33\theta_B + 14.306\theta_C + 0.0709\,\Delta_h = 5.53 \quad . \qquad (B)$$

For shear equilibrium:

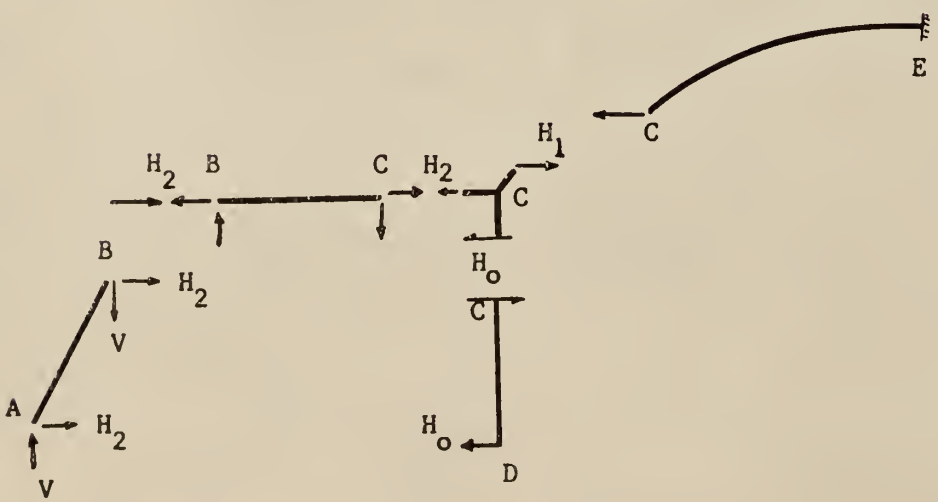


Fig. 21. Shear Equilibrium.

$$H_1 = \frac{EIC}{Lr}\left(\theta_C - \theta_E + \frac{\Delta_h}{y_2}\right) - H_1^F = \frac{E \times 70}{80 \times 51.2}\left(\theta_C + \frac{\Delta_h}{31.99}\right) - 8.67$$

$$= E(0.0171\ \theta_C + 0.000535\ \Delta_h) - 8.67$$

$$H_o = \frac{M_{DC} + M_{CD}}{15} = E(0.267\ \theta_C - 0.0356\ \Delta_h)$$

$$H_2 = \frac{1}{15}(M_{AB} + M_{BA} - V \cdot 10) = E(-0.133\ \theta_B - 0.3556\ \theta_C - 0.045137\ \Delta_h)$$

$$H_1 - H_2 - H_o = 0 \quad ; \quad \text{or}$$

$$0.133\ \theta_B + 0.1057\ \theta_C + 0.081272\ \Delta_h = +8.67 \quad . \qquad (C)$$

Solving the simultaneous equations (A) , (B) , and (C) , the solution is obtained:

$$\theta_B = -.12713063/E \quad ; \quad \theta_C = +0.33039816/E \quad ; \quad \Delta_h = +108.3343/E \ .$$

Therefore, the results are:

$$M_{AB} = -26.34^{k-'} \ , \qquad M_{BA} = -27.76^{k-'} \ ,$$

$$M_{BC} = +27.76^{k-'} \ , \qquad M_{CB} = +35.2^{k-'} \ ,$$

$$M_{CD} = -28.48^{k-'} \ , \qquad M_{DC} = -28.04^{k-'} \ ,$$

$$M_{CE} = +7.05^{k-'} \ , \qquad M_{EC} = -43.03^{k-'} \ .$$

CONCLUSIONS

It is evident that the column analogy method gives the combined effects of sidesway and joint rotation. In frame analysis, such as gable frames and arch problems, this method provides a direct and easily understood approach. The comparison of the results of the column analogy method and the slope-deflection method reveals that they agree very closely. The latter takes considerably more time and may lead to confusion concerning the signs of the rotations and the sidesways of the structure; while the former gives the desired results more directly.

Although this report does not contain a numerical illustration of computing the ordinates of influence lines, this method is a good tool for computing the influence line ordinates for continuous frames with variable cross-section.

ACKNOWLEDGMENT

The author wishes to express his deep appreciation to his major professor, Dr. Robert R. Snell, for his aid, advice, suggestions, and encouragement during the preparation of this report.

NOTATIONS

X, Y, Z = the redundant forces, horizontal, vertical, and rotation respectively.

d'_x , d'_y , d'_z = the horizontal, vertical, and rotation displacements due to the applied load.

d_{xx} , d_{xy} , d_{xz} = the horizontal, vertical, and rotational deflections, respectively, due to X = 1.

d_{yx} , d_{yy} , d_{yz} = the horizontal, vertical, and rotational deflections due to Y = 1.

d_{zx} , d_{zy} , d_{zz} = the horizontal, vertical, and rotational deflections due to Z = 1.

L = the length of the member.

EI = flexural rigidity.

A = total elastic area = $\int \frac{ds}{EI}$

I_x , I_y , I_{xy} = moment of inertia of the elastic area about x- , y-axis and product of inertia about x- and y-axis.

i = moment of inertia of a single member.

M = the resulting bending moment at any point of a structure.

M_x , M_y = the statical moments of elastic load about x- and y-axis.

M_i = the indeterminate moment of the structure.

P = elastic load = $\int \frac{M_s \, ds}{EI}$

C_1 = the stiffness coefficient at the near end of the member CE with respect to rotation at the near end.

C_4^{CE} = the stiffness coefficient at the end C of the member CE with respect to horizontal displacement at C.

C_2 = the stiffness coefficient at the far end of the member with respect to rotation at the near end.

C_4^{EC} = the stiffness coefficient at the end E of the member CE with respect to horizontal displacement at C.

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A STUDY OF YU's EXTENSION OF THE COLUMN ANALOGY METHOD FOR CONTINUOUS FRAME AND ARCH ANALYSIS

by

HUI-PANG CHEN

Diploma, Taipei Institute of Technology, 1963

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Civil Engineering

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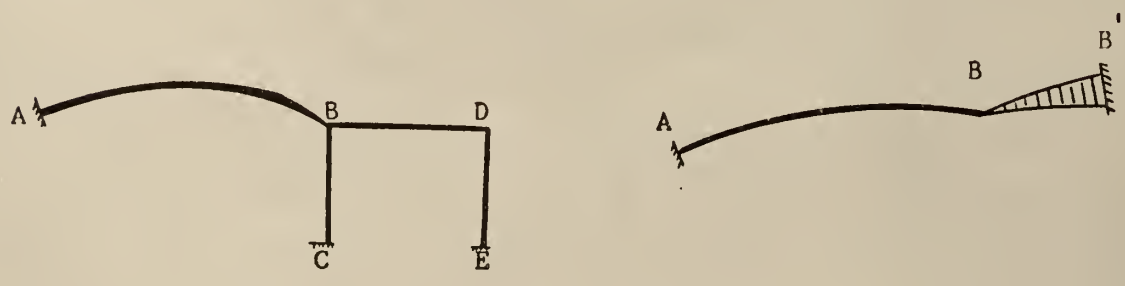
1967

The column analogy, as introduced by Professor H. Cross[1] in 1930, is applicable to beams, arches, and bends of not more than three redundant elements. This report presents an extension by the column analogy method which may be applicable directly to continuous frames, beams and arches. Although these structures may have more than three redundants, this method is still applicable if a proper extension is made. The ideas presented in this report show that the column analogy is a convenient tool for analysis, especially, for variable sections.

The characteristics of this method are such that the two structures in Fig. A are equivalent as far as the redundant elements in span AB are concerned. Span AB is assumed to be supported at B by the "elastic foundation BCDE" which is converted to an equivalent elastic member BB'. The extended method is based on this type of substitution and is useful in many analysis problems.



(a) Original Structure (b) Substituted Structure

Fig. A. Illustration Of The Extended Method.