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Composite Beam with Warpage for Explicit Finite Element Simulation

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COMPOSITE BEAM WITH WARPAGE FOR EXPLICIT FINITE ELEMENT SIMULATION

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by

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ABSTRACT

This study presents the warpage analysis in thin-walled beams of arbitrary open cross section subjected to dynamic loads. Strength analysis has also been conducted for laminated composite beams under static loading conditions. The beam has seven displacement degrees of freedom at each node and the element formulation is based on Vlasov theory of thin-walled beams. Coupling between the force and moment resultants, and the transverse shear deformation have been accounted in the development of laminated composite beam theory. Hellinger-Reissner mixed variational principle is used in element formulation, with an augmented Lagrangian to impose the constraint condition on the rotational degree of freedom. A lumped mass matrix for the beam element has been derived, and central difference scheme is used for explicit time integration. The convergence of 2-node and 3-node thin-walled beam finite elements is studied and the results presented. An eigenvalue analysis is also performed using the lumped mass matrix. Several examples of dynamic loading are studied, and the time history results are compared with implicit time integration results obtained using 3D shell models in ANSYS. Results are also presented for various laminate stacking sequences.

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NOTATION

A	-	Area of Cross section
t	-	Wall thickness of the beam
E	-	Young's Modulus
G	-	Shear Modulus
J	-	St.Venant's torsion constant
u^o, v^o, w^o	-	Axial and transverse displacements
$\phi_x^o, \phi_y^o, \phi_z^o$	-	Rotation components about x, y, & z axis respectively
θ^o	-	Rate of twist
ω	-	Sectorial Area
N_x	-	Axial force
Q_y, Q_z	-	Transverse shear forces
M_t, M_y, M_z	-	Moments about x, y, & z axis
B_ω	-	Bi-moment
ε_x^o	-	Axial strain
$\gamma_{xy}^o, \gamma_{xz}^o$	-	Transverse shear strain
$\kappa_t^o, \kappa_y^o, \kappa_z^o$	-	Curvatures
ψ^o	-	Warping strain parameter
S_y, S_z	-	First moment of area
I_y, I_z, I_{yz}	-	Second, and product moments of area about y & z axis
S_ω	-	Sectorial static moment
$I_{\omega y}, I_{\omega z}$	-	Sectorial linear moments about y and z axis
I_ω	-	Second sectorial moment
Y_s, Z_s	-	Coordinates of shear center
u, v, w	-	Plate displacements
V	-	Total strain energy
U^c	-	Complementary strain energy
λ	-	Lagrange parameter
ε	-	Penalty parameter

$[F]$ - Flexibility matrix

$[S]$ - Strain-displacement matrix

N_1, N_2 - Displacement shape functions

$N_1^{'}$ - Stress shape functions

$[k]$ - Stiffness matrix

A_{ij}, B_{ij}, D_{ij} - Equivalent modulus of multi-directional laminates

$\bar{A}_{ij}, \bar{B}_{ij}, \bar{D}_{ij}$ - Modified laminate stiffness coefficients

Π - Potential energy functional

$\varepsilon_x, \varepsilon_s$ - Membrane normal strains

γ_{xs} - Membrane shear strain

$\kappa_x, \kappa_s, \kappa_{xs}$ - Membrane curvatures

N_x, N_{xs} - Membrane normal and shear force resultants

M_x, M_{xs} - Membrane moment resultants

T_ω, T_s - Flexural twist, St.Venant Torque

$[m]$ - Mass matrix

$\{H\}$ - Beam force resultants vector

$\{X\}$ - Nodal degrees of freedom vector

$[c]$ - Damping matrix

$\{r\}^{\text{int}}$ - Internal force vector

$W_{inertia}$ - Work done by inertia forces

W_{int} - Work done by internal forces

ρ - Mass density of the material

$X, X^{'}$ - Longitudinal tensile and compressive strengths of the laminate

$Y, Y^{'}$ - Transverse tensile and compressive strengths of the laminate

S - Shear Strength of the laminate

INTRODUCTON

1.1 General Introduction and Literature Survey

Thin-walled beams are shell structures that have dimensions satisfying the proportions $t/d \le 0.1$ and $d/l \le 0.1$, where t is the thickness of the shell, d the characteristic dimension of the cross section, and l its length. A few examples of such structures in practical use are bridge supports, aircraft wings, building frames etc. Vlasov [1] proposed the theory of bending and torsion in thin-walled isotropic beams in 1930. The distinguishing feature of these structures apart from their spatial dimensions is that they undergo longitudinal displacement or warpage when subjected to torsional loads. This out-of-plane displacement is given by law of sectorial areas proposed by Vlasov. Also central to the Vlasov theory is the concept of sectorial properties, flexural twist and bi-moment.

Based on the Vlasov theory, a number of displacement-based finite element models were developed for static and free vibration analysis of thin-walled beams of constant rigid arbitrary open cross section. In [14] a hybrid formulation has been presented, and the element developed takes into account sharp variations in geometry. In [13], Kawai et al present a formulation for beam of variable cross section and suggest a cubic variation for the torsional degree of freedom. Chandhary [18] presented the exact sectorial-area theory for calculation of torsional stiffness matrix of prismatic beams. Calculation of sectorial properties of the cross section forms an important part of the analysis of thin-walled beams. Reviews on the methods of computation of the sectorial properties can be found in ref [1], [3], [7], and [8]. In [7] an approach is presented that is convenient to implement

using a computer program. Subsequently Coyette [3] has suggested a further simplification of this approach.

Composite materials are made of two or more distinctive constituents that are mechanically combined to produce a material, which has properties, superior to each of its constituents. Common applications of composites are in structures that require high stiffness to weight ratio, corrosion resistance, greater fatigue tolerance, high temperature resistance, and impact resistance. The important feature in the design of general laminated composites is the coupling between the extensional, transverse, and torsional deformations. The isotropic theory of thin-walled beams can be extended to laminated composite beams using the approach proposed by Gjelsvik[6]. Bauld & Tzeng[20] proposed a Vlasov type theory on similar lines for beams made of symmetric laminated fiber reinforced composites. In mid-plane symmetric laminates the coupling between the force and moment resultants is eliminated. The shearing strain in the middle surface was treated as negligible, and the contour stress is considered to be small compared to the axial stress. Further each laminate is assumed to obey Kirchoff's theory. Gupta [21] presented a 2-node anisotropic thin-walled beam finite element with 8 degrees of freedom at each node using one-dimensional first order Hermitian interpolation polynomials. Chandra and Chopra [22] present a Vlasov type theory for analyzing composite I-beams wherein the transverse shear and axial-twist couplings are considered. The resulting beam has 9 forces and 7 beam displacement variables.

Explicit time integration is used in the dynamic analysis of these beams because of their inherent nature to treat the problem with modest memory and processor requirements. The displacements at any instant in time are expressed in terms of corresponding

historical information and using a lumped mass matrix the equation of motions are uncoupled. This obviates the need to store the complete global stiffness matrix and makes the computation of the internal force vector easy.

In the strength analysis of laminated composites it is necessary to determine the ply in which failure occurs. Strength analysis could be conducted either on a macro scale or micro scale. In the micro-mechanical analysis the laminated composites are analyzed w.r.t the mechanical properties of each of its constituents. In macro-mechanical analysis the effective properties of each ply is determined and each ply is treated as a homogenous anisotropic material. The stresses and strains in each ply are calculated and subsequently a failure criterion is applied to determine the first-ply failure.

1.2 **Outline of the Study**

In the present study an attempt is made to study the warpage in laminated composite beams subjected to impact loading, and strength of laminated composite beams under static loading conditions. The study is presented in three stages. A 2-node beam finite element model for isotropic material is derived using mixed variational principle as proposed by Noor[2]. The beam has seven displacement degrees of freedom at each node. Lumped mass matrix developed using HRZ lumping scheme considers the effects of rotary inertia, and internal force vector is also derived for explicit finite element simulation. Following this, a 2-node thin-walled laminated composite beam finite element is derived using the approach proposed by Gjelsvik[6]. Coupling between axial and bending deformations, effects of transverse shear deformation have been taken into account, and the contour is assumed to be rigid. The results are validated with analytical solutions where available and also with results from ANSYS using 3D shell models. Strength analysis of laminated composite beams is performed by computing the strains in each ply of each segment of the cross section and subsequently applying the Tsai-Wu failure criterion to determine first-ply failure.

ISOTROPIC THEORY OF THIN-WALLED BEAMS

2.1 Law of Sectorial Areas

The cross-section of thin-walled beams exhibit out-of-plane displacement in addition to the axial and transverse displacement, and rotations about the three coordinates. This out-of-plane displacement or warping is given by the law of sectorial areas proposed by Vlasov [1]. The law of sectorial areas as proposed by Vlasov is based on the following assumptions:

- The cross section of the beam is perfectly rigid in its plane.
- The shear strain in the middle surface is negligible.

The longitudinal and transverse displacements for a thin-walled beam derived using the law of sectorial area is:

$$u(x,y,z) = u^o(x) + \varphi_y^o(x)z - \varphi_z^o(x)y - \omega(y,z)\theta^o \tag{2.1}$$

$$v(x,y,z) = v^o(x) - \varphi_x^o(x)z \tag{2.2}$$

$$w(x,y,z) = w^o(x) + \varphi_x^o(x)y \tag{2.3}$$

The second and third terms in (2.1) are due to bending in the y and z planes respectively. Equation (2.1) is a more general expression and the Euler-Bernoulli beam theory is a particular case of this. The difference is in the fourth term of the equation which gives the sectorial warping of the cross section. The quantity $\omega(y,z)$ is called the sectorial area and θ is the rate of twist or torsional warping $(\theta = \frac{\partial \varphi_x}{\partial x})$. The sectorial area is computed along the cross section contour with respect to a pole and a sectorial origin. It is defined as twice the area swept along the contour of the cross section by the radius

joining the pole and any point along the center-line of the contour. The general formula for the sectorial area is given below, and is illustrated using the Fig 2.1.

$$\omega_{DOS} = \int_0^s d\omega(s) = \int_0^s h(s) \quad ds$$

The first subscript D denotes the point called "pole of sectorial coordinates" from which lines radiate to every point on the centerline of the contour. The second subscript denotes the point of intersection of the initial radius with the centerline of the contour, and the third subscript denotes a point along the centerline of the contour. s is the sectorial coordinate. For a particular case of a channel section shown in Fig 2.2, twice the area swept by the initial radius DO along the lower straight line segment (shown shaded) gives the sectorial area of that segment. The sectorial area of the channel section is the sum of all such areas as we move along the entire contour.

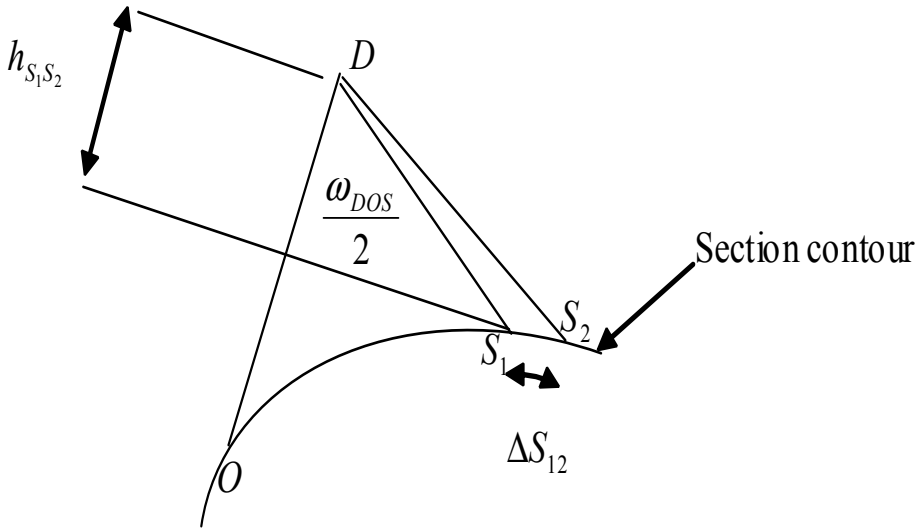


Fig 2.1: Sectorial area representation

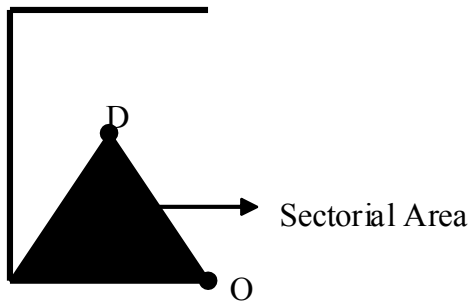


Fig 2.2: Computing the sectorial area for a channel section

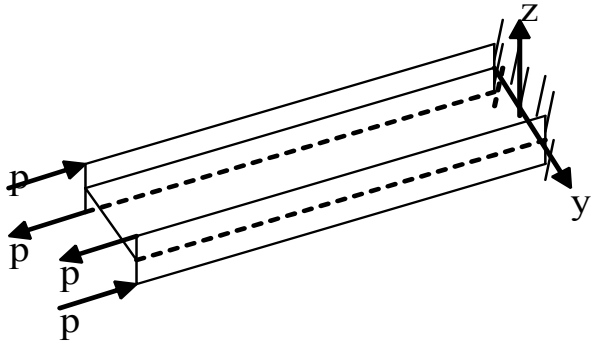


Fig 2.3: Pictorial representation of bi-moment

2.2 Flexural Twist and Bi-Moment

The St.Venant's theory of torsion assumes that the cross section is free of longitudinal stresses when subjected to torsion. This theory does not hold for thin-walled beams and according to Vlasov the twisting moment on the cross section of a thin-walled beam results in bending moments, which give rise to longitudinal stresses, in addition to the rotation. Vlasov called the component of twisting moment that causes bending a "Flexural Twist". This flexural twist causes the bimoment and hence the additional longitudinal stresses in a thin-walled beam when subjected to torsion. Thus the total torsional moment in the case of thin-walled beams is a sum of two components, the warping component and the St.Venant component.

$$T = T_{\omega} + T_{s}$$

T_s is given by the St.Venant theory

T_{ω} is the warping torque that comes from the last term in (2.1)

When the beam bends in the z-plane the loads at the four corners of the I-beam will be symmetric about the y-axis and vice versa. Under pure central extension, the forces are would be symmetric w.r.t both the axes. The I-beam will be subjected to a bi-moment when the forces at the corners of the flanges are skew-symmetric w.r.t both the cross-sectional axes as shown in Fig 2.3.

2.3 Sectorial Properties

The procedure of finding the stresses in a thin-walled beam using the theory of thin-walled beams involves the calculation of the sectorial properties of the cross section in addition to the bending properties. These are the sectorial area ω, sectorial static

moment S_ω, sectorial product moments $I_{\omega y}$ & $I_{\omega z}$, and the sectorial second moment I_ω. The expressions for the sectorial properties of cross sections are:

$$S_\omega = \int_A \omega \, dA \qquad I_\omega = \int_A \omega^2 \, dA \tag{2.4}$$

$$I_{\omega y} = \int_A \omega y \, dA \qquad I_{\omega z} = \int_A \omega z \, dA \tag{2.5}$$

For complex shapes or for the purpose of implementation using a computer program, it would be simpler to assume the cross section to be connected by a series of straight segments and the properties evaluated by summing up the value for each segment [3]. The relevant expressions for an arbitrary open cross section in $y-z$ plane are given below.

Area of cross section, $S_t = \sum S_i$

First Moment about Y-axis, $S_y = \sum S_i \quad (z_m + z_n)/2$

First Moment about Z-axis, $S_z = \sum S_i \quad (y_m + y_n)/2$

Moment of inertia about Y-axis, $I_y = \sum S_i \quad (z_m^2 + z_m z_n + z_n^2)/2$

Moment of inertia about Z-axis, $I_z = \sum S_i \quad (y_m^2 + y_m y_n + y_n^2)/2$

Product moment of inertia,

$$I_{yz} = \sum S_i \quad (y_m z_m + y_n z_n)/3 + \sum S_i \quad (y_m z_n + y_n z_m)/6$$

Y-coordinate of center of gravity, $y_g = \dfrac{S_z}{S_t}$

Z-coordinate of center of gravity, $z_g = \dfrac{S_y}{S_t}$ $\qquad (2.6a-h)$

The inertial properties relative to axes passing through the center of gravity are

$$I_y = I_y - y_g * S_z$$

$$I_z = I_z - z_g * S_y$$

$$I_{yz} = I_{yz} - y_g * z_g * S_t$$

To calculate the torsional properties, the origin of the coordinate system is shifted to the center of gravity. Y_i, Z_i , are the coordinates of the end points of straight line segments in the new coordinate system. The end nodes of the segments are numbered in a way that sectorial coordinate of the first node of the segment is known when the element is to be processed. The sectorial coordinate of the very first node of the cross section is taken as zero. Thus the sectorial coordinate of the end node of any segment is $\omega_n = \omega_m + \Delta\omega_{mn}$, where $\Delta\omega$ is the sectorial area swept by the line joining the pole and the point 'm', along the segment till the point 'n'.

$$Y_i = y_i - y_g$$

$$Z_i = z_i - z_g$$

Sectorial static moment, $S_\omega = \sum S_i \quad (\omega_m + \omega_n)/2$

Sectorial linear moment about Y-axis,

$$I_{\omega Y} = \sum S_i \quad [\omega_m(2Z_m + Z_n) + \omega_n(2Z_n + Z_m)]/6$$

Sectorial linear moment about Z-axis,

$$I_{\omega Z} = \sum S_i \quad [\omega_m(2Y_m + Y_n) + \omega_n(2Y_n + Y_m)]/6$$

Sectorial moment of inertia, $I_{\omega o} = \sum S_i \quad (\omega_m^2 + \omega_m\omega_n + \omega_n^2)/3$

Torsional moment of inertia, $K = \sum (l_i t_i)/3 \qquad (2.7a-e)$

m, n are the start and end nodes of each segment. l_i, t_i are the length and thickness of the i^{th} segment

The coordinates of the shear center are obtained using the relations:

$$Y_s = \frac{I_{YZ} * I_{\omega Z} - I_Z * I_{\omega Y}}{I_{YZ}^2 - I_Y * I_Z}$$

$$Z_s = \frac{I_Y * I_{\omega Z} - I_{YZ} * I_{\omega Y}}{I_{YZ}^2 - I_Y * I_Z} \qquad (2.8a-b)$$

The principal sectorial moment of inertia I_ω is,

$$I_\omega = I_{\omega o} + I_{\omega Z} * Z_s - I_{\omega Y} * Y_s - S_\omega^2 / S_t \qquad (2.9)$$

and the principal sectorial coordinates are given by,

$$W_i = -\omega_i - Z_s * Y_i + Y_s * Z_i + S_\omega / S_t \qquad (2.10)$$

The shear center is defined as that point through which an external transverse force produces no torsion, i.e. the beam is in a state of pure bending. In other words, any load that is offset from the line of shear centers causes bending as well as torsion. The shear center is also called the principal pole. The sectorial coordinates calculated with the pole at the shear center are called the principal sectorial coordinates. The sectorial moment of inertia calculated with the pole at shear center and the principal radius as the initial radius is called the principal sectorial moment of inertia. The line of shear centers is as important in the theory of thin-walled beams as the centroidal axis is in the theory of plane sections. This line coincides with the line of centroids for cross sections which have two axis of symmetry. For asymmetric cross sections the shear center will be shifted from the centroid along the axis of symmetry.

LAMINATED COMPOSITES THIN-WALL BEAM THEORY

3.1 Coordinate System and Laminate theory fundamentals

The development presented is in line with the theory of thin-walled isotropic beams by Gjelsvik[6]. The essence of Gjelsvik's theory is in expressing the beam force and moment resultants in terms of plate stress resultants. Three co-ordinate systems are used:(a) A right-handed Cartesian co-ordinate system (x, y, z) with origin at the centroid of the beam cross section, (b) A right-handed curvilinear co-ordinate system (x, n, s) at the mid-surface of the contour, and (c) A contour co-ordinate system s with the origin at some point along the contour. These coordinate systems are illustrated in Fig 3.1.

The axial, normal and tangential displacements respectively of a point on the contour of the cross section are,

$$u(s,x) = u^o + z\phi_y^o - y\phi_z^o - \omega\theta^o \tag{3.1}$$

$$v(s,x) = v^o\ Cos\theta + w^o\ Sin\theta - q(s)\phi_x^o \tag{3.2}$$

$$w(s,x) = v^o\ Sin\theta - w^o\ Cos\theta + r(s)\phi_x^o \tag{3.3}$$

u, v, w are the displacements of a point on the contour of the beam cross section. $u^o, v^o, w^o, \phi_x^o, \phi_y^o, \phi_z^o, \& \theta^o$ are the beam deformations as a function of the axial coordinate x. $q, r, \& \theta$ are functions of contour coordinate s and are shown in Fig 3.1. ω is called the sectorial area, and is defined as twice the area swept along the contour of the cross section by the radius joining the pole and any point along the center-line of the contour. The general expression for sectorial area is,

$$\omega_{DOS} = \int_0^s d\omega(s) = \int_0^s r(s)\quad ds$$

The relation between six independent plate force resultants and membrane strains and curvatures for laminated composites is,

$$\begin{Bmatrix} N_x \\ N_s \\ N_{zs} \\ M_z \\ M_s \\ M_{zs} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & B_{11} & B_{12} & B_{13} \\ & A_{22} & A_{23} & B_{21} & B_{22} & B_{23} \\ & & A_{33} & B_{31} & B_{32} & B_{33} \\ & Sym & & D_{11} & D_{12} & D_{13} \\ & & & & D_{22} & D_{23} \\ & & & & & D_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_s \\ \gamma_{xs} \\ \kappa_x \\ \kappa_s \\ \kappa_{zs} \end{Bmatrix} \tag{3.4}$$

where,

$$A_{ij} = \sum_{k=1}^{n} Q_{ij}^{k} (h_k - h_{k-1}) = h \sum_{k=1}^{n} Q_{ij}^{k} (\frac{h_k}{h} - \frac{h_{k-1}}{h}) = h\, a_{ij}$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{N} Q_{ij}^{k} (h_k^2 - h_{k-1}^2) = \frac{h^2}{2} \sum_{k=1}^{n} Q_{ij}^{k} (\frac{h_k^2}{h^2} - \frac{h_{k-1}^2}{h^2}) = h^2\, b_{ij}$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N} Q_{ij}^{k} (h_k^3 - h_{k-1}^3) = \frac{h^3}{3} \sum_{k=1}^{n} Q_{ij}^{k} (\frac{h_k^3}{h^3} - \frac{h_{k-1}^3}{h^3}) = h^3\, d_{ij}$$

Q_{ij}^{k} are the lamina stiffness coefficients for the k^{th} laminate layer. h is the thickness of the laminate and h_k & h_{k-1} are the coordinates along the $n-$axis illustrated in Fig 3.2.

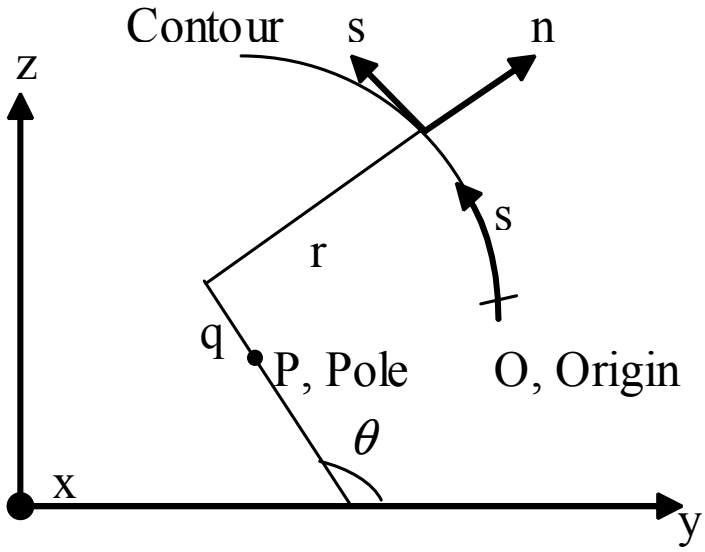


Fig. 3.1. Illustration of rectangular coordinate system (x, y, z), plate coordinate system (x, n, s), and contour coordinate s

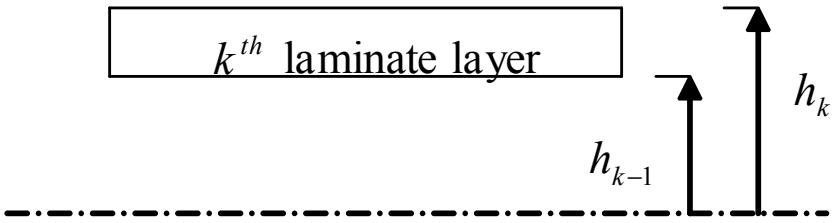


Fig. 3.2. Cross-section of a laminate.

3.2 Expressions for beam force and moment resultants

The plate membrane strains and curvatures are related to the plate displacement components u, v, w by the following expressions,

Membrane normal strain in the x direction, $\varepsilon_x = \dfrac{\partial u}{\partial x}$ (3.5a)

Membrane normal strain in the contour s direction, $\varepsilon_s = \dfrac{\partial w}{\partial s}$ (3.5b)

Membrane shear strain, $\gamma_{xs} = \dfrac{\partial u}{\partial s} + \dfrac{\partial w}{\partial z}$ (3.5c)

Curvatures - $\kappa_x = \dfrac{\partial^2 v}{\partial x^2}$

$$\kappa_s = \frac{\partial^2 v}{\partial s^2} \quad (3.5d)$$

$$\kappa_{xs} = 2\frac{\partial^2 v}{\partial x \partial s}$$

Substituting for the displacements u, v, w in $3.5(a-d)$, we get the plate strains and curvatures in terms of beam displacements.

$$\varepsilon_x = \frac{\partial u}{\partial x} = \varepsilon_x^o + z\,\kappa_z^o - y\,\kappa_y^o - \omega\psi^o \quad (3.6a)$$

$$\varepsilon_s = \frac{\partial w}{\partial s} = 0 \quad (3.6b)$$

$$\gamma_{xs} = \frac{\partial u}{\partial s} + \frac{\partial w}{\partial z} = \gamma_{xy}^o\, Cos\theta + \gamma_{xz}^o\, Sin\theta \quad (3.6c)$$

$$\kappa_x = \frac{\partial^2 v}{\partial x^2} = -q\frac{\partial^2 \phi_x^o}{\partial s^2} = q\psi^o \quad (3.6d)$$

$$\kappa_s = \frac{\partial^2 v}{\partial s^2} = -\frac{\partial^2 q}{\partial s^2}\phi_x^o = 0 \quad (3.6e)$$

$$\kappa_{xs} = 2\frac{\partial^2 v}{\partial x \partial s} = -2\kappa_t^o \qquad (3.6f)$$

where,

$$\varepsilon_x^o = \frac{du^o}{dx} \qquad \gamma_{xy}^o = \frac{dv^o}{dx} - \phi_z^o \qquad \gamma_{xz}^o = \frac{dw^o}{dx} + \phi_y^o$$

$$\kappa_y^o = \frac{d\phi_z^o}{dx} \qquad \kappa_z^o = \frac{d\phi_y^o}{dx} \qquad \kappa_t^o = \frac{d\phi_x^o}{dx} \qquad \psi^o = \frac{d\theta^o}{dx}$$

The assumption in the isotropic thin-walled beams that the contour of the cross section does not deform implies that $\varepsilon_s = \kappa_s = 0$. This assumption only gives accurate results for laminates for which the coupling terms $A_{13}, A_{23}, D_{13}, D_{23}$ are zero ([24], [25], [26] & [27]). Therefore, it is assumed that only the stress resultants $N_s \,\&\, M_s$ are negligible. From equation (3.4),

$$A_{21}\varepsilon_x + A_{22}\varepsilon_s + A_{23}\gamma_{sx} + B_{21}\kappa_x + B_{22}\kappa_s + B_{23}\kappa_{sx} = 0 \qquad (3.7a)$$

$$B_{12}\varepsilon_x + B_{22}\varepsilon_s + B_{32}\gamma_{sx} + D_{21}\kappa_x + D_{22}\kappa_s + D_{23}\kappa_{sx} = 0 \qquad (3.7b)$$

Solving for ε_s from the above equations,

$$\varepsilon_s = -\frac{(A_{21}D_{22} - B_{12}B_{22})\varepsilon_x + (A_{23}D_{22} - B_{32}B_{22})\gamma_{sx} + (B_{21}D_{22} - D_{21}B_{22})\kappa_x + (B_{23}D_{22} - D_{23}B_{22})\kappa_{sx}}{(A_{22}D_{22} - B_{22}^2)} \qquad ...(3.8a)$$

and,

$$\kappa_s = -\frac{(A_{22}B_{12} - A_{21}B_{22})\varepsilon_x + (A_{22}B_{32} - A_{23}B_{22})\gamma_{sx} + (A_{22}D_{21} - B_{21}B_{22})\kappa_x + (A_{22}D_{23} - B_{23}B_{22})\kappa_{sx}}{(A_{22}D_{22} - B_{22}^2)} \qquad ...(3.8b)$$

Since the contour stress resultants N_s, M_s are negligible, substituting $(3.8a)\&(3.8b)$ in (3.4) the four stress resultants $N_x, M_x, N_{sx}\&M_{sx}$ are expressed in terms of the four strains components $\varepsilon_x, \kappa_x, \gamma_{sx}, \kappa_{sx}$.

$$\begin{Bmatrix} N_x \\ N_{sx} \\ M_x \\ M_{sx} \end{Bmatrix} = \begin{bmatrix} \overline{A_{11}} & \overline{A_{13}} & \overline{B_{11}} & \overline{B_{13}} \\ \overline{A_{31}} & \overline{A_{33}} & \overline{B_{31}} & \overline{B_{33}} \\ \overline{B_{11}} & \overline{B_{13}} & \overline{D_{11}} & \overline{D_{13}} \\ \overline{B_{31}} & \overline{B_{33}} & \overline{D_{31}} & \overline{D_{33}} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \gamma_{sx} \\ \kappa_x \\ \kappa_{sx} \end{Bmatrix} \tag{3.9}$$

The coefficients in (3.9) are given by,

$$\overline{A_{11}} = A_{11} + \frac{2A_{12}B_{12}B_{22} - A_{12}^2 D_{22} - B_{12}^2 A_{22}}{A_{22}D_{22} - B_{22}^2} \tag{3.10a}$$

$$\overline{A_{13}} = A_{13} + \frac{A_{12}(B_{22}B_{23} - A_{23}D_{22}) + B_{12}(A_{23}B_{22} - B_{23}A_{22})}{A_{22}D_{22} - B_{22}^2} \tag{3.10b}$$

$$\overline{A_{33}} = A_{33} + \frac{2A_{23}B_{22}B_{23} - A_{23}^2 D_{22} - B_{23}^2 A_{22}}{A_{22}D_{22} - B_{22}^2} \tag{3.10c}$$

$$\overline{B_{11}} = B_{11} + \frac{A_{12}(B_{22}D_{12} - B_{12}D_{22}) + B_{12}(B_{12}B_{22} - D_{12}A_{22})}{A_{22}D_{22} - B_{22}^2} \tag{3.10d}$$

$$\overline{B_{13}} = B_{13} + \frac{A_{12}(B_{22}D_{23} - B_{23}D_{22}) + B_{12}(B_{22}B_{23} - D_{23}A_{22})}{A_{22}D_{22} - B_{22}^2} \tag{3.10e}$$

$$\overline{B_{31}} = B_{31} + \frac{A_{23}(B_{22}D_{12} - B_{12}D_{22}) + B_{23}(B_{12}B_{22} - D_{12}A_{22})}{A_{22}D_{22} - B_{22}^2} \tag{3.10f}$$

$$\overline{B_{33}} = B_{33} + \frac{A_{23}(B_{22}D_{23} - B_{23}D_{22}) + B_{23}(B_{22}B_{23} - D_{23}A_{22})}{A_{22}D_{22} - B_{22}^2} \tag{3.10g}$$

$$\overline{D_{11}} = D_{11} + \frac{2B_{12}B_{22}D_{12} - B_{12}^2 D_{22} - D_{12}^2 A_{22}}{A_{22}D_{22} - B_{22}^2} \tag{3.10h}$$

$$\overline{D_{13}} = D_{13} + \frac{B_{12}(B_{22}D_{23} - B_{23}D_{22}) + D_{12}(B_{23}B_{22} - D_{23}A_{22})}{A_{22}D_{22} - B_{22}^2} \tag{3.10i}$$

$$\overline{D_{33}} = D_{33} + \frac{2B_{22}B_{23}D_{23} - B_{23}^2 D_{22} - D_{23}^2 A_{22}}{A_{22}D_{22} - B_{22}^2} \qquad (3.10j)$$

While the $\overline{A}\,\&\,\overline{D}$ matrices are symmetric, the matrix $\overline{B}$ is not symmetric.

The shear strain in the middle surface of each lamina γ_{xs} is not zero. All higher order derivatives of the beam displacement components are neglected. Further q is treated as a linear function of the contour coordinate s .

The relations between plate and beam force resultants derived by Gjelsvik are,

$$N = \int_c N_x \, ds \qquad (3.11a)$$

$$Q_y = \int_c N_{sx} \, Cos\theta \, ds \qquad (3.11b)$$

$$Q_z = \int_c N_{sx} \, Sin\theta \, ds \qquad (3.11c)$$

$$M_z = -\int_c (N_x \, y - M_x \, Sin\theta) \, ds \qquad (3.11d)$$

$$M_y = \int_c (N_x \, z - M_x \, Cos\theta) \, ds \qquad (3.11e)$$

$$M_t = T_\omega + T_s = \int_c (\omega \frac{\partial N_x}{\partial x} + q \, \frac{\partial M_x}{\partial x}) \, ds - \int_c (M_{sx} + M_{xs}) \, ds \qquad (3.11f)$$

$$B_\omega = -\int_c (N_x \, \omega + M_x \, q) \, ds \qquad (3.11g)$$

N is the axial force, $Q_y \,\&\, Q_z$ are the shear forces, $M_y \,\&\, M_z$ are the bending moments, M_t is the axial twist and B_ω is the bi-moment. The steps in the derivation of the beam force and moment resultants are summarized.

$$N = \int N_x \, ds = \int (\overline{A}_{11} \, \varepsilon_x + \overline{A}_{13} \, \gamma_{xs} + \overline{B}_{11} \, \kappa_x + \overline{B}_{13} \, \kappa_{xs}) \, ds$$

$$\Rightarrow \varepsilon_x^o \int a_{11}\, hds + \gamma_{xy}^o \int (a_{13}\, Cos\theta)\, hds + \gamma_{xz}^o \int (a_{13}\, Sin\theta)\, hds - \kappa_y^o \int (a_{11}\, y)\, hds + \kappa_z^o \int (a_{11}\, z)\, hds$$
$$- \kappa_t^o \int (2\, b_{13}\, h)\, hds - \psi^o \int (a_{13}\, \omega - b_{11}\, q\, h)\, hds \qquad (3.12a)$$

$$M_y = \int (N_x\, z + M_x\, Cos\theta)\, ds$$

$$\Rightarrow \int ((\overline{A}_{11} z + \overline{B}_{11} Cos\theta)\, \varepsilon_x + (\overline{A}_{13} z + \overline{B}_{13} Cos\theta)\, \gamma_{xs} + (\overline{B}_{11} z + \overline{D}_{11} Cos\theta)\, \kappa_x + (\overline{B}_{13} z + \overline{D}_{13} Cos\theta)\, \kappa_{xs})\, ds$$
$$\Rightarrow \varepsilon_x^o \int (a_{11} z + b_{11} h Cos\theta)\, hds + \gamma_{xy}^o \int (a_{13} z + b_{13} h Cos\theta)\ Cos\theta\, hds + \gamma_{xz}^o \int (a_{13} z + b_{13} h Cos\theta)\ Sin\theta\, hds$$
$$- \kappa_y^o \int (a_{11} yz + b_{11} yh Cos\theta)\, hds + \kappa_z^o \int (a_{11}\, z^2 + b_{11} zh Cos\theta)\, hds - \kappa_t^o \int (2\, b_{13} zh + 2 d_{13} h^2 Cos\theta)\, hds$$
$$- \psi^o \int (a_{11}\, \omega z + b_{11} \omega h Cos\theta - b_{11} z\, q\, h - d_{11} h^2 q Cos\theta)\, hds \qquad (3.12b)$$

$$M_z = -\int (N_x\, y - M_x\, Sin\theta)\, ds$$

$$\Rightarrow -\int ((\overline{A}_{11} y - \overline{B}_{11} Sin\theta)\, \varepsilon_x + (\overline{A}_{13} y - \overline{B}_{13} Sin\theta)\, \gamma_{xs} + (\overline{B}_{11} y + \overline{D}_{11} Sin\theta)\, \kappa_x + (\overline{B}_{13} y + \overline{D}_{13} Sin\theta)\, \kappa_{xs})\, ds$$
$$\Rightarrow -\varepsilon_x^o \int (a_{11} y - b_{11} h Sin\theta)\, hds - \gamma_{xy}^o \int (a_{13} y - b_{13} h Sin\theta)\ Cos\theta\, hds - \gamma_{xz}^o \int (a_{13} y - b_{13} h Sin\theta)\ Sin\theta\, hds$$
$$+ \kappa_y^o \int (a_{11} y^2 - b_{11} yh Sin\theta)\, hds - \kappa_z^o \int (a_{11}\, yz - b_{11} zh Sin\theta)\, hds + \kappa_t^o \int (2\, b_{13} yh - 2 d_{13} h^2 Sin\theta)\, hds$$

$$+ \psi^o \int (a_{11}\, \omega y - b_{11} \omega h Sin\theta - b_{11} y\, q\, h + d_{11} h^2 q Sin\theta)\, hds \qquad (3.12c)$$

$$M_\omega = -\int (N_x\, \omega - M_x\, q)\, ds$$

$$\Rightarrow -\int ((\overline{A}_{11} \omega - \overline{B}_{11} q)\, \varepsilon_x + (\overline{A}_{13} \omega + \overline{B}_{13} q)\, \gamma_{xs} + (\overline{B}_{11} \omega + \overline{D}_{11} q)\, \kappa_x + (\overline{B}_{13} \omega + \overline{D}_{13} q)\, \kappa_{xs})\, ds$$

$$\Rightarrow -\varepsilon_x^o \int (a_{11} \omega + b_{11} hq)\, hds - \gamma_{xy}^o \int (a_{13} \omega + b_{13} hq)\ Cos\theta\, hds - \gamma_{xz}^o \int (a_{13} \omega + b_{13} hq)\ Sin\theta\, hds$$

$$+ \kappa_y^o \int (a_{11} \omega y + b_{11} yhq)\, hds - \kappa_z^o \int (a_{11}\, \omega z + b_{11} zhq)\, hds - \kappa_t^o \int (2\, b_{13} \omega h + 2 d_{13} h^2 q)\, hds$$

$$+ \psi^o \int (a_{11}\, \omega^2 - d_{11} h^2 q^2)\, hds \qquad (3.12d)$$

$$T = T_\omega + T_s = \int (\omega \frac{\partial N_x}{\partial x} + q \frac{\partial M_x}{\partial x})\, ds - 2\int M_{sx}\, ds$$

$$\Rightarrow -\varepsilon_x^o \int (2b_{31}h)\, hds - \gamma_{xy}^o \int (2b_{33}hCos\theta)\ hds - \gamma_{xz}^o \int (2b_{33}hSin\theta)\ hds$$

$$-\kappa_y^o \int ((a_{13}\omega + b_{13}hq)\, Cos\theta + b_{31}yh)hds + \kappa_z^o \int ((a_{13}\ \omega + b_{13}hq)\, Sin\theta + b_{31}zh)hds + \kappa_t^o \int (4d_{33}h^2)\, hds$$
$$-\psi^o \int (4b_{13}\omega h)\, hds \qquad (3.12e)$$

$$Q_y = \int_c N_{sx}\, Cos\theta\, ds$$

$$\Rightarrow \varepsilon_x^o \int (a_{31}\, Cos\theta)\, hds + \gamma_{xy}^o \int (a_{33}\, Cos^2\theta)\, hds + \gamma_{xz}^o \int (a_{33}\, Sin\theta\, Cos\theta)\, hds - \kappa_y^o \int (a_{31}yCos\theta)\, hds$$
$$+\kappa_z^o \int (a_{31}z\, Cos\theta)\, hds - \kappa_t^o \int (2b_{33}hCos\theta)\ hds - \psi^o \int (a_{31}\omega - b_{31}qh)Cos\theta\, hds$$

$$(3.12f)$$

$$Q_z = \int_c N_{sx}\, Sin\theta\, ds$$

$$\Rightarrow \varepsilon_x^o \int (a_{31}\, Sin\theta)\, hds + \gamma_{xy}^o \int (a_{33}\, Sin\theta\, Cos\theta)\, hds + \gamma_{xz}^o \int (a_{33}\, Sin^2\theta)\, hds - \kappa_y^o \int (a_{31}ySin\theta)\, hds$$

$$+\kappa_z^o \int (a_{31}z\, Sin\theta)\, hds - \kappa_t^o \int (2b_{33}hSin\theta)\ hds - \psi^o \int (a_{31}\omega - b_{31}qh)Sin\theta\, hds$$

$$(3.12g)$$

The above relations are expressed in matrix form,

$$\begin{Bmatrix} N \\ Q_y \\ Q_z \\ M_z \\ M_y \\ M_t \\ B_\omega \end{Bmatrix} = [C] \begin{Bmatrix} \varepsilon_x^o \\ \gamma_{xy}^o \\ \gamma_{xz}^o \\ \kappa_y^o \\ \kappa_z^o \\ \kappa_t^o \\ \psi^o \end{Bmatrix} \qquad (3.13)$$

$[C]$ is the constitutive equation matrix for a beam

$$[C]=\begin{bmatrix} \int a_{11} & \int a_{13}c & \int a_{13}s & -\int a_{13}y & \int a_{13}z & -\int 2b_{13}h & -\int (a_{11}\omega - b_{11}qh) \\ & \int a_{33}c^2 & \int a_{33}sc & -\int a_{13}yc & \int a_{13}zc & -\int 2b_{33}hc & -\int (a_{13}\omega - b_{13}qh)c \\ & & \int a_{33}s^2 & -\int a_{13}ys & \int a_{13}zs & -\int 2b_{33}hs & -\int (a_{13}\omega + b_{13}hq)s \\ & & & \int (a_{11}y^2 - b_{11}yhs) & \int (a_{11}yz - b_{11}yhs) & 2\int (b_{13}y - d_{13}hs)h & \int (a_{11}\omega\, y - b_{11}\omega\, hs \\ & & & & & & -b_{11}yqh + d_{11}qh^2 s) \\ & & Symm & & \int (a_{11}z^2 + b_{11}zhc) & -2\int (b_{13}z + d_{13}hc)h & \int (a_{11}\omega\, z + b_{11}\omega\, hc \\ & & & & & & -b_{11}zqh - d_{11}qh^2 c) \\ & & & & & 4\int d_{33}h^2 & -4\int b_{13}\omega\, h \\ & & & & & & \int (a_{11}\omega^2 - d_{11}q^2h^2) \end{bmatrix}$$

where, $\int X = \int_c X\ hds$

FINITE ELEMENT FORMULATION

A modified form of Hellinger-Reissner mixed variational principle is used in the formulation of this thin-walled beam element [2]. The important steps in the derivation of the stiffness matrices are summarized. The modified functional is -

$$\Pi = \Pi_{HR} + \int_0^l \lambda(\partial\varphi_x^o - \theta^o)\, dx - \frac{1}{2\varepsilon}\int_0^l \lambda^2\, dx \tag{4.1}$$

$$\Pi_{HR} = \int_0^l (V - U^c)\, dx \tag{4.2}$$

Π_{HR} is the functional from the Hellinger-Reissner mixed variational principle. The second term containing λ, which is the Lagrangian parameter, enforces the constraint condition on the rotation degree of freedom. The last term with the penalty parameter ε, is a regularization term.

The beam strain-displacement relations are,

$$\begin{aligned}
\varepsilon_x^o &= \partial u^o & \kappa_y^o &= \partial\varphi_z^o & \kappa_z^o &= \partial\varphi_y^o \\
\gamma_{xy}^o &= \partial v^o - \varphi_z^o & \gamma_{xz}^o &= \partial w^o + \varphi_y^o & & \\
\kappa_t^o &= \partial\varphi_x^o & \psi^o &= \partial\theta^o & & \qquad ..(4.3)
\end{aligned}$$

The relationship between internal forces and strain components is derived as follows.

$$\begin{Bmatrix} N_x \\ M_z \\ M_y \\ B_\omega \end{Bmatrix} = \sigma_x \int \begin{Bmatrix} 1 \\ -y \\ z \\ \omega \end{Bmatrix} \mathrm{dA} = \mathrm{E} \int \begin{Bmatrix} 1 \\ -y \\ z \\ \omega \end{Bmatrix} \begin{bmatrix} 1 & -\mathrm{y} & \mathrm{z} & \omega \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \kappa_y^o \\ \kappa_z^o \\ \psi^o \end{Bmatrix} \mathrm{dA}$$

$$\begin{Bmatrix} N_x \\ M_z \\ M_y \\ B_\omega \end{Bmatrix} = \mathrm{E} \begin{bmatrix} \mathrm{A} & -\mathrm{S_z} & \mathrm{S_y} & \mathrm{S}_\omega \\ & \mathrm{I_z} & -\mathrm{I_{yz}} & -\mathrm{S}_{\omega z} \\ \text{Symm} & & \mathrm{I_y} & \mathrm{S}_{\omega y} \\ & & & \mathrm{I}_\omega \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \kappa_y^o \\ \kappa_z^o \\ \psi^o \end{Bmatrix} \tag{4.4}$$

$$\begin{Bmatrix} Q_y \\ Q_z \\ M_t \end{Bmatrix} = \int \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\mathrm{z} & \mathrm{y} \end{bmatrix} \begin{Bmatrix} \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix} \mathrm{dA} = \mathrm{G} \begin{bmatrix} \mathrm{A} & 0 & -\mathrm{S_y} \\ 0 & \mathrm{A} & \mathrm{S_z} \\ -\mathrm{S_y} & \mathrm{S_z} & J \end{bmatrix} \begin{Bmatrix} \gamma_{xy}^o \\ \gamma_{xz}^o \\ \kappa_t^o \end{Bmatrix} \tag{4.5}$$

The expression for total strain energy is,

$$V = \begin{Bmatrix} N_x \\ M_z \\ M_y \\ B_\omega \end{Bmatrix}^T \begin{Bmatrix} \varepsilon_x^o \\ \kappa_y^o \\ \kappa_z^o \\ \psi^o \end{Bmatrix} + \begin{Bmatrix} Q_y \\ Q_z \\ M_t \end{Bmatrix}^T \begin{Bmatrix} \gamma_{xy}^o \\ \gamma_{xz}^o \\ \kappa_t^o \end{Bmatrix}$$

$$\Rightarrow \; V = \begin{Bmatrix} N_x \\ M_z \\ M_y \\ B_\omega \end{Bmatrix}^T \begin{bmatrix} \partial & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial \end{bmatrix} \begin{Bmatrix} u^o \\ v^o \\ w^o \\ \phi_z^o \\ \phi_y^o \\ \phi_x^o \\ \theta^o \end{Bmatrix}$$

$$+ \begin{Bmatrix} Q_y \\ Q_z \\ M_t \end{Bmatrix}^T \begin{bmatrix} 0 & \partial & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \partial & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial & 0 \end{bmatrix} \begin{Bmatrix} u^o \\ v^o \\ w^o \\ \phi_z^o \\ \phi_y^o \\ \phi_x^o \\ \theta^o \end{Bmatrix}$$

$$\Rightarrow V = \begin{Bmatrix} N_x \\ Q_y \\ Q_z \\ M_z \\ M_y \\ M_t \\ B_\omega \end{Bmatrix}^T \begin{bmatrix} \partial & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \partial & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \partial & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial \end{bmatrix} \begin{Bmatrix} u^o \\ v^o \\ w^o \\ \phi_z^o \\ \phi_y^o \\ \phi_x^o \\ \theta^o \end{Bmatrix} \tag{4.6}$$

The expression for complementary strain energy is,

$$U^c = \frac{1}{2E} \begin{Bmatrix} N_x \\ M_z \\ M_y \\ B_\omega \end{Bmatrix}^T \begin{Bmatrix} \varepsilon_x^o \\ \kappa_y^o \\ \kappa_z^o \\ \psi^o \end{Bmatrix} + \frac{1}{2G} \begin{Bmatrix} Q_y \\ Q_z \\ M_t \end{Bmatrix}^T \begin{Bmatrix} \gamma_{xy}^o \\ \gamma_{xz}^o \\ \kappa_t^o \end{Bmatrix}$$

$$\Rightarrow U^c = \frac{1}{2E} \begin{Bmatrix} N_x \\ M_z \\ M_y \\ B_\omega \end{Bmatrix}^T \begin{bmatrix} \mathrm{A} & -\mathrm{S_y} & \mathrm{S_z} & -\mathrm{S}_\omega \\ & \mathrm{I_z} & -\mathrm{I_{yz}} & \mathrm{I}_{\omega z} \\ \text{Symm} & & \mathrm{I_y} & -\mathrm{I}_{\omega y} \\ & & & \mathrm{I}_\omega \end{bmatrix}^{-1} \begin{Bmatrix} N_x \\ M_z \\ M_y \\ B_\omega \end{Bmatrix}$$

$$+ \frac{1}{2G} \begin{Bmatrix} Q_y \\ Q_z \\ M_t \end{Bmatrix}^T \begin{bmatrix} \mathrm{A} & 0 & -\mathrm{S_y} \\ 0 & \mathrm{A} & \mathrm{S_z} \\ -\mathrm{S_y} & \mathrm{S_z} & J \end{bmatrix}^{-1} \begin{Bmatrix} Q_y \\ Q_z \\ M_t \end{Bmatrix}$$

$$\Rightarrow U^c = \frac{1}{2}\begin{Bmatrix} N_x \\ Q_y \\ Q_z \\ M_z \\ M_y \\ M_t \\ B_\omega \end{Bmatrix}^T \begin{bmatrix} \mathrm{E\,A} & 0 & 0 & -\mathrm{E\,S_z} & \mathrm{E\,S_y} & 0 & -\mathrm{E\,S_\omega} \\ & \mathrm{G\,A} & 0 & 0 & 0 & -\mathrm{G\,S_y} & 0 \\ & & \mathrm{G\,A} & 0 & 0 & \mathrm{G\,S_z} & 0 \\ & & & \mathrm{E\,I_z} & -\mathrm{E\,I_{yz}} & 0 & \mathrm{E\,I_{\omega z}} \\ \text{Symm} & & & & \mathrm{E\,I_y} & 0 & -\mathrm{E\,I_{\omega y}} \\ & & & & & \mathrm{G\,J} & 0 \\ & & & & & & \mathrm{E\,I_\omega} \end{bmatrix}^{-1} \begin{Bmatrix} N_x \\ Q_y \\ Q_z \\ M_z \\ M_y \\ M_t \\ B_\omega \end{Bmatrix} \quad (4.7)$$

The second and the third terms in the functional given in (4.1) can be expressed as,

$$\int_0^l \{\lambda\}\left(\partial\phi_x^o - \theta^o\right)dx = \int_0^l \begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial & -1 \end{bmatrix} \begin{Bmatrix} u^o \\ v^o \\ w^o \\ \phi_z^o \\ \phi_y^o \\ \phi_x^o \\ \theta^o \end{Bmatrix} dx \qquad (4.8)$$

and

$$\int_0^l \{\lambda\}^2\, dx = \int_0^l \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix} dx \qquad (4.9)$$

Substituting the expressions (4.6),(4.7),(4.8),&(4.9) in (4.1) we get,

$$\Pi = \Pi_{HR} + \int_0^l \lambda(\partial\phi_x^o - \theta^o)\, dx - \frac{1}{2}\varepsilon\int_0^l \lambda^2\, dx$$

$$= \{H\}^T[S]\{X\} - \frac{1}{2}\{H\}^T[F]\{H\} + \{\lambda\}^T[Q]\{X\} - \frac{1}{2\varepsilon}\{\lambda\}^T[P]\{\lambda\} \qquad (4.10)$$

Internal force vector, $\{H\}^T = \lfloor \mathrm{N_{xi}} \quad \mathrm{Q_{yi}} \quad \mathrm{Q_{zi}} \quad \mathrm{M_{yi}} \quad \mathrm{M_{zi}} \quad \mathrm{M_{xi}} \quad \mathrm{B_{\omega i}} \rfloor$

Displacement vector, $\{X\}^T = \lfloor \mathrm{u_i^o} \quad \mathrm{v_i^o} \quad \mathrm{w_i^o} \quad \phi_{zi}^o \quad \phi_{\mathrm{yi}}^o \quad \phi_{xi}^o \quad \theta_i^o \rfloor$

Vector of Lagrange parameters, $\{\lambda\}^T = \lfloor \lambda_{1i} \quad \lambda_{2i} \rfloor$

Strain-displacement matrix, $[S] = \int_0^l N_i' \begin{bmatrix} \partial & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \partial & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \partial & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial \end{bmatrix} N_j \, dx$

Flexibility Matrix,

$$[F] = \int_0^l N_i' N_j' \begin{bmatrix} \mathrm{E\,A} & 0 & 0 & -\mathrm{E\,S_z} & \mathrm{E\,S_y} & 0 & -\mathrm{E\,S_\omega} \\ & \mathrm{G\,A} & 0 & 0 & 0 & -\mathrm{G\,S_y} & 0 \\ & & \mathrm{G\,A} & 0 & 0 & \mathrm{G\,S_z} & 0 \\ & & & \mathrm{E\,I_z} & -\mathrm{E\,I_{yz}} & 0 & \mathrm{E\,I_{\omega z}} \\ \text{Symm} & & & & \mathrm{E\,I_y} & 0 & -\mathrm{E\,I_{\omega y}} \\ & & & & & \mathrm{G\,J} & 0 \\ & & & & & & \mathrm{E\,I_\omega} \end{bmatrix}^{-1} dx$$

$$[Q] = \int N_i' \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial & -1 \end{bmatrix} N_j \, dx$$

$$[P] = \int_0^l N_i' \quad N_j' \, dx$$

N_i' are the stress shape functions, and N_i are the displacement shape functions.

$$N_1 = 1 - \frac{x}{l} \qquad N_2 = \frac{x}{l} \qquad N_1' = 1$$

Stress continuity is not enforced at the element boundary. The value of the penalty parameter ε in this study is taken to be equal to the product EA. It must be noted that since this is a one-dimensional beam element all loads on the beam are assumed to be applied at the centroid of the cross section.

The expression (4.10) is a variational functional in three field variables $\{H\}^T, \{X\}^T$, and $\{\lambda\}^T$. Minimizing this expression w.r.t $\{H\}^T, \{X\}^T$, and $\{\lambda\}^T$ we get,

$$\frac{\partial \Pi}{\partial \{H\}^T} = 0 \Rightarrow [S]\{X\} = [F]\{H\} \tag{4.11}$$

$$\frac{\partial \Pi}{\partial \{X\}^T} = 0 \Rightarrow \{H\}^T[S] = \{\lambda\}^T[Q] \tag{4.12}$$

$$\frac{\partial \Pi}{\partial \{\lambda\}^T} = 0 \Rightarrow [Q]\{X\} = \frac{-1}{\varepsilon}[P]\{\lambda\} \tag{4.13}$$

$$-\varepsilon[P]^{-1}[Q]\{X\} = \{\lambda\} \tag{4.14}$$

$$[F]^{-1}[S]\{X\} = \{H\} \tag{4.15}$$

$$[S]^T\{H\} = [Q]^T\{\lambda\} \tag{4.16}$$

Eliminating the unknowns $\{H\}$ and $\{\lambda\}$ from (4.14),(4.15),&(4.16) we get the expression for element stiffness matrix as, $[k] = ([S]^T[F]^{-1}[S] + \varepsilon[Q]^T[P]^{-1}[Q])$ (4.17)

The finite element equation for the beam element can now be written as $[k]\{X\} = \{f\}$.

The expression (4.17) is rearranged in the form,

$$[k] = [S]^{T}_{\text{mod}} [F]^{-1}_{\text{mod}} [S]_{\text{mod}} \tag{4.18}$$

where,

$$[F]_{\text{mod}} = \int N_i^{'} \quad N_j^{'} \begin{bmatrix} EA & 0 & 0 & -ES_z & ES_y & 0 & -ES_\omega & 0 & 0 \\ 0 & GA & 0 & 0 & 0 & -GS_y & 0 & 0 & 0 \\ 0 & 0 & GA & 0 & 0 & GS_z & 0 & 0 & 0 \\ -ES_z & 0 & 0 & EI_z & -EI_{yz} & 0 & EI_{\omega z} & 0 & 0 \\ ES_y & 0 & 0 & EI_{yz} & EI_y & 0 & -EI_{\omega y} & 0 & 0 \\ 0 & -GS_y & GS_z & 0 & 0 & GJ & 0 & 0 & 0 \\ ES_\omega & 0 & 0 & EI_{\omega z} & -EI_{\omega y} & 0 & EI_\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon \end{bmatrix}^{-1} dx$$

$$[S]_{\text{mod}} = \int N_i^{'} \begin{bmatrix} \partial & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \partial & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \partial & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial & -1 \end{bmatrix} N_j \, dx$$

DYNAMIC RESPONSE

The governing equation for dynamic response of the structure/medium is

$$[m]\{\ddot{d}\} + [c]\{\dot{d}\} + \{r^{\text{int}}\} = \{r^{ext}\} \tag{5.1}$$

$[m]$ and $[c]$ are the element mass and damping matrices respectively.

$\{r^{\text{int}}\}$ is the element internal force vector.

$\{r^{ext}\}$ is the element external force vector.

$\{\dot{d}\}$ & $\{\ddot{d}\}$ are the vectors of nodal velocities and accelerations respectively.

5.1 Explicit Direct Time Integration

Generally, in direct time integration the dynamic equation of motion for a system is written as,

$$[M]\{\ddot{D}\}_n + [C]\{\dot{D}\}_n + \{R^{\text{int}}\}_n = \{R^{ext}\}_n \tag{5.2}$$

The subscript n denotes the value at time $n\Delta t$, Δt being the time increment. If the material is linear elastic, the internal force vector at any instant can be evaluated using the stiffness matrix as $[K]\{D\}_n$. However in explicit time integration the internal force vector is generally obtained by calculating the strains and thereby stresses, from the nodal displacements.

Central Difference scheme of explicit time integration is used in the present study. The method approximates the velocity and acceleration by a truncated Taylor series. Neglecting the higher order terms from the Taylor series approximation of $\{\dot{D}\}_{n+1}$ and $\{\ddot{D}\}_{n+1}$,

$$\{\dot{D}\}_n = \frac{1}{2\ \Delta t}(\{D\}_{n+1} - \{D\}_{n-1}) \tag{5.3}$$

$$\{\ddot{D}\}_n = \frac{1}{\Delta t^2}(\{D\}_{n+1} - 2\{D\}_n + \{D\}_{n-1}) \tag{5.4}$$

Substituting approximations $(5.3)\&(5.4)$ in (5.2), the expression for displacement at the next time instant is,

$$(\frac{1}{\Delta t^2}[M] + \frac{1}{2\ \Delta t}[C])\{D\}_{n+1} = \{R^{ext}\}_n - \{R^{\text{int}}\}_n + \frac{1}{\Delta t^2}[M](2\{D\}_n - \{D\}_{n-1}) + \frac{1}{2\ \Delta t}[C]\{D\}_{n-1} \tag{5.5}$$

If damping is neglected this equation simplifies to,

$$\frac{1}{\Delta t^2}[M]\{D\}_{n+1} = \{R^{ext}\}_n - \{R^{\text{int}}\}_n + \frac{1}{\Delta t^2}[M](2\{D\}_n - \{D\}_{n-1})$$

This expression can further be simplified as given below if a diagonal mass matrix is used.

$$D^i_{n+1} = \frac{\Delta t^2}{M_{ii}}(R^{ext\,i}_n - R^{\text{int}\,i}_n) + \frac{1}{\Delta t^2}(2D^i_n - D^i_{n-1}) \tag{5.6}$$

the superscript i refers to the i^{th} d.o.f.

The above equation requires that the condition $\Delta t \leq 2/\omega_{\max}$, be satisfied for the solution to be stable. $\omega_{\max}$ is the highest natural frequency of $\det([K] - \omega^2[M]) = 0$.

5.2 Mass Matrix

From the equation for dynamic response of a system the mass matrix is computed from the work done by the inertia forces,

$$W_{inertia} = \int_V \{\delta U\}^T \rho \{\ddot{U}\} dV \tag{5.7}$$

where,

$$\{U\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [L] \begin{Bmatrix} u^o \\ v^o \\ w^o \\ \phi_z^o \\ \phi_y^o \\ \phi_x^o \\ \theta^o \end{Bmatrix}$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 & -y & z & 0 & -\omega \\ 0 & 1 & 0 & 0 & 0 & -z & 0 \\ 0 & 0 & 1 & 0 & 0 & y & 0 \end{bmatrix}$$

The mass matrix defined above is the consistent mass matrix because the shape functions used to obtain a discrete representation of the mass are the same as the displacement shape functions. Now,

$$\{U\} = [L][N]\{X\}$$

$$\Rightarrow \begin{Bmatrix} \ddot{U} \end{Bmatrix} = [L][N]\{\ddot{X}\}$$

$$\Rightarrow \{\delta U\}^T = \{\delta X\}^T [N]^T [A]^T$$

$$\Rightarrow [m] = \{\delta X\}^T \int_V [N]^T [L]^T \rho [L][N] dV \{\ddot{X}\}$$

ρ is the mass density of the material, and V the volume of the element. $[N]$ is the shape function matrix.

$$[L]^T[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -y & 0 & 0 \\ z & 0 & 0 \\ 0 & -z & y \\ -\omega & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -y & z & 0 & -\omega \\ 0 & 1 & 0 & 0 & 0 & -z & 0 \\ 0 & 0 & 1 & 0 & 0 & y & 0 \end{bmatrix}$$

$$\Rightarrow [L]^T[L] = \begin{bmatrix} 1 & 0 & 0 & -y & z & 0 & -\omega \\ 0 & 1 & 0 & 0 & 0 & -z & 0 \\ 0 & 0 & 1 & 0 & 0 & y & 0 \\ -y & 0 & 0 & y^2 & -yz & 0 & \omega y \\ z & 0 & 0 & -yz & z^2 & 0 & -\omega z \\ 0 & -z & y & 0 & 0 & z^2+y^2 & 0 \\ -\omega & 0 & 0 & \omega y & -\omega z & 0 & \omega^2 \end{bmatrix}$$

$$\Rightarrow [m] = \int_0^l [N]^T \rho \left(\int_A [L]^T[L]\, dA \right) [N]\, dx$$

Let the integral over the area of the cross section of the beam be expressed as,

$$[Z] = \int_A [L]^T[L]\, dA = \begin{bmatrix} A & 0 & 0 & -S_z & S_y & 0 & -S_\omega \\ 0 & A & 0 & 0 & 0 & -S_y & 0 \\ 0 & 0 & A & 0 & 0 & S_z & 0 \\ -S_z & 0 & 0 & I_z & -I_{yz} & 0 & I_{\omega z} \\ S_y & 0 & 0 & -I_{yz} & I_y & 0 & -I_{\omega y} \\ 0 & -S_y & S_z & 0 & 0 & I_y + I_z & 0 \\ -S_\omega & 0 & 0 & I_{\omega z} & -I_{\omega y} & 0 & I_\omega \end{bmatrix}$$

The final form of the consistent mass matrix is,

$$[m] = \rho \int_0^l N_i \, N_j \, [Z] \, dx \tag{5.8}$$

Lumped Mass Matrix

The lumped mass matrix of the beam element for central difference scheme has been derived using the HRZ lumping scheme [5]. Outline of the steps followed is as follows:

1. Compute the consistent mass matrix
2. Compute the total mass of the element
3. Sum the diagonal coefficients of the consistent mass matrix associated with translational d.o.f only.
4. Compute a scale factor defined as the ratio of the total mass to the sum as calculated in step 3
5. Multiply all diagonal coefficients of the consistent mass matrix by this scale factor and assign zero to all other coefficients

5.3 Internal Force Vector

Internal work due to straining of the material is $\{W^{\text{int}}\} = \int_V [\delta\varepsilon]^T \{\sigma\} \, dV$ (5.9)

$[\delta\varepsilon]^T$ is the strain in the element due to any arbitrary displacement $\{\delta d\}$, and $\{\sigma\}$ is the resulting stress vector. The stress-strain relation for an isotropic material and the strain-displacement relation are given by (5.10) & (5.11) below,

$$\{\sigma\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} E & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{xy} \\ \varepsilon_{xz} \end{Bmatrix} = [C]\{\varepsilon\} \tag{5.10}$$

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{xy} \\ \varepsilon_{xz} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -y & z & 0 & -\omega \\ 0 & 1 & 0 & 0 & 0 & -z & 0 \\ 0 & 0 & 1 & 0 & 0 & y & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \gamma_{xy}^o \\ \gamma_{xz}^o \\ \kappa_y^o \\ \kappa_z^o \\ \kappa_t^o \\ \psi^o \end{Bmatrix} \tag{5.11}$$

$$\begin{Bmatrix} \varepsilon_x^o \\ \gamma_{xy}^o \\ \gamma_{xz}^o \\ \kappa_y^o \\ \kappa_z^o \\ \kappa_t^o \\ \psi^o \end{Bmatrix} = \begin{bmatrix} \partial & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \partial & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \partial & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial \end{bmatrix} \begin{Bmatrix} u^o \\ v^o \\ w^o \\ \phi_z^o \\ \phi_y^o \\ \phi_x^o \\ \theta^o \end{Bmatrix}$$

$$\begin{Bmatrix} u^o \\ v^o \\ w^o \\ \phi_z^o \\ \phi_y^o \\ \phi_x^o \\ \theta^o \end{Bmatrix} = [N]\{d\}$$

$$\{d\} = \begin{bmatrix} u_i^o & v_i^o & w_i^o & \phi_{zi}^o & \phi_{yi}^o & \phi_{ti}^o & \theta_i^o \end{bmatrix}^T$$

The subscript i indicates the value of that d.o.f at the i^{th} node.

The following notation is introduced,

$$[L] = \begin{bmatrix} 1 & 0 & 0 & -y & z & 0 & -\omega \\ 0 & 1 & 0 & 0 & 0 & -z & 0 \\ 0 & 0 & 1 & 0 & 0 & y & 0 \end{bmatrix}$$

$$[I] = \begin{bmatrix} \partial & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \partial & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \partial & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial \end{bmatrix}$$

Expressing strain vector in (5.11) using the above notation,

$$\{\varepsilon\} = [L][I][N]\{d\}$$

$$[B] = [I][N]$$

$$\{\varepsilon\} = [L][B]\{d\}$$

$$\Rightarrow \{W^{\text{int}}\} = \{\delta d\}^T \int_V [B]^T [L]^T [C][L][B]\{d\}\, dV$$

From this the internal force can be expressed as,

$$\{r^{\text{int}}\} = \int_V [B]^T [L]^T \quad [C] \quad [L][B] \quad \{d\} \quad dV$$

$$= \int_l [B]^T \quad (\int_A [L]^T [C][L] \quad dA) \quad [B] \quad \{d\} \quad dx$$

Evaluating the integral over the cross sectional area,

$$\int_A [L]^T [C][L]\, dA = \begin{bmatrix} EA & 0 & 0 & -ES_z & ES_y & 0 & -ES_\omega \\ 0 & GA & 0 & 0 & 0 & -GS_y & 0 \\ 0 & 0 & GA & 0 & 0 & GS_z & 0 \\ -ES_z & 0 & 0 & EI_z & -EI_{yz} & 0 & EI_{\omega z} \\ ES_y & 0 & 0 & -EI_{yz} & EI_y & 0 & -EI_{\omega y} \\ 0 & -GS_y & GS_z & 0 & 0 & G(I_y + I_z) & 0 \\ -ES_\omega & 0 & 0 & EI_{\omega z} & -EI_{\omega y} & 0 & EI_\omega \end{bmatrix}$$

$$\{r^{\text{int}}\} = \int_l [B]^T \begin{bmatrix} EA & 0 & 0 & -ES_z & ES_y & 0 & -ES_\omega \\ 0 & GA & 0 & 0 & 0 & -GS_y & 0 \\ 0 & 0 & GA & 0 & 0 & GS_z & 0 \\ -ES_z & 0 & 0 & EI_z & -EI_{yz} & 0 & EI_{\omega z} \\ ES_y & 0 & 0 & -EI_{yz} & EI_y & 0 & -EI_{\omega y} \\ 0 & -GS_y & GS_z & 0 & 0 & G(I_y + I_z) & 0 \\ -ES_\omega & 0 & 0 & EI_{\omega z} & -EI_{\omega y} & 0 & EI_\omega \end{bmatrix} [B]\{d\}\, dx \quad (5.12)$$

Enforcing the constraint condition on the twist d.o.f, and introducing the penalty parameter, the $[B]$ matrix and the internal force vector are modified as,

$$[B] = \begin{bmatrix} \partial & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \partial & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \partial & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \partial \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial & -1 \end{bmatrix} [N]$$

$$\{r^{\text{int}}\} = \int_l [B]^T \begin{bmatrix} EA & 0 & 0 & -ES_z & ES_y & 0 & ES_\omega & 0 & 0 \\ 0 & GA & 0 & 0 & 0 & -GS_y & 0 & 0 & 0 \\ 0 & 0 & GA & 0 & 0 & GS_z & 0 & 0 & 0 \\ -ES_z & 0 & 0 & EI_z & -EI_{yz} & 0 & -ES_{\omega z} & 0 & 0 \\ ES_y & 0 & 0 & EI_{yz} & EI_y & 0 & ES_{\omega y} & 0 & 0 \\ 0 & -GS_y & GS_z & 0 & 0 & GJ & 0 & 0 & 0 \\ ES_\omega & 0 & 0 & -ES_{\omega z} & ES_{\omega y} & 0 & EI_\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varepsilon \end{bmatrix} [B]\{d\}\, dx$$

$$\rightarrow (5.13)$$

Following the procedure above, except that the $[C]$ matrix for the laminated composites will substitute the $[C]$ matrix for the isotropic case, derives the internal force vector for the laminated composite beams.

FAILURE ANALYSIS

6.1 Tsai-Wu Failure Criterion

Tsai-Wu failure criterion is used to check for first-ply failure in laminated composite beams. The expression to be satisfied for failure to occur is

$$F_{xx}\sigma_x^2 + 2F_{xy}\sigma_x\sigma_y + F_{yy}\sigma_y^2 + F_{ss}\sigma_s^2 + F_x\sigma_x + F_y\sigma_y = 1 \qquad (6.1)$$

The following relations relate the strength parameters to the strength of the material.

$$F_{xx} = \frac{1}{XX'} \qquad F_{yy} = \frac{1}{YY'} \qquad F_{ss} = \frac{1}{S^2}$$

$$F_x = \frac{1}{X} - \frac{1}{X'} \qquad F_y = \frac{1}{Y} - \frac{1}{Y'} \qquad F_{xy} = \frac{-1}{2}\sqrt{F_{xx}F_{yy}}$$

X =Longitudinal Tensile Strength

X' =Longitudinal Compressive Strength

Y =Transverse Tensile Strength

Y' =Transverse Compressive Strength

S =Shear Strength

6.2 Plate Strain - Beam Strain Relations

Once the beam displacements are obtained the beam strains can be evaluated using the beam strain-displacement relation. The membrane strains at the centerline of any segment of the cross section can be obtained using the relations,

$$\varepsilon_x = \varepsilon_x^o + z\,\kappa_z^o - y\,\kappa_y^o - \omega\psi^o$$

$$\gamma_{xs} = \gamma_{xy}^o\, Cos\theta + \gamma_{xz}^o\, Sin\theta$$

$$\kappa_x = -q\frac{\partial^2 \phi_x^o}{\partial s^2} = q\,\psi^o$$

$$\kappa_{xs} = -2\,\kappa_t^o \qquad \text{....6.2}(a-d)$$

Using these strains the off-axis strains in each ply can be evaluated using the relations $(6.3)\&(6.4)$ below,

$$\varepsilon_x = \varepsilon_x^c + \Delta h\,\kappa_x^c \qquad (6.3)$$

$$\gamma_{xs} = \gamma_{sx}^c + \Delta h\,\kappa_{sx}^c \qquad (6.4)$$

$\varepsilon_x^c, \gamma_{sx}^c, \kappa_x^c, \kappa_{sx}^c$ are the strains at the centerline of the membrane of each segment from $6.2(a-d)$ (also refer Fig 7.20). Since from assumptions strain and curvature in the contour directions $\varepsilon_s \,\&\, \kappa_s$ are treated negligible, we have only two strains on each ply $\varepsilon_x \,\&\, \gamma_{sx}$. The stresses in the ply directions can be calculated using the stress-strain relations for the laminate and subsequently failure criterion can be applied to determine the first-ply failure.

NUMERICAL RESULTS

7.1 Validation of isotropic thin-walled beam model

A convergence study of the 2 and 3 node beam elements is first presented with the simple cantilever beam problem. Further four examples are solved and results are compared with the analytical solutions and/or 3D shell model in ANSYS. The shell element used is SHELL63, which has six d.o.f's at each of its four nodes. The material properties are, $E = 6.895\times10^{10}\ Pa$, $\nu = 0.32$, $\rho = 2600\ {Kg}/{m^3}$. The following problems are solved: (a) Clamped I-section beam under static torsional loading at the mid section (b) Free vibrations of a straight cantilever beam of channel section (c) Cantilever beam of Z-section under impulse load (d) Cantilever beam of I-section under impulsive load.

Convergence study

The problem selected is a simple cantilever beam of channel section subjected to a unit tip moment. The cross section of the beam and the loading are shown in Fig 7.1 and 7.2. The analytical solution for the tip rotation is given in [9]. The error in rate of twist is 0.005% and 0.001% for two 2-node and 3-node beam elements respectively.

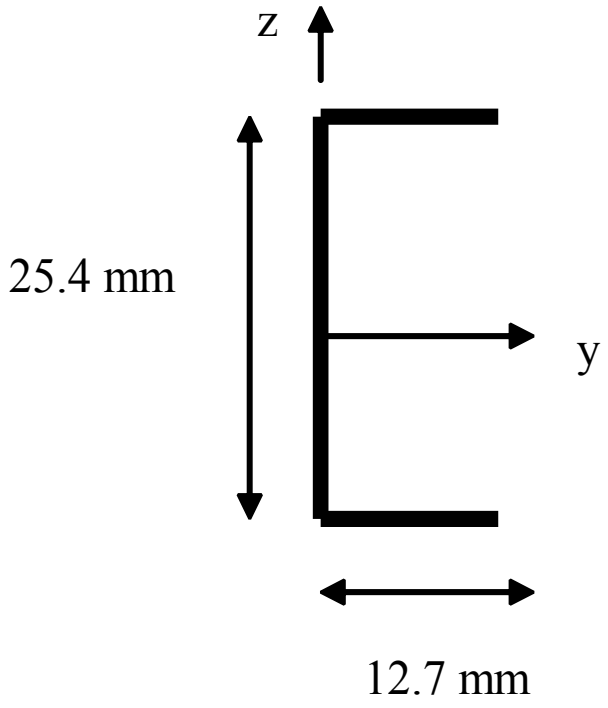


Fig 7.1: C-section geometry

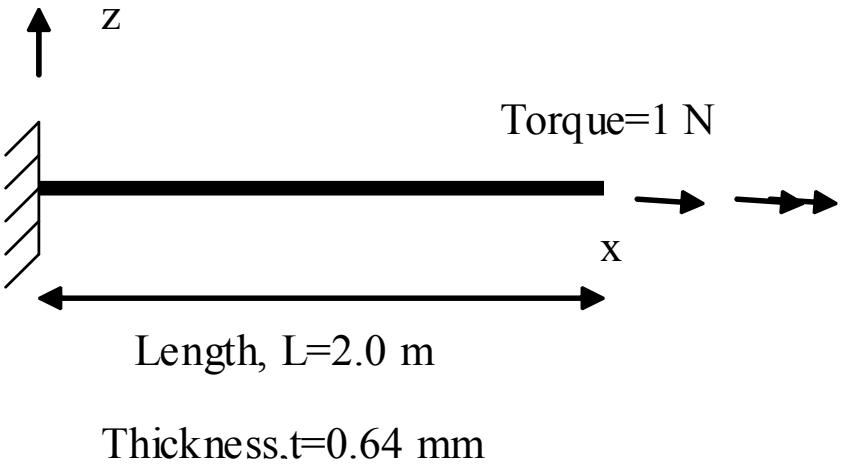


Fig 7.2: Left end fixed, unit torque at the free end

Clamped I-section beam under static torsional loading at the mid span

The cross section of the beam and loading are shown in the Fig 7.3 and 7.4, and the 3D ANSYS shell model used for validation is shown in Fig 7.5, and the deformed shape in Fig 7.6. The longitudinal axis is divided into sixteen 2-node beam elements. In ANSYS the d.o.f's of the nodes at the mid-section are constrained with the d.o.f of the node at the centroid of the mid-span on which a unit torsional moment is applied. Results are presented in Table 7.1.

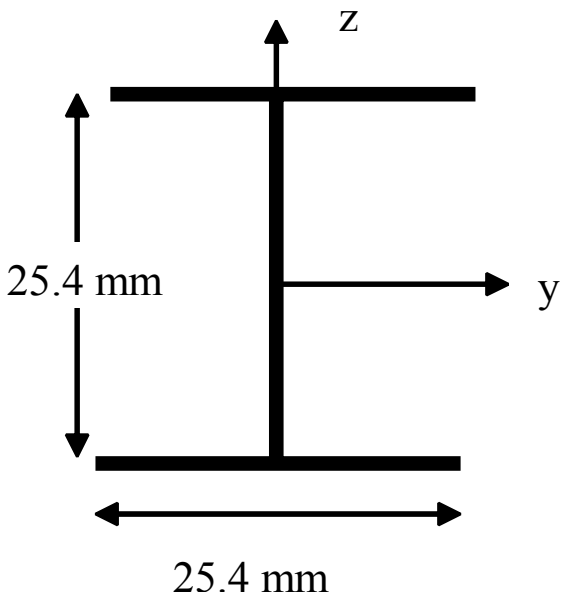
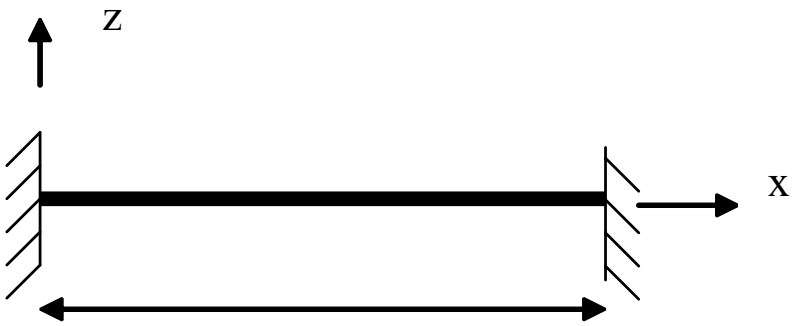


Fig 7.3: I-section geometry



Length, L=1.016 m

Thickness, t=0.635 mm

Fig 7.4: Clamped beam, unit torque at mid- span

Fig 7.5: ANSYS model

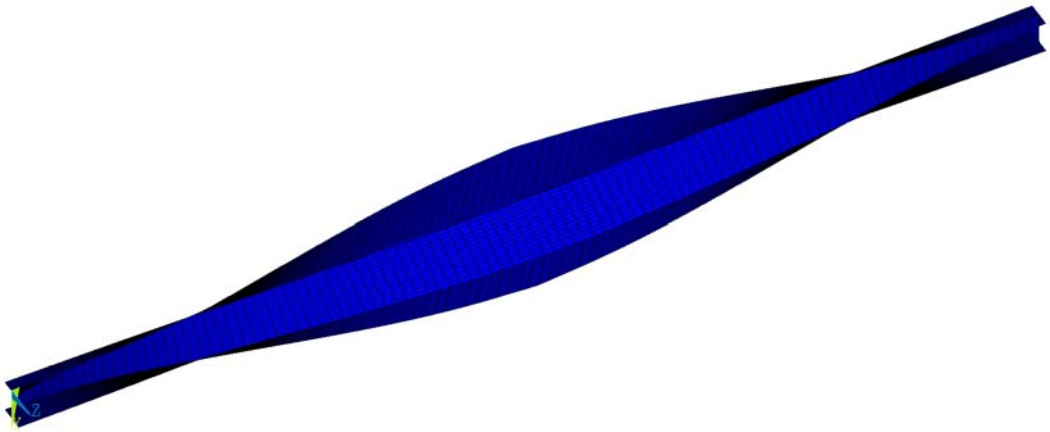


Fig 7.6: Deformed shape the I-section in ANSYS

Table 7.1: The axial twist at the mid-span and the rate of twist at the quarter-span locations obtained from the code, the analytical solution, and ANSYS.

D.o.f	Code	Analytical Solution	ANSYS
Twist at mid-span	0.22761	0.23023	0.23378
Rate of twist	0.67673	0.67347	-

Free vibrations of a straight cantilever beam of channel section

The geometry of the C-section is the same as used for the convergence study, except that the length of the beam is 1.016 m in this case. Sixteen beam elements are used to mesh the longitudinal axis of the C-beam. Table 7.2 lists the first six natural frequencies from the code and the 3D model.

Table 7.2: Minimum six eigenvalues for a symmetric channel section obtained from code and 3D shell model in ANSYS

Mode	Minimum frequencies for a cantilever beam, Hz	
	Ω_{shell}	Ω_{beam}
1	11.454	11.425
2	21.838	21.904
3	42.162	42.260
4	71.623	71.404
5	87.865	88.233
6	187.37	187.86

Cantilever beam of Z-section under dynamic loading

Geometry and loading are shown in Fig 7.7 and 7.8. The 3D shell model and the deformed shape are shown in Fig 7.9 and 7.10. The critical time step is calculated from the maximum natural frequency, and thereafter the stable time step is calculated using a Courant number of 0.966. A transverse unit impulsive load is applied for a time period of 0.05 sec. In the case of implicit time integration performed in ANSYS, the load is applied in two load steps. In the first load step a unit transverse load is applied in $1\mu s$ in one sub-step. In the second load step the same load is applied for 0.05 sec over 100 sub-steps. Fig 7.12 shows the time response due to this load obtained from the code and from the 3D shell model in ANSYS. The geometry of this z-section is such that the centroid and the shear center coincide. Therefore there is no axial twist and warping of the cross section.

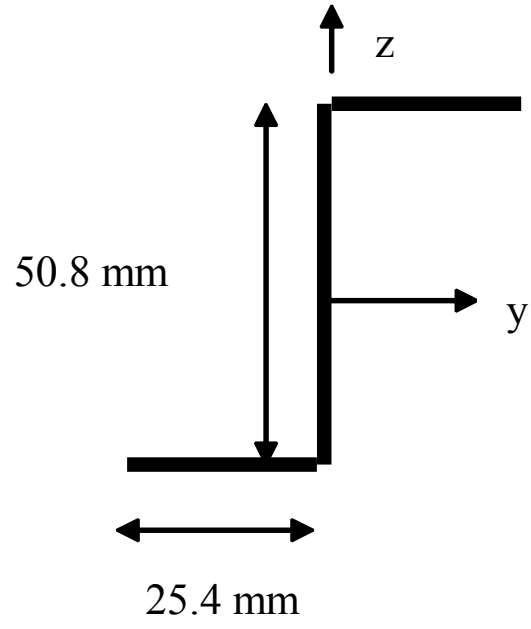


Fig 7.7: Z-section geometry

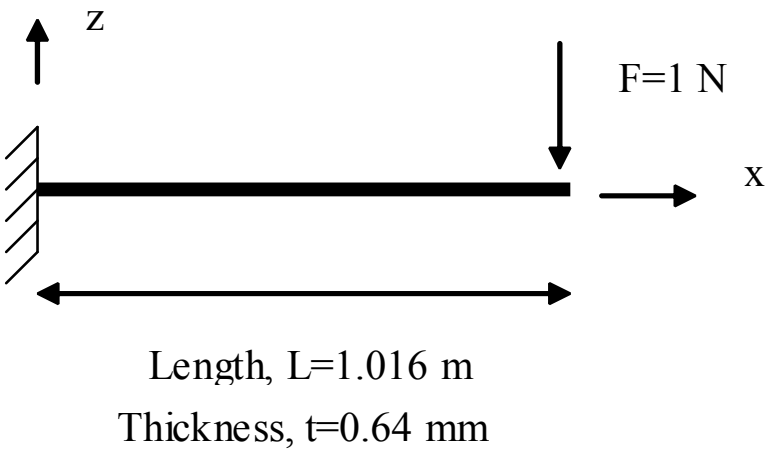


Fig 7.8: Transverse step load in the z direction

Fig 7.9: ANSYS model

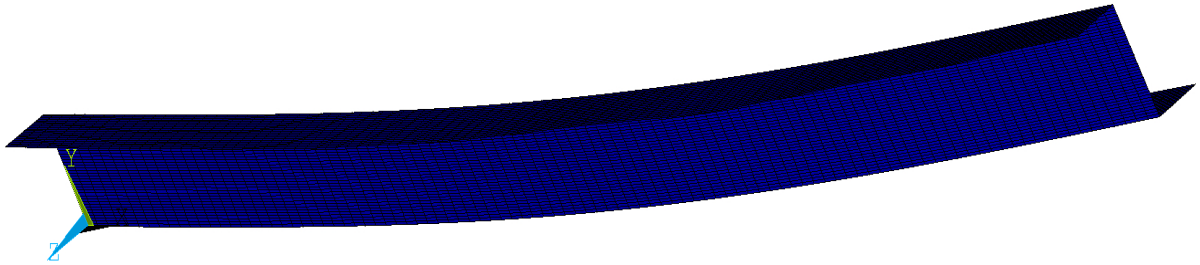


Fig 7.10: Deformed shape of the z-section cantilever beam after transverse impulsive load for 0.05 sec in the z-direction

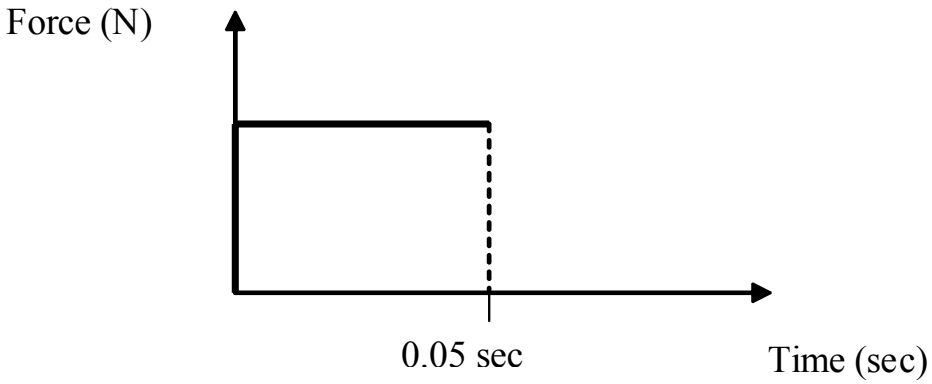


Fig 7.11: Load Vs Time curve for dynamic loading

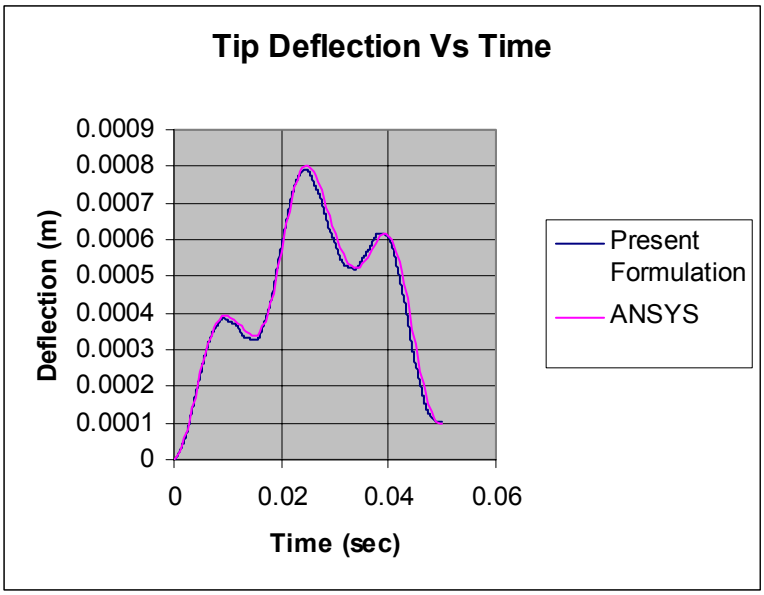


Fig 7.12: Time history of the transverse deflection of the z-beam

Cantilever beam of I-section under dynamic loading

The geometry of the I-section and the 3D shell model in ANSYS are shown in Fig 7.3 and 7.5. A torsional unit impulsive load is applied at the free end for a time period of 0.05 sec. Fig 7.14 shows the time response due to this load obtained from the code and from the 3D shell model in ANSYS. The deformed shape of the beam is shown in the Fig 7.13. This is a case of pure torsion since the centroid coincides with the shear center. All other d.o.f except for axial twist, rate of twist, and axial displacement are zero. Fig 7.15 also shows the time history of the warpage degree of freedom. In the case of the 3D ANSYS model, the time histories of the twist degree of freedom at two nodes- one at the free end of the beam and other at a distance of $\Delta x = 0.0762 \;\; m$ from the free end are collected. Using the expression $\frac{\theta_2 - \theta_1}{\Delta x}$, the warpage at each time step is computed and plotted.

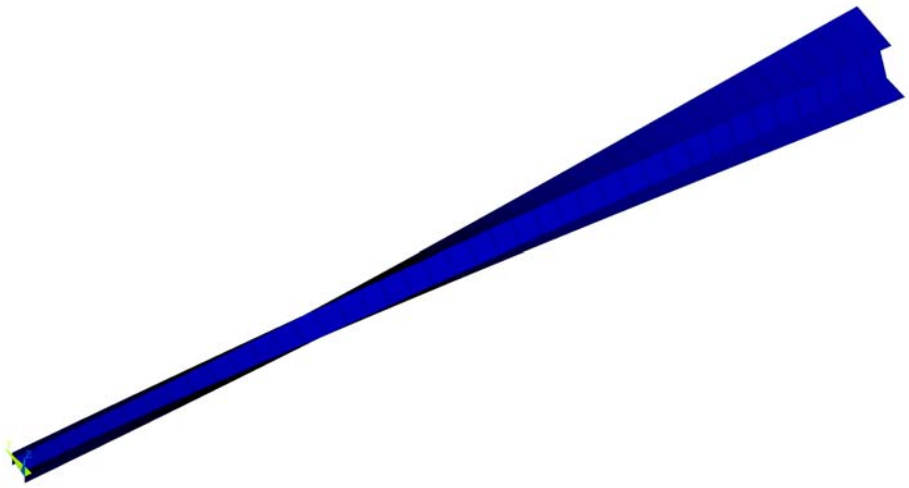


Fig 7.13: Deformed shape of the I-beam under torsional impulsive load

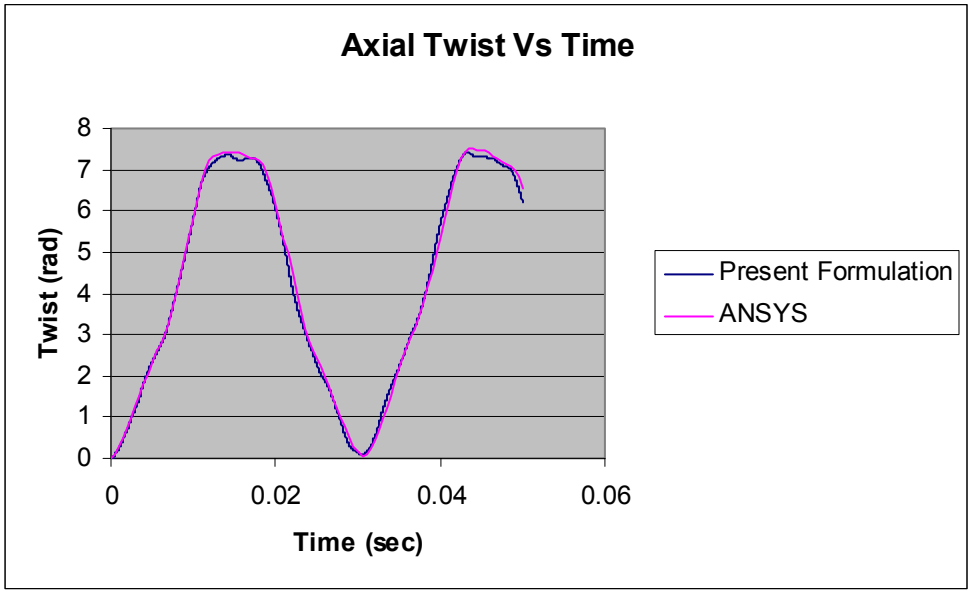


Fig 7.14: Time response of twist d.o.f at the free end

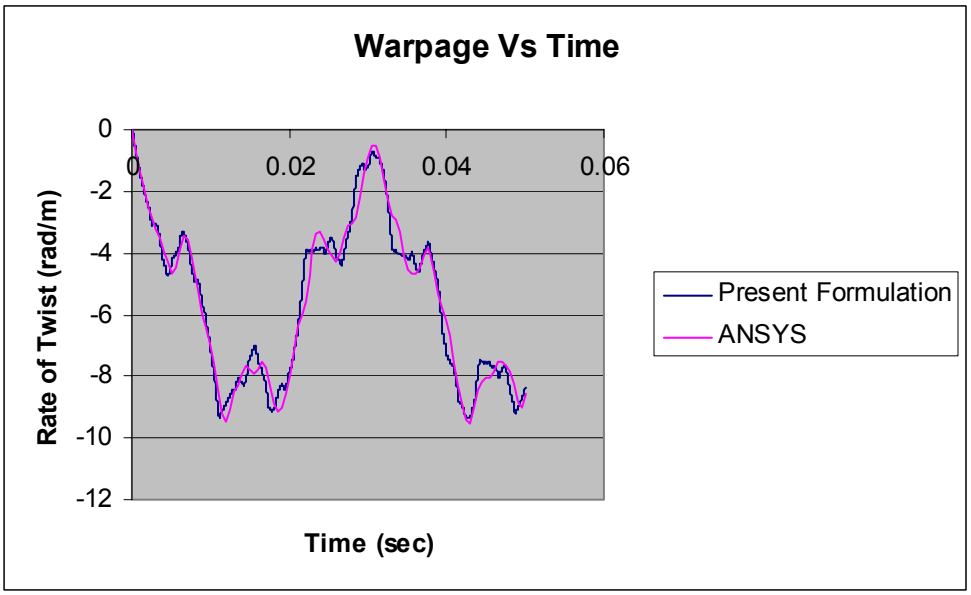


Fig 7.15: Time response of rate of twist d.o.f at the free end from the FORTRAN code

7.2 Validation of laminated composite thin-walled beam model

The formulation for the laminated is implemented in a FORTRAN computer code. Different stacking sequences and cross sections have been experimented with and the results are presented. Solutions are compared with 3D shell models in ANSYS. The linear layered structural shell SHELL99 with 8 nodes and 6 degrees of freedom at each node is used for meshing the model in ANSYS. The material is T300/5208 and the properties are listed in the strength analysis section. The following examples are presented:

1. Cantilever C- beam with a laminate lay-up $[0/90]_T$ under unit static vertical load
2. Free Vibrations of I-section beam with a laminate lay-up $[0/45/-45/90]_S$
3. Cantilever channel beam with a $[0/45/-45/90]_S$ lay-up under impulsive loading
4. Cantilever I-beam with a laminate lay-up $[45/-45]_S$ under impulsive loading
5. Cantilever I-beam with a laminate lay-up $[45/-45]_s$ and $[45/-45]_{5s}$ under unit static vertical load

Cantilever channel section beam under unit vertical load at the free end

The cross section and the load case are shown in Fig 7.1 and 7.16. The beam is divided into 16 elements along the longitudinal $x-$axis. At the free end of the shell model in ANSYS, a point is created at the centroid of the cross section and meshed with a 3D mass element with very low specified inertia properties. By applying the constraint equation, the degrees of freedom of all nodes on the cross section are constrained with the degrees of freedom of the mass element. Since for a C-section the shear center and centroid do not coincide, there will be twist along with the deflection due to the vertical load at the centroid. The results are presented in the Table 7.3.

Table 7.3. Vertical Deflection and twist due to a unit transverse vertical load on a channel section made of a $[0/90]_T$ laminate.

D.O.F	Present formulation	ANSYS
Vertical Deflection (m)	0.002309	0.00233
Axial Twist (rad)	-0.1494	-0.1514

Free Vibrations of a I-beam

Geometry of the I-beam is shown in Fig. 7.3 and the ANSYS model in Fig 7.5. The laminate stacking sequence is quasi-isotropic, i.e. $[0/45/-45/90]_S$. The first six natural frequencies from the shell model and the code are presented in Table 7.4.

Table 7.4. Minimum six eigenvalues for a symmetric I-section with quasi-isotropic lay-up obtained from code and 3D shell model in ANSYS

Mode	Minimum frequencies for a cantilever beam, Hz	
	Ω_{shell}	Ω_{beam}
1	16.801	17.550
2	30.957	32.317
3	31.362	32.775
4	104.96	109.21
5	127.47	133.58
6	193.25	201.85

Cantilever c-section beam under impulsive loading

The geometry and the ANSYS model are those used for the static case. The laminate lay-up is quasi-isotropic. An impulsive transverse load is applied at the free end of the cantilever beam for duration of 0.05 sec. For implicit time integration in ANSYS the load is applied in 2 load steps. In the first load step a unit load is applied at the tip in one sub-step for $1\,\mu s$. Loading in the second load step is for 0.05 sec over 100 sub-steps. In explicit time integration the stable time is obtained by multiplying the critical time step from the maximum eigenvalue by a Courant number 0.966. Fig 7.17 and 7.18 are the plots of deflection Vs time and rate of twist Vs time at the free end. The rate of twist in the ANSYS model is calculated by picking the values of axial twist of 2 nodes along the length of the beam and applying the formula $\frac{\theta_2 - \theta_1}{\Delta x}$, Δx is the distance between the 2 nodes selected.

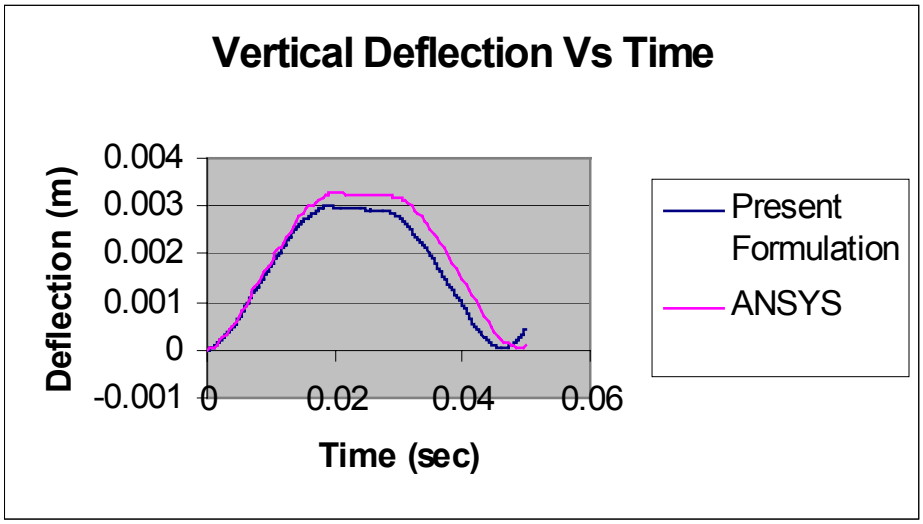


Fig 7.16: Time response of Vertical deflection at the free end from the FORTRAN code and ANSYS

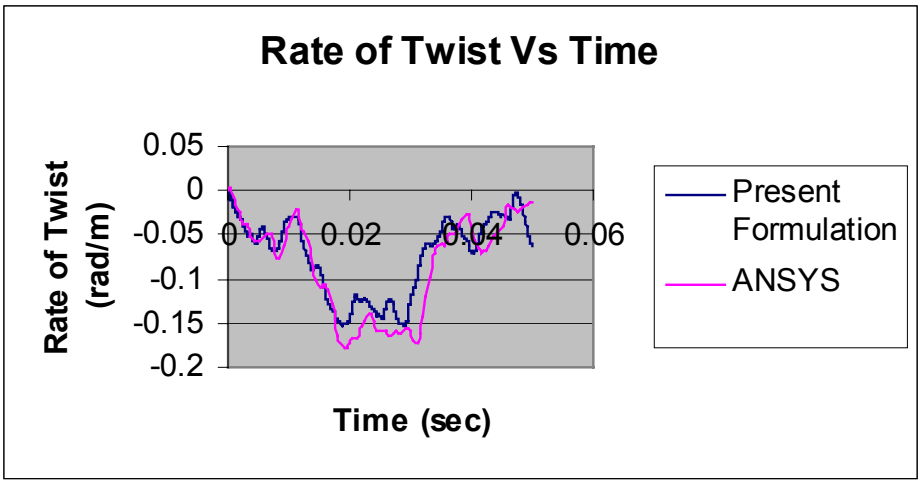


Fig 7.17: Time response of rate of twist from the FORTRAN code and ANSYS

Cantilever I-section beam under impulsive loading

The geometry of the I-section is the same as the one used for the free vibration analysis. The laminate is made up of 4 plies with angle-ply lay-up-one ply at 45^o and other at -45^o. The load in this case is unit impulsive torsional load at the free end. The number of load steps and the duration of each load step are the same as in the above case. Fig 7.19 and 7.20 are the plots of rotational degree of freedom and rate of twist w.r.t time at the tip.

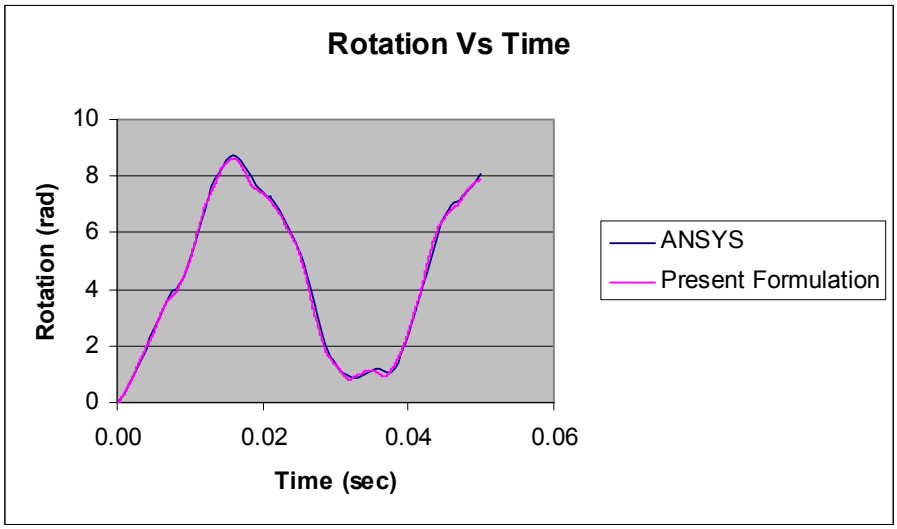


Fig 7.18: Time response of twist at the tip from the FORTRAN code and ANSYS

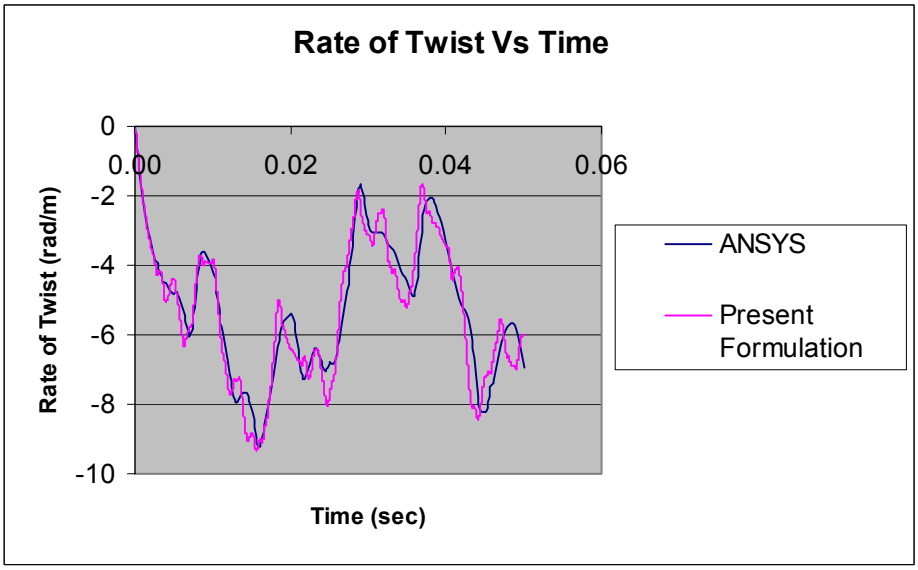


Fig 7.19: Time response of rate of twist from the FORTRAN code and ANSYS

Cantilever I-beam with a laminate lay-up $[45/-45]_s$ and $[45/-45]_{5s}$ under unit static vertical load

Results are presented in the table below.

Table 7.5. Axial twist from FORTRAN code and ANSYS for I-section beam made of angle ply laminates and subjected to torque at the free end

Laminate	Present formulation	ANSYS
$[45/-45]_s$	4.45	4.56
$[45/-45]_{5s}$	2.8	2.9

7.3 Validation of strength prediction

A simple cantilever beam-bending problem is chosen to determine first-ply failure load. An I-beam with a laminate lay-up $[0/90/45/-45]_s$ is subjected to a unit tip load. The strains from the code are compared with analytical results.

From [23] the expression for the effective EI for the I-beam is,

$$EI_{eff} = \frac{h^3}{12A_{11,web}^{-1}} + 2b_f\left(\frac{\xi_1^2}{A_{11,flange}^{-1}} + \frac{1}{D_{11,flange}^{-1}}\right)$$

Substituting the values in the equation above EI_{eff}=463.65 $N-m^2$. The maximum moment for this case is Wl at the fixed end, where W is the applied load at the tip and l the length of the beam. The deflection at the tip and the curvature for the case of a cantilever beam are given by,

Deflection, $\delta = \frac{Wl^3}{3EI_{eff}}$ = 0.754 E-3

Curvature, $\kappa = \frac{M}{EI_{eff}}$ = 0.219 E-2

From the code the corresponding values are 0.756 E-3 and 0.212 E-2 .

The strength values for a T300/5208 laminate are,

Longitudinal tensile strength, X = 1500 MPa

Longitudinal compressive strength, X' = 1500 MPa

Transverse tensile strength, Y = 40 MPa

Transverse compressive strength, Y' = 246 MPa

Shear strength, S = 68 MPa

The program computes the strains at the center of each ply in each segment of the cross section (refer Fig 7.20 below). The strain at the centerline of the segment 1 is computed as follows.

The origin is at the centroid of the cross section. The coordinates of the point at the middle of the segment 1 are y = -0.00635 and z = -0.0127. From expressions $6.2(a-d)$ the strains at the centerline of this segment for a unit tip load are,

ε_x = 2.78 E-5 and $\gamma_{xs} = 0$

Corresponding values from the code are, ε_x = 2.697 E-5 and γ_{xs} = -2.315 E-14

The strength parameters for this laminate are,

$F_{XX} = 0.44\ (GPa)^{-2}$, $F_{YY} = 0.44\ (GPa)^{-2}$, $F_X = 0$, $F_Y = 0$

F_{XY} = -0.22 $(GPa)^{-2}$, $F_{SS} = 216.2\ (GPa)^{-2}$

The first-ply failure is found to occur at the clamped end of the I-beam, for a load of 301 N in the outer ply i.e. -45^o ply of the 5th segment.

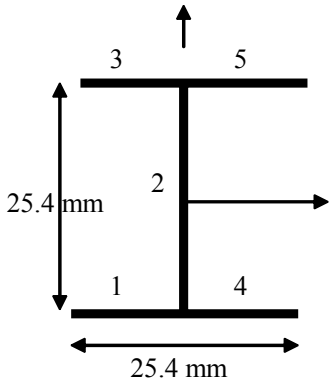


Fig 7.20. I section in strength analysis

The second example is an I-beam with fixed ends and a tip load at the center span. The laminate lay-up is $[0/90]_s$.

The value of EI_{eff} = 583.37 $N-m^2$ is calculated as in the first case. From [9] the deflection at the center of the beam is $\delta = \frac{Wl^3}{192EI_{eff}}$ = 0.936 E-5, and from the code δ = 0.114 E-4. The maximum moment at the center-span of the beam is $\frac{Wl}{8}$ = 0.127 $N-m$.

$(\kappa_x)_{formula} = \frac{M}{EI_{eff}}$ = 2.17 E-4 and $(\kappa_x)_{code}$ = 1.905 E-4

The strength parameters for the laminate material T300/5208 are given in the first example. The strains at the centerline of segment 1 are calculated as in the first case.

From computations, ε_x = 0.275 E-5 and $\gamma_{xs} = 0$

From code, ε_x = 0.242 E-5, γ_{xs} = -1.56 E-14

Failure for this configuration occurs at the clamped end for a load of 2194 N in the outer ply i.e. 0^o ply of the segment 1.

The third example is an I-section cantilever beam with a ply lay-up $[45/-45]_s$ and loaded by a unit force at the tip. The dimensions are shown in the figure below and the length of the beam is 0.3 m. The elastic properties are,

$E_1 = 181.0\ GPa$, $E_2 = 10.3\ GPa$, $G = 7.17\ GPa$ and $\nu = 0.28$

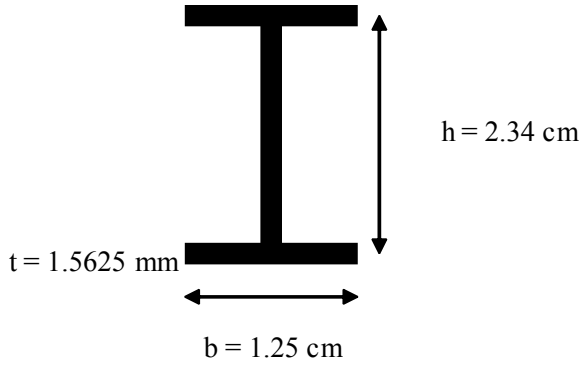


Fig 7.21. Geometry of I-section

The tip deflection and rotation from ANSYS are 0.124 mm and 0.6194 rad respectively. The corresponding values from the code are 0.126 mm and 0.629 rad. The comparison of strains near the fixed end obtained from the code and ANSYS is shown below.

	ANSYS	FORTRAN code
Axial strain in segment 1	0.47 e-5	0.44 e-5
Shear strain in segment 2	0.139 e-5	0.14 e-5

CONCLUSION

A finite element formulation of a 2-node laminated composite beam element based on Vlasov theory for constant arbitrary open cross section thin-walled beams has been presented. The formulation is developed using Hellinger-Reissner mixed variational principle. The beam has seven displacement degrees of freedom at each node. Warpage analysis under dynamic loading and failure analysis under static loading are studied. Several examples are presented for static, eigenvalue and time history analysis which show good correlation between the results from the code and ANSYS/analytical solutions. From the several examples worked out for validation of this formulation, it has been observed that although the time step calculated from the highest natural frequency is very small, the time taken by this explicit finite element code is significantly less than the time taken by ANSYS which uses implicit time integration. In the case of angle-ply laminates it is observed that the assumption that the contour strains $\varepsilon_s \,\&\, \kappa_s$ are negligible does not give accurate results. However, with more layers the effect of the coupling terms $D_{13} \,\&\, D_{23}$ is not significant as they approach zero and hence better results are obtained. Another alternative is to include the strains $\varepsilon_s \,\&\, \kappa_s$, and this has been implemented in the formulation.

RECOMMENDATION

A Finite Element for thin-wall isotropic and laminated composite closed cross section beams can be formulated using Vlasov theory. Woven composites are finding wide applications in aerospace applications for high fracture toughness and impact resistance. The volume of literature on woven composite thin-wall beams is very limited. The present theory can be extended to the analysis of beams made of woven composite materials. Although the formulation is comprehensive, the validation of strength prediction of the laminated composite beams presented in this study is primitive. A better validation can be done under dynamic loading either with results from simulations using a Finite Element package which gives a good failure prediction in composites, or experimental results, or any pre-published source if it exists. Implicit time integration could also be used with this formulation for certain kinds of problems where loading is quasi-static since larger time steps are allowed in such methods and they are unconditionally stable.

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