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UNIVERSITY OF CALIFORNIA, SAN DIEGO

Stationary Tropical Landau-Ginzburg Potential for the Complex Projective Plane

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

D. Peter Overholser

Committee in charge:

Professor Mark Gross, Chair
Professor Kenneth Intriligator
Professor Julius Kuti
Professor Dragos Oprea
Professor Justin Roberts

2013

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Chair

University of California, San Diego

2013

EPIGRAPH

Q: It has beauties.

A: The machine.

Q: Yes. We construct these machines not because we confidently expect them to do what they are designed to do– change the government in this instance–but because we intuit a machine, out there, glowing like a shopping center...

Donald Barthelme, *The Explanation*

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ACKNOWLEDGEMENTS

I'd like to thank my advisor, Mark Gross, for his patient and insightful guidance, and Rosanna for her support and encouragement. Also, thanks to Angela Hicks and Emily Leven for their suggestions in the proof of Claim 3.3.18.

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ABSTRACT OF THE DISSERTATION

Stationary Tropical Landau-Ginzburg Potential for the Complex Projective Plane

by

D. Peter Overholser

Doctor of Philosophy in Mathematics

University of California, San Diego, 2013

Professor Mark Gross, Chair

A modification of Gross's tropically based Landau-Ginzburg potential is explored in the context of the mirror symmetry of $\mathbb{P}^2$. This modification relies on techniques developed by Markwig and Rau and leads to novel conjectural methods for the tropical computation of certain Gromov-Witten invariants.

Chapter 1

Background

Though not yet completely understood or even rigorously defined, the phenomenon known as mirror symmetry has, for some time, been at the center of an active and rich area of mathematical and physical research. The field was born in an observation by physicists in an exploration of the Calabi-Yau threefolds required in the formulation superstring theory; the manifolds seemed to come in couples where the Hodge numbers $h^{1,2}$ and $h^{1,1}$ were interchanged [2]. Implications of this observation were not immediately apparent, but a convincing demonstration of the power of an underlying "mirror" relationship came with Candelas, de La Ossa, Green, and Parke's prediction of the number of rational curves of any degree on the quintic threefold. [3] These numbers were of some classical interest in algebraic geometry, and the work of Candelas et al. predicted results far beyond the reach of then current techniques. This work sparked an explosion of interest among mathematicians; the last two decades have seen the growth of a field that continues to draw surprising connections between disparate areas of physics and mathematics.

In the context of this discussion, mirror symmetry will consist of a relationship between two constructions (the A- and B-models) associated to a particular manifold X. For a broader discussion or more detail, see [4] [11]. On one side of the mirror, we have the A-model. This landscape concerns counts of rational curves on X satisfying certain intersection, tangency, and genus requirements. These "counts," called Gromov-Witten invariants, can be used to perturb the usual cup

product on the cohomology of X into something called quantum cohomology, a construction whose operations can then be compiled into a particularly nice object called a Frobenius manifold.

If X is a Fano variety (such as $\mathbb{P}^n$), the complementary, on B-model side of the mirror lies something called a Landau-Ginzburg model. This consists of a pair $(\hat{X}, W)$, where $\hat{X}$ is a variety and $W : \hat{X} \to \mathbb{C}$ a regular function called a *Landau-Ginzburg potential*. Through Barannikov's technique of semi-infinite variation of Hodge structures, one can again recover a Frobenius manifold. Mirror symmetry dictates that the Frobenius manifolds arising in the A- and B-models should be the same.

Happily, and somewhat suggestively, when $X = \mathbb{P}^2$, both sides of the mirror are intrinsically susceptible to analysis from a field known as tropical geometry [11]. Tropical geometry is a relatively recent and active area of research (see [6] for an overview), so named for the location of its invention, Brazil. The techniques of the field have a peculiar strength in translating difficult geometric questions into problems of combinatorics. The first insight into tropical geometry's descriptive power for the A-model came from Mikhalkin's proof that one can compute certain Gromov-Witten invariants for $\mathbb{P}^2$ by counting objects called tropical curves [16]. The ease with which these invariants can now be computed and the conceptual insight given by the tropical point of view has inspired many attempts to generalize the result. Gathmann, Markwig, Kerber, Rau and others have made significant progress in this regard, establishing not only methods for the tropical computation of certain *descendant* Gromov-Witten invariants (intuitively, "counts" of curves satisfying particular incidence and *tangency* conditions), but also an intersection theory on a relevant moduli space [7] [15]. Unfortunately, their results are unable to address certain types of descendent invariants.

I began my thesis work by investigating a result [9] (as exposited out by Pandharipande and Lee [17] [12]) in which Givental presents a method for computing the Frobenius manifolds encoding data about higher genus invariants using genus 0 information when $X = \mathbb{P}^n$. It soon became clear that a tropical interpretation of these techniques would require a thorough understanding and extension

of the results of Gathmann et al. on the tropical computation of descendant invariants. This extension became the subject of my thesis, which is a fusion of the techniques of Markwig and Rau with those of Gross.

1.1 Background

Gromov-Witten Invariants

The concept of Gromov-Witten invariants is central to what follows, so I will make an effort to explain them in some greater detail. A Gromov-Witten invariant associated to a a manifold X is denoted as follows:

$$\langle \alpha_1, \dots \alpha_n \rangle_{g,\beta} \in \mathbb{Q}$$

In this definition, $\alpha_i \in H^*(X, \mathbb{Q})$, g is a non-negative integer, and $\beta \in H_2(X, \mathbb{Z})$. The idea is to count maps of rational curves with n marked points $\{x_i\}$ to X where x_i maps to an object dual to α_i in X, the image of the curve represents β, and the curve has a finite automorphism group. These are the so called *stable maps*. More formally, we start with the following definition.

Definition 1.1.1 (Stable Maps)**.** Let X be a variety, and

$$f : (C, x_1, \dots, x_n) \to X$$

such that C is a proper connected reduced nodal algebraic curve, $x_1, \dots, x_n \in C$ are distinct points not coinciding with any nodes of C w. An automorphism of f is an automorphism ϕ of $(C, x_1, \dots, x_n)$ such that $f \circ \phi = f$. If f has a finite automorphism group, then we say that it is a *stable n-pointed map to* X.

In genus 0, the finiteness condition is easy to check; it is equivalent to the requirement that each component of the normalization of C on which f is constant has at least three points which are nodes or marked.

If $\beta \in H_2(X, \mathbb{Z})$, we say that a stable n-pointed map f *represents* β if $f_*([C]) = \beta$, where $[C]$ is the fundamental class of the curve. The moduli space $\mathcal{M}_{g,n}(X, \beta)$ of n-pointed rational curves of genus g mapping to to β constructed

and compactified to $\bar{\mathcal{M}}_{g,n}(X,\beta)$, a proper Deligne-Mumford stack. Unfortunately, this procedure doesn't always result in a space of the dimension predicted by Riemann-Roch, so we concern ourselves with an object of the right dimension, the so called virtual fundamental class,

$$[\bar{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \in H_{2d}(\bar{\mathcal{M}}_{g,n}(X,\beta),\mathbb{Q})$$

where d is the expected dimension of $\bar{\mathcal{M}}_{g,n}(X,\beta)$, given by $n+(\dim_{\mathbb{C}}X-3)(1-g)+\int_\beta c_1(\mathcal{T}_X)$. Procedures for constructing the virtual fundamental class are quite subtle and outside the scope of this discussion. For more information, see [1] [13].

If $f:(C,x_1,\ldots,x_n)\to X$ is stable map, we define a map $ev:\bar{\mathcal{M}}_{g,n}(X,\beta)\to X^n$ by $ev([f])=(f(x_1),\ldots,f(x_n))$. We can then give the definition:

$$\langle\alpha_1,\ldots\alpha_n\rangle_{g,\beta}=\int_{[\bar{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} ev^*(\alpha_1\times\cdots\times\alpha_n)$$

This is the convenient interpretation of "counts of rational curves." In general, though of great classical interest, these numbers are quite difficult to compute. One powerful result of mirror symmetry is the translation of this near-impossible calculation to a subtly connected integral on the B-side of the mirror.

To further complicate the picture, there is a useful extension of the Gromov-Witten invariant, called the *gravitational descendent* Gromov-Witten invariant:

$$\langle\psi^{j_1}\alpha_1,\ldots\psi^{j_n}\alpha_n\rangle_{g,\beta}=\int_{[\bar{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} \psi_1^{j_1}\cup\ldots\cup\psi_n^{j_n}\cup ev^*(\alpha_1\times\cdots\times\alpha_n)$$

Here we've attached a natural line bundle $\mathcal{L}_i$ to $\bar{\mathcal{M}}_{g,n}(X,\beta)$ associated to each marked point x_i whose fiber at a point $[(C,x_1,\ldots,x_n)]$ is the cotangent line $\mathfrak{m}_{x_i}/\mathfrak{m}_{x_i}^2$, where $\mathfrak{m}_{x_i}\subseteq\mathcal{O}_{C,x_i}$ is the maximal ideal. Then we can define $\psi_i=c_1(\mathcal{L}_i)\in H^2(\bar{\mathcal{M}}_{g,n}(X,\beta),\mathbb{Q})$.

These may seem unnecessarily complicated way of encoding tangency conditions, but this formulation ends up satisfying a number of nice relationships that make their computation quite a bit easier. In fact, due to an array of identities known as the divisor axiom, fundamental class axiom, and topological recursion relationship, one can completely compute the *genus 0* descendent Gromov-Witten invariants from the ordinary Gromov-Witten invariants.

Tropical Geometry

Tropical geometry has a number of attractive interpretations. One can view it as a combinatorial study of "corners" of piecewise linear functions, and neglect what immediately follows. However, as we are discussing counts of curves in $\mathbb{P}^2$, a question of algebraic geometry, we'll roughly follow Gathmann's description of "amoebas" [6]. Here we take a curve C in $\mathbb{P}^2$, restrict it to $(\mathbb{C}^*)^2$, and map the result to $\mathbb{R}^2$ as follows:

$$Log(z_1, z_2) = (-\log|z_1|, -\log|z_2|)$$

The resulting image should be a blob in the plane with several well-defined legs sprouting out in various directions (an amoeba). See Figure 1.1.1 for an example.

The next goal is to isolate the amoeba's "skeleton," represented by the dotted

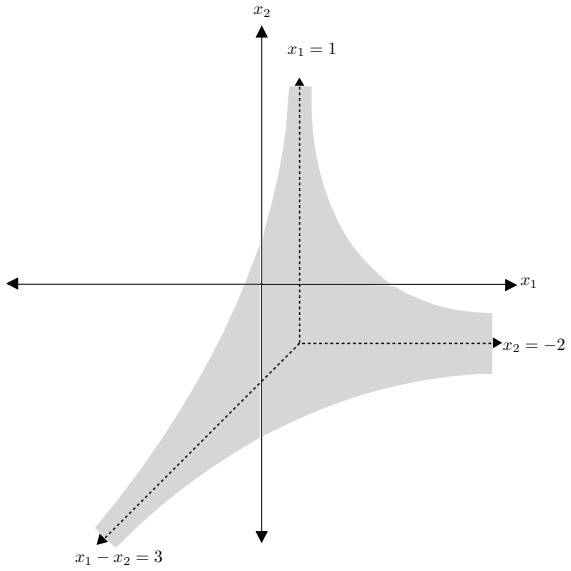

Figure 1.1.1: Amoeba of $C = \{(z_1, z_2)|e^1 z_1 + e^{-2} z_2 = 1\}$.

lines in Figure 1.1.1. One way to do this is to introduce a family of curves varying with positive real parameter t. For our particular example, one would set C_t to be the curve given by $C = \{(z_1, z_2)|tz_1 + t^{-2} z_2 = 1\}$. For each value of t, we also define the map $Log_t(z_1, z_2)$ from $(\mathbb{C}^*)^2$ to $\mathbb{R}^2$:

$$Log_t(z_1, z_2) = (-\log_t|z_1|, -\log_t|z_2|) = (-\frac{\log|z_1|}{\log t}, -\frac{\log|z_2|}{\log t})$$

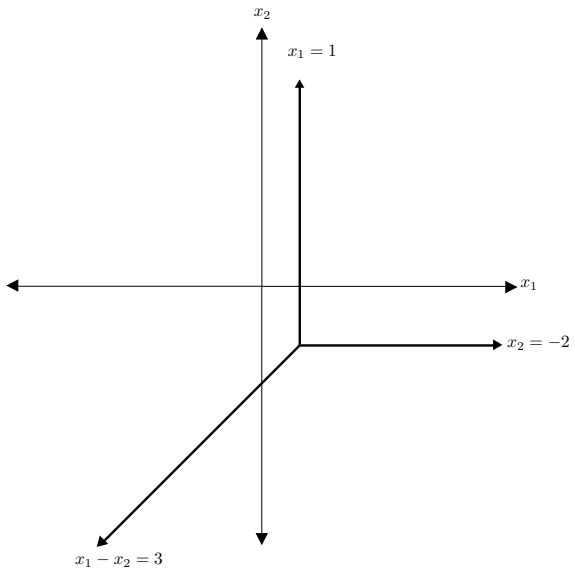

Figure 1.1.2: Tropical curve corresponding to amoeba above.

If one examines the image of C_t under Log_t, the amoeba retracts to its skeleton as t approaches ∞. Another way to think about this is that, in our example, we are zooming out on the picture while rescaling the axes to keep the center of the amoeba at $(1, -2)$. The limiting situation is the piecewise linear geometric realization of a graph hinted at by the "skeleton" of the amoeba, as seen in Figure 1.1.2, which is the tropical curve associated to the plane curve C.

Thus algebraic curves in $\mathbb{P}^2$ are related to piecewise linear *tropical curves* in $\mathbb{R}^2$. Because this process is unwieldy, it is desirable to abstract from this relationship. For our purposes, we'll define tropical curves in $\mathbb{R}^2$ in a more combinatorial language, following Mikhalkin [16].

1.1.1 Marked tropical curves

Let $\bar{\Gamma}$ be the topological realization of a graph with no bivalent vertices. Let Γ_1 be the set of edges, Γ_0 the set of vertices. Define Γ to be $\bar{\Gamma}$ without its univalent vertices. Note that Γ generally will have non-compact edges, which we gather into a set Γ_1^∞. Assign a weight function $w : \Gamma_1 \to \mathbb{Z}^{\geq 0}$ such that $w(\Gamma_1^\infty) \subseteq \{0, 1\}$ and

$w^{-1}(0) \subseteq \Gamma_1^\infty$. Assign a label x_i to each of the weight 0 edges using an inclusion

$$\{x_1, \ldots, x_n\} \hookrightarrow \Gamma_\infty^{[1]}$$
$$x_i \mapsto E_{x_i}$$

The data $(\Gamma, x_1, \ldots, x_n)$ constitutes a *marked graph*. A marked graph can be given a geometric manifestation using the following definition.

Definition 1.1.2 (Marked parametrized tropical curve)**.** A *marked parametrized tropical curve* is a continuous map $h : (\Gamma, x_1, \ldots, x_n) \to \mathbb{R}^n$ satisfying:

- If $E \in \Gamma_1^\infty$ and $w(E) = 0$, then $h|_E$ is constant. That is, h collapses labeled edges. On other edges, $h|_E$ is a proper embedding of E into a line of rational slope in $\mathbb{R}^n$.

- Let V be a vertex of Γ, and $E_1, \ldots E_m$ be the edges adjacent to V. Let $v(E_i)$ be a primitive vector pointing away from $h(V)$ along the direction of $h(E_i)$. Then
$$\sum_{i=1}^{m} w(E_i)v(E_i) = 0.$$
This is known as the balancing condition.

In the following, we will conflate a collapsed edge with its label. That is, if $h : (\Gamma, x_1, \ldots, x_n) \to \mathbb{R}^n$ is a marked parametrized tropical curve, we write $h(x_i) = h(E_{x_i})$.

Due to the dependence on the map, it is clear that two different marked parametrized tropical curves could have the same image and should morally be considered the same "curve". In order to formalize this relationship, we say that two parametrized tropical curves $h : (\Gamma, x_1, \ldots, x_n) \to \mathbb{R}^n$ and $h' : (\Gamma', x_1', \ldots, x_n') \to \mathbb{R}^n$ are *equivalent* if there is a homeomorphism $\phi : \Gamma \to \Gamma'$ with $\phi(E_{x_i}) = E_{x_i'}$ for each i and $h = h' \circ \phi$. We can then define a *marked tropical curve* to be an equivalence class of parametrized marked tropical curves.

1.1.2 Significance

These objects are relatively easy to work with, and the amoeba technique sketched above might lead one to hope that counts of algebraic plane curves

(Gromov-Witten invariants of $\mathbb{P}^2$) could be somehow recovered from counts of tropical curves.

In fact, as mentioned before, Mikhalkin has shown that certain Gromov-Witten invariants associated to $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are calculable using enumerative tropical geometric techniques in $\mathbb{R}^2$ [16]. In what follows, we denote by T_i the positive generator of $H^{2i}(\mathbb{P}^2, \mathbb{Z})$. Note that T_2 is Poincaré dual to a point. In the case of $\mathbb{P}^2$, these methods make it possible to compute invariants of the type $\langle \overbrace{T_2, \ldots T_2}^{n} \rangle_{g,d}$, which morally counts genus g, degree d curves passing through n points in $\mathbb{P}^2$, by counting genus g tropical curves with marked points (collapsed edges) mapping to n points in general position in $\mathbb{R}^2$, where the unbounded, uncollapsed edges of the curve are translates of the 1-dimensional skeleton of the toric fan of $\mathbb{P}^2$. Tropical curves are concretely combinatorial objects, making this result quite appealing; not only does it facilitate the actual calculation of Gromov-Witten invariants but provides conceptual insight.

This success has inspired quite a bit of related research. Gathmann, Markwig, Rau, Allerman, and others have had success in generalizing these results, with Markwig and Rau presenting a method for computing $\mathbb{P}^2$ Gromov-Witten invariants of the type:

$$\langle \psi^{j_1} T_{i_1}, \ldots \psi^{j_n} T_{i_n} \rangle_{0,d}$$

when $j_k = 0$ if $i_k \neq 2$ [15].

In other words, they constructed tropical counting methods that give the same result as classical Gromov-Witten invariants when ψ-classes are only attached to point conditions. One key ingredient here was Mikhalkin's result that attaching a ψ-class to a marked point corresponds to increasing the valency of the vertex connected to this marked point (remember that marked points are actually collapsed non-compact edges.) Another was the formulation of the correct multiplicity to assign to each tropical curve, that is, how to count its contribution to the total sum of the relevant Gromov-Witten invariant. Several powerful techniques were developed for this result, including a particular intersection theory on the relevant moduli space of tropical curves. Some insight into the success of these techniques can be found in Gibney and Maclagan's description of the geometric relationship

between the relevant tropical moduli and classical moduli spaces [8].

Gross developed tropical techniques for computing different types of descendent Gromov-Witten invariants in his exploration of mirror symmetry of $\mathbb{P}^2$. In particular, his techniques for invariants of the type $\langle \psi^k T_2, T_2, \dots T_2 \rangle_{0,d}$ agree with Markwig and Rau's result, while his tropical counting methods for invariants of the type $\langle \psi^k T_1, T_2, \dots T_2 \rangle_{0,d}$ and $\langle \psi^k T_0, T_2, \dots T_2 \rangle_{0,d}$ address new classes of invariants. Tropical computation of invariants of this type involve mysterious definitions for the multiplicity of the relevant tropical curves. One important point here is that Gross's argument is not only significant for the expansion of the classes of Gromov-Witten invariants susceptible to tropical computation, but also provides a convincing argument that tropical geometry is intrinsically the natural language with which to express the mirror relationship in question.

In his account, a perturbation of the Landau-Ginzburg potential W is defined using counts of objects called "tropical disks". An integral is performed on this perturbation of W, giving quantities that, because of mirror symmetry, reflect the actual Gromov-Witten invariant counts on the A-model side of the mirror, as we will see in the next section.

1.2 Mirror symmetry for $\mathbb{P}^2$

This thesis is a merging of the techniques of Markwig and Rau with those of Gross as sketched above. Gross's exploration of the mirror symmetry of $\mathbb{P}^2$ provide the essential context and significance of what follows, so a brief description review of his description of the mirror symmetry for $\mathbb{P}^2$ will be given here.

1.2.1 Frobenius Manifolds

Recall the brief mention of Frobenius manifolds as the object most central to our discussion of mirror symmetry. For much more on these objects, see [14].

Definition 1.2.1. A pre-Frobenius structure on a complex manifold $\mathcal{M}$ is a triple of data $(\nabla, g, \mathcal{A})$ where

- $\nabla : \mathcal{T}_M \to \mathcal{T}_\mathcal{M} \otimes \Omega^1_\mathcal{M}$ is a flat connection.
- g is a metric on $\mathcal{M}$, a symmetric pairing $g : S^2(\mathcal{T}_\mathcal{M}) \to \mathcal{O}_\mathcal{M}$ inducing an isomorphism $\mathcal{T}_\mathcal{M} \cong \mathcal{T}^*_\mathcal{M}$. Furthermore, g must be compatible with ∇:

$$d(g(X,Y)) = g(\nabla X, Y) + g(X, \nabla Y)$$

- $\mathcal{A} : \mathcal{S}^3(\mathcal{T}_\mathcal{M}) \to \mathcal{O}_\mathcal{M}$ is a symmetric tensor.

These structures allow us to define a multiplication $\cdot$ on each tangent space of $\mathcal{M}$ by

$$\mathcal{A}(X,Y,Z) = g(X \cdot Y, Z)$$

where X, Y and Z are vector fields

Definition 1.2.2. A pre-Frobenius structure on a manifold $\mathcal{M}$ qualifies as a Frobenius manifold if

- The product $\cdot$ is associative.
- Locally on $\mathcal{M}$, there is a potential function $\mathcal{F}$ such that

$$\mathcal{A}(X,Y,Z) = XYZ\mathcal{F}$$

1.2.2 A model

Gromov-Witten invariants can, in a fairly straightforward way, be used to define a Frobenius manifold that encodes a structure known as quantum cohomology. We'll confine our discussion to the concrete example of $X := \mathbb{P}^2$, but the discussion can be easily generalized to any non-singular Fano variety X.

First, define $\mathcal{M} := \operatorname{Spec} \mathbb{C}[[y_0, y_1, y_2]]$. This will be the fabric of our Frobenius manifold. Let T_i be a generator for $H^{2i}(\mathbb{P}^2, \mathbb{C})$ and let

$$\gamma := y_0 T_0 + y_1 T_1 + y_2 T_2$$

Then we can give the following important definition, the *Gromov-Witten potential of* X.

$$\Phi := \sum_{k=0}^{\infty} \sum_{\beta \in H_2(X,\mathbb{Z})} \frac{1}{k!} \langle \gamma^k \rangle_{0,\beta}$$

Take the metric to be constant on $\mathcal{M}$, with

$$g(\partial_{y_i}, \partial_{y_j}) := \int_{\mathbb{P}^2} T_i \cup T_j$$

and the connection ∇ given by the flat sections ∂_{y_i}. In addition, define

$$\mathcal{A}(\partial_{y_i}, \partial_{y_j}, \partial_{y_k}) := \partial_{y_i}\partial_{y_j}\partial_{y_k}\Phi$$

If we choose T_1 to live in the $H^2(\mathbb{P}^2, \mathbb{R})$ with $\int_\beta T_1 \geq 0$ for every class β represented by a stable curve, we can set $e^{y_1} = \kappa_1$ and note that Φ is now an element of $\mathbb{C}[\kappa_1][[y_0, y_2]]$. In this situation, we can define a larger Frobenius manifold $\overline{\mathcal{M}} := \operatorname{Spec} \mathbb{C}[\kappa_1][[y_0, y_2]]$.

The Frobenius manifold arrived at in this way has a very rich structure, and in fact encodes all genus 0 Gromov-Witten invariants (descendent or not) of $\mathbb{P}^2$. One of the more interesting structures derived here is something known as Givental's J-function, wich arises naturally on the B-model side of the picture. On the A-model side, it is a natural result of the following definition, a "twisting" of our original connection by the multiplicative structure of $\mathcal{M}$.

Definition 1.2.3 (Dubrovin Connection)**.** If $\mathcal{M}$ is a pre-Frobenius manifold with a vector field E and $d_0 \neq 0$, we can define a connection $\hat{\nabla}$ on the vector bundle $p_1^*\mathcal{T}_\mathcal{M}$ on $\mathcal{M} \times \mathbb{C}^\times$, where $p_1 : \mathcal{M} \times \mathbb{C}^\times \to \mathcal{M}$ is the projection onto the first factor. Let $\hbar$ be the coordinate on $\mathbb{C}^\times$. Then we define:

$$\hat{\nabla}_X(Y) = \nabla_X(Y) + \hbar^{-1} X \circ Y$$
$$d_0\hat{\nabla}_{\hbar\partial_\hbar} = \hbar\partial_\hbar Y - \hbar^{-1} E \circ Y + Gr_E(Y)$$

where Gr_E is the $\mathcal{O}_\mathcal{M}$-linear map defined on flat vector fields Y by $Y \to [E, Y]$. This "twisted" connection is known as the Dubrovin connection.

The Dubrovin connection allows us to define the *quantum differential equation*, solutions of which are closely related to the formulation of Givental's J-function.

Definition 1.2.4 (Quantum differential equation)**.** The quantum differential equation on our frobenius manifold is given by

$$\hat{\nabla}_{\partial_i} s = 0,\ i = 0, 1, 2$$

Hopefully now the reader has some hint of how the following (central) definition is related to our fundamental objects of study.

Definition 1.2.5 (Givental's J-function for $\mathbb{P}^2$)**.** Let T_i generate $H^{2i}(X,\mathbb{Z})$ with T_i positive. Then $J_{\mathbb{P}^2}:\mathcal{M}\times\mathbb{C}^\times\to H^*(\mathbb{P}^2,\mathbb{C})$ is defined as follows:

$$J_{\mathbb{P}^2}(y_0,y_1,y_2,\hbar) := e^{\frac{y_0T_0+y_1T_1}{\hbar}}\cup\left(T_0+\sum_{i=0}^{2}\left(y_2\hbar^{-1}\delta_{2,i}\right.\right.$$
$$\left.\left.\sum_{d\geq 1}\sum_{\nu\geq 0}\langle T_2^{3d+i-2-\nu},\psi^\nu T_{2-i}\rangle_{0,d}\hbar^{-(\nu+2)}e^{dy_1}\frac{y_2^{3d+i-2-\nu}}{(3d+i-2-\nu)!}\right)T_i\right)$$

We can define the functions J_i by the decomposition of J:

$$J_{\mathbb{P}^2}=\sum_{i=0}^{n}J_iT_i$$

The content of the Barannikov's statement of mirror symmetry for $\mathbb{P}^2$ comes down to the equality of these functions with something that arises naturally on the B-model side of the picture.

1.2.3 B model

Here we follow the summary of Barannikov's results [18] as given in [10]. The mirror of $\mathbb{P}^2$ is the manifold $\hat{X}:=V(x_0x_1x_2-1)\subseteq\mathrm{Spec}[x_0,x_1,x_2]$ along with a regular function $W:\hat{X}\to\mathbb{C}$ given by $W=x_0+x_1+x_2$. The latter object is called a Landau-Ginzburg potential, while the pair $(\hat{X},W)$ is called a Landau-Ginzburg model.

We consider the universal unfolding of W parametrized by the moduli space $\mathrm{Specf}\,\mathbb{C}[[t_0,t_1,t_2]]$

$$W_{\mathbf{t}}:=\sum_{i=0}^{2}W^it_i,$$

and the local system $\mathcal{R}$ on $\mathcal{M}\times\mathbb{C}^\times$ whose fiber at a point $(\mathbf{t},\hbar)$ whose fiber at a point is the relative homology group $H_n(\hat{X},Re(W_{\mathbf{t}}/\hbar)\ll 0)$. With this setup, Barannikov shows the following, the results of a technique known as semi-infinite variation of Hodge structures. First, there is a unique choice of the following data:

- A (multi-valued) basis of sections of $\mathcal{R}$, Ξ_0, Ξ_1, Ξ_2, with Ξ_i uniquely defined modulo $\Xi_0, \ldots, \Xi_{i-1}$.

- A section s of $\mathcal{R}^\vee \otimes_{\mathbb{C}} \mathcal{O}_{\mathcal{M}\times\mathbb{C}}$ defined by integration of a family of holomorphic forms on $\hat{X} \times \mathcal{M} \times \mathbb{C}^\times$ of the form

$$e^{W_{\mathbf{t}}/\hbar} f \mathrm{dlog} x_1 \wedge \mathrm{dlog} x_2$$

  where $\hbar$ is the coordinate on $\mathbb{C}$ and f is a regular function on $\hat{X} \times \mathcal{M} \times \mathbb{C}^\times$ with $f|_{\hat{X}\times\{0\}\times\mathbb{C}^\times} = 1$ and which extends to a regular function on $\hat{X} \times \mathcal{M} \times (\mathbb{C}^\times \cup \{\infty\})$.

- The monodromy associated with $\hbar \to \hbar e^{2\pi i}$ in $\mathcal{R}$ is given, in the constructed basis, by $\exp((3 * 2\pi i N)$, where

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

- A fiber of $\mathcal{R}^\vee$ is identified with the ring $\mathbb{C}[\alpha]/(\alpha^3)$, with α^i dual to Ξ_i. The selected section s of $\mathcal{R}^\vee \otimes \mathcal{O}_{\mathcal{M}\times\mathbb{C}^\times}$ gives us an element of each fiber of $\mathcal{R}^\vee$, which we write as

$$s(\mathbf{t}, \hbar) = \sum_{i=0}^{2} \alpha^i \int_{\Xi_i} e^{W_{\mathbf{t}}/\hbar} f \mathrm{dlog} x_1 \wedge \mathrm{dlog} x_2$$

  We require that we can write

$$s(\mathbf{t}, \hbar) = \hbar^{-(3\alpha)} \sum_{i=0}^{2} \phi_i(\mathbf{t}, \hbar)(\alpha\hbar)^i$$

  for functions ϕ_i satisfying

$$\phi_i(\mathbf{t}, \hbar) = \delta_{0,i} + \sum_{j=1}^{\infty} \phi_{i,j}(\mathrm{t})\hbar^{-j}$$

  for $0 \leq i \leq 2$. In the above,

$$\hbar^{-3\alpha} = \sum_{i=0}^{2} \frac{(3)^i}{i!}(-\log\hbar)^i \alpha^i,$$

  which absorbs the multi-valuedness of the integrals.

As a result of these conditions, if we set $y_i(\mathbf{t}) = \phi_{i,1}(\mathbf{t})$, the functions y_i form a set of coordinates on $\mathcal{M}$, $\lim_{\hbar\to\infty} \hbar^i \phi_i(0,\hbar) = \delta_{0,i}$, and we are able to state the following:

Theorem 1.2.6 (Mirror symmetry for $\mathbb{P}^2$)**.** Given the above setup, on the $\mathbb{C}$ vector space $\mathbb{C}[[y_0, y_1, y_2, \hbar^{-1}]]$,

$$J_i = \phi_i$$

1.2.4 Tropical Connection

The basis of this thesis is Gross's use of tropical geometry to construct an alternate universal unfolding of W, called W_k, this time with a very concrete connection to flat coordinates. Mirror symmetry then dictates that the coefficients recovered through Barranikov's period integrals on this function should correspond to classical Gromov-Witten invariants appearing in the A-model description of the J-function. Not only does this result yield new methods for the tropical computation of the invariants appearing in the J-function for $\mathbb{P}^2$, but more importantly demonstrates the striking simplicity with which this formulation of mirror symmetry can be expressed through the techniques of tropical geometry. It seems as if tropical geometry is the most natural language with which to express this particular relationship, and there is some hope that the ideas will be extended to more general classes of objects.

1.3 Overview of thesis

Following this very concrete tropical interpretation of mirror symmetry and the tropical curve-counting results of Markwig and Rau, this thesis will expand the tools of Gross in [10] to construct a new potential $W_{k,n}$ that encodes data concerning Gromov-Witten invariants for which tropical counting methods are unknown. Evaluation of Barranikov's period integrals confirms previously conjectured methods of tropical counts for an expanded class of invariants. These results, where overlapping, are in agreement with Markwig and Rau and a range of computed examples.

Chapter 2

Stationary Potential $W_{k,n}$

The idea in this chapter is to generalize the discussion of trivalent disks and curves of particular Maslov index (as in [11]) to a similar analysis of objects inspired by Markwig and Rau's techniques. In what follows, restrictions are placed on the valency of vertices based on the number of ψ-classes associated to attached marked points.

2.1 Definitions

Let $M := \mathbb{Z}^2$, $M_{\mathbb{R}} := M \otimes \mathbb{R}$, and T_i a positive generator of $H^{2i}(\mathbb{P}^2, \mathbb{Z})$.

Definition 2.1.1 (Σ)**.** Let Σ be the toric fan of $\mathbb{P}^2$ in $M_{\mathbb{R}}$. More explicitly, let $\hat{\rho}_0 := (-1,-1) \in M_{\mathbb{R}}$, $\hat{\rho}_1 := (1,0) \in M_{\mathbb{R}}$, $\hat{\rho}_2 := (0,1) \in M_{\mathbb{R}}$, $\rho_i = \{x \in M_{\mathbb{R}} | x = r\hat{\rho}_i \text{ for some } r \geq 0\}$, and $\sigma_{i,j} := \{x \in M_{\mathbb{R}} | x = r_1\hat{\rho}_i + r_2\hat{\rho}_j \text{ for some } r_1, r_2 \geq 0\}$. Then, Σ is the set of rational convex polygons in $M_{\mathbb{R}}$ given by

$$\Sigma = \{\{0\}, \rho_0, \rho_1, \rho_2, \sigma_{0,1}, \sigma_{1,2}, \sigma_{2,0}\}.$$

We stratify Σ by dimension, defining $\Sigma^{[0]} := \{\{0\}\}$, $\Sigma^{[1]} := \{\rho_0, \rho_1, \rho_2\}$, and $\Sigma^{[2]} := \{\sigma_{0,1}, \sigma_{1,2}, \sigma_{2,0}\}$.

Denote by T_Σ the free abelian group generated by $\Sigma^{[1]}$. For $\rho \in \Sigma^{[1]}$, denote

by v_ρ the corresponding generator in T_Σ. There is a natural map

$$r : T_\Sigma \to M_\mathbb{R} \tag{2.1.1}$$

$$v_\rho \mapsto \hat{\rho}. \tag{2.1.2}$$

2.2 Tropical curves and disks

Recall the definition of the marked tropical curves given in the first chapter. Here we introduce some more structure related to these objects.

The *combinatorial type* of a marked tropical curve $h : (\Gamma', x_1, \ldots, x_d) \to M_\mathbb{R}$ is defined as the homeomorphism class of $\overline{\Gamma}$, the marked points, and weights along with, for each pair $V \in \Gamma^{[0]}$ and edge E attached to V, a primitive tangent vector pointing along the image of E away from V.

Definition 2.2.1 (Tropical curves in X_Σ)**.** Recall the definition of the marked tropical curve from Chapter 1. A marked tropical curve h is *in* X_Σ if, for each unmarked unbounded edge $E \in \Gamma_\infty^{[1]}$, $h(E)$ is a translate of some $\rho \in \Sigma^{[1]}$.

Definition 2.2.2 (Degree of a tropical marked curve)**.** If h is a marked tropical curve in X_Σ, the *degree* of h, notated $\Delta(h)$, is defined to be

$$\Delta(h) := \sum_{\rho \in \Sigma^{[1]}} d_\rho v_\rho$$

where d_ρ is the number of unbounded edges of Γ that are mapped to translates of ρ by h.

The following definition, inspired by [11] [15], will be central to what follows, as it defines the moduli spaces of the objects that we will eventually count to compute tropical Gromov-Witten invariants. This definition was

Definition 2.2.3 ($\mathcal{M}^{\text{trop}}_{\Delta,I}(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k, \psi^\nu S)$)**.** Let $P_1, \ldots, P_k \in M_\mathbb{R}$ be general. Let $S \subseteq M_\mathbb{R}$, $I = \{i_1, \ldots, i_n\} \subseteq \{1, \ldots, k\}$. Define

$$\mathcal{M}^{\text{trop}}_{\Delta,I}(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k, \psi^\nu S)$$

to be the moduli space of rational $(n+1)$-pointed tropical curves in X_Σ, $h : (\Gamma, p_1, \ldots, p_n, x) \to M_\mathbb{R}$ such that

1. $h(p_j) = P_{i_j}$, $1 \leq i_1 < \ldots < i_n \leq k$.

2. If E_x shares a vertex V_k with E_{p_k}, then

$$Val(V_k) = 3 + r_{i_k} + \nu$$

   and the valency of the vertex V_j attached to E_{p_j} for $j \neq k$ is given by

$$Val(V_j) = 3 + r_{i_j}$$

3. Otherwise, the valency of the vertex V_x attached to E_x is given by $Val(V_x) = \nu + 3$ and $Val(V_j) = 3 + r_{i_j}$ for $1 \leq j \leq n$.

4. $h(x) \in S$.

5. The weight of each unbounded edge of Γ is either 0 or 1.

Lemma 2.2.4. For $P_1, \ldots, P_k \in M_{\mathbb{R}}$ general with $I = \{i_1, \ldots, i_n\} \subseteq \{1, \ldots, k\}$,

1. $\mathcal{M}^{\text{trop}}_{\Delta,I}(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k, \psi^{r_\nu}M_{\mathbb{R}})$ is a polyhedral complex of $|\Delta| - n - \nu - \sum_{j=1}^n r_{i_j}$.

2. $\mathcal{M}^{\text{trop}}_{\Delta,I}(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k, \psi^{r_\nu}C)$ is a polyhedral complex of $|\Delta| - n - \nu - \sum_{j=1}^n r_{i_j} - 1$ for C a general translate of a tropical curve in $M_{\mathbb{R}}$.

3. $\mathcal{M}^{\text{trop}}_{\Delta,I}(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k, \psi^{r_\nu}Q)$ is a polyhedral complex of $|\Delta| - n - \nu - \sum_{j=1}^n r_{i_j} - 2$ for Q a general point in $M_{\mathbb{R}}$.

Proof. This follows the proof of Lemma 5.11 in [11], changing number of bounded edges to be $|\Delta| + n + 1 - (\nu + \sum_{j=1}^n r_{i_j} + 3)$. □

Our strategy for counting these objects involves an object similar to a tropical curve in X_Σ, the tropical disk. These are much the same as tropical curves, except that they each have one edge truncated by a univalent vertex, and should be thought of as pieces of marked tropical curves that have been torn apart at one vertex.

Definition 2.2.5 (Tropical disks)**.** Let $\overline{\Gamma}$ be a weighted, connected finite graph without bivalent vertices, with the additional choice of a univalent vertex V_{out} adjacent to a unique edge E_{out}. Let

$$\Gamma' := (\overline{\Gamma} \setminus \overline{\Gamma}_\infty^{[0]}) \cup \{V_{out}\} \subseteq \overline{\Gamma}.$$

Suppose that Γ' is a tree with one compact external edge and a number of non-compact external edges. Then a *parametrized d-pointed tropical disk* in $M_\mathbb{R}$ with domain Γ' is:

- A choice of inclusion $\{p_1, \ldots, p_d\} \hookrightarrow \Gamma_\infty^{[1]} \setminus \{E_{out}\}$, written $p_i \to E_{p_i}$.
- A weight function $w : \Gamma'^{[1]} \to \mathbb{Z}_{\geq 0}$ with $w(E) = 0$ if and only if $E = E_{p_i}$ for some i and $w(E) = 1$ for all other edges in $\Gamma_\infty'^{[1]}$.
- A continuous map $h : \Gamma' \to M_\mathbb{R}$ satisfying the conditions for tropical curves laid out in the introduction, except that there is no balancing condition at the univalent vertex V_{out}.

An isomorphism of parametrized tropical disks between

$$h_1 : (\Gamma_1', p_1, \ldots, p_d) \to M_\mathbb{R} \text{ and } h_2 : (\Gamma_2', p_1, \ldots, p_d) \to M_\mathbb{R}$$

is a homemorphism $\Phi : \Gamma_1' \to \Gamma_2'$ respecting marked edges and weights, such that $h_1 = h_2 \circ \Phi$. Just as with marked tropical curves, we refer to an isomorphism class of parametrized marked tropical disks a *marked tropical disk*.

The combinatorial type of a marked tropical disk $h : (\Gamma', p_1, \ldots, p_d) \to M_\mathbb{R}$ is defined as the homeomorphism class of $\overline{\Gamma}$, the marked points, weights, and V_{out}, along with, for each pair $V \in \Gamma^{[0]}$ and edge E attached to V, a primitive tangent vector pointing along the image of E away from V.

Definition 2.2.6 (Descendent tropical disks of type I)**.** Let $1 \leq i_1 < \ldots < i_d \leq k$, let $I = \{i_j\}_j$, and $h : (\Gamma', p_1, \ldots, p_d) \to M_\mathbb{R}$ be a tropical disk in X_Σ with the further restriction that the valency of each vertex V is given by $r_V + 3$, where r_V is the sum of r_{i_j} over all p_i attached to V. Then we call h a descendent tropical disk of type I.

Definition 2.2.7 (Descendent tropical disks in $(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k)$ with boundary Q)**.** A descendent tropical disk in $(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k)$ is a d-pointed descendent tropical disk $h : (\Gamma, p_1, \ldots, p_d) \to M_\mathbb{R}$ of type I for some $I \subseteq \{1, 2, \ldots, k\}$ with $|I| = d$, $h(p_j) = P_{i_j}$, and $h(E)$ is a translate of some $\rho \in \Sigma^{[1]}$ for each $E \in \Gamma_\infty^{[1]}$ with $w(E) = 1$.

As seen in Lemma 2.2.4, these tropical objects come in moduli spaces whose dimensions are easily computed. We make the following definition to develop a shorthand for speaking about the type of disks which belong in a moduli spaces of particular dimension.

Definition 2.2.8 (Flexibility)**.** Let $h : (\Gamma', p_1, \ldots, p_d) \to M_\mathbb{R}$ be a descendent tropical disk of type $I = \{i_1, \ldots, i_d\}$ in $(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k)$. Define the flexibility of h as

$$F(h) := |\Delta(h)| - d - \sum_{j=1}^{d} r_{i_j}$$

Lemma 2.2.9. If $P_1, \ldots, P_k, Q$ are chosen in general position, the set of descendent tropical disks h in $(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k)$ with boundary Q and $F(h) = n$ is an $n-1$ dimensional polyhedral complex. The set of such disks with $F(h) = n$ and arbitrary boundary is an $n+1$ dimensional polyhedral complex.

Proof. Fix a combinatorial type of descendent tropical disk of type I with degree Δ. Let V_i be the vertex attached to p_i. If the combinatorial type is general, all vertices are trivalent except for V_{out} (univalent) and V_i ($(3 + r_{i_j})$-valent) for $1 \leq j \leq d$. A graph of this type has $|\Delta| + d$ unbounded edges and thus $|\Delta| + d - 1 - \sum_{j=1}^{d} r_{i_j}$ bounded edges. If we select a location for $h(V_{out}) \in M_\mathbb{R}$ and an affine length for each bounded edge, this determines a descendent tropical disk of type I, $h : \Gamma' \to M_\mathbb{R}$, and produces a cell in the moduli space $\mathcal{M}_{\Delta,d}^{disk,I}(X_\Sigma)$ of all descendent tropical disks of type I of degree Δ. The closure of this cell is $(\mathbb{R}_{\geq 0})^{|\Delta|+d-1-\sum_{j=1}^{d} r_{i_j}} \times M_\mathbb{R}$. There are only a finite number of combinatorial types of disks of a given degree, so $\mathcal{M}_{\Delta,d}^{disk,I}(X_\Sigma)$ is a finite polyhedral complex of dimension $|\Delta| + d + 1 - \sum_{j=1}^{d} r_{i_j}$. We have a piecewise linear map $ev : \mathcal{M}_{\Delta,d}^{disk,I}(X_\Sigma) \to M_\mathbb{R}^d$ on this space given by evaluation on p_i, explicitly $h \to (h(p_1), \ldots, h(p_d))$. Let $E \subset \mathcal{M}_{\Delta,d}^{disk,I}(X_\Sigma)$ be the

union of cells mapping under ev to cells of codimension ≥ 1 in $M^d_{\mathbb{R}}$. Because E is a closed subset of $\mathcal{M}^{disk,I}_{\Delta,d}(X_\Sigma)$, $h(E) \subset M^d_{\mathbb{R}}$ is closed. If $(P_{i_1}, \ldots, P_{i_d}) \in M^d_{\mathbb{R}} \setminus h(E)$ then $h^{-1}(P_{i_1}, \ldots, P_{i_d})$ is a codimension $2d$ subset of $\mathcal{M}^{disk,I}_{\Delta,d}(X_\Sigma)$, and thus is of dimension $|\Delta| - d + 1 - \sum_{j=1}^{d} r_{i_j} = F(h) + 1$. Of course, fixing Q decreases the dimension by 2, giving $F(h) - 1$.

□

Definition 2.2.10 ((Semi)rigidity)**.** Let h be a descendent tropical disk in

$$(X_\Sigma, \psi^{r_1} P_1, \ldots, \psi^{r_k} P_k)$$

with boundary Q. We say h is semirigid if $F(h) = 1$ and rigid if $F(h) = 0$.

Now for a result concerning the amount with which we "count" each of these tropical disks. Note that the definition is inherently local, allowing us to neatly extend the result to descendent tropical curves when the need arises. From this point on, we will fix Σ as the fan associated to $\mathbb{P}^2$.

Definition 2.2.11 (Multiplicity for tropical disks, $lab(h)$)**.** Let P_i, Q be chosen in general position. Let h be a semirigid descendent tropical disk in

$$(X_\Sigma, \psi^{r_1} P_1, \ldots, \psi^{r_k} P_k)$$

with boundary Q. Define $ev'(h) : \mathcal{M}^{disk,I}_{\Delta,d}(X_\Sigma) \to M^{d+1}_{\mathbb{R}}$, by

$$ev'(h) = (h(V_{out}), h(p_1), \ldots, h(p_d))$$

For each vertex $v \in Vert(h)$, define $n_i(v)$ to be the number of unbounded rays emanating from v in the direction m_i. Define

$$lab(h) := \prod_{v \in Vert(h)} \frac{1}{n_0(v)! n_1(v)! n_2(v)!}$$

Because we've chosen our points in general position, h should contain no collapsed bounded edges and be on the interior of a cell in $\mathcal{M}^{disk,I}_{\Delta,d}(X_\Sigma)$. Therefore, ev' should be linear in a neighborhood of h, and we can define define

$$Mult(h) := |\det(ev')| lab(h).$$

Now that we've thoroughly defined the tropical disk, we can give the following lemma, which hints at their importance for the construction of tropical curves.

Lemma 2.2.12. Let $P_1, \ldots, P_k \in M_{\mathbb{R}}$ be general and $S \subseteq M_{\mathbb{R}}$ a subset. Let $h \in \mathcal{M}^{\text{trop}}_{\Delta,I}(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k, \psi^{\nu}S)$. Let $\Gamma'_1, \ldots, \Gamma'_l$ denote the closures of each of the connected components of $\Gamma \setminus E_x$, with h_i being the respective restrictions of h. Each disk h_i is viewed as being marked by those points $p \in \{p_1, \ldots, p_n\}$ with $E_p \subseteq \Gamma'_i$. If E_x doesn't share a vertex with any of the E_{p_i}, then $l = \nu + 2$. In the special case that E_{p_J} shares a vertex with E_x, we discard the edge E_{p_J} from discussion as well, so we have disks $h_1, \ldots, h_{\nu + r_{i_J} + 1}$.

1. If $S = M_{\mathbb{R}}$ and $|I| = |\Delta| - \nu - \sum_{j=1}^{n} r_{i_j}$, then either:

   (a) E_x does not share a vertex with any E_{p_i}. In this case each of the disks h_i are semirigid for all but two choices of i. For these two choices of i, h_i is rigid.

   (b) E_x shares a vertex with E_{p_J}. In this case, h_i is semirigid for all choices of i.

2. If $S = C$, a general translate of a tropical curve in $M_{\mathbb{R}}$, and $|I| = |\Delta| - \nu - \sum_{j=1}^{n} r_{i_j} - 1$, then h_i is semirigid for all but one choice of i, and is rigid for this choice.

3. If $S = Q$, a general point in $M_{\mathbb{R}}$, and $|I| = |\Delta| - \nu - \sum_{j=1}^{n} r_{i_j} - 2$, then h_i is semirigid for all i.

Proof. By Lemma 2.2.4, each of the moduli spaces in consideration are zero dimensional. Thus none of the h_i can be deformed while preserving their endpoint. By

the generality of the P_i and lemma 2, we must have $F(h_i) \leq 1$ for each i. Observe:

$$\begin{aligned}\sum_i F(h_i) &= \sum_i |\Delta(h_i)| - \left(\sum_{p_j \in h_i} (1 + r_{i_j})\right) \\ &= \begin{cases} |\Delta| - \sum_{j=1}^n r_{i_j} - (n-1) + r_{i_J}, \text{ Case (1)(b)} \\ |\Delta| - \sum_{j=1}^n r_{i_j} - n \text{ otherwise} \end{cases} \\ &= \begin{cases} \nu, \text{ Case (1)(a)} \\ \nu + r_{ij} + 1, \text{ Case (1)(b)} \\ \nu + 1, \text{ Case 2} \\ \nu + 2, \text{ Case 3} \end{cases}\end{aligned}$$

There are $\nu + 2$ disks in each case except for (1)(b), when there are $\nu + r_{i_J} + 1$ disks. □

Our analysis of tropical curves also requires another, slightly different, object. These will serve as the building pieces for the tropical disks which in turn form the tropical curves.

Definition 2.2.13 (Descendent tropical trees of type I)**.** Given $(\psi^{r_1} P_1, \ldots, \psi^{r_k} P_k)$, X_Σ and $1 \leq i_1 < \ldots < i_d \leq k$, let $I = \{i_j\}_j$, and $h : (\Gamma, p_1, \ldots, p_d) \to M_\mathbb{R}$ be a d-pointed tropical curve with the following properties

- $h(p_j) = P_{i_j}$ for each $i \in I$
- The valency of each vertex V is given by $r_V + 3$, where r_V is the sum of r_{i_j} over all p_i attached to V.
- A choice of $E_{out} \in \Gamma_\infty^{[1]}$ such that for any $E \in \Gamma_\infty^{[1]} \setminus E_{out}$, $h(e)$ is either a point or translate of some $\rho \in \Sigma^{[1]}$
- $w(E) = 1$ for all $E \in \Gamma_\infty^{[1]}$ that are not marked or chosen as E_{out}. This is the feature distinguishing the tropical tree from the tropical disk. If a tree were to have it's edge truncated at a point Q, it would be a disk.

Then we call h a *descendent tropical tree of type I*.

Note that flexibility and rigidity can be defined in the same way for tropical trees as was done for tropical disks. A descendent tropical disk can be formed from a descendent tropical tree of the same flexibility by the truncation of the outgoing edge E_{out} at a point $Q \in M_{\mathbb{R}}$, and a rigid tropical disk can be converted to an equally flexible tropical tree by infinitely extending the image of the bounded edge leading to Q. Thus each descendent tropical tree h with $F(h) = n$ corresponds to a 1-dimensional family of descendent tropical disks with arbitrary boundary Q and $F(h) = n$. Therefore, by 2.2.4, the set of descendent tropical trees with $F(h) = n$ in $(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k)$ is an n-dimensional polyhedral complex and the terms semirigid and rigid are appropriately evocative.

Definition 2.2.14 (Multiplicity for tropical trees)**.** Let h be a rigid descendent tropical tree in $(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k)$ with P_i chosen in general position. Define $M_{\Delta,d}^{tree,I}(X_\Sigma)$ to be the set of descendent tropical trees of type I. Define $ev'' : M_{\Delta,d}^{tree,I}(X_\Sigma) \to M_{\mathbb{R}}^d$ by $ev''(h) = (h(p_1), \ldots, h(p_d))$. By abuse of notation, we define

$$lab(h) := \prod_{v \in Vert(h)} \frac{1}{n_0(v)!n_1(v)!n_2(v)!}$$

where here E_{out} doesn't contribute to the product, and

$$Mult(h) := |\det(ev'')|lab(h)$$

If h is a descendent tropical tree or disk in $(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k)$, let $I(h) \subseteq \{1, \ldots, k\}$ be its defining set. Now that we have defined the notions of (semi)rigid disks and trees and their multiplicities, we will examine how to glue these objects together. These results will become important in the construction of *scattering diagrams* in a later section.

Lemma 2.2.15. Suppose h_1 and h_2 are rigid descendent tropical trees in

$$(X_\Sigma, \psi^{r_1^1}P_1, \ldots, \psi^{r_k^1}P_k) \text{ and } (X_\Sigma, \psi^{r_1^2}P_1, \ldots, \psi^{r_k^2}P_k),$$

respectively, with P_i chosen in general position. Name their respective outgoing edges E_1 and E_2, respectively, with $E_i = m_i - \mathbb{R}_{\geq 0}r(\Delta(h_i))$, $r_i \in T_\Sigma$, $m_i \in M_{\mathbb{R}}$

for $i = 1, 2$. Define $I_i := I(h_i)$ and suppose $I_1 \cap I_2 = \varnothing$ and $E_1 \cap E_2 = z \in M_{\mathbb{R}} \setminus \{P_1, \ldots, P_k\}$. Then h_1 and h_2 can be joined at z to form a new rigid descendent tropical tree h_3 in $(X_\Sigma, \psi^{r_1^3} P_1, \ldots, \psi^{r_k^3} P_k)$, where $r_i^3 = r_i^j$ if $i \in I_i$, with outgoing edge $E_3 = z - \mathbb{R}_{\geq 0} r(m_1 + m_2)$ and $Mult(h^3) = |r(\Delta(h_1)) \wedge r(\Delta(h_1))| Mult(h_1) Mult(h_2)$, where we've identified $\bigwedge^2 \mathbb{Z} \cong \mathbb{Z}$ using, say, the map $e_1 \wedge e_2 \mapsto 1$.

Proof. Note that h_3 meets all requirements for status as a rigid descendent tropical tree in $(X_\Sigma, \psi^{r_1^3} P_1, \ldots, \psi^{r_k^3} P_k)$ as $0 = F(h_1) + F(h_2) = F(h_3)$. The multiplicity argument can be facilitated by certain explicit choices of coordinates on $M_{\Delta,d}^{tree, I(h_i)}(X_\Sigma)$ for $1 \leq i \leq 3$. For each $i = 1, 2$, pick as the first coordinate the location in $M_{\mathbb{R}}$ of the vertex attached to E_i, and the other coordinates as the affine lengths of the various bounded edges.

Let $r(\Delta(h_1)) := v = (v_1, v_2)$, $r(\Delta(h_2)) := w = (w_1, w_2)$. Because we assume general position and $E_1 \cap E_2 \neq \varnothing$, then $|v \wedge w| \neq 0$ and $A := \begin{pmatrix} w_1 & v_1 \\ w_2 & v_2 \end{pmatrix}$ is invertible. Given our above choices of coordinates,

$$Mult(h_1) = \left| \det \begin{pmatrix} 1 & 0 & x_1(l_1) \\ 0 & 1 & y_1(l_1) \\ \vdots & \vdots & \vdots \\ 1 & 0 & x_{n_1}(l_1) \\ 0 & 1 & y_{n_1}(l_1) \end{pmatrix} \right| lab(h_1) \tag{2.2.1}$$

where $I(h_1) = \{i_1, \ldots i_{n_1}\}$ and $x_j(l)$ (resp. $y_j(l_1)$) is a vector giving the dependence of the x (resp. y) coordinate of the jth term of $ev''(h)$ on the lengths l_1 of the bounded edges of the tree defined by h_1. Likewise

$$Mult(h_2) = \left| \det \begin{pmatrix} 1 & 0 & x_{n_1+1}(l_2) \\ 0 & 1 & y_{n_1+1}(l_2) \\ \vdots & \vdots & \vdots \\ 1 & 0 & x_{n_1+n_2}(l_2) \\ 0 & 1 & y_{n_1+n_2}(l_2) \end{pmatrix} \right| lab(h_2) \tag{2.2.2}$$

where $I(h_2) = \{i_{n_1+1}, \ldots i_{n_1+n_2}\}$ and $x_j(l)$ (resp. $y_j(l)$) is a vector giving the dependence of the x (resp. y) coordinate of the $j - n_1$-st term of $ev''(h)$ on the

lengths l_2 of the bounded edges of the tree defined by h_2. Note that the base point of h_i is given by $z - r(m_i)s_i$ for $i = \{1, 2\}$ where s_i is the affine length of the bounded edge created from E_i. If we choose the coordinates on the moduli space as (location of the base point z, l_1', s_1, s_2, l_2') where l_i' are the lengths of the set of bounded edges of h_3 descended from h_i, then

$$Mult(h_3) = \left| \det \begin{pmatrix} 1 & 0 & x_1(l_1) & v_1 & 0 & 0 \\ 0 & 1 & y_1(l_1) & v_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & x_{n_1}(l_1) & v_1 & 0 & 0 \\ 0 & 1 & y_{n_1}(l_1) & v_2 & 0 & 0 \\ 1 & 0 & 0 & 0 & w_1 & x_{n_1+1}(l_2) \\ 0 & 1 & 0 & 0 & w_2 & y_{n_1+1}(l_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & w_1 & x_{n_1+n_2}(l_2) \\ 0 & 1 & 0 & 0 & w_2 & y_{n_1+n_2}(l_2) \end{pmatrix} \right| lab(h_3) \qquad (2.2.3)$$

Subtracting v_1 copies the first column and v_2 of the second from the fourth gives

$$Mult(h_3) = \left| \det \begin{pmatrix} 1 & 0 & x_1(l_1) & 0 & 0 & 0 \\ 0 & 1 & y_1(l_1) & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & x_{n_1}(l_1) & 0 & 0 & 0 \\ 0 & 1 & y_{n_1}(l_1) & 0 & 0 & 0 \\ 1 & 0 & 0 & -v_1 & w_1 & x_{n_1+1}(l_2) \\ 0 & 1 & 0 & -v_2 & w_2 & y_{n_1+1}(l_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & -v_1 & w_1 & x_{n_1+n_2}(l_2) \\ 0 & 1 & 0 & -v_2 & w_2 & y_{n_1+n_2}(l_2) \end{pmatrix} \right| lab(h_3) \quad (2.2.4)$$

$$= \left| \det \begin{pmatrix} 1 & 0 & x_1(l_1) \\ 0 & 1 & y_1(l_1) \\ \vdots & \vdots & \vdots \\ 1 & 0 & x_{n_1}(l_1) \\ 0 & 1 & y_{n_1}(l_1) \end{pmatrix} \right| lab(h_1) \left| \det \begin{pmatrix} v_1 & w_1 & x_{n_1+1}(l_2) \\ v_2 & w_2 & y_{n_1+1}(l_2) \\ \vdots & \vdots & \vdots \\ v_1 & w_1 & x_{n_1+n_2}(l_2) \\ v_2 & w_2 & y_{n_1+n_2}(l_2) \end{pmatrix} \right| lab(h_2) \quad (2.2.5)$$

Let M be the second matrix in equation 2.2.5 and M' the matrix in equation 2.2.1. Subdivide $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A is defined above, $M' = \begin{pmatrix} A' & B \\ C' & D \end{pmatrix}$ where $A' := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$\begin{aligned} \det M &= \det A(\det D - \det(C'A'^{-1}B)) = \det A'(\det D - \det(CA^{-1}B)) \\ &= \det A(\det A'[\det D - \det(CA^{-1}B)]) = \det A \cdot \det M'. \end{aligned}$$

Of course, $Mult(h_2) = |\det M'| lab(h_2)$, so we have

$$\begin{aligned} Mult(h_3) &= Mult(h_1)|\det M| lab(h_2) = Mult(h_1)|\det A||\det M' lab(h_2) \\ &= |v \wedge w| Mult(h_1) Mult(h_2) \end{aligned}$$

□

The next lemma shows how to join a semirigid tropical disk with a rigid tree to form a new semirigid disk. We will later introduce a method called *broken lines* that will help us understand disks formed in this way.

Lemma 2.2.16. Suppose h_2 is a semirigid descendent tropical disk in

$$(X_\Sigma, \psi^{r_1^2} P_1, \ldots, \psi^{r_k^2} P_k)$$

with basepoint Q and h_1 is a rigid descendent tropical tree in

$$(X_\Sigma, \psi^{r_1^1} P_1, \ldots, \psi^{r_k^1} P_k)$$

with P_i chosen in general position. Let E_1 be the outgoing edge of h_1 with $E_1 = v_1 - \mathbb{R}_{\geq 0} r(\Delta(h_1))$, and the edge incoming to V_{out} defined by $-r(\Delta(h_2))$. Define $I_i := I(h_i)$, and suppose $I_1 \cap I_2 = \varnothing$ and $Q \in E_1$. Then h_1 and h_2 can be joined at Q to form a new semirigid descendent tropical disk h_3 in $(X_\Sigma, \psi^{r_1^3} P_1, \ldots, \psi^{r_k^3} P_k)$ with $r_j^3 = r_j^i$ if $j \in I_i$, basepoint $Q' = Q - \epsilon r(\Delta(h_1 + h_2))$ for $\epsilon > 0$ small, and $Mult(h_3) = |r(\Delta(h_1)) \wedge r(\Delta(h_2))| Mult(h_1) Mult(h_2)$.

Proof. Note that h_3 meets all requirements for status as a semirigid descendent tropical disk in $(X_\Sigma, \psi^{r_1} P_1, \ldots, \psi^{r_k} P_k)$ as $1 = F(h_1) + F(h_2) = F(h_3)$. Again, we select certain explicit choices of coordinates on $M_{\Delta,d}^{tree, I(h_1)}(X_\Sigma)$ and $M_{\Delta,d}^{disk, I(h_i)}(X_\Sigma)$ for $i = 2, 3$. For each h_1 pick as the first coordinate the location in $M_\mathbb{R}$ of the vertex attached to to E_1, and the other coordinates as the affine lengths of the various bounded edges. For h_2, choose the first coordinate as the location in $M_\mathbb{R}$ of the base point and the latter coordinates the affine lengths of its bounded edges. Then:

$$Mult(h_1) = \left| \det \begin{pmatrix} 1 & 0 & x_1(l_1) \\ 0 & 1 & y_1(l_1) \\ \vdots & \vdots & \vdots \\ 1 & 0 & x_{n_1}(l_1) \\ 0 & 1 & y_{n_1}(l_1) \end{pmatrix} \right| lab(h_1) \tag{2.2.6}$$

where $I(h_1) = \{i_1, \ldots i_{n_1}\}$ and $x_j(l)$ (resp. $y_j(l_1)$) is a vector giving the dependence of the x (resp. y) coordinate of the jth term of $ev''(h)$ on the lengths l_1 of the

bounded edges of the tree defined by h_1. Likewise

$$Mult(h_2) = \left| \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & x_{n_1+n_2}(l_2) \\ 0 & 1 & y_{n_1+n_2}(l_2) \end{pmatrix} \right| lab(h_2) \tag{2.2.7}$$

where $I(h_2) = \{i_{n_1+1}, \ldots i_{n_1+n_2}\}$ and $x_j(l)$ (resp. $y_j(l)$) is a vector giving the dependence of the x (resp. y) coordinate of the $j - n_1$-st term of $ev'(h)$ on the lengths l_2 of the bounded edges of the tree defined by h_2. We will select coordinates on the moduli space for h_3 as follows. The first two coordinate are the location of the vertex V joining h_1 to h_2, followed by the lengths of the bounded edges descended from h_2, followed by the affine length of the bounded edge truncated from E_1, followed by the affine length of the edge joining Q' and V, followed by the lengths of the bounded edges coming from h_2. Order the marked points of h_3 so that those descended from h_1 precede those coming from h_2. Let the $r(m_1) := v = (v_1, v_2)$ and $r(m_2) := w = (w_1, w_2)$.

$$Mult(h_3) = \left| \det \begin{pmatrix} 1 & 0 & x_1(l_1) & v_1 & 0 & 0 \\ 0 & 1 & y_1(l_1) & v_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & x_{n_1}(l_1) & v_1 & 0 & 0 \\ 0 & 1 & y_{n_1}(l_1) & v_2 & 0 & 0 \\ 1 & 0 & 0 & 0 & w_1 + v_1 & 0 \\ 0 & 1 & 0 & 0 & w_2 + v_2 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & 0 & x_{n_1+n_2}(l_2) \\ 0 & 1 & 0 & 0 & 0 & y_{n_1+n_2}(l_2) \end{pmatrix} \right| lab(h_3)$$

Subtracting v_1 copies of the first column and v_2 copies of the second column from

the fourth gives:

$$Mult(h_3) = \left| \det \begin{pmatrix} 1 & 0 & x_1(l_1) & 0 & 0 & 0 \\ 0 & 1 & y_1(l_1) & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & x_{n_1}(l_1) & 0 & 0 & 0 \\ 0 & 1 & y_{n_1}(l_1) & 0 & 0 & 0 \\ 1 & 0 & 0 & -v_1 & w_1+v_1 & 0 \\ 0 & 1 & 0 & -v_2 & w_2+v_2 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & -v_1 & 0 & x_{n_1+n_2}(l_2) \\ 0 & 1 & 0 & -v_2 & 0 & y_{n_1+n_2}(l_2) \end{pmatrix} \right| lab(h_3)$$

The properties of $lab(h)$ and the block lower triangular form of the above matrix give us the desired result. □

2.3 The family of stationary tropical Landau-Ginzburg potentials

Recalling the description of the significance of the Landau-Ginzburg potential in the first chapter, we now construct a modified version of the tropically defined potential given in [11].

Definition 2.3.1 ($R_{k,n}$)**.** For each $P_i \in \{P_1, \ldots, P_k\}$ associate the variable u_i in the ring:

$$R_{k,n} := \frac{\mathbb{C}[u_1, t_1, \ldots, u_k, t_k]}{(t_1^n, u_1^2, \ldots, t_k^n, u_k^2)}$$

Definition 2.3.2 ($u_{I(h)}$)**.** Let h be a descendent tropical disk or tree in

$$(X_\Sigma, \psi^{r_1} P_1, \ldots, \psi^{r_k} P_k).$$

Then

$$u_{I(h)} := \prod_{i \in I(h)} u_i t_i^{r_i}$$

Definition 2.3.3 ($Mono(h)$)**.** Let h be a semirigid descendent tropical disk or rigid descendent tropical tree. Then

$$Mono(h) := Mult(h)u_{I(h)}z^{\Delta(h)} \in \mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,n}[[y_0]]$$

where $z^{\Delta(h)} \in \mathbb{C}[T_\Sigma]$ is the monomial associated to $\Delta(h)$

Definition 2.3.4 ($W_{k,n}(Q)$)**.** We define the k-pointed n-descendent (stationary) Landau
Ginzburg potential as

$$W_{k,n}(Q) := y_0 + \sum_h Mono(h) \in R_{k,n}$$

where the sum is over all semirigid $h \in \bigcup_{(r_1,\dots,r_k)\in\mathbb{Z}_{\geq 0}}(X_\Sigma, \psi^{r_1}P_1, \dots, \psi^{r_k}P_k)$ with basepoint Q. This sum has a finite number of non-zero terms given the expected dimension of the moduli space of these objects and the fact that monomials with power of t_i greater than n are equivalent to $0 \in R_{k,n}$.

2.3.1 B-model tropical moduli

Here we define the moduli space relevant to this picture. We will address the construction in some generality, but the reader should keep in mind our specific case of $X = \mathbb{P}^2$. In what follows, we fix $M := \mathbb{Z}^n$, $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. Fix a complete fan Σ in $M_{\mathbb{R}}$ with X_Σ a non-singular toric variety. Let T_Σ be the free abelian group generated by the elements of the one-dimensional cones of Σ, $\Sigma^{[1]}$, with $\rho \in \Sigma^{[1]}$ represented by v_ρ in T_Σ. Let $r : T_\Sigma \to M$ be the map given by $r(v_\rho) = \hat{\rho}$, where $\hat{\rho}$ is the primitive generator of ρ in M. In our case of $X_\Sigma = \mathbb{P}^2$, we have the fan Σ defined at the beginning of this chapter. As the assumption of non-singularity implies the surjectivity of r, we have the following exact sequence:

$$0 \longrightarrow K_\Sigma \longrightarrow T_\Sigma \xrightarrow{r} M \to 0$$

with K_σ the kernel of r. Dualizing over $\mathbb{Z}$ gives

$$0 \to N \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(T_\Sigma, \mathbb{Z}) \longrightarrow \mathrm{Pic}\, X_\Sigma \longrightarrow 0$$

Tensoring with $\mathbb{C}^\times$ gives the sequence

$$0 \to N \otimes \mathbb{C}^\times \to \mathrm{Hom}(T_\Sigma, \mathbb{C}^\times) \xrightarrow{\kappa} \mathrm{Pic}X_\Sigma \otimes \mathbb{C}^\times \to 0$$

defining κ, which will end up providing the family of mirrors to X_Σ.
The *Kähler moduli space* of X_Σ is defined

$$\mathcal{M}_\Sigma := \mathrm{Pic}X_\Sigma \otimes \mathbb{C}^\times = \mathrm{Spec}\mathbb{C}[K_\Sigma]$$

Of course, this is very simple in our case, with $K_\Sigma \cong \mathbb{Z}$. Note that κ, by definition, is now a map:

$$\kappa : \mathrm{Spec}\mathbb{C}[T_\Sigma] \to \mathcal{M}_\Sigma$$

A fiber of κ over a closed point of $\mathcal{M}_\Sigma$ is isomorphic to $\mathrm{Spec}\mathbb{C}[M]$. We can pass to the universal cover $\tilde{\mathcal{M}}_\Sigma := \mathrm{Pic}X_\Sigma \otimes \mathbb{C}$ of $\mathcal{M}_\Sigma$ with the map $\tilde{\mathcal{M}}_\Sigma \to \mathcal{M}_\Sigma$ given by $D \otimes y \mapsto D \otimes \exp(y)$.
Finally, set

$$\check{X}_\Sigma := \mathrm{Hom}(T_\Sigma, \mathbb{C}^\times) \times_{\mathcal{M}_\Sigma} \tilde{\mathcal{M}}_\Sigma,$$

pulling back the family given by κ to $\tilde{\mathcal{M}}_\Sigma$.

Define the k^{th}*-order thickening* of the Kähler moduli space $\tilde{\mathcal{M}}_\Sigma$ to be the ringed space

$$\tilde{\mathcal{M}} :_{\Sigma,k} := (\tilde{\mathcal{M}}_\Sigma, \mathcal{O}_{\Sigma,k})$$

where $\mathcal{O}_{\Sigma,k}(U)$ for $U \subseteq \tilde{\mathcal{M}}_\Sigma$ given by expressions of the form

$$\sum_{\substack{n=0 \\ I \subseteq \{1,\ldots,k\}}}^{\infty} f_{n,I} y^n u_I$$

where $u_I \in R_{k,n}$, $f_{n,I}$ is a holomorphic function on U for each n and I and there are only a finite number of terms for each n.

The k^{th}*-order thickening of the mirror family* $\check{X}_{\Sigma,k} := (\check{X}_\Sigma, \mathcal{O}_{\check{X}_{\Sigma,k}})$ is defined similarly, giving us a family

$$\kappa : \check{X}_{\Sigma,k} \to \tilde{\mathcal{M}}_{\Sigma,k}$$

We can now see the relevance of this discussion to our earlier constructions; $W_{k,n}(Q)$ is, by construction, a regular function on $\check{X}_{\Sigma,k}$. We can think of this map as providing a family of Landau-Ginzburg potentials.

The sheaf of relative differentials $\Omega^1_{\check{X}_{\Sigma,k}/\check{M}_{\Sigma,k}}$ is canonically isomorphic to the trivial locally free sheaf $M \otimes_{\mathbb{Z}} \mathcal{O}_{\check{X}_{\Sigma,k}}$, with $m \otimes 1$ corresponding to the differential

$$\mathrm{dlog} m := \frac{d(z^{\overline{m}})}{z^{\overline{m}}}$$

where $\overline{m}$ is any lift of $m \in M$ and $\mathrm{dlog} z^{\overline{m}}$ is well defined as a relative differential independent of the choice of the lift. Thus, a choice of generator $\bigwedge^2 M \cong \mathbb{Z}$ determines a nowhere-vanishing relative holomorphic two-form Ω, which is, up to sign, canonical. Explicitly, if e_1, e_2 is a positively oriented basis of M, we choose

$$\Omega := \mathrm{dlog} e_1 \wedge \mathrm{dlog} e_2$$

2.3.2 Automorphisms

In this section we define a particular set of automorphisms of $\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,n}[[y_0]]$.

Definition 2.3.5 (module of log derivations)**.** The module of log derivations of $\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,n}[[y_0]]$ is given by

$$\Theta(\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,n}[[y_0]]) := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,n}[[y_0]]) = (\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,n}[[y_0]]) \otimes_{\mathbb{Z}} N$$

An element of the module of log derivations $f \otimes n \in (\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,n}[[y_0]]) \otimes_{\mathbb{Z}} N$ acts as a derivation on $\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,n}[[y_0]]$ over $\mathbb{C}[K_\Sigma] \otimes_{\mathbb{C}} R_{k,n}[[y_0]]$ by

$$(f \otimes n)(z^m) = f\langle n, r(m)\rangle z^m$$

If $\sigma \in m_{R_{k,n}} \Theta(\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,n}[[y_0]])$ where $m_{R_{k,n}} = (u_1, \ldots, u_k) \in R_{k,n}$, then define:

$$\exp(\sigma) \in Aut(\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,n}[[y_0]])$$

by

$$\exp(\sigma(a)) := a + \sum_{i=1}^{\infty} \sigma^i(a)$$

There is a natural Lie bracket defined on this module:

$$\begin{aligned}[z^m\partial_n, z^{m'}\partial_{n'}] :=& z^m\partial_n(z^{m'}\partial_{n'} - z^{m'}\partial_{n'}(z^m))\partial_n \\ =& z^{m+m'}(\langle n, r(m'))\rangle\partial_{n'} - \langle n', r(m)\rangle\partial_n).\end{aligned}$$

Now we can define

$$\mathfrak{v}_{\Sigma,k} := \bigoplus_{m\in T_\Sigma, r(m)\neq 0} z^m(m_{R_{k,n}} \otimes r(m)^\perp) \subseteq \Theta(\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,n}[[y_0]])$$

It is easy to see that $\mathfrak{v}_{\Sigma,k}$ is closed under the Lie bracket, and is thus a Lie subalgebra of the module of log derivations. Because of this fact, following set is a group under the multiplication given by the Baker-Campbell-Hausdorff formula.

Definition 2.3.6 ($\mathbb{V}_{\Sigma,k}$)**.**

$$\mathbb{V}_{\Sigma,k} := \{\exp(\sigma)|\sigma \in \mathfrak{v}_{\Sigma,k}\}$$

Finally, as $\mathfrak{v}_{\Sigma,k}$ is generated by elements of the form $cu_I z^m \otimes n$, $\mathbb{V}_{\Sigma,k}$ is generated by elements of the form $\exp(cu_I z^m\otimes)$, whose action is given by:

$$\exp(cu_I z^m \otimes n)(z^{m'}) = z^{m'}(1 + cu_I\langle n, r(m')\rangle z^m)$$

The generators of this group preserve our choice of Ω; in fact, the original version of this group was defined as a group of Hamiltonian symplectomorphisms. The idea is contained in the following definition.

Definition 2.3.7 (X_f)**.** Given an element of $f \in \mathbb{C}[T_\Sigma] \otimes R_{k,n}[[y_0]])$, we define X_f as the corresponding Hamiltonian vector field with respect to Ω, that is, the vector field satisfying $\iota(X_f)\Omega = df$. In this definition, we are considering the relative differential with respect to κ, so df is an element of $(\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,n}[[y_0]]) \otimes_{\mathbb{Z}} M$. We write $f \otimes m$ as $f\mathrm{dlog}m$ and $d(z^m) = z^m\mathrm{dlog}r(m)$.

Given the above, we can explicitly describe X_{z^m} for $m \in M$. First, identify N with M using the isomorphism $\bigwedge^2 M \cong \mathbb{Z}$ given by the definition of Ω. For $m \in M$, define $X_m \in N$ by $X_m(m') = m \wedge m' \in \bigwedge^2 M \cong \mathbb{Z}$.

Lemma 2.3.8. If $f \in m_{R_{k,n}}(\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,n}[[y_0]])$ and $\theta \in \mathbb{V}_{\Sigma,k}$, then

$$\theta \circ X_f \circ \theta^{-1} = X_{\theta(f)}$$

Proof. See [11], Lemma 5.21. □

2.3.3 Scattering diagrams

Here we develop a tool for understanding the dependence of $W_{k,n}(Q)$ on $Q \in M_{\mathbb{R}}$. Although well suited for this purpose, it turns out that these objects have broader application, including some relation to cluster algebras. In the following, we again closely follow the presentation in [11], modifying where necessary.

Definition 2.3.9 (Scattering diagram)**.** Fix $k \geq 0$.

1. A *ray* or *line* is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$ such that

   - $\mathfrak{d} \in M_{\mathbb{R}}$ is given by
   $$\mathfrak{d} = m_0' - \mathbb{R}_{\geq 0} r(m_0)$$
   if $\mathfrak{d}$ is a ray and
   $$\mathfrak{d} = m_0' - \mathbb{R} r(m_0)$$
   if $\mathfrak{d}$ is a line, where $m_0' \in M_{\mathbb{R}}$ and $m_0 \in T_\Sigma$ with $r(m_0) \neq 0$. The set $\mathfrak{d}$ is the *support* of the ray or line. If $\mathfrak{d}$ is a ray, then m_0' is called the *initial point* and is denoted $Init(\mathfrak{d})$.
   - $f_{\mathfrak{d}} \in \mathbb{C}[z^{m_0}] \otimes_{\mathbb{C}} R_{k,n}[[y_0]]$
   - $f_{\mathfrak{d}} \equiv 1 \mod (u_1, \ldots, u_k) z^{m_0}$

2. A *scattering digram* $\mathfrak{D}$ is a finite collection of lines and rays.

If $\mathfrak{D}$ is a scattering diagram, we write

$$Supp(\mathfrak{D}) := \bigcup_{\mathfrak{d} \in \mathfrak{D}} \mathfrak{d} \subseteq M_{\mathbb{R}}$$

and

$$Sing(\mathfrak{D}) := \bigcup_{\mathfrak{d} \in \mathfrak{D}} \partial \mathfrak{d} \cup \bigcup_{\substack{\mathfrak{d}_1, \mathfrak{d}_2 \\ \dim \mathfrak{d}_1 \cap \mathfrak{d}_2 = 0}} \mathfrak{d}_1 \cap \mathfrak{d}_2$$

where $\partial \mathfrak{d} = \{Init(\mathfrak{d})\}$ if $\mathfrak{d}$ is a ray, and is empty if it is a line.

Definition 2.3.10 ($\theta_{\gamma,\mathfrak{D}} \in \mathbb{V}_{\Sigma,k}$)**.** Given a scattering diagram $\mathfrak{D}$ and smooth immersion $\gamma : [0,1] \to M_{\mathbb{R}} \setminus Sing(\mathfrak{D})$ whose endpoints are not in $Supp(\mathfrak{D})$ and with

γ intersecting $Supp(\mathfrak{D})$ transversally, we can use this information to define a ring automorphism $\theta_{\gamma,\mathfrak{D}} \in \mathbb{V}_{\Sigma,k}$. First, we find numbers

$$0 < t_1 \leq t_2 \leq \ldots \leq t_s < 1$$

and elements $\mathfrak{d}_i$ such that $\gamma(t_i) \in \mathfrak{d}_i$, $\mathfrak{d}_i \neq \mathfrak{d}_j$ if $i \neq j$ and s is taken to be as large as possible to account for all elements of $\mathfrak{D}$ that are crossed by γ. For each $i \in \{1, \ldots, s\}$, define $\theta_{\gamma,\mathfrak{d}_i} \in \mathbb{V}_{\Sigma,k}$ to be the automorphism with action

$$\theta_{\gamma,\mathfrak{d}_i}(z^m) = z^m f_{\mathfrak{d}_i}^{\langle n_0, r(m)\rangle}$$
$$\theta_{\gamma,\mathfrak{d}_i}(d) = d$$

for $m \in T_\Sigma$, $d \in R_{k,n}[[y_0]]$, where $n_0 \in N$ is chosen to be primitive, annihilating the tangent space to $\mathfrak{d}_i$ and satisfying

$$\langle n_0, \gamma'(t_i)\rangle < 0$$

Then $\theta_{\gamma,\mathfrak{D}} := \theta_{\gamma,\mathfrak{d}_s} \circ \cdots \circ \theta_{\gamma,\mathfrak{d}_1}$, where composition is taken from right to left.

Next we give the definition for the particular scattering diagram we'll be spending our time with.

Definition 2.3.11 ($\mathfrak{D}_n(\Sigma, P_1, \ldots, P_k)$)**.** Let $P_1, \ldots, P_k$ be chosen in general position. Define $\mathfrak{D}_n(\Sigma, P_1, \ldots, P_k)$ to be the scattering diagram which contains one ray for each rigid tropical descendent tree h in $(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k)$ for any choice of $r_i \leq n$. Note that this is a finite set for generally chosen P_i. The ray corresponding to h is of the form $(\mathfrak{d}, f_\mathfrak{d})$, where

- $\mathfrak{d} = h(E_{out})$
- $f_\mathfrak{d} = 1 + w_\Gamma(E_{out})Mult(h)z^{\Delta(h)}u_{I(h)}$

The behavior of this particular scattering diagram has one key property that will be used many times in what follows.

Lemma 2.3.12. Let $P_1, \ldots, P_n$ be generally chosen. If

$$P \in Sing(\mathfrak{D}_n(\Sigma, P_1, \ldots, P_k))$$

is a singular point with $P \notin \{P_1, \ldots, P_k\}$ and γ_p is a small loop around P, then $\sigma_{\gamma_p,\mathfrak{D}_n(\Sigma,P_1,\ldots,P_k))} = Id$.

Proof. See [11], Proposition 5.28 for the complete argument. A simpler situation that demonstrates the heart of the argument will be given. Consider the case in which only two rays, say $\mathfrak{d}_1$ and $\mathfrak{d}_2$, in the scattering diagram pass through (and do not begin) at $P \in Sing(\mathfrak{D}_n(\Sigma, P_1, \ldots, P_k))$ with $P \notin \{P_1, \ldots, P_k\}$. Assuming that the trees associated to $\mathfrak{d}_1$ and $\mathfrak{d}_2$ pass through no common points, Lemma 4 2.2.15 shows that one glue them at P to form a new rigid tropical tree whose outgoing edge gives a ray $\mathfrak{d}_3$ with initial point P in the scattering diagram. Then

$$\sigma_{\gamma_p, \mathfrak{D}_n(\Sigma, P_1, \ldots, P_k))} = \sigma^{-1}_{\gamma_p, \mathfrak{d}_1} \circ \sigma_{\gamma_p, \mathfrak{d}_2} \circ \sigma_{\gamma_p, \mathfrak{d}_3} \circ \sigma_{\gamma_p, \mathfrak{d}_1} \circ \sigma^{-1}_{\gamma_p, \mathfrak{d}_2} = Id.$$

The last equality can be seen by direct computation. □

2.3.4 Broken lines

Once we have assembled a scattering diagram, the semirigid disks with a particular endpoint Q can be found by analyzing objects called broken lines. The definition is rather involved, but the idea is quite simple. One begins with a line of slope equal to one of elements of $\Sigma^{[1]}$ in $M_{\mathbb{R}}$ far away from the point of interest, Q, and far outside of the chosen general points P_i. One labels the line with the polynomial associated to its element of T_Σ, and begins traveling along the line (in the direction opposite that specified by the monomial) until reaching an element $(\mathfrak{d}, f_\mathfrak{d}) \in \mathfrak{D}$. At this point, you can either choose to bend the line in a fashion dictated by $(\mathfrak{d}, f_\mathfrak{d})$ while appropriately adjusting the attached monomial, or continue on undisturbed. If, miraculously, you end up running into Q after some time, you've discovered a *broken line with endpoint Q*. More precisely:

Definition 2.3.13 (Broken lines)**.** A *broken line* is a continuous proper piecewise linear map

$$\beta : (-\infty, 0] \to M_{\mathbb{R}}$$

with endpoint $Q = \beta(0)$, along with some additional data. Let

$$-\infty = t_0 < t_1 < \cdots < t_n = 0$$

be the smallest set of real numbers such that $\beta|_{(t_{i-1}, t_i)}$ is linear. Then, for each $1 \leq i \leq n$, we are given the additional data of a monomial $c_i z^{m_i^\beta} \in \mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,n}[[y_0]]$ with $m_i^\beta \in T_\Sigma \setminus K_\Sigma$, satisfying:

1. For each i, $r(m_i^\beta) = -\beta'(t)$ for $t \in (t_{i-1}, t_i)$.

2. $m_1^\beta = v_\rho$ for some $\rho \in \Sigma^{[1]}$ and $c_1 = 1$.

3. $\beta(t_i) \in Supp(\mathfrak{D}_n(\Sigma, P_1, \ldots, P_k)) \setminus Sing(\mathfrak{D}_n(\Sigma, P_1, \ldots, P_k))$ for $1 \leq i \leq n$.

4. If $\beta \in \mathfrak{d}_1 \cap \cdots \cap \mathfrak{d}_s$, then $c_{i+1} z^{m_{i+1}^\beta}$ is a term in

$$(\theta_{\beta,\mathfrak{d}_1} \circ \cdots \circ \theta_{\beta,\mathfrak{d}_1})(c_i z^{m_i^\beta})$$

More explicitly, suppose that $f_{\mathfrak{d}_j} = 1 + c_{\mathfrak{d}_j} z^{m_{\mathfrak{d}_j}}$, $1 \leq j \leq s$, with $c_{\mathfrak{d}_j}^2 = 0$, and $n \in N$ is primitive, orthogonal to all of the $\mathfrak{d}_j$'s, chosen so that

$$\begin{aligned}(\theta_{\beta,\mathfrak{d}_s} \circ \cdots \circ \theta_{\beta,\mathfrak{d}_1})(c_i z^{m_i^\beta}) &= c_i z^{m_i^\beta} \prod_{j=1}^{s} (1 + c_{\mathfrak{d}_j} z^{m_{\mathfrak{d}_j}})^{\langle n, r(m_i^\beta)\rangle} \\ &= c_i z^{m_i^\beta} \prod_{j=1}^{s} (1 + \langle n, r(m_i^\beta)\rangle c_{\mathfrak{d}_j} z^{m_{\mathfrak{d}_j}})\end{aligned}$$

Then we must have

$$c_{i+1} z^{m_{i+1}^\beta} = \prod_{j \in J} \langle n, r(m_i^\beta)\rangle c_{\mathfrak{d}_j} z^{m_{\mathfrak{d}_j}}$$

for some $J \subseteq \{1, \ldots, s\}$ We interpret this as β being bent at time t_i by $\mathfrak{d}_j$ for $j \in J$.

We will also need to examine the variation of broken lines as their endpoint Q varies. That is the aim of the following two definitions.

Definition 2.3.14 (Family of broken lines)**.** A *family of broken lines* consists of:

- A continuous map $B : (-\infty, 0] \times I \to M_{\mathbb{R}}$ with I an interval in $\mathbb{R}$.

- Continuous functions $t_0, t_1, \ldots, t_n : I \to [-\infty, 0]$ such that $-\infty = t_0(s) < t_1(s) < \ldots < t_n(s) = 0$ for all $s \in I$. These functions define the the bending points of the broken line for any point s in the deformation.

- Monomials $c_i z^{m_i^B}$ for $1 \leq i \leq n$ Note that this implies that the basic directions of the broken line remain unchanged through the deformation.

The data satisfies the property that $B|_{(-\infty,0]\times\{s\}}$ is a broken line for any $s \in I$.

As is easy to imagine, there are situations which limit the deformation of a given broken line. One such occurrence might be that two of the bends converge, contracting a segment of the broken line. We would like to develop a language for speaking about these situations.

Definition 2.3.15 (Degenerate broken line)**.** A *degenerate broken line* is a limit of broken lines which bends at a point of $\mathrm{Sing}(\mathfrak{D}_n(\Sigma, P_1, \ldots, P_k))$. More precisely, it is a continuous proper map $\beta : (-\infty, 0] \to M_{\mathbb{R}}$ (which functions as a limiting "degenerate" curve) that comes along with data $-\infty = t_0 < t_1 \leq \ldots \leq t_n = 0$ and monomials $c_i z^{m_i^B}$ such that there exists

- A continuous map $B : (-\infty, 0] \times [0, 1] \to M_{\mathbb{R}}$.
- Continuous functions $\bar{t}_0, \ldots \bar{t}_n : [0, 1] \to [-\infty, 0]$ such that

$$-\infty = \bar{t}_0(s) \leq \cdots \leq \bar{t}_n(s) = 0$$

  for all $s \in [0, 1]$ with strict inequality for $s \neq 1$.

- $B_s := B_{(-\infty,0]\times\{s\}}$ (along with data $c_i z^{m_i^B}$) is a broken line for $s \neq 1$
- $\beta = B_1$ and $t_i = \bar{t}_i(1)$ for all i.
- $\beta(t_i) \in \mathrm{Sing}(\mathfrak{D}_n(\Sigma, P_1, \ldots, P_k))$ for some i.

Proposition 2.3.16. If $Q \notin \mathrm{Supp}(\mathfrak{D}_n(\Sigma, P_1, \ldots, P_k))$ is general, then there is a one-to-one correspondence between broken lines with endpoint Q and semirigid disks with boundary Q. In addition, if β is a broken line corresponding to a disk h, and cz^m is the monomial associated to the last segment of β, then

$$cz^m = \mathrm{Mono}(h)$$

Proof. The argument is identical to that appearing in [11] Proposition 5.32 except for the claim below.

Claim 2.3.17. Let $h : \Gamma' \to M_{\mathbb{R}}$ be a semirigid descendent tropical disk with basepoint Q' in $(X_\Sigma, \psi^{r_1} P_1, \ldots, \psi^{r_k} P_k)$, with P_i chosen in general position. Suppose furthermore that all vertices of Γ' except for V_{out} (bivalent) and V_i ($r_{i_j} + 3$-valent) are trivalent and h cannot be deformed continuously in a family of semirigid disks with endpoint Q'. Then there is a uniquely determined subset $\Xi = \Xi(h) \subseteq \overline{\Gamma}$ which is a union of edges of $\overline{\Gamma}$ and is homeomorphic to $[-\infty 0]$, connecting some point in $\overline{\Gamma}_\infty^{[0]} \setminus \{V_{out}\}$ to V_{out}, satisfying:

- Ξ is disjoint from ∂E_{p_i} for all i.
- The restriction of h to the closure of any connected component to $\Gamma' \setminus \Xi$ is a rigid descendent tropical disk.

Proof. We proceed inductively on the number of vertices of Γ'. The base case is trivial. For the induction step, let Γ' have outgoing edge E_{out} with vertices V_{out} and V, $h(V_{out}) = Q'$.

The first step is to show that V is disjoint from E_{p_i} for all i. Suppose, for contradiction, $V = P_i$. Then V is $r_i + 3$-valent (counting E_{p_i} and E_{out}). The remaining incoming edges of V can be thought of as the outgoing edges of descendent tropical disks h_j with basepoint P_i. Then $1 = F(h) = -r_i - 1 \sum_{j=1}^{r_i+1} F(h_j)$, so $r_i + 2 = \sum_{j=1}^{r_i+1} F(h_j)$, and $F(h_j) > 1$ for at least one h_j. This is a contradiction, as lemma 1 shows that such a disk could be deformed continuously while preserving the endpoint P_i, and thus h would also be deformable.

The rest of the proof of the claim is as in [11]. □

□

2.4 Wall Crossings for Q

In this section, we will examine the dependence of $W_{k,n}(Q)$ on Q, showing that changing this basepoint simply alters $W_{k,n}$ by an element of $\mathbb{V}_{\Sigma,k}$. First, we prove a simple lemma necessary for the larger result. It demonstrates how a rigid tropical tree can be pulled apart at a marked point to form a set of semirigid tropical disks.

Lemma 2.4.1. Let $h_1 \dots h_n$ be descendent semirigid tropical disks with boundary P_0 in $(X_\Sigma, \psi^{r_1}P_1, \dots, \psi^{r_k}P_k)$ with $P_0, \dots, P_n$ in general position. If $I(h_i) \cap I(h_j) = \varnothing$ for all $i \neq j$ and $r(\Delta(h_1) + \cdots + \Delta(h_n)) \neq 0$, then the disks h_i can be joined at P_0 to give a rigid descendent tropical tree h_0 in $(X_\Sigma, \psi^{n-1}P_0, \psi^{r_1}P_1, \dots, \psi^{r_k}P_k)$ with outgoing edge $P_0 - \mathbb{R}r(\Delta(h_1) + \cdots + \Delta(h_n))$. Let $M_i \subseteq \{h_i\}$ be the set of our original disks which are simply outgoing edges in the direction $\hat{\delta}_i$. Then

$$Mult(h_0) = \frac{1}{|M_0|!|M_1|!|M_2|!} \prod_{i \in \{1,\dots,n\}} Mult(h_i)$$

Proof. It is easy to see that the resulting tree is rigid. Order the coordinates on $M_{\Delta,d}^{tree,I(h_0)}(X_\Sigma)$ as follows. Start with the location of $h_0(p_0)$, followed by the lengths of the bounded edges of h_1, then h_2, and so on. Order the output so that first comes $h_0(p_0)$, then the locations of of the images of the marked points descended from h_1, then those from h_2, and so on. Similarly order the coordinates on $M_{\Delta,d}^{tree,I(h_i)}(X_\Sigma)$ for $i \neq 0$. Let $n_i = |I(h_i)|$, and l_i the set of lengths of bounded edges of h_i. Then

$$Mult(h_i) = \left| \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & x_{n_1+\cdots+n_{i-1}+1}(l_i) \\ 0 & 1 & y_{n_1+\cdots+n_{i-1}+1}(l_i) \\ \vdots & \vdots & \vdots \\ 1 & 0 & x_{n_1+\cdots+n_{i-1}+n_i}(l_i) \\ 0 & 1 & y_{n_1+\cdots+n_{i-1}+n_i}(l_i) \end{pmatrix} \right| lab(h_i) := |\det M_i| lab(h_i)$$

for $i \neq 0$ and

$$Mult(h_0) = \left| \det \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & x_1(l_1) & 0 & \cdots & 0 \\ 0 & 1 & y_1(l_1) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & 0 & \cdots & 0 \\ 1 & 0 & x_{n_1}(l_1) & 0 & \cdots & 0 \\ 0 & 1 & y_{n_1}(l_1) & 0 & \cdots & 0 \\ 1 & 0 & 0 & x_{n_1+1}(l_2) & \cdots & 0 \\ 0 & 1 & 0 & y_{n_1+1}(l_2) & \cdots & 0 \\ \vdots & \vdots & 0 & \vdots & \cdots & 0 \\ 1 & 0 & 0 & x_{n_1+n_2}(l_2) & \cdots & 0 \\ 0 & 1 & 0 & y_{n_1+n_2}(l_2) & \cdots & 0 \\ \vdots & \vdots & 0 & 0 & \ddots & 0 \\ 1 & 0 & 0 & 0 & \cdots & x_{n_1+\cdots+n_{n-1}+1}(l_n) \\ 0 & 1 & 0 & 0 & \cdots & y_{n_1+\cdots+n_{n-1}+1}(l_n) \\ \vdots & \vdots & 0 & 0 & \cdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & x_{n_1+\cdots+n_n}(l_n) \\ 0 & 1 & 0 & 0 & \cdots & y_{n_1+\cdots+n_n}(l_n) \end{pmatrix} \right| lab(h_0)$$

Let the matrix above be called M_0. Again, by lower block triangularity of M_i, $|\det M_0| = \prod_{i\in\{0,\ldots,n\}} |\det M_i|$. We're only left $lab(h_0)$ to worry about, but it is clear by definition that $lab(h_0) = \frac{1}{|M_0|!|M_1|!|M_2|!} \prod_{i\in\{1,\ldots,n\}} lab(h_i)$.

□

Lemma 2.4.2. Let h_0 be a rigid tropical tree in $(X_\Sigma, \psi^{\nu-1}P_0, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k)$ with E_{out} attached to E_{p_i} and $I(h_1) = \{i_0, \ldots, i_n\}$. Then, by splitting h_0 at P_0, we can form ν semirigid descendent tropical disks with endpoint P_0 in

$$(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k).$$

Proof. Call the ν tropical disks formed by the above procedure $h_1, \ldots, h_\nu$. Each

$F(h_i) \leq 1$ as h_0 is rigid. Note $F(h_0) = |\Delta(h_0)| - 1 - (\nu - 1) - \sum_{i=1}^{n} 1 + r_{i_j} = 0$, so

$$\sum_{i=1}^{n} F(h_i) = \sum_{i}^{n} |\Delta(h_i)| - \left(\sum_{p_j \in h_i} (1 + r_{i_j}) \right) = \nu$$

Thus $F(h_i) = 1$ for all $i \in \{1, \ldots, n\}$. □

Theorem 2.4.1. If Q, $Q' \in M_{\mathbb{R}} \setminus \mathrm{Supp}(\mathfrak{D}_n(\Sigma, P_1, \ldots, P_k))$ are general, and γ is a path connecting Q and Q' for which $\theta_{\gamma, \mathfrak{D}_n(\Sigma, P_1, \ldots, P_k)}(W_{k,n}(Q))$ is defined, then $\theta_{\gamma, \mathfrak{D}_n(\Sigma, P_1, \ldots, P_k)}(W_{k,n}(Q)) = W_{k,n}(Q')$.

Proof. As in much of the remainder of this thesis, this proof follows its counterpart (Theorem 5.35) in [11]. The only significant new material is found in dealing with the situation in which a broken line degenerates at a marked point. This arrangement follows a larger theme, the fact that the analysis of our scattering diagrams is identical to the behavior of Gross's scattering diagrams as long as we stay away from marked points. Let $\mathfrak{D} := \mathfrak{D}_n(\Sigma, P_1, \ldots, P_k)$, and U the union of $\mathfrak{D}$ with the images of all degenerate broken lines with arbitrary endpoint. Of course, $dim(U) \leq 1$. A broken line B is semirigid, and can generally be deformed continuously by translating its endpoint. As the endpoint moves, the terminal segment is pulled along with it and the translation will generally propagate continuously through the length of the broken line. We may run into a problem, however, if one of the bends of the line hits a singular point of $\mathfrak{D}$. In this case, B converges to a degenerate broken line. Therefore, B should deform continuously as long as we keep its endpoint within a single component of $M_{\mathbb{R}} \setminus U$. Under continuous deformations, the monomial associated to B remains constant, so $W_{k,n}(Q_1) = W_{k,n}(Q_2)$ as long Q_1 and Q_2 share a component of $M_{\mathbb{R}} \setminus U$.

Let u_1, u_2 be adjacent connected components of $M_{\mathbb{R}} \setminus U$ and $L = \overline{u}_1 \cap \overline{u}_2$, and assume that $dim(L) = 1$. Situate $Q_i \in u_i$ near to L, with $\gamma : [0, 1] \to M_{\to} R$ a short general path from Q_1 to Q_2 crossing L exactly once at $\gamma(s_0)$. As γ is general, U is a manifold nearby $\gamma(s_0)$. Our analysis now focuses on how different types of broken lines propagate as their endpoint crosses over L. We will distinguish broken lines by the direction of their terminal segment.

Let n_0 be a primitive vector annihilating the tangent space to L at $\gamma(s_0)$ and pointing toward Q_2, and $\mathfrak{B}(Q_i)$ be the set of broken lines ending at Q_i. For $\beta \in \mathfrak{B}(Q_i)$, set $m_\beta = \beta_*(-\frac{\partial}{\partial t}|_{t=0})$, a vector pointing away from Q_i along the terminal segment of β. Now we partition $\mathfrak{B}(Q_i) = \mathfrak{B}^+(Q_i) \cup \mathfrak{B}^0(Q_i) \cup \mathfrak{B}^-(Q_i)$ where $\beta \in \mathfrak{B}^+(Q_i)$ if $\langle n_0, m_\beta \rangle > 0$, with $\mathfrak{B}^0(Q_i)$ and $\mathfrak{B}^-(Q_i)$ defined likewise. Then set $W_{k,n}(Q_i)^+$ to be the contribution to $W_{k,n}(Q_i)$ from elements of $\mathfrak{B}^+(Q_i)$. Define $W_{k,n}(Q_i)^0$ and $W_{k,n}(Q_i)^-$ analogously. We aim to show

$$\theta_{\gamma,\mathfrak{D}}(W_{k,n}(Q_1)^-) = W_{k,n}(Q_2)^- \tag{2.4.1}$$

$$\theta^{-1}_{\gamma,\mathfrak{D}}(W_{k,n}(Q_2)^+) = W_{k,n}(Q_2)^+ \tag{2.4.2}$$

$$W_{k,n}(Q_1)^0 = W_{k,n}(Q_2)^0 \tag{2.4.3}$$

By the nature of $\theta_{\gamma,\mathfrak{D}}$, $\theta_{\gamma,\mathfrak{D}}(W_{k,n}(Q_1)^0) = W_{k,n}(Q_1)^0$, so these three equalities will prove the theorem.

Proof of (2.4.1) and (2.4.2) Note that (2.4.1) follows from (2.4.2) by interchanging the roles of Q_1 and Q_2. Now, we're assuming that Q_1 is very close to L, so if $\beta \in (B)(Q_1)$, $\beta([t_{n-1}, 0])$ interesects L if and only if $\beta \in \mathfrak{B}^+(Q_i)$. Let $\beta \in \mathfrak{B}^-(Q_1)$ with terminal monomial cz^m. Note that we can deform β continuously along γ through γ_{s_0}, for if it converged to a degenerate broken line, it would have to do so at L, and in doing so would violate the fact that U is a manifold near γ_{s_0}. Let $\theta_{\Sigma,\mathfrak{D}}(cz^m) = \sum_{i=1}^{l} c_i z^{m_i}$, and β' be the deformation of β with endpoint γ_{s_0}. For each $i \in \{1, \ldots, l\}$, we can form a new broken line β_i with monomial $c_i z^{m_i}$ by appending short segments of direction $-r(m_i)$ to β'. These are broken lines with endpoint in u_2, and can thus be continuously deformed into membership in $\mathfrak{B}^-(Q_2)$. Clearly, any $\beta \in \mathfrak{B}^-(Q_2)$ arises in this way, so we have (2.4.1).

Proof of (2.4.3). Here we partition $\mathfrak{B}(Q_i)^0 = \bigsqcup_{j=1}^{l} \mathfrak{B}^i_j$ and show that for each $j \in \{1, \ldots, l\}$, $\mathfrak{B}^1_j$ and $\mathfrak{B}^2_j$ make equal contributions to $W_{k,n}(Q_1)$ and $W_{k,n}(Q_2)$ respectively. We also assume that a broken line with endpoint $\gamma(s_0)$ passes through at most one singular point. The general case follows by a simple induction argument.

Suppose $\beta_1 \in \mathfrak{B}(Q_1)^0$ deforms continuously to $\beta_2 \in \mathfrak{B}(Q_2)^0$. In this case, each β_i will appear in a one element set, say $\mathfrak{B}^i_j$, and each $\mathfrak{B}^i_j$ will make the same contribution to $W_{k,n}(Q_i)$.

If $\beta \in \mathfrak{B}(Q_1)^0$ cannot be continuously deformed to an element of $\mathfrak{B}(Q_2)^0$, then it must deform to a degenerate broken line when the basepoint reaches γ_{s_0}. In other words, there is a map $B : (-\infty, 0] \times [0, s_0] \to M_{\mathbb{R}}$ such that $B|_{(-\infty,0]\times\{0\}} = \beta$, $B|_{(-\infty,0]\times[0,s_0)}$ is a continuous deformation, and $B|_{(-\infty,0]\times\{s_0\}}$ is a degenerate broken line bending at $P \in Sing(\mathfrak{D})$, with $P = B(\hat{t}_j(s_0), s_0)$. There are two cases to examine: $P \in \{P_1, \ldots, P_k\}$ and $P \notin \{P_1, \ldots, P_k\}$.

Case where $P = P_m \in \{P_1, \ldots, P_k\}$. This is the case that requires a different approach than that seen in [11]. Let us begin with a broken line β as discussed in the previous paragraph. Because β deforms to a degenerate line bending at P_m, we know that it must bend along exactly one ray $\mathfrak{d}_0$ emanating from P_m. Assume that the function attached to d_0 has a monomial containing $t_m^{r_m}$. By construction, $\mathfrak{d}_0$ is produced by a rigid descendent tropical tree, which, by Lemma 2.4.2, is constructed from $r_m + 1$ semirigid descendent tropical disks with endpoint P_i, called $h_1, \ldots h_{r_m+1}$. Also note that $B|_{(-\infty,\bar{t}_j(s_0)]\times\{s_0\}}$ is a broken line ending at P_m, corresponding to a semirigid disk h_0 with endpoint P_m. The monomials of the h_i for $i \in 0, \ldots, r_m + 1|$ are compatible in the sense that none of the disks pass through a common marked point (except P_m). This set of data, a broken line degenerating at P_m and a ray emanating from P_m along which it bends, will define a set of broken lines on each side of the wall defined by the degeneration of β; we will show that these give equal contributions to $W_{k,n}(Q_i)^0$.

The first step is to note that, by Lemma 2.4.1, we can form a rigid tropical tree g_j for each $0 \leq j \leq r_m + 1$ by joining all of the h_i except for h_j at P_i and extending an unbounded outgoing edge d_j as dictated by the balancing condition. See Figure 2.4.1.

Some of the disks h_i may be translates of the rays $\Sigma^{[1]}$, and their labelling ambiguity will contribute to the multiplicity of the trees from which they are assembled. Let $M_i \subseteq \{h_0, \ldots h_{r_m+1}\}$ be the set of disks which are unbounded edges headed out in the direction $\hat{\rho}_i$ from P_m. The polynomial associated to the

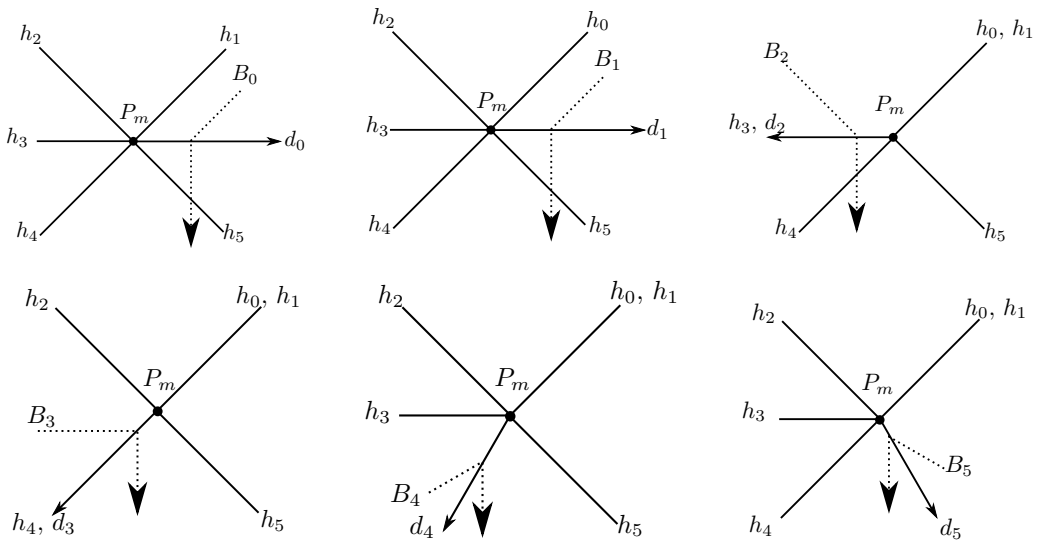

Figure 2.4.1: Example of mechanics of Theorem 2 where $P \in \{P_1, \ldots, P_k\}$.

ray $\mathfrak{d}_i$ generated by the edge E^i_{out} of g_i is then given by:

$$\begin{aligned} f_{\mathfrak{d}_i} &= 1 + w_\Gamma(E^i_{out}) Mult(g_i) z^{\Delta(g_i)} u_{I(g_i)} \\ &= 1 + w_\Gamma(E^i_{out}) u_m \frac{1}{|M_0 \setminus \{h_i\}|! |M_1 \setminus \{h_i\}|! |M_2 \setminus \{h_i\}|!} \prod_{l \neq i} \mathrm{Mono}(h_l) \end{aligned}$$

Next, note that each choice of $0 \leq i \leq r_m + 1$ where $w(d_i) \neq 0$ gives rise to a broken line B_i bending along $\mathfrak{d}_i$; this is the broken line that arises from the concatenation of the broken line associated with h_i (see Proposition 2.3.16) and $B|_{[\bar{t}_j(s_0),0]} \times \{s_0\}$ (see Figure 2.4.1). Observe that, in this notation, $\beta = B_0$. In the exceptional case where $\mathfrak{d}_i$ is collapsed (as dictated by the balancing condition), note that the line B_i does not bend between the h_i and $B|_{[\bar{t}_j(s_0),0]} \times \{s_0\}$ portion. In this case, we treat the situation as if the broken line has a virtual bend; this line will not converge to a degenerate broken line and will deform continuously to contribute the same quantity on both sides of the wall defined by the degeneration of β.

Note that the side of the wall that each B_i inhabits is dictated by the sign of $v(d_i) \wedge v(B|_{[\bar{t}_j(s_0),0] \times \{s_0\}})$, where $v(d_i)$ gives the direction vector of d_i and $v(B|_{[\bar{t}_j(s_0),0] \times \{s_0\}})$ gives the direction vector for the outgoing piece of the broken line. Also, $v(B|_{[\bar{t}_j(s_0),0] \times \{s_0\}})$ is given by $-\sum_{j=0}^{r_m+1} r(\Delta(h_j))$. Our goal is to show

that the contribution to the monomial from broken lines B_i for which $v(d_i) \wedge v(B|_{[\bar{t}_j(s_0),0]\times\{s_0\}})$ is negative is equal to the contribution of those for which it is positive. Let $I^- := \{l \in \{0,\dots,r_m+1\} \mid \left(\sum_{l\neq i} r(\Delta(h_l)\right) \wedge r(\Delta(h_i)) < 0\}$ under the identification of $\bigwedge^2 M_{\mathbb{R}}$ with $\mathbb{Z}$, with I^0 and I^+ defined analogously. Some of the semirigid disks h_i may be copies of the same translate of a ray from $\Sigma^{[1]}$, and thus the set I^- may index multiple copies of the same broken line. To account for this, define $\overline{I}^-$ to be the partition of I^- by the equivalence relation $i \sim j$ if $h_i \equiv h_j$. If $i \neq j$, the only way this can happen is if $h_i, h_j \in M_k$ for some k. Define $\overline{I}^+$ likewise.

The monomial obtained from the bend of B_i at d_i is given by

$$w(d_i)\langle n_i, r(\Delta(h_i)\rangle \mathrm{Mono}(h_i)\mathrm{Mono}(g_j) =$$
$$w(d_i)\langle n_i, r(\Delta(h_i)\rangle \mathrm{Mono}(h_i) u_m \frac{1}{|M_0 \setminus \{h_i\}|!|M_1 \setminus \{h_i\}|!|M_2 \setminus \{h_i\}|!} \prod_{l\neq i} \mathrm{Mono}(h_l) =$$
$$w(d_i)\langle n_i, r(\Delta(h_i)\rangle u_m \frac{1}{|M_0 \setminus \{h_i\}|!|M_1 \setminus \{h_i\}|!|M_2 \setminus \{h_i\}|!} \prod_{l} \mathrm{Mono}(h_l),$$

where n_i is a vector perpendicular to d_i chosen so that $w(d_i)\langle n_i, r(\Delta(h_i)\rangle = |\left(\sum_{l\neq i} r(\Delta(h_l)\right) \wedge r(\Delta(h_i))|$ (as the vector defining d_i is given by $\sum_{l\neq i} r(\Delta(h_l)$. We then wish to show that:

$$\sum_{[i]\in\overline{I}^-} w(d_i)\langle n_i, r(\Delta(h_i)\rangle u_m \frac{1}{|M_0 \setminus \{h_i\}|!|M_1 \setminus \{h_i\}|!|M_2 \setminus \{h_i\}|!} \prod_{l} \mathrm{Mono}(h_l)$$
$$=$$
$$\sum_{[i]\in\overline{I}^+} w(d_i)\langle n_i, r(\Delta(h_i)\rangle u_m \frac{1}{|M_0 \setminus \{h_i\}|!|M_1 \setminus \{h_i\}|!|M_2 \setminus \{h_i\}|!} \prod_{l} \mathrm{Mono}(h_l).$$

The i for which $v(d_i) \wedge v(B|_{[\bar{t}_j(s_0),0]\times\{s_0\}}) = 0$ are precisely those for which the edge E_{out} is collapsed. Therefore, the contribution to $W_{k,n}(Q_i)^0$ from I^0 is automatically equal for $i = \{1,2\}$ as these broken lines can be deformed continuously past the wall defined by the degeneracy of β.

The result then follows from some basic observations.

$$\begin{aligned}
0 &= \left(\sum_{j=0}^{r_m+1} r(\Delta(h_j))\right) \wedge \left(\sum_{j=0}^{r_m+1} r(\Delta(h_j))\right) \\
&= \sum_{j=0}^{r_m+1} r(\Delta(h_j)) \wedge \left(\sum_{l=0}^{r_m+1} r(\Delta(h_l))\right) \\
&= \sum_{j=0}^{r_m+1} r(\Delta(h_j)) \wedge \left(\sum_{l\neq j} r(\Delta(h_l)\right) \\
&= \sum_{j\in I^-} r(\Delta(h_j)) \wedge \left(\sum_{l\neq j} r(\Delta(h_l)\right) + \sum_{j\in I^+} r(\Delta(h_j)) \wedge \left(\sum_{l\neq j} r(\Delta(h_l)\right) \\
&+ \sum_{j\in I^0} r(\Delta(h_j)) \wedge \left(\sum_{l\neq j} r(\Delta(h_l)\right) \\
&= \sum_{j\in I^-} r(\Delta(h_j)) \wedge \left(\sum_{l\neq j} r(\Delta(h_l)\right) + \sum_{j\in I^+} r(\Delta(h_j)) \wedge \left(\sum_{l\neq j} r(\Delta(h_l)\right)
\end{aligned}$$

Then

$$-\sum_{j\in I^-} r(\Delta(h_j)) \wedge \left(\sum_{l\neq j} r(\Delta(h_l)\right) = \sum_{j\in I^+} r(\Delta(h_j)) \wedge \left(\sum_{l\neq j} r(\Delta(h_l)\right)$$

so

$$\sum_{j\in I^-} \left| r(\Delta(h_j)) \wedge \left(\sum_{l\neq j} r(\Delta(h_l)\right)\right| = \sum_{j\in I^+} \left| r(\Delta(h_j)) \wedge \left(\sum_{l\neq j} r(\Delta(h_l)\right)\right|$$

and

$$\sum_{j\in I^-} w(d_j)\langle n_j, r(\Delta(h_i)\rangle = \sum_{j\in I^+} w(d_j)\langle n_j, r(\Delta(h_i)\rangle$$

thus

$$\begin{aligned}
&\sum_{j\in I^-} w(d_j)\langle n_j, r(\Delta(h_i)\rangle u_m \frac{1}{|M_0|!|M_1|!|M_2|!} \prod_l \mathrm{Mono}(h_l) \qquad (2.4.4)\\
&= \sum_{j\in I^+} w(d_j)\langle n_j, r(\Delta(h_i)\rangle u_m \frac{1}{|M_0|!|M_1|!|M_2|!} \prod_l \mathrm{Mono}(h_l)
\end{aligned}$$

This is almost the equality we want. To finish, note that if $h_i \in M_k$, $\#[i] = M_k$.

Otherwise, $\#[i] = 1$. Therefore

$$\begin{aligned}
&\sum_{[i]\in\bar{I}^-} w(d_i)\langle n_i, r(\Delta(h_i)\rangle u_m \frac{1}{|M_0 \setminus \{h_i\}|!|M_1 \setminus \{h_i\}|!|M_2 \setminus \{h_i\}|!} \prod_l \mathrm{Mono}(h_l) \\
&= \sum_{i\in I^-} w(d_i)\langle n_i, r(\Delta(h_i)\rangle u_m \frac{1}{\#[i]|M_0 \setminus \{h_i\}|!|M_1 \setminus \{h_i\}|!|M_2 \setminus \{h_i\}|!} \prod_l \mathrm{Mono}(h_l) \\
&= \sum_{i\in I^-} w(d_i)\langle n_i, r(\Delta(h_i)\rangle u_m \frac{1}{|M_0|!|M_1|!|M_2|!} \prod_l \mathrm{Mono}(h_l) \\
&= \sum_{i\in I^+} w(d_i)\langle n_i, r(\Delta(h_i)\rangle u_m \frac{1}{|M_0|!|M_1|!|M_2|!} \prod_l \mathrm{Mono}(h_l) \\
&= \sum_{i\in I^-} w(d_i)\langle n_i, r(\Delta(h_i)\rangle u_m \frac{1}{\#[i]|M_0 \setminus \{h_i\}|!|M_1 \setminus \{h_i\}|!|M_2 \setminus \{h_i\}|!} \prod_l \mathrm{Mono}(h_l) \\
&= \sum_{[i]\in\bar{I}^+} w(d_i)\langle n_i, r(\Delta(h_i)\rangle u_m \frac{1}{|M_0 \setminus \{h_i\}|!|M_1 \setminus \{h_i\}|!|M_2 \setminus \{h_i\}|!} \prod_l \mathrm{Mono}(h_l).
\end{aligned}$$

Thus we see that the contributions to $W_{k,n}$ from our set of broken lines is equal on either side of the wall. Note that deforming any of the B_i (except those in I^0) to degeneration at P_m will give exactly the same set of broken lines, showing that broken lines degenerating at P_m (for a particular deformation of Q) can be partitioned into sets which give equal contributions to $W_{k,n}(Q_i)^0$ for $i \in \{1, 2\}$.

Case where $P \notin \{P_1, \ldots, P_k\}$ The argument for this case is identical to that presented in [11]. □

2.5 Wall Crossings for P_i

We follow [11], Theorem 5.39, with some modifications where necessary. Most of the material is identical except for Case 2 of Claim 2.5.3. As the points $P_1, \ldots, P_k$ vary, the scattering diagram encoding the rigid disks will also vary. In order to better understand this type of variation, we construct families of scattering diagrams varying in one dimension.

Definition 2.5.1 (Scattering diagrams in $M_{\mathbb{R}} \times L$)**.** Let $L \subseteq \mathbb{R}$ be a closed interval. Our ambient space will for this section be $M_{\mathbb{R}} \times L$. Let π_1 be the projection onto $M_{\mathbb{R}}$ and π_2 the projection onto L. A scattering diagram in $M_{\mathbb{R}} \times L$ is a finite set $\mathfrak{D}$ consisting of pairs $(\mathfrak{d}, f_{\mathfrak{d}})$ such that:

- $\mathfrak{d} \subseteq M_{\mathbb{R}} \times L$ is a polyhedron of dimension 2 such that $\pi_2(\mathfrak{d})$ is one dimension. Furthermore, there is a one dimension subset $\mathfrak{b} \subseteq M_{\mathbb{R}} \times L$ and an element $m_0 \in T_\Sigma$ such that
$$\mathfrak{d} = \mathfrak{b} - \mathbb{R}_{\geq 0}(r(m_0), 0)$$
- $f_{\mathfrak{d}} \in \mathbb{C}[z^{m_0}] \otimes_{\mathbb{C}} R_{k,n}[[y_0]]$
- $f \cong 1 \mod (u_1, \ldots, u_k) z^{m_0}$.

Define
$$\mathrm{Sing}(\mathfrak{D}) = \bigcup_{\mathfrak{d} \in \mathfrak{D}} \partial \mathfrak{d} \cup \bigcup_{\substack{\mathfrak{d}_1, \mathfrak{d}_2 \in \mathfrak{D} \\ \dim(\mathfrak{d}_1 \cap \mathfrak{d}_2) = 1}} \mathfrak{d}_1 \cap \mathfrak{d}_2$$

Because this set of singularities is 1-dimensional, we will stratify it into dimensional components. Let $Interstices(\mathfrak{D})$ be the finite set of points where $\mathrm{Sing}(\mathfrak{D})$ is not a manifold, and $Joints(\mathfrak{D})$ the set of closures of connected components of $\mathrm{Sing}(\mathfrak{D}) \setminus Interstices(\mathfrak{D})$. If the image of a joint j under π_2 is a point, then j is called *horizontal*, and we define all joints that are not horizontal to be *vertical*.

One can define $\theta_{\gamma,\mathfrak{D}} \in \mathbb{V}_{\sigma,k}$ for a path $\gamma \in (M_{\mathbb{R}} \times L) \setminus \mathrm{Sing}(\mathfrak{D})$ in an analogous fashion to what was done in the usual version of scattering diagrams. As before, this automorphism will be the composition of a set defined by the components of the scattering diagram that the path crosses. Assume $\gamma(t_i) = (\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}$, and

passes from one side to the other. Choose $n_0 \in N$ primitive so that $\langle n_0, r(m_0)\rangle = 0$ and n_0 is smaller on the side of $\mathfrak{d}$ that γ passes into. Then,

$$\theta_{\gamma,\mathfrak{d}}(z^m) = z^m f_{\mathfrak{d}}^{\langle n_0, r(m)\rangle}.$$

A *broken line* in $M_{\mathbb{R}} \times L$ is an object that is simply a broken line in one of the scattering diagrams in the varying family. More precisely, it is a map $\beta : (-\infty, 0] \to M_{\mathbb{R}} \times L$ along with data $t_0 < \cdots < t_n$ and monomials $c_i z^{m_i^\beta}$ such that:

1. $\pi_2 \circ \beta = P \in L$.
2. $\pi_1 \circ \beta$ is a broken line (as defined before) with respect to the scattering diagram $\mathfrak{D}_P$ in $M_{\mathbb{R}}$ given, after identifying $M_{\mathbb{R}} \times P$ with $M_{\mathbb{R}}$, by

$$\mathfrak{D}_P := \{(\mathfrak{d} \cap (M_{\mathbb{R}} \times \{P\}), f_{\mathfrak{d}}) | (\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D} \text{ such that } \mathfrak{d} \cap (M_{\mathbb{R}} \times \{P\}) \neq \emptyset\}$$

Theorem 2.5.1. Let W and W' be $W_k(Q)$ for two different choices of general points $P_1, \ldots, P_k$ and $P_1', \ldots, P_k'$. Then

$$W' = \theta(W)$$

for some $\theta \in \mathbb{V}_{\Sigma,k}$.

Proof. This proof will proceed by induction on k, noting that the base case of $k = 1$ is equivalent to Theorem 2.4.1. It is also clearly enough to show this result in the case that only P_1 moves, as we can then successively use the same result to move the other points.

Consider a set of points in general position, $P_1^1, \ldots, P_k, P_1^2$, and a line segment L joining P_1^1 to P_1^2. Our goal, of course, will be to show that moving P_1^1 to P_1^2 transforms $W_{k,n}$ by an element of $\mathbb{V}_{\Sigma,k}$. For all but a finite number of points $P_1 \in L$, we can assume that $P_1, P_2, \ldots, P_k$ are sufficiently general so ensure that $Trees_n(\Sigma, P_1, P_2, \ldots, P_k)$ is a finite set and all elements are of general type (that is, all vertices are trivalent except possibly those connected to marked points). This gives rise to a family of scattering diagrams $\mathfrak{D}_n(\Sigma, P, P_2, \ldots, P_k)$ for $P \in L$ general. These can be assembled into a scattering diagram $\tilde{\mathfrak{D}} = \tilde{\mathfrak{D}}_n(\Sigma, L, P_2, \ldots, P_k)$ in $M_{\mathbb{R}} \times L$, determined by the property that for $P \in L$ general,

$$\mathfrak{D}_n(\Sigma, P, P_2, \ldots, P_k) =$$
$$\left\{(\tilde{\mathfrak{d}} \cap (M_{\mathbb{R}} \times \{P\}), f_{\tilde{\mathfrak{d}}}) | (\tilde{\mathfrak{d}}, f_{\tilde{\mathfrak{d}}}) \in \tilde{\mathfrak{D}}_n \text{ such that } \tilde{\mathfrak{d}} \cap (M_{\mathbb{R}} \times \{P\}) \neq \emptyset\right\}$$

Because we wish to understand out the dependence of $W_{k,n}$ on the location of $P_1 \in L$, we write $W_{k,n}(Q, P_1)$ to denote the function $W_{k,n}(Q)$ arising from the scattering diagram $\mathfrak{D}_n(P_1, P_2, \ldots, P_k)$. We wish to show that if γ is a general path in $M_{\mathbb{R}} \times L$ joining (Q, P_1^1) to (Q, P_1^2),

$$W_{k,n}(Q, P_1^2) = \theta_{\gamma, \tilde{\mathfrak{D}}}(W_{k,n}(Q, P_1^1)).$$

Observe that it is enough to show

1. $W_{k,n}(Q, P)$ is constant for $(Q, P) \in M_{\mathbb{R}} \times L$ varying within a connected component of $(M_{\mathbb{R}} \times L) \setminus Supp(\tilde{\mathfrak{D}})$.

2. For two such connected components separated by a wall $(\tilde{\mathfrak{d}}, f_{\tilde{\mathfrak{d}}})$ and points (Q, P), (Q', P') on either side of the wall, we have

   $$W_{k,n}(Q', P') = \theta_{\gamma, \tilde{\mathfrak{D}}}(W_{k,n}(Q, P))$$

   for a short path γ connecting (Q, P) to (Q', P').

In fact, once (1) is shown, Theorem 2.4.1 already implies (2). To see this, recall that there are no walls in $\tilde{\mathfrak{D}}$ projecting to points in L, so we can always choose our points (Q, P) and (Q', P') on opposite sides of a wall with $P = P'$. But this is precisely the case shown in the previous theorem.

In order to show (1), we will use similar techniques to those in analysis of the dependence of $W_{k,n}$ on the location of Q. Take (Q, P) and (Q', P') general, sharing a connected component of $(M_{\mathbb{R}} \times L) \setminus Supp(\tilde{\mathfrak{D}})$, and move from (Q, P) to (Q', P') along a general path γ. Consider broken lines in $M_{\mathbb{R}} \times L$ with endpoint $\gamma(t)$. As t varies, we can continuously deform a broken line with endpoint $\gamma(t)$ unless one of the bends of the line converges to a singular point of $\tilde{\mathfrak{D}}$. Such a family of broken lines traces out a two-dimesnional subset of $M_{\mathbb{R}} \times L$, and γ can be chosen sufficiently generally such that none of the broken lines converge to broken lines

passing through interstices of $\tilde{\mathfrak{D}}$, as interstices are codimension three. However, joints may be unavoidable, and an analysis of the effects of passing through a joint takes some care.

The first thing to notice is that we have already analyzed (in the proof of Theorem 2.4.1) the situation in which a broken line passes through a vertical joint. To see this, notice that we may assume that γ has been chosen so that at time t_0, when a broken line passes through a vertical joint, $\pi_2(\gamma(t))$ remains constant for t in a small neighborhood of t_0. This gives us precisely the situation discussed in the proof above.

Thus, the only novel situation is that in which a broken line passes through a horizontal joint, which can occur when two or more walls come together, but also can be a result of a new wall sprouting out from a newly possible combinatorial type of tropical tree as P_1 varies. For our purposes, it is enough to show that if $\mathfrak{j}$ is a horizontal joint and $\gamma_{\mathfrak{j}}$ is a small loop in $M_{\mathbb{R}} \times L$ around the joint, then $\theta_{\gamma_{\mathfrak{j}}} = Id$.

To see this, observe that if $\mathfrak{j}$ projects to $P \in L$, $\mathfrak{j}$ is contained in some polygons $\tilde{\mathfrak{d}}_1, \ldots \tilde{\mathfrak{d}}_n \in \tilde{\mathfrak{D}}$ and necessarily, for $P' \in L$ near P, $\tilde{\mathfrak{d}}_i \cap M_{\mathbb{R}} \times \{P'\}$ is either a ray parallel to $\mathfrak{j}$ or it is empty. Thus, as $P_1 \in L$ moves from one side of P to the other, a set $\mathfrak{d}_1, \ldots, \mathfrak{d}_p$ of parallel rays in $\mathfrak{D}_n(\Sigma, P_1, P_2, \ldots, P_k)$ come together to yield the joint $\mathfrak{j}$ and then breaks apart into a different set of parallel rays $\mathfrak{d}'_1, \ldots, \mathfrak{d}'_{p'}$ on the other side. If we let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be the scattering diagrams $\mathfrak{D}_n(\Sigma, P_1, P_2, \ldots, P_n)$ chosen for P_1 very near to, but on opposite sides of, P, we can use these objects to get a handle on the nature of $\theta_{\gamma_{\mathfrak{j}}, \tilde{\mathfrak{D}}}$. In particular, letting γ be a short line segment (with $\pi_2(\gamma) = P$) crossing $\mathfrak{j}$, we can write

$$\begin{aligned}\theta_{\gamma,\mathfrak{D}_1} =& \theta_{\gamma,\mathfrak{d}_1} \circ \cdots \circ \theta_{\gamma,\mathfrak{d}_p} \\ \theta_{\gamma,\mathfrak{D}_2} =& \theta_{\gamma,\mathfrak{d}'_1} \circ \cdots \circ \theta_{\gamma,\mathfrak{d}'_{p'}}.\end{aligned}$$

The ordering of these automorphisms is immaterial as they commute. As $\theta_{\gamma_{\mathfrak{j}}, \tilde{\mathfrak{D}}} = \theta_{\gamma,\mathfrak{D}_2} \circ \theta^{-1}_{\gamma,\mathfrak{D}_1}$ and (we are assuming) $\theta_{\gamma_{\mathfrak{j}}, \tilde{\mathfrak{D}}} = Id$, we have $\theta_{\gamma,\mathfrak{D}_2} = \theta_{\gamma,\mathfrak{D}_1}$. Thus, broken lines will behave exactly the same way near $\mathfrak{j}$ on both sides of P, and it is enough to show that $\theta_{\gamma_{\mathfrak{j}}, \tilde{\mathfrak{D}}} = Id$ for horizontal joints $\mathfrak{j}$. The proof of this requires some subtle work and is the heart of the theorem.

The idea as follows. For $I \subseteq \{1, \ldots, k\}$, define:

$$Ideal(I) := \langle u_i | i \notin I \rangle \subseteq \mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,n}[[y_0]].$$

Then use induction to show, for each horizontal joint $\mathfrak{j}$,

Lemma 2.5.2. For $k' \geq 0$ and $|I| = k'$, we have

$$\gamma_{\mathfrak{j},\tilde{\mathfrak{D}}} \equiv Id \mod Ideal(I)$$

.

Proof. The base case for $k' = 0$ is trivial because all automorphisms are trivial modulo the ideal $(u_1, \ldots, u_k)$. Assume the claim for all $k'' < k'$. Fix a set I with $|I| = k'$, and an orientation on $M_{\mathbb{R}}$ so that if any joint $\mathfrak{j}$ is given an orientation, this determines the orientation of a loop $\gamma_{\mathfrak{j}}$ around $\mathfrak{j}$. By our earlier observation, if $\mathfrak{j}$ is horizontal, $\theta_{\gamma_{\mathfrak{j}},\tilde{\mathfrak{D}}}$ consists of a composition of automorphisms associated to parallel rays. Applying the definition of our automorphisms,

$$\theta_{\gamma_{\mathfrak{j}},\tilde{\mathfrak{D}}}(z^{m''}) = f_{\mathfrak{j}}^{\langle n_{\mathfrak{j}}, r(m'')\rangle} z^{m''}$$

for some $n_{\mathfrak{j}}$ primitive, zero on the tangent space to $\mathfrak{j}$, and

$$f_{\mathfrak{j}} \in \mathbb{C}[\{m \in T_\Sigma | r(m) \text{ is tangent to } \mathfrak{j}\}] \otimes_{\mathbb{C}} R_{k,n}[[y_0]].$$

Note that $f_{\mathfrak{j}}$ depends on the choice of direction (sign) of $n_{\mathfrak{j}}$. Assume we have chosen these consistently so that if any two joints $\mathfrak{j}$, $\mathfrak{j}'$ share the same tangent space, $n_{\mathfrak{j}} = n_{\mathfrak{j}'}$.

To prove the claim, we need to show that $f_{\mathfrak{j}} \equiv 1 \mod Ideal(I)$. Fix $m \in T_\Sigma$. For each horizontal joint $\mathfrak{j}$, let the term in $f_{\mathfrak{j}} \mod Ideal(I)$ involving z^m be $c_{m,\mathfrak{j}} z^m$. By the induction hypothesis, $c_{m,\mathfrak{j}} = \overline{c}_{m,\mathfrak{j}} \prod_{i \in I} u_i t_i^{r_i}$ for some $\overline{c}_{m,\mathfrak{j}} \in \mathbb{C}$, as $c_{m,\mathfrak{j}} \equiv 1 \mod Ideal(I')$ for any $I' \subsetneq I$.

First observe that if $\overline{c}_{m,\mathfrak{j}} \neq 0$, then $r(m) \neq 0$. Indeed, note that $f_{\mathfrak{j}}$ is a product of factors of the form $(1 + c_{m'} z^{m'})^{\pm 1}$ with $r(m') \neq 0$ by the construction of $\theta_{\gamma_{\mathfrak{j}},\tilde{\mathfrak{D}}}$. Then $\log f_{\mathfrak{j}}$ is a sum of expressions of the form $\pm \log(1 + c_{m'} z^{m'})$. After expanding this out using the (finite) Taylor series, we see that $\log f_{\mathfrak{j}} = \sum \pm c_{m'} z^{m'}$

with $r(m') \neq 0$ for every m' appearing in this sum. By definition, modulo $Ideal(I)$, $\log f_{\mathfrak{j}} = \sum_{m \in T_\Sigma} c_{m,\mathfrak{j}} z^m$. By comparing these expansions, we see that $c_{m,\mathfrak{j}} = 0$ if $r(m) = 0$, and of course then $\overline{c}_{m,\mathfrak{j}} = 0$.

The plan now is to fix a single $m \in T[\Sigma]$ with $r(m) \neq 0$ and show that $\overline{c}_{m,\mathfrak{j}} = 0$ for all horizontal joints $\mathfrak{j}$. In order to facilitate what follows, we define $\overline{c}_{m,\mathfrak{j}} = 0$ for vertical joints $\mathfrak{j}$. Note that $\overline{c}_{m,\mathfrak{j}}$ depends on the orientation of $\mathfrak{j}$– a change of orientation of $\mathfrak{j}$ changes the direction of $\gamma_{\mathfrak{j}}$, replacing $f_{\mathfrak{j}}$ with $f_{\mathfrak{j}}^{-1}$ and changing the sign of $\overline{c}_{m,\mathfrak{j}}$. Thus, we can view

$$\mathfrak{j} \mapsto \overline{c}_{m,\mathfrak{j}}$$

as a 1-chain for the one dimensional simplicial complex $Sing(\mathfrak{D})$. This map depends on an implicit orientation on $\mathfrak{j}$.

Claim 2.5.3. $\mathfrak{j} \mapsto \overline{c}_{m,\mathfrak{j}}$ is a 1-cycle.

Proof. We need to check the 1-cycle condition at each interstice of $\tilde{\mathfrak{D}}$, so let $(Q, P) \in Interstices(\tilde{\mathfrak{D}})$. Consider a small two-sphere S in $M_{\mathbb{R}} \times L$ with center (Q, P), Suppose that $x_1, \ldots, x_s \in S$ are distinct points such that

$$\{x_1, \ldots, x_s\} = \bigcup_{\mathfrak{j} \in Joints(\tilde{\mathfrak{D}})} S \cap \mathfrak{j}.$$

Choose a base-point $y \in S \setminus Supp(\tilde{\mathfrak{D}})$, small counterclockwise loops γ_i around x_i in S, and paths β_i joining y with the base-point of γ_i in such a way that

$$\beta_1 \gamma_1 \beta_1^{-1} \cdots \beta_s \gamma_s \beta_s^{-1} = 1.$$

Then, recalling that $\theta_{\gamma,\tilde{\mathfrak{D}}}$ only depends on the homotopy type of the path γ in $M_{\mathbb{R}} \times L \setminus Sing(\tilde{\mathfrak{D}})$, we see that

$$\theta_{\beta_s}^{-1} \circ \theta_{\gamma_s} \circ \theta_{\beta_s} \circ \cdots \circ \theta_{\beta_1}^{-1} \circ \theta_{\gamma_1} \circ \theta_{\beta_1} = Id$$

where we've suppressed the subscripts $\tilde{\mathfrak{D}}$. There are two cases requiring different analysis.

Case 1. In this case, (Q, P) does not satisfy $Q \in \{P, P_2, \ldots, P_k\}$. Then, by Lemma 2.3.12, $\theta_{\gamma_i} = Id$ for each γ_i which is a loop around a vertical joint

containing (Q,P). On the other hand, modulo $Ideal(I)$, for γ_i around a horizontal joint $\mathfrak{j}_i$, $f_{\mathfrak{j}_i}$ is of the form $1+(\cdots)\prod_{i\in I}u_i$. Then, by defintion, $\theta_{\gamma_\mathfrak{j}}$ commutes, modulo $Ideal(I)$, with any element of $\mathbb{V}_{\Sigma,k}$, as $u_j\prod_{i\in I}u_i\cong 0 \mod Ideal(I)$ for any $\mathfrak{j}$. Thus, θ_{γ_i} commutes with θ_{β_i} and our earlier expression becomes

$$\prod\theta_{\gamma_i}\cong 1 \mod Ideal(I).$$

where the product is over all γ_i circling horizontal joints. If we apply this to a monomial $z^{m''}$, we get

$$\prod f_{\mathfrak{j}_i}^{\langle n_{\mathfrak{j}_i},r(m'')\rangle}z^{m''} \mod Ideal(I),$$

which, after expansion and reading off the coefficients of $z^{m+m''}$, gives the identity

$$\sum\langle n_{\mathfrak{j}_i},r(m'')\rangle\overline{c}_{m,\mathfrak{j}_i}=0$$

for any $m''\in T_\Sigma$. A monomial z^m can only appear in $f_{\mathfrak{j}_i}$ if $r(m)$ is tangent to $\mathfrak{j}_i$, so the only horizontal joints containing (Q,P) with $\overline{c}_{m,\mathfrak{j}}\neq 0$ are those joints contained in the affine line $(Q,P)+\mathbb{R}(r(m),0)$. Let $s\in\{0,1,2\}$ be the number of joints satisfying this criteria and containing (Q,P). Clearly, if $s=0$, $\overline{c}_{m,\mathfrak{j}}=0$. If $s=1$, with $\mathfrak{j}$ the only such joint, it follows from the above equality that $\overline{c}_{m,\mathfrak{j}}=0$. If $s=2$, with $\mathfrak{j}_1,\mathfrak{j}_2$ the two joints, the same equality implies that $\overline{c}_{m,\mathfrak{j}_1}=\overline{c}_{m,\mathfrak{j}_1}$ if j_1 and j_2 are oriented in the same direction, which shows that the 1-cycle condition holds at (Q,P).

Case 2. This is the only case that differs significantly from the proof presented in [11]. Suppose we have an interstice (Q,P) satisfying $Q\in\{P,P_2,\ldots,P_k\}$, with $Q=P_i$, where we write $i=1$ if $Q=P_1$. Suppose there are two vertical joints, $\mathfrak{j}_1$ and $\mathfrak{j}_2$, with endpoint (Q,P), $\mathfrak{j}_1,\mathfrak{j}_2\subseteq\{P_i\}\times L$ if $i\neq 1$ and $\mathfrak{j}_1,\mathfrak{j}_2=\{(P',P')|P'\in L\}\subseteq M_\mathbb{R}\times L$ if $i=1$. Take a base point y near x_1 and assume β_1 is a constant path so that $\theta_{\beta_1}=\mathrm{Id}$. The argument will be the same as that appearing in the previous case once we show:

$$\theta_{\beta_2}^{-1}\circ\theta_{\gamma_2}\circ\theta_{\beta_2}\circ\theta_{\gamma_1}=\mathrm{Id}$$

Consider $\mathfrak{D}_n(\Sigma,P',P_2,\ldots,P_k)$ for $P'\in L$, with P' near but not equal to P. The rays emanating from P_i (P' if $i=1$) whose monomial contains a factor of t_i^r are in

one-to-one correspondence with the terms of $\frac{1}{(r+1)!}\left[W_{k-1,n}(P_i,P')-y_0\right]^{r+1}$ where $W_{k-1,n}(P_i,P')$ denotes $W_{k-1,n}(P_i)$ computed using the marked points

$$P',P_2,\ldots,P_{i-1},P_{i+1},\ldots,P_k$$

if $i\neq 1$ and $P_2,\ldots,P_k$ if $i=1$. In particular, given a term cz^m in

$$\frac{1}{(r_i+1)!}\left[W_{k-1,n}(P_i,P')-y_0\right]^{r+1},$$

the corresponding ray carries the function $1+u_it_i^{r_i}cw(m)z^m$, where $w(m)$ is the index of $r(m)$. If γ is a simple loop around P_i, the contribution to $\sigma_{\gamma,\mathfrak{D}_n(\Sigma,P',P_2,\ldots,P_k)}$ from such a ray is $\exp(\pm X_{u_it_i^{r_i}cz^m})$. All automorphisms attached to the rays emanating from P_i commute, so, defining $\sigma_{\gamma,r,\mathfrak{D}_n(\Sigma,P',P_2,\ldots,P_k)}$ to be the automorphism resulting from γ crossing rays whose attached function's monomial includes a factor of t_i^r, we see

$$\sigma_{\gamma,r,\mathfrak{D}_n(\Sigma,P',P_2,\ldots,P_k)}=\prod\exp(\pm X_{u_it_i^rcz^m})=\exp\left(\pm X_{\frac{u_it_i^r}{(r+1)!}[W_{k-1}(P_i,P')-y_0]^{r+1}}\right)$$

Here the product is over all terms cz^m appearing in $(W_{k-1,n}(P_i,P')-y_0)^{r+1}$. Then $\sigma_{\gamma,\mathfrak{D}_n(\Sigma,P',P_2,\ldots,P_k)}=\prod_{r=0}^n\sigma_{\gamma,r,\mathfrak{D}_n(\Sigma,P',P_2,\ldots,P_k)}$. Furthermore, if $P'\in\pi_2(\mathfrak{j}_1)\setminus\{P\}$ and $P''\in\pi_2(\mathfrak{j}_2)\setminus\{P\}$, then, by our original inductive assumption applied to $k-1$ points if $i\neq 1$, and our previous theorem if $i=1$,

$$W_{k-1,n}(P_i,P'')=\theta_{\beta_2}(W_{k-1,n}(P_i,P'))$$

so

$$\theta_{\beta_2}\left(\frac{u_i}{(r+1)!}\left[W_{k-1}(P_i,P')-y_0\right]^{r+1}\right)=\frac{u_i}{(r+1)!}\left[W_{k-1}(P_i,P'')-y_0\right]^{r+1}$$

Thus,

$$\begin{aligned}\sigma_{\gamma_2,r,\mathfrak{D}_n(\Sigma,P'',P_2,\ldots,P_k)}=&\exp\left(\pm X_{\frac{u_it_i^r}{(r+1)!}[W_{k-1}(P_i,P'')-y_0]^{r+1}}\right)\\=&\exp\left(\pm X_{\frac{u_it_i^r}{(r+1)!}\left[\theta_{\beta_2}(W_{k-1}(P_i,P'))-y_0\right]^{r+1}}\right)\\=&\exp\left(\pm X_{\theta_{\beta_2}\left(\frac{u_it_i^r}{(r+1)!}[W_{k-1}(P_i,P')-y_0]^{r+1}\right)}\right)\end{aligned}$$

Then, by Lemma 2.3.8,

$$\sigma_{\gamma_2,r,\mathfrak{D}_n(\Sigma,P'',P_2,\dots,P_k)} = \left(\theta_{\beta_2} \circ \sigma_{\gamma_1,r,\mathfrak{D}_n(\Sigma,P',P_2,\dots,P_k)} \circ \theta_{\beta_2}^{-1}\right)^{-1}$$

the last inverse on the right since γ_1 and γ_2 are homotopic to loops in $M_{\mathbb{R}} \times P'$ and $M_{\mathbb{R}} \times P''$ with opposite orientations. As $\theta_{\gamma_2} = \prod_{r=0}^{n} \theta_{\gamma_2,r,\mathfrak{D}_n(\Sigma,P'',P_2,\dots,P_k)}$ and $\theta_{\gamma_1} = \prod_{r=0}^{n} \theta_{\gamma_2,r,\mathfrak{D}_n(\Sigma,P',P_2,\dots,P_k)}$, we get

$$\theta_{\gamma_2} = (\theta_{\beta_2} \circ \theta_{\gamma_1} \circ \theta_{\beta_2}^{-1})^{-1}.$$

□

To finish the proof of the claim, and hence the theorem, we reproduce the conclusion to [11] Theorem 5.39. Note that a cycle σ given by $\mathfrak{j} \mapsto \overline{c}_{m,\mathfrak{j}}$ is in fact zero. Indeed, picking a given joint with $\overline{c}_{m,\mathfrak{j}} \neq 0$, the fact that σ is a cycle implies that the line containing $\mathfrak{j}$ can be written as a union of joints $\mathfrak{j}'$ with orientation compatible with that on $\mathfrak{j}$, and $\overline{c}_{m,\mathfrak{j}'} = \overline{c}_{m,\mathfrak{j}}$. However, there must be one joint j' contained in this line which is unbounded in the direction of $r(m)$. This is a contradiction, as none of the polyhedra of $\tilde{\mathfrak{D}}$ containing $\mathfrak{j}'$ can involve a monomial of the form z^m, since a ray carrying the monomial z^m is unbounded only in the direction of $-r(m)$. Thus $\overline{c}_{m,\mathfrak{j}} = 0 = \overline{c}_{m,\mathfrak{j}'}$ as desired.

□

□

Chapter 3

Integrals

3.1 Tropical Gromov-Witten Invariants

Here we define the invariants that the integral will eventually calculate, tropical versions of the Gromov-Witten invariants introduced in the first chapter. In this tropical account of events, the integral can be thought of as a device for gluing together the semirigid and rigid disks catalogued in the Landau-Ginzburg potential and counting the resulting curves with the appropriate multiplicities.

Definition 3.1.1 (Tropical GW Invariants)**.** Fix general points Q and $P_1, P_2 \ldots \in M_{\mathbb{R}}$. Let L be the tropical line with vertex Q, which is the tropical curve given by attaching unbounded rays in the direction of $(-1-1)$, $(1,0)$, and $(0,1)$ to Q. For a tropical curve h in $\mathbb{P}^2$ with a marked point x, let V_x be the vertex attached to E_x, and denote by $h_1, \ldots, h_n$ the set of descendent tropical disks obtained by removing V_x and E_x from Γ, with the outgoing edge of h_i called $E_{i,out}$. Let $m(h_i) := w(E_{i,out})m^{prim}(h_i) = -r(\Delta(h_i))$, where m^{prim} is a primitive vector tangent to $h_i(E_{i,out})$ pointing away from $h(x)$. Recall the definition of $n_i(V_x)$ as the number of unbounded edges attached to the vertex V_x (the vertex to which E_x is attached)

which are mapped to unbounded rays in the direction $\hat{\rho}_i$. Define:

$$\begin{aligned} Mult_x^0(h) =& \frac{1}{n_0(V_x)!n_1(V_x)!n_2(V_x)!} \\ Mult_x^1(h) =& -\frac{\sum_{k=1}^{n_0(V_x)} \frac{1}{k} + \sum_{k=1}^{n_1(V_x)} \frac{1}{k} \sum_{k=1}^{n_0(V_x)} \frac{1}{k}}{n_0(V_x)!n_1(V_x)!n_2(V_x)!} \\ Mult_x^2(h) =& \frac{\left(\sum_{l=0}^{2} \sum_{k=1}^{n_l(V_x)} \frac{1}{k}\right)^2 + \sum_{l=0}^{2} \sum_{k=1}^{n_l(V_x)} \frac{1}{k^2}}{2n_0(V_x)!n_1(V_x)!n_2(V_x)!} \end{aligned}$$

We can now define the invariants $\langle \psi^{r_1} P_1 \ldots, \psi^{r_k} P_k, \psi^\nu S \rangle_{0,d}^{trop}$. Let $\hat{I} = \{1, \ldots, k\}$. We separate the instances depending on the dimension of S.

1. When $3d - 2 - \nu - \sum_{i=1}^{k} r_k - k = 0$, we define

   $$\langle \psi^{r_1} P_1 \ldots, \psi^{r_k} P_k, \psi^\nu Q \rangle_{0,d}^{trop}$$

   to be

   $$\sum_h Mult(h)$$

   where the sum is over all $h \in \mathcal{M}^{\text{trop}}_{\Delta,\hat{I}}(X_\Sigma, \psi^{r_1} P_1, \ldots, \psi^{r_k} P_k, \psi^\nu Q)$ with

   $$Mult(h) := Mult_x^0(h) \prod_i Mult(h_i).$$

2. When $3d - 1 - \nu - \sum_{i=1}^{k} r_k - k = 0$, we define

   $$\langle \psi^{r_1} P_1 \ldots, \psi^{r_k} P_k, \psi^\nu L \rangle_{0,d}^{trop}$$

   to be

   $$\sum_h Mult(h)$$

   where the sum is over all descendent marked tropical rational curves satisfying one of the following conditions:

   - $\nu \geq 0$ and

     $$h \in \mathcal{M}^{\text{trop}}_{\Delta,\hat{I}}(X_\Sigma, \psi^{r_1} P_1, \ldots, \psi^{r_k} P_k, \psi^\nu L).$$

Furthermore, $|I| = d$, and no unbounded edge of Γ having a common vertex with E_x other than E_x maps into the connected component of $L \setminus \{Q\}$ containing $h(x)$. By lemma 2.3.8, there is precisely one j, $1 \leq j \leq \nu + 2$ such that h_j is rigid. Suppose that the connected component of $L \setminus \{Q\}$ is $Q + \mathbb{R}_{\geq 0}\hat{\rho}_i$. Then we define:

$$Mult(h) := |m(h_j) \wedge \hat{\rho}_i| Mult_x^0(h) \prod_i Mult(h_i).$$

- $\nu \geq 1$ and
$$h \in \mathcal{M}^{\text{trop}}_{\Delta,\hat{I}}(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k, \psi^{\nu-1}Q).$$
In this case,
$$Mult(h) := Mult_x^1(h) \prod_i Mult(h_i).$$

3. When $3d - \nu - \sum_{i=1}^k r_k - k = 0$, we define
$$\langle \psi^{r_1}P_1 \ldots, \psi^{r_k}P_k, \psi^\nu M_{\mathbb{R}} \rangle_{0,d}^{trop}$$
to be
$$\sum_h Mult(h)$$
where the sum is over all descendent marked tropical rational curves satisfying one of the following conditions:

- $\nu \geq 0$,
$$h \in \mathcal{M}^{\text{trop}}_{\Delta,\hat{I}}(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k, \psi^\nu M_{\mathbb{R}})$$
and E_x does not share a vertex with any of the E_{p_i}'s. Furthermore, no unbounded edge of Γ having a common vertex with E_x other than E_x maps into the connected component of $M_{\mathbb{R}} \setminus L$ containing $h(x)$. By Lemma 2.3.8, there are precisely two j_1, j_2 with $1 \leq j_1 < j_2 \leq \nu + 2 + r_i$ such that h_{j_i} is rigid. Then we define:

$$Mult(h) := |m(h_{j_1}) \wedge m(h_{j_2})| Mult_x^0(h) \prod_i Mult(h_i).$$

- $\nu \geq 0$,
$$h \in \mathcal{M}^{\text{trop}}_{\Delta,\hat{I}}(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k, \psi^\nu M_\mathbb{R})$$
and E_x shares a vertex with E_{p_i}. Suppose R unbounded edges of Γ having a common vertex with E_x other than E_x and E_{p_i} map into the connected component of $M_\mathbb{R} \setminus L$ containing $h(x)$. Then we define:
$$Mult(h) := \binom{r_i + \nu - R}{\nu} Mult^0_x(h) \prod_i Mult(h_i).$$

- $\nu \geq 1$ and
$$h \in \mathcal{M}^{\text{trop}}_{\Delta,\hat{I}}(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k, \psi^\nu M_\mathbb{R})$$
Furthermore, no unbounded edge of Γ having a common vertex with E_x other than E_x maps into the connected component of $L \setminus \{Q\}$ containing $h(x)$. By Lemma 2.3.8, there is precisely one j, $1 \leq j \leq \nu + 2$ such that h_j is rigid. Suppose that the connected component of $L \setminus \{Q\}$ is $Q + \mathbb{R}_{\geq 0} m_i$. Then we define:
$$Mult(h) := |m(h_j) \wedge m_i| Mult^1_x(h) \prod_i Mult(h_i).$$

- $\nu \geq 2$ and
$$h \in \mathcal{M}^{\text{trop}}_{\Delta,\hat{I}}(X_\Sigma, \psi^{r_1}P_1, \ldots, \psi^{r_k}P_k, \psi^{\nu-2}Q)$$
In this case,
$$Mult(h) := Mult^2_x(h) \prod_i Mult(h_i)$$

In all cases $S = \{Q\}, L$ or $M_\mathbb{R}$, we define, for $\sigma \in \Sigma$,

$$\langle \psi^{r_1}P_1 \ldots, \psi^{r_k}P_k, \psi^\nu S \rangle^{trop}_{d,\sigma}$$

where we only count contributions coming from curves h where $h(x) \in Int(\sigma)$.

These definitions can be made more compact through an expansion of the evaluation map discussed earlier to include maps of divisors.

3.2 Examples

As we will stop short of proving the equality between our tropical invariants and their classical counterparts, a few examples are included to provide evidence for this conjecture.

3.2.1 Example 1

Below, in Figure 3.2.1, are the tropical curves relevant to the calculation of $\langle \psi^6 P_1, \psi^2 M_{\mathbb{R}} \rangle_{0,3}^{trop}$, whose classical counterpart is $\langle \psi^6 T_2, \psi^2 T_0 \rangle_{0,3} = 1/216$ (as computed by Gathmann's computer program Growi [5]). By following the definitions given above, the reader can verify that this example gives the correct quantity. The edges in the diagrams have been perturbed from their actual direction for the sake of clarity. Note that the curve in the middle figure contains a bounded edge of weight two which must be factored into its computation of multiplicity.

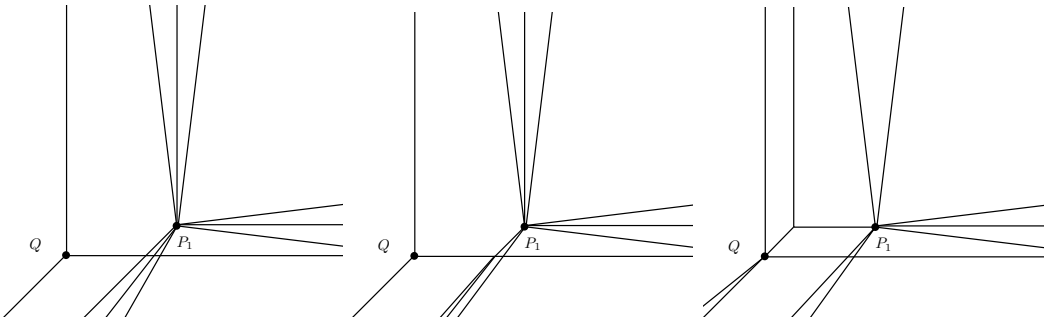

Figure 3.2.1: Graphs contributing to $\langle \psi^6 P_1, \psi^2 M_{\mathbb{R}} \rangle_{0,3}^{trop}$. From left to right, the contributions involve $Mult_x^0(h)$, $Mult_x^1(h)$, and $Mult_x^2(h)$.

3.2.2 Example 2

Below, in Figure 3.2.2, are the tropical curves relevant to the calculation of $\langle \psi^7 P_1, \psi M_{\mathbb{R}} \rangle_{0,3}^{trop}$, whose classical counterpart is $\langle \psi^7 T_2, \psi T_0 \rangle_{0,3}^{trop} = -\frac{1}{216}$ (as computed by Gathmann's computer program Growi [5]). In these examples the collapsed edge marking x is shown.

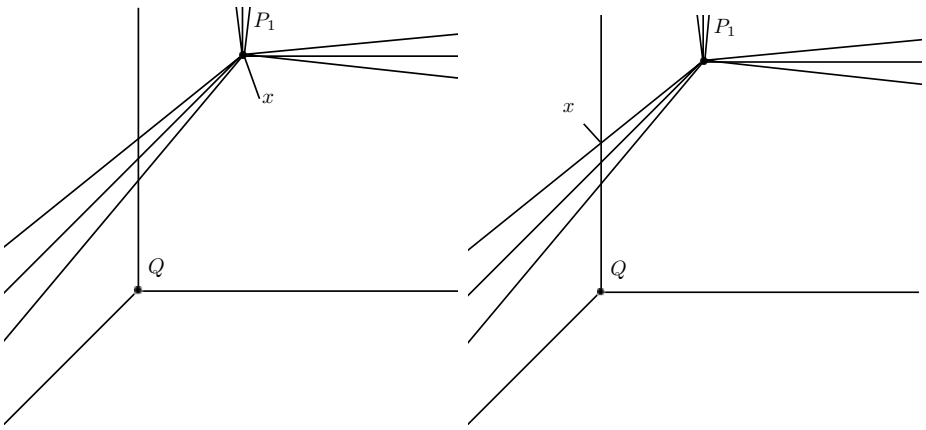

Figure 3.2.2: Graphs contributing to $\langle \psi^7 P_1, \psi M_{\mathbb{R}} \rangle_{0,3}^{trop}$ involving $Mult_x^0(h)$ (on the left) and $Mult_x^1(h)$ (on the right).

3.2.3 Example 3

The tropical curves relevant to the calculation of $\langle \psi^2 P_1, \psi P_2, \psi M_{\mathbb{R}} \rangle_{0,2}^{trop}$ are pictured below. This invariant's classical counterpart is $\langle \psi^2 T_2, \psi T_2, \psi T_0 \rangle_{0,2}^{trop} = 0$ (as computed by the Dilaton axiom [see next chapter]). Again, the collapsed edge marking x is shown.

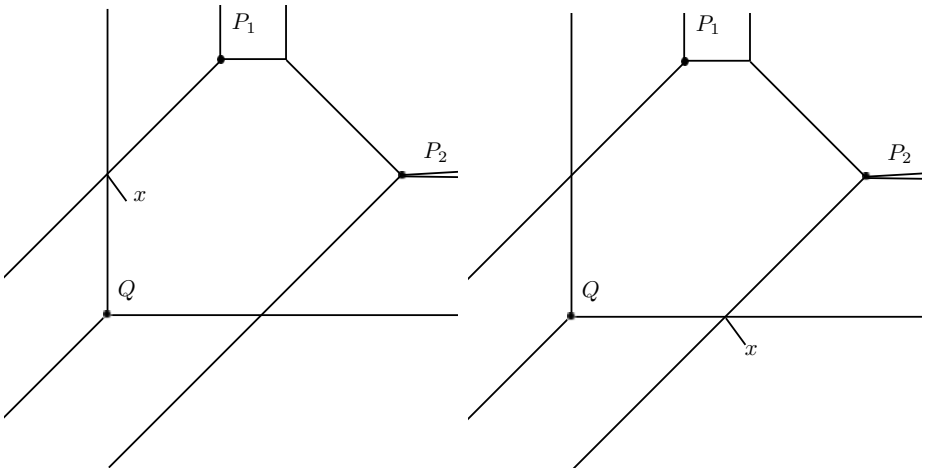

Figure 3.2.3: Graphs contributing to $\langle \psi^2 P_1, \psi P_2, \psi M_{\mathbb{R}} \rangle_{0,2}^{trop}$ involving $Mult_x^1(h)$.

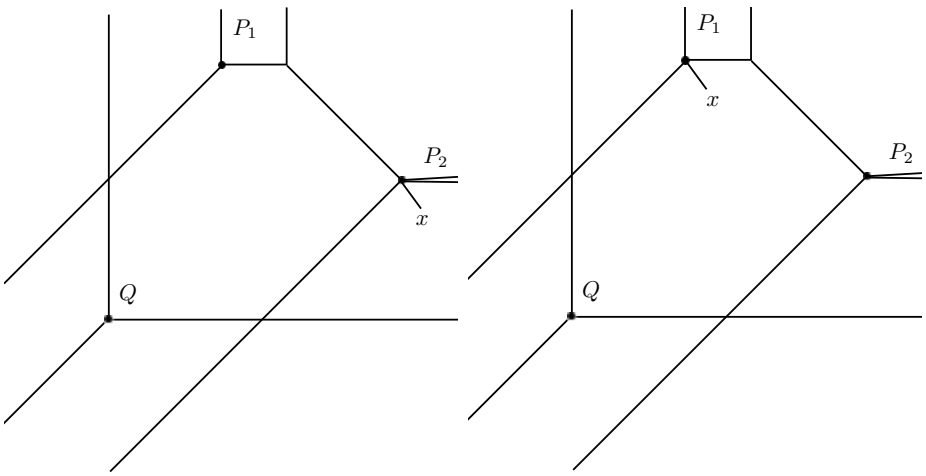

Figure 3.2.4: Graphs whose contributions involve $Mult_x^0(h)$.

3.3 Evaluation of Integrals

The material in this section follows very closely that of Section 5.5 in [11]. In this section we will do the work of evaluating the integrals of our potential $W_{k,n}$ and see that the result naturally encodes our tropical Gromov-Witten invariants. See the final chapter for the final result of the integral. Some of the invariants recovered are known from Gross' work and Theorem 1.2.6, while our conjectural tropical invariants are also recovered. The only major modifications in methods occur in the proof of Lemma 3.3.16.

Lemma 3.3.1. Let $\sigma \in \mathbb{V}_{\Sigma,k}$, $(u,h) \in \widetilde{\mathcal{M}}_{\Sigma,k} \times \mathbb{C}^\times$ and suppose that f is in the ideal generated by $(u_1,\ldots,u_k)$ in $\mathbb{C}[T_\Sigma] \otimes_{\mathbb{C}} R_{k,n}[[y_0]]$. Then, for any cycle

$$\Xi \in H_2(\kappa^{-1}(u), \mathrm{Re}(W_0(Q)/\hbar) \ll 0, \mathbb{C}),$$

we have

$$\int_\Xi e^{\frac{(W_0+f)}{\hbar}}\Omega = \int_\Xi e^{\frac{\theta(W_0+f)}{\hbar}}\Omega.$$

Proof. See [11] Lemma 5.40. □

Lemma 3.3.2. For $\Xi \in H_2(\kappa^{-1}(u), \mathrm{Re}(W_0(Q)/\hbar) \ll 0, \mathbb{C})$, the integral

$$\int_\Xi e^{\frac{W_{k,n}}{\hbar}}\Omega$$

is independent of the choice of general Q, $P_1,\ldots,P_k$.

Proof. Follows immediately from the previous lemma and Theorems 2.4.1 and 2.5.1. □

This key result will show (after computation of our integral) that our definitions of tropical Gromov-Witten invariants do not depend on the choice of general points $P_1 \dots P_k$ and thus make sense as true invariants.

Lemma 3.3.3. Restricting to $x_0x_1x_2 = y_1$, we have

$$\sum_{i=0}^{2} \alpha^i \int_{\Xi_i} e^{(x_0+x_1+x_2)/\hbar} x_0^{n_0} x_1^{n_1} x_2^{n_2} = \hbar^{-3\alpha} e^{\alpha y_1} \sum_{i=0}^{2} \psi_i(n_0, n_1, n_2)\alpha^i$$

where α and Ξ_i are as defined in Section 1.2.3 (recall that α^i is dual to Ξ_i) and

$$\psi_i(n_0, n_1, n_2) = \sum_{d=0}^{\infty} D_i(d, n_0, n_1, n_2)\hbar^{-(3d-n_0-n_1-n_2)} e^{dy_1}$$

with D_i defined below.

If $d \geq n_0, n_1, n_2$

$$D_0(d, n_0, n_1, n_2) = \frac{1}{(d-n_0)!(d-n_1)!(d-n_2)!}$$

$$D_1(d, n_0, n_1, n_2) = -\frac{\sum_{k=1}^{d-n_0} \frac{1}{k} + \sum_{k=1}^{d-n_1} \frac{1}{k} + \sum_{k=1}^{d-n_2} \frac{1}{k}}{(d-n_0)!(d-n_1)!(d-n_2)!}$$

$$D_2(d, n_0, n_1, n_2) = -\frac{\left(\sum_{l=1}^{2} \sum_{k=1}^{d-n_l} \frac{1}{k}\right)^2 + \left(\sum_{l=1}^{2} \sum_{k=1}^{d-n_l} \frac{1}{k^2}\right)}{(d-n_0)!(d-n_1)!(d-n_2)!}$$

If $n_0 > d \geq n_1, n_2$,

$$D_0(d, n_0, n_1, n_2) = 0$$

$$D_1(d, n_0, n_1, n_2) = \frac{(-1)^{n_0-d-1}(n_0-d-1)!}{(d-n_1)!(d-n_2)!}$$

$$D_2(d, n_0, n_1, n_2) = \frac{(-1)^{d-n_0}(n_0-d-1)!}{(d-n_1)!(d-n_2)!} \left(\sum_{k=1}^{n_0-d-1} \frac{1}{k} + \sum_{k=1}^{d-n_1} \frac{1}{k} + \sum_{k=1}^{d-n_2} \frac{1}{k} \right)$$

with analogous expressions if $n_1 > d$ or $n_2 > d$.

If $n_0, n_1 > d \geq n_2$,

$$D_0(d, n_0, n_1, n_2) = 0$$

$$D_1(d, n_0, n_1, n_2) = 0$$

$$D_2(d, n_0, n_1, n_2) = \frac{(-1)^{n_0+n_1}(n_0-d-1)!(n_1-d-1)!}{(d-n_2)!}$$

with analogous expressions if $n_1, n_2 > d \geq n_0$ or $n_0, n_2 > d \geq n_1$.
Finally, if $n_0, n_1, n_2 > d$,

$$D_i(d, n_0, n_1, n_2) = 0.$$

Proof. See [11], Lemma 5.43. □

Definition 3.3.4 ($B(c)$)**.** Let $c = \prod_{i=1}^{k} u_i^{\epsilon_i} t_i^{r_i} \in R_{k,n}$ where $\epsilon_i \in \{0, 1\}$ and $r_i \in \{0, \ldots, n\}$. Then

$$B(c) := \sum_{i=1}^{k} r_i.$$

Definition 3.3.5 (L_i^d)**.** Fix general $P_1, \ldots, P_k$. For general Q, let $S_{k,n}$ be the finite set of triples (c, ν, m) with $c \in R_{k,n}$, $\nu \geq 0$ an integer, and $m \in T_\Sigma$ such that:

$$e^{(W_{k,n}(Q) - W_{0,n}(Q))/\hbar} = \sum_{(c,\nu,m) \in S_{k,n}} c\hbar^{-\nu} z^m,$$

with each term $c\hbar^{-\nu} z^m$ of the form $\hbar^{-\nu} \prod_{i=1}^{\nu} \mathrm{Mono}(\mathrm{h_i})$ for $h_1, \ldots, h_\nu$ semirigid descendent tropical disks with boundary Q.
Then

$$L_i^d = L_i^d(Q) := \sum_{(c,\nu,m) \in S_{k,n}} c\hbar^{-(3d+\nu-|m|)} D_i(d, m).$$

Definition 3.3.6 ($\tilde{\sigma}_d$)**.** Let T_Σ be generated by v_i for $i \in \{0, 1, 2\}$. For each cone $\sigma \in \Sigma$, σ is the image under rf of a proper face $\tilde{\sigma}$ of the cone $K \subseteq T_\Sigma \otimes \mathbb{R}$ generated by v_0, v_1, v_2 (the first octant). For $d \geq 0$, define $K_d \subseteq K$ the cube

$$K_d = \left\{ \sum_{i=0}^{2} n_i v_i | 0 \leq n_i \leq d \right\}$$

and for $\sigma \in \Sigma$

$$\tilde{\sigma}_d := (\tilde{\sigma} + K_d) \setminus \bigcup_{\tau \subsetneq \sigma, \tau \in \Sigma} (\tilde{\tau} + K_d).$$

where $+$ denotes the Minkowski sum.

Definition 3.3.7 ($L_{i,\sigma}^d$)**.** For $\sigma \in \Sigma$, define

$$L_{i,\sigma}^d = L_{i,\sigma}^d(Q) := \sum_{(c,\nu,m) \in S_{k,n},\, m \in \tilde{\sigma}_d} c\hbar^{-(3d+\nu-|m|)} D_i(d, m).$$

Lemma 3.3.8. $L_i^d = \sum_{\sigma \in \Sigma} L_{i,\sigma}^d$.

Proof. Follows immediately from definitions. □

Lemma 3.3.9. Let $\{0\} \neq \omega \in \Sigma$, and $v \in w$ be non-zero. Then

$$\lim_{s \to \infty} L_{i,\omega}^d(Q + sv) = 0.$$

Proof. This result is perhaps the key trick to computing these integrals. As with much of the rest of this thesis, the proof of this lemma translates directly from the one given in Gross' work as the argument relies on the geometric properties of scattering diagrams and broken lines, and not on the rules used to construct the scattering diagram. These geometric properties are unchanged in our slightly modified scattering diagrams. For details on the proof of the claim, see [11], Lemma 5.51. □

Definition 3.3.10 ($L_{i,\gamma,w\to\tau}^d$)**.** Let $\mathfrak{D} = \mathfrak{D}_n(\Sigma, P_1, \ldots, P_k)$. Let C_1 and C_2 be connected components of $M_{\mathbb{R}} \setminus \mathfrak{D}$ with $dim(\bar{C}_1 \cap \bar{C}_2) = 2$. Pick general points Q_i in C_i, and let γ be a general path from Q_1 to Q_2 intersecting $Supp(\mathfrak{D})$ exactly once at $\gamma(t_0)$, a nonsingular point of $Supp(\mathfrak{D})$. Let $\mathfrak{d} \in \mathfrak{D}$ contain $\gamma(t_0)$, and let $n_{\mathfrak{d}}$ be a primitive vector perpendicular to $\mathfrak{d}$ pointing toward Q_1.

Suppose that $f_{\mathfrak{d}} = 1 + c_{\mathfrak{d}} z^{m_{\mathfrak{d}}}$. Select $w, \tau \in \Sigma$ with $dim(\tau) = dim(w) + 1$ and $w \in \tau$. Note that there is a unique index $j \in \{0, 1, 2\}$ such that $m_j \in \tau$ but $m_j \notin w$. Call this index $j(w, t)$.

Define

$$L_{i,\mathfrak{d},\gamma,w\to\tau}^d := \sum_{(c,\nu,m)} \langle n_{\mathfrak{d}}, \hat{\rho}_{j(w,\tau)} \rangle c_{\mathfrak{d}} c D_i(d, m + m_{\mathfrak{d}} + v_{j(w,\tau)}) h^{-(\nu + 3d - |m + m_{\mathfrak{d}}|)},$$

where we sum over all (c, ν, m) in $S_{k,n}(Q)$ satisfying $m + m_{\mathfrak{d}} \in \tilde{w}_d$ but $m + m_{\mathfrak{d}} + t_{j(w,\tau)} \in \tilde{\tau}_d$. If (c, ν, m) satisfies these condition, then we say that $c\hbar^{-\nu} z^m$ contributes to $L_{i,\mathfrak{d},\gamma,w\to\tau}^d$.

Define

$$L_{i,\gamma,w\to\tau}^d := \sum_{\mathfrak{d}} L_{i,\mathfrak{d},\gamma,w\to\tau}^d$$

where $\mathfrak{d}$ ranges over all rays of $\mathfrak{D}$ containing $\gamma(t_0)$. In order to define this operation for a general path γ, break it up into segments of the type outlined above and sum over these.

Lemma 3.3.11. Let γ_j be the straight line path joining Q with $Q + s\hat{\rho}_j$ for $s \gg 0$. Let $\gamma_{j,j+1}$ be the loop based at Q which passes linearly from Q to $Q + s\hat{\rho}_j$, takes a large circular arc to $Q + s\hat{\rho}_{j+1}$, and then proceeds linearly from $Q + s\hat{\rho}_{j+1}$ to Q. Here we take j modulo 3, and $\gamma_{j,j+1}$ is always a counterclockwise loop. Let $\sigma_{j,j+1} = \rho_j + \rho_{j+1}$ a two dimensional cone in Σ. Then

$$L_i^d(Q) = L_{i,\{0\}}^d(Q) - \sum_{j=0}^{2} L_{i,\gamma_j,\{0\}\to\rho_j}^d - \sum_{j=0}^{2} L_{i,\gamma_{j,j+1},\rho_{j+1}\to\sigma_{j,j+1}}^d.$$

Proof. This result is the place where Lemma 3.3.9 comes in handy, allowing us to divide the results of the integral into pieces which have nice tropical interpretations. For details, see [11], Lemma 5.54. □

Lemma 3.3.12.

$$\begin{aligned} &L_{i,\{0\}}^d(Q) \\ &\quad = \delta_{0,d}\delta_{0,i} + \sum_{\nu \geq i} \sum_{\substack{I \subseteq \{1,\dots,k\} \\ I = \{i_1,\dots,i_l\} \\ i_1 < \dots < i_l \\ r_1 + \dots + r_l = 3d - 2 + i - \nu - l}} \langle \psi^{r_1} P_{i_1}, \dots, \psi^{r_l} P_{i_l}, \psi^{\nu} S \rangle_{d,\{0\}}^{\mathrm{trop}} u_{I,r} h^{-(\nu+2-i)} \end{aligned}$$

where $S = Q$, L, or $M_{\mathbb{R}}$ when $i = 0$, 1, or 2 respectively, $r_i \in \mathbb{Z}_{\geq 0}$, and $u_{I,r} = \prod_{j=1}^{l} u_{i_j} t^{r_j}$.

Proof. This lemma is very similar to that given in Gross' proof, with slight alterations arising due to the possibility of overvalent (more than trivalent) vertices arising away from Q. If $d = 0$, then the only element $(c, \nu, m) \in S_{k,n}$ contributing to $L_{i,\{0\}}^d$ is $(1, 0, 0)$. This term contributes 1 if $i = 0$, and 0 otherwise, accounting for the term $\delta_{0,d}\delta_{0,i}$.

If $d \neq 0$, (c, ν, m) with $m = \sum_{i=0}^{2} n_i v_i$ contributes only if $n_0, n_1, n_2 \leq d$.

Write

$$c\hbar^{-\nu} z^m = \hbar^{-\nu} \prod_{i=1}^{\nu} \mathrm{Mono}(h_i) = \hbar^{-\nu} \prod_{i=1}^{\nu} z^{\Delta(h_i)} u_{I(h_i)}$$

for h_i semirigid disks with boundary Q. Let Γ be the graph achieved by identifying the outgoing vertices of h_i and adding $(d-n_0)+(d-n_1)+(d-n_2)+1$ outgoing edges. One edge is collapsed to a marked point x at Q, while the rest are divided so that $d-n_i$ are mapped to the ray $Q+\mathbb{R}\rho_i$. Note that Γ is a balanced curve.

The contribution of $c\hbar^{-\nu}z^m$ to $L^d_{i,\{0\}}$ is then

$$\begin{aligned}&\hbar^{-(3d+\nu-n_0-n_1-n_2)}D_i(d,n_0,n_1,n_2)\prod_{j=1}^{\nu}\mathrm{Mult}(h_i)u_{I(h_i)}\\ =&\hbar^{-(3d+\nu-n_0-n_1-n_2)}D_i(d,n_0,n_1,n_2)u_{I(\Gamma)}Mult^i_x(\Gamma)\end{aligned}$$

Suppose that $I(\Gamma)=\{i_1,\dots,i_{3d-2+i-\nu'-B(c)}\}$ for some ν'. Let $O(h)=\sum_{i\in I(h)}r_i$. Then

$$\begin{aligned}Val(V_x)-1&=\sum_{i=1}^{\nu}(|\Delta(h_i)-\#I(h_i)-O(h_i))+(d-n_0)+(d-n_1)+(d-n_2)\\&=3d-(3d-2+i-\nu'-B(c))-B(c)=\nu'+2-i\end{aligned}$$

Then the curve defined by h contributes precisely the expected contribution, as given in Definition 34, to $\langle\psi^{r_1}P_{i_1},\dots,\psi^{r_l}P_{i_l},\psi^{\nu}S\rangle^{\mathrm{trop}}_{d,\{0\}}u_{I,r}h^{-(\nu+2-i)}$. Conversely, any curve Γ contributing to $\langle\psi^{r_1}P_{i_1},\dots,\psi^{r_l}P_{i_l},\psi^{\nu}S\rangle^{\mathrm{trop}}_{d,\{0\}}u_{I,r}h^{-(\nu+2-i)}$ can be broken up at V_x to a collection of semirigid disks with endpoint Q. For each $i\in\{0,1,2\}$, at most d of these semirigid disks are in the direction ρ_i, defining the nonnegative numbers n_i. The remaining semirigid disks $h_1,\dots,h_\nu$ all have marked points, and thus there is a term $h^{-\nu}\prod_{i=1}^{\nu}\mathrm{Mult}(h_i)$ in the expansion of $\exp((W_{k,n}(Q)-W_{0,n}(Q))/\hbar)$ making the same contribution to $L^d_{i,\{0\}}(Q)$ that h makes to

$$\langle\psi^{r_1}P_{i_1},\dots,\psi^{r_l}P_{i_l},\psi^{\nu}S\rangle^{\mathrm{trop}}_{d,\{0\}}u_{I,r}h^{-(\nu+2-i)}.$$

Thus we've shown that each term contributing to $L^d_{i,\{0\}}(Q)$ corresponds the correct contribution of a tropical curve, and every tropical curve contributing to the invariant arises in such a fashion. $\square$

Lemma 3.3.13.

$$-L^d_{i,,\gamma_j,\{0\}\to\rho_j} = \sum_{\nu\geq i-1} \sum_{\substack{I\subseteq\{1,\dots,k\}\\ I=\{i_1,\dots,I_l\}\\ i_1<\dots<i_l\\ r_1+\dots+r_l=3d-2+i-\nu-l}} \langle \psi^{r_1}P_{i_1},\dots,\psi^{r_l}P_{i_l},\psi^\nu S\rangle^{\mathrm{trop}}_{d,\rho_j} u_{I,r}h^{-(\nu+2-i)}$$

where $S = Q$, L, or $M_{\mathbb{R}}$ when $i = 0$, 1, or 2 respectively, $r_i \in \mathbb{Z}_{\geq 0}$, and $u_{I,r} = \prod_{j=1}^{l} u_{i_j}t^{r_j}$.

Proof. Both sides are zero if $i = 0$, so we assume $i \geq 1$. Without loss of generality, we set $j = 0$ and consider $L^d_{i,,\gamma_j,\{0\}\to\rho_0}(Q)$, which is a sum of contribution from points in the intersection of $P \in Q + (\rho_0 \setminus \{0\})$ with $Supp(\mathfrak{D})$. Suppose $P \in Q + (\rho_0 \setminus \{0\}) \cap \mathfrak{d}$ and $f_{\mathfrak{d}} = 1 + c_{\mathfrak{d}}z^{m_{\mathfrak{d}}}$. Consider the contribution to $L^d_{i,,\gamma_j,\{0\}\to\rho_0}(Q)$ from a small segment γ of γ_0, from Q_1 to Q_2 crossing only $\mathfrak{d}$. It may be that $\mathfrak{d}$ shares its support at P with some other rays in $\mathfrak{D}$, but the action of these rays will not present any additional complications. Slicing $\mathfrak{d}$ at P, we arrive at a rigid tropical disk $h_0 : \Gamma'_0 \to M_{\mathbb{R}}$ based at P. We can write

$$f_{\mathfrak{d}} = 1 + w_{\Gamma'_0}(E_{out,0})Mult(h_0)z^{\Delta(h_0)}u_{I(h_0)}$$

A term $c\hbar^{-\nu}z^m$ in $\exp(W_{k,n}(Q_1) - W_{0,n}(Q)/\hbar)$ represents ν distinct semirigid tropical descendent disks with boundary Q_1, $h_1,\dots,h_\nu$, each with at least one marked point, and

$$cz^m\hbar^{-\nu} = \hbar^{-\nu}\prod_{i=1}^{\nu} Mult(h_i)z^{\Delta(h_i)}u_{I(h_i)}$$

Revisiting the definition of $L^d_{i,,\gamma_j,\{0\}\to\rho_j}$, we see that $cz^m\hbar^{-\nu}$ contributes iff $m + m_{\mathfrak{d}} = \sum_{i=0}^{n}\Delta(h_i) = dv_0 + n_1v_1 + n_2v_2$ with $n_1, n_2 \leq d$ and $\prod_{i=0}^{\nu} u_{I(h_i)} \neq 0$. Assume $cz^m\hbar^{-\nu}$ contributes; because of the latter condition, $h_1,\dots,h_\nu$ deform to semirigid disks with endpoint P. This data can be assembled into a tropical curve– identifying the outgoing vertices of Γ_i as with the vertex V_{out} from the rigid disk h_0 and adding on $2d - n_1 - n_2 + 1$ unbounded edges to the same vertex to form Γ. A map $h : \Gamma \to M_{\mathbb{R}}$ can be defined in the obvious fashion on all of Γ except for the unbounded edges; it should map $d - n_i$ of them into the direction of ρ_i for

$i = 1, 2$ and collapse the final one into a marked point x at P. Observe that Γ is descendent tropical curve and $Val(V_{out}) = \nu + 2 + 2d - n_1 - n_2$.

The contribution to $-L^d_{i,,\gamma_j,\{0\}\to\rho_j}$ from this term is, by definition,

$$-\langle n_{\mathfrak{d}}, \hat{\rho}_0\rangle u_{I(h)} w_{\Gamma_1}(E_{out,1}) \prod_{i=0}^{\nu} Mult(h_i) D_i(d, d+1, n_1, n_2) h^{-(\nu+3d-(d+n_1+n_2))}$$

Setting $m(h_0) = r(m_{\mathfrak{d}})$, the previous expression is equal to the following

$$|m(h_0) \wedge m_0| u_{I(h)} D_i(d, d+1, n_1, n_2) \hbar^{-(Val(V_{out})-2)} \prod_{i=0}^{\nu} Mult(h_i)$$

Using the same reasoning as above, suppose we have an element $h \in \mathcal{M}^{trop}_{\Delta_d, 3d-i-\nu}(P^{r_1}_{j_1}, \ldots, P^{r_l}_{j_l}, \psi^\nu S)$ where $S = Q, L, M$ when $i = 0, 1, 2$, respectively, $I(h) = \{j_1, \ldots, j_{3d-2+i-\nu'-B(c)}\}$ for some ν' (where $B(c) = \sum_{m=1}^{l} r_m$), $Val(V_{out}) - 2 = \nu' + 2 - i$. In addition, $D_i(d, d+1, n_1, n_2)$ is exactly $Mult^{i-1}_x(h)$, which tells us that the above expression should be precisely the amount that h contributes to $\langle \psi^{r_1} P_{i_1}, \ldots, \psi^{r_l} P_{i_l}, \psi^\nu S\rangle^{\text{trop}}_{d,\rho_j} u_{I,r} h^{-(\nu'+2-i)}$. It is easy to check that any curve h contributing to $\langle \psi^{r_1} P_{i_1}, \ldots, \psi^{r_l} P_{i_l}, \psi^\nu S\rangle^{\text{trop}}_{d,\rho_j} u_{I,r} h^{-(\nu'+2-i)}$ with $h(E_x) = P$ must arise in the manner described above [11], Lemma 5.56.

□

Lemma 3.3.14. For each point $P \in Sing(\mathfrak{D})$, let γ_P be a small counterclockwise loop around P, small enough so that it doesn't go around any other point of $Sing(\mathfrak{D})$. Then

$$L^d_{i,\gamma_{j,j+1},\rho_{j+1}\to\sigma_{j,j+1}} = \sum_{P\in Sing(\mathfrak{D})\cap(Q+\sigma_{j,j+1})} L^d_{i,\gamma_P,\rho_{j+1}\to\sigma_{j,j+1}}$$

Proof. See [11], Lemma 5.57. This result follows using the same methods in Gross' work, as it occurs away from all marked points at which the behavior of our scattering diagram differs from that explored in [11]. □

Lemma 3.3.15. Let $P \in Sing(\mathfrak{D}) \cap (Q + \sigma_{j,j+1})$, and suppose that

$$P \notin \{P_1, \ldots, P_k\}.$$

Then

$$-L^d_{i,\gamma_P,\rho_{j+1}\to\sigma_{j,j+1}} = \sum_{\nu\geq 0}\sum_h Mult(h)u_{i(h)}\hbar^{-(\nu+2-i)}$$

where the sum is over curves h contributing to $\langle\psi^{r_1}P_{i_1},\ldots,\psi^{r_l}P_{i_l},\psi^\nu S\rangle^{\text{trop}}_{d,\sigma_{j,j+1}}$ for $I\subseteq\{1,\ldots,k\}$, $I=\{i_1,\ldots,i_l\}$, $i_1<\ldots<i_l$ with $r_1+\ldots+r_l=3d-2+i-\nu-l$ and $h(x)=P$.

Proof. See [11], Lemma 5.58, employing the slightly altered terminology and methods of the previous results. □

Lemma 3.3.16. Let $P\in Sing(\mathfrak{D})\cap(Q+\sigma_{j,j+1})$, and suppose that

$$P=P_l\in\{P_1,\ldots,P_k\}$$

Then

$$-L^d_{i,\gamma_P,\rho_{j+1}\to\sigma_{j,j+1}} = \sum_{m=1}^{n} u_l(-t_l)^{m-1}\hbar^m\delta_{d,0}\delta_{2,i} + \sum_{\nu\geq 0}\sum_h Mult(h)u_{i(h)}\hbar^{-(\nu+2-i)}$$

where the sum is over curves h contributing to $\langle\psi^{r_1}P_{i_1},\ldots,\psi^{r_s}P_{i_s},\psi^\nu S\rangle^{\text{trop}}_{d,\sigma_{j,j+1}}$ for $I\subseteq\{1,\ldots,k\}$, $I=\{i_1,\ldots,i_s\}$, $i_1<\ldots<i_s$ with $r_1+\ldots+r_s=3d-2+i-\nu-s$ and $h(x)=P_l$.

Proof. Here we assume $i=2$, and write

$$L_{P,j}=L^d_{2,\gamma_P,\rho_{j+1}\to\sigma_{j,j+1}}$$

Choose a basepoint Q' near P_l. As discussed before, sets of $r+1$ semirigid disks with endpoint Q' not bending near P_l should be (roughly) in one-to-one correspondence with rays in $\mathfrak{D}$ emanating from P_l whose monomial contains t_l^r. Recall that a rigid tree with r powers of ψ on P_l will be $r+3$-valent at V_l (the vertex associated to the edge collapsing to P_l). One of the edges attached to V_l is the outgoing edge of the tree, and one is collapsed to P_l. This leaves $r+1$ semirigid disks attached to P_l, disks which should correspond to broken lines not bending near P_l. See Lemma 2.4.2.

More precisely, sets of semirigid disks $\{h_1, \ldots, h_{r+1}\}$ with endpoint Q', $\prod_i Mono(h_i) \neq 0$, and $\sum_i r(\Delta(h_i)) \neq 0$ are in one to one correspondence with rays in $\mathfrak{D}$ with attached monomial containing t_l^r. Fortunately, such sets are naturally recovered from $\exp(W_{k,n}(Q') - W_{0,n}(Q'))$.

To find terms from $\exp(W_{k,n}(Q') - W_{0,n}(Q'))$ contributing to $L_{P,j}$, we should examine those not containing the factor u_l, as any term that does will not produce any new terms as we cross a scattering diagram ray emanating from P_l. We consider a term $c\hbar^{-\nu} z^{\hat{n}}$ of the form:

$$c\hbar^{-\nu} z^{\hat{n}} = \hbar^{-\nu} \prod_{p=1}^{\nu} Mult(h_p) z^{\Delta(h_p)} u_{I(h_p)},$$

where each of the disks h_p with boundary Q' is given by $h_P : \Gamma_p \to M_{\mathbb{R}}$. Note that our analysis will also have to consider the rigid trees containing semirigid disks corresponding to unbounded rays (translated copies of $r(v_i)$) emanating from P_l. The next goal is to analyze the contribution to $L_{P,j}$ from terms including the factor t_l^r. Call this quantity $L_{P,j,r}$.

We will write $\hat{n} = \sum_{i=1}^{\nu} \Delta(h_i) = \sum_{j=0}^{2} n_j v_j$ and choose the normal vectors $n_{\mathfrak{d}}$ to each ray $\mathfrak{d}$ emanating from P_l such that they point in the direction opposite to γ_P' when γ_P crosses $\mathfrak{d}$. With this choice and the standard identification of $\bigwedge^2 M$ with $\mathbb{Z}$, supposing that $\mathfrak{d}$ descends from a tropical tree $h : \Gamma \to M_{\mathbb{R}}$, we have $w_\Gamma(E_{out}) n_{\mathfrak{d}} = X_{r(\Delta(h))}$.

The term $c\hbar^{-\nu} z^{\hat{n}}$ will contribute to $L_{P,j,r}$ when γ_P crosses rays whose corresponding rigid tree contains exactly $r+1$ semirigid disks joined at P_l. The relevant rays can be enumerated as follows. First select $s \leq \nu$ of our semirigid disks with basepoint Q', attach $r - s + 1$ unbounded rays in the primary directions, and use the balancing principle to extend the appropriate contribution $\mathfrak{d}$ to $\mathfrak{D}$.

More explicitly, select $\{h_{i_1}, \ldots h_{i_s}\} \subseteq \{h_1, \ldots h_\nu\}$ and m_j copies of the ray parallel to $\hat{\rho}_j$ for $0 \leq j \leq 2$ such that $s + m_0 + m_1 + m_2 = r + 1$. Let $\tilde{n} := \sum_{j=1}^{s} \Delta(h_{i_j}) + \sum_{j=0}^{2} m_i v_i$ and $r(\tilde{n}) := w_{\tilde{n}} v_{\tilde{n}}$, where $v_{\tilde{n}}$ is primitive. These choices will produce a ray $\mathfrak{d} \in \mathfrak{D}$ with attached function

$$f_{\mathfrak{d}} = 1 + w_{\tilde{n}} \prod_{j=1}^{s} Mono(h_{i_j}) z^{v_0 m_0 + v_1 m_1 + v_2 m_2} \frac{1}{m_0! m_1! m_2!}.$$

Let $c'h^{-(\nu-s)}z^{n'} := \hbar^{-(\nu-s)}\prod_{h\in\{h_1,\dots h_\nu\}\setminus\{h_{i_1},\dots h_{i_s}\}} Mono(h)$. It is easy to see that the term $c'h^{-(\nu-s)}z^{n'}$ will generate a contribution of $c\hbar^{-\nu}z^{\hat{n}}$ to $L_{P,j,r}$ upon crossing $\mathfrak{d}$, and this contribution will occur exactly when $n_{j+2}+m_{j+2} \leq d = n_j+m_j < n_{j+1}+m_{j+1}$. For simplicity of exposition, we set $j=0$ in what follows. The quantity of the contribution is then, by definition

$$\langle n_{\mathfrak{d}}, \hat{\rho}_0\rangle D_2(d, n_0+m_0+1, n_1+m_1, n_2+m_2)\cdot \\ \hbar^{-(\nu-s+3d-(n_0+m_0+n_1+m_1+n_2+m_2))}\frac{1}{m_0!m_1!m_2!}.$$

Noting that $\sum_{j=0}^{2} m_j = r+1-s$ and recalling the isomorphism of $\bigwedge^2 M$ with $\mathbb{Z}$, we see that the above becomes

$$(r(\tilde{n})\wedge\hat{\rho}_0)D_2(d, n_0+m_0+1, n_1+m_1, n_2+m_2)\cdot \\ \hbar^{-(\nu-r-1+3d-(n_0+n_1+n_2))}\frac{1}{m_0!m_1!m_2!}$$

Our goal is to now sum this contribution over all choices of s, $\{h_{i_1},\dots h_{i_s}\}\subseteq\{h_1,\dots h_\nu\}$, m_0, m_1, and m_2. These should exhaust the set of relevant rays emanating from P_l that γ_P crosses, and should thus calculate the total contribution. After a little rearrangement, the sum becomes the following, where $k = r+1-d+n_0$ and $|\hat{n}| = n_0+n_1+n_2$:

$$\frac{\hbar^{-(\nu-r-1+3d-(n_0+n_1+n_2))}}{(d-n_0)!}\sum_{s=0}^{r+1}\sum_{m_1+m_2=k-s}\frac{(-1)^{m_1+n_1+d+1}(n_1+m_1-d-1)!}{m_1!m_2!(d-n_2-m_2)!}\cdot \\ \left(\binom{\nu-1}{s-1}(n_2-n_1)+\binom{\nu}{s}(m_2-m_1)\right). \qquad (3.3.1)$$

In this sum, we are taking any terms involving factorials with negative arguments to be 0. The $(d-n_0)!$ factor comes from the fact that $m_0+n_0=d$, the first summation sets the number of disks from $h_1,\dots,h_\nu$ we're selecting for the rigid disk that gives us $\mathfrak{d}$, the terms $m_1!$ and $m_2!$ are self explanatory, while the remainder of the factorials in the sum come from the definition of D_2. The last terms are from the wedge product and the multiplicities arising from the selection of an s-element subset from a ν element set. The $\binom{\nu-1}{s-1}$ term results from a count of the number of choices of s-disks from the set of ν disks includes a particular edge. Summing this over all possible choices of disks gives the factor of (n_2-n_1). In order to prove the lemma, we'll need an ugly sublemma.

Lemma 3.3.17. Let $d > 0$, $\nu, n_0, n_1, n_2, r \in \mathbb{Z}_{\geq 0}$ with $n_2, n_0 \leq d$. Set $k = r + 1 - d + n_0$, $|\hat{n}| = n_0 + n_1 + n_2$. Then

$$\frac{1}{(d-n_0)!}\sum_{s=0}^{r+1}\sum_{m_1+m_2=k-s}\frac{(-1)^{m_1+n_1+d+1}(n_1+m_1-d-1)!}{m_1!m_2!(d-n_2-m_2)!}\cdot \left(\binom{\nu-1}{s-1}(n_2-n_1)+\binom{\nu}{s}(m_2-m_1)\right) = -\frac{1}{(d-n_0)!(d-n_1)!(d-n_2)!}\binom{\nu+3d-|\hat{n}|-1-((d-n_0)+(d-n_1))}{r-((d-n_0)+(d-n_1))}$$

where all terms are taken to be 0 if they involve any factorials with negative arguments.

Proof. The left hand side of the statement can be rewritten:

$$\frac{1}{(d-n_0)!(d-n_2)!}\sum_{m_1+m_2+s=k}\frac{(-1)^{m_1+n_1+d+1}(n_1+m_1-d-1)!}{m_1!}\binom{d-n_2}{m_2}\cdot \left(\binom{\nu-1}{s-1}(n_2-n_1)+\binom{\nu}{s}(m_2-m_1)\right)$$

There are two cases to distinguish: either $n_1 \geq d+1$ or $n_1 < d+1$. In the former, the sum (up to a sign that won't end up mattering) in the statement should be the coefficient of x^k in the expansion of:

$$(x+1)^{d-n_2}x(x+1)^{\nu-1}\frac{1}{(x+1)^{n_1-d}}(n_2-n_1)+\left(x\frac{d}{dx}\left((x+1)^{d-n_2}\right)\right)(x+1)^{\nu}\cdot \frac{1}{(x+1)^{n_1-d}}-(x+1)^{d-n_2}(x+1)^{\nu}\left(x\frac{d}{dx}\left(\frac{1}{(x+1)^{n_1-d}}\right)\right)$$

Letting $F(x) = x(x+1)^{2d-n_2-n_1+\nu-1}(n_1-d-1)!$, the above simplifies to

$$(n_2-n_1)F(x)+(d-n_2)F(x)+(n_1-d)F(x)=0.$$

Note that here we didn't make use of the statement that $d > 0$ In the other case, define $g_n(x)$ for $n \geq 1$ where $\frac{d^m}{dx^m}g_n(x) = g_{n-m}(x)$ and $g_0(x) = \frac{1}{1+x}$. Such a set can be defined recursively as follows:

$$g_1(x) = \log(1+x)$$
$$g_n(x) = \frac{(1+x)^{(n-1)}}{((n-1)!)^2}\left((n-1)!\log(1+x)-k_n\right)$$

where $k_{n+1} = (n-1)! + k_n(n)$. Note that $g_n = \frac{n}{x+1} g_{n+1} + \frac{(x+1)^{n-1}}{(n)!}$. The utility of these functions comes in their expansion about $x = 0$. By integrating the power series expansion of $log(1+x)$, it's easy to see that

$$g_n(x) = -\frac{k_n x^0}{((n-1)!)^2 0!} - \frac{k_{n-1} x^1}{((n-2)!)^2 1!} - \cdots - \frac{k_1 x^{n-1}}{((0)!)^2 (n-1)!} + \frac{x^n}{(n)_n} - \frac{x^{n+1}}{(n+1)_n} + \frac{x^{n+2}}{(n+2)_n} - \cdots .$$

where $(r)_k$ is the Pochhammer symbol denoting the k-th falling factorial of r.

Let H_n denote the n-th harmonic number. Note that $k_1 = 0$, $k_2 = H_1$, and, by induction, $k_n = H_{n-1}(n-1)!$.

We will also be using a somewhat strange operation on the power series expansions of our functions. Let $\overbrace{f}^{m}$ denote the function arrived at by neglecting all terms of the power series expansion of f with exponent less than m. For example,

$$\overbrace{g_n}^{n} = \frac{x^n}{(n)_n} - \frac{x^{n+1}}{(n+1)_n} + \frac{x^{n+2}}{(n+2)_n} - \cdots .$$

This tool removes all powers of g_n which are not attached to falling factorials, thus allowing us to write out the generating function for our sum when $d > n_1 + 1$.

In particular, the sum on the left hand side is the coefficient of x^k in the

expansion of the following (about $x = 0$):

$$x(x+1)^{d-n_2}(x+1)^{\nu-1}\overbrace{g_{d+1-n_1}(x)}^{d+1-n_1}(n_2-n_1)+\left(x\frac{d}{dx}\left((x+1)^{d-n_2}\right)\right)(x+1)^{\nu}\overbrace{g_{d-n_1+1}(x)}^{d+1-n_1}$$
$$-(x+1)^{d-n_2}(x+1)^{\nu}\overbrace{\left(x\frac{d}{dx}\left(g_{d+1-n_1}(x)\right)\right)}^{d+1-n_1}=$$
$$x(x+1)^{d-n_2}(x+1)^{\nu-1}\overbrace{g_{d+1-n_1}(x)}^{d+1-n_1}(n_2-n_1)+x(d-n_2)(x+1)^{d-n_2-1}(x+1)^{\nu}\overbrace{g_{d+1-n_1}(x)}^{d+1-n_1}$$
$$-(x+1)^{d-n_2}(x+1)^{\nu}\overbrace{(xg_{d-n_1}(x))}^{d+1-n_1}=$$
$$x(x+1)^{d-n_2}(x+1)^{\nu-1}\overbrace{g_{d+1-n_1}(x)}^{d+1-n_1}(n_2-n_1)+x(d-n_2)(x+1)^{d-n_2-1}(x+1)^{\nu}\overbrace{g_{d+1-n_1}(x)}^{d+1-n_1}$$
$$-(x+1)^{d-n_2}(x+1)^{\nu}\overbrace{\left(x\frac{d-n_1}{x+1}g_{d+1-n_1}(x)-\frac{x(x+1)^{(d-n_1-1)}}{(d-n_1)!}\right)}^{d+1-n_1}=$$
$$x(x+1)^{d-n_2}(x+1)^{\nu-1}\overbrace{g_{d+1-n_1}(x)}^{d+1-n_1}(n_2-n_1)+x(d-n_2)(x+1)^{d-n_2-1}(x+1)^{\nu}\overbrace{g_{d+1-n_1}(x)}^{d+1-n_1}-$$
$$(x+1)^{d-n_2}(x+1)^{\nu}\overbrace{\left(x\frac{d-n_1}{x+1}g_{d+1-n_1}(x)\right)}^{d+1-n_1}-(x+1)^{d-n_2}(x+1)^{\nu}\overbrace{\left(\frac{x(x+1)^{(d-n_1-1)}}{(d-n_1)!}\right)}^{d+1-n_1}=$$
$$x(x+1)^{d-n_2}(x+1)^{\nu-1}\overbrace{g_{d+1-n_1}(x)}^{d+1-n_1}(n_2-n_1)+x(d-n_2)(x+1)^{d-n_2-1}(x+1)^{\nu}\overbrace{g_{d+1-n_1}(x)}^{d+1-n_1}-$$
$$(x+1)^{d-n_2}(x+1)^{\nu}\overbrace{\left(x\frac{d-n_1}{x+1}g_{d+1-n_1}(x)\right)}^{d+1-n_1}=$$
$$x(x+1)^{d-n_2}(x+1)^{\nu-1}\overbrace{g_{d+1-n_1}(x)}^{d+1-n_1}(n_2-n_1)+x(d-n_2)(x+1)^{d-n_2-1}(x+1)^{\nu}\overbrace{g_{d+1-n_1}(x)}^{d+1-n_1}-$$
$$(d-n_1)(x+1)^{d-n_2}(x+1)^{\nu}\overbrace{\left(x\frac{1}{x+1}g_{d+1-n_1}(x)\right)}^{d+1-n_1}$$

Before we proceed, we need a way to move the term $x\frac{1}{x+1}$ from beneath the brace. Toward this end, we will compare $\overbrace{\left(x\frac{1}{x+1}g_{d+1-n_1}(x)\right)}^{d+1-n_1}$ with $x\frac{1}{x+1}\overbrace{(g_{d+1-n_1}(x))}^{d+1-n_1}$ by calculating these in terms of a product of power series expansions. Let the coefficients of the expansion of $g_n(x)$ define a_i and b_i such that:

$$g_n(x)=\sum_{i=0}^{n-1}a_ix^i+\sum_{i=n}^{\infty}b_ix^i$$

and they are 0 otherwise. Then

$$\frac{1}{1+x}g_n(x) = \sum_{i=0}^{\infty}\left(\sum_{c=0}^{i}(a_{i-c}+b_{i-c})(-1)^c\right)x^i$$

Define $Q = a_{n-1} - a_{n-2} + \cdots + (-1)^{n-1}a_0$. Then

$$\begin{aligned}\overbrace{\frac{x}{1+x}g_n(x)}^{n} &= \sum_{i=n-1}^{\infty}\left(\sum_{c=0}^{i}(a_{i-c}+b_{i-c})(-1)^c\right)x^{i+1} \\ &= \sum_{i=n-1}^{\infty}\left((-1)^{i+n-1}Q + \sum_{c=0}^{i}(b_{i-c})(-1)^c\right)x^{i+1}.\end{aligned}$$

Alternately,

$$\begin{aligned}\frac{x}{1+x}\overbrace{g_n(x)}^{n} &= \sum_{i=0}^{\infty}\left(\sum_{c=0}^{i}(b_{i-c})(-1)^c\right)x^{i+1} \\ &= \sum_{i=n-1}^{\infty}\left(\sum_{c=0}^{i}(b_{i-c})(-1)^c\right)x^{i+1}.\end{aligned}$$

Thus

$$\overbrace{\frac{x}{1+x}g_n(x)}^{n} = \frac{x}{1+x}\overbrace{g_n(x)}^{n} + \frac{x^n}{1+x}Q.$$

Then the expression for our generating function becomes

$$x(x+1)^{d-n_2}(x+1)^{\nu-1}\overbrace{g_{d+1-n_1}(x)}^{d+1-n_1}(n_2-n_1) + x(d-n_2)(x+1)^{d-n_2-1}(x+1)^{\nu}\overbrace{g_{d+1-n_1}(x)}^{d+1-n_1} -$$
$$(d-n_1)(x+1)^{d-n_2}(x+1)^{\nu}\frac{x}{x+1}\overbrace{g_{d+1-n_1}}^{d+1-n_1} - (d-n_1)(x+1)^{d-n_2}(x+1)^{\nu}\frac{x^{d-n_1+1}}{x+1}Q =$$
$$-(d-n_1)(x+1)^{d-n_2}(x+1)^{\nu}\frac{x^{d-n_1+1}}{x+1}Q$$

In order to finish the lemma, we'll have to find Q. Start by noting that

$$\begin{aligned}a_i &= \frac{-k_{d-n_1+1-i}}{((d-n_1-i)!)^2 i!} = -\frac{H_{d-n_1-i}(d-n_1-i)!}{((d-n_1-i)!)^2 i!} \\ &= -\frac{1}{(d-n_1)!}\binom{d-n_1}{d-n_1-i}H_{d-n_1-i}.\end{aligned}$$

Thus,

$$Q = \sum_{i=0}^{d-n_1}(-1)^i a_{d-n_1-i} = \frac{-1}{(d-n_1)!}\sum_{i=0}^{d-n_1}(-1)^i H_i\binom{d-n_1}{i}$$

Define $\hat{Q} = (d-n_1)!Q = \sum_{i=0}^{d-n_1}(-1)^i H_i\binom{d-n_1}{i}$. We claim that $\hat{Q} = \frac{-1}{d-n_1}$. Assuming this result, we see that this turns our generating function into

$$\frac{-1}{(d-n_1)!}(x+1)^{d-n_2}(x+1)^{\nu-1}x^{d-n_1+1}.$$

The coefficient in of x^k for this final quantity is just $\frac{1}{(d-n_1)!}\binom{\nu+d-n_2-1}{k-1-d+n_1}$, which gives us the desired equality.

Now for the claimed result about Q.

Claim 3.3.18. $\sum_{i=0}^{d-n_1}(-1)^i H_i\binom{d-n_1}{i} = \frac{-1}{d-n_1}$

Proof. Thanks to Angela Hicks and Emily Leven for their suggestions on the proof of this claim.

Define $R_n := \sum_{i=0}^{n}(-1)^i H_i\binom{n}{i}$. We will examine $R_n - R_{n-1}$.

$$\begin{aligned}
R_n - R_{n-1} &= \sum_{i=0}^{n}(-1)^i H_i\binom{n}{i} - \sum_{i=0}^{n-1}(-1)^i H_i\binom{n-1}{i} \\
&= \sum_{i=0}^{n}(-1)^i H_i\binom{n}{i} - \sum_{i=0}^{n-1}(-1)^i H_i\binom{n-1}{i} \\
&= \sum_{i=0}^{n}(-1)^i H_i\left(\binom{n}{i} - \binom{n-1}{i}\right) \\
&= \sum_{i=1}^{n}(-1)^i (\frac{1}{i} + H_{i-1})\left(\binom{n-1}{i-1}\right) \\
&= \sum_{i=1}^{n}(-1)^i \frac{1}{i}\binom{n-1}{i-1} + \sum_{i=1}^{n}(-1)^i H_{i-1}\binom{n-1}{i-1} \\
&= \sum_{i=1}^{n}(-1)^i \frac{1}{n}\binom{n}{i} - \sum_{i=0}^{n-1}(-1)^i H_i\binom{n-1}{i} \\
&= \frac{1}{n}\sum_{i=1}^{n}(-1)^i\binom{n}{i} - R_{n-1}
\end{aligned}$$

It is a well known fact that $\sum_{i=0}^{n}(-1)^i\binom{n}{i} = 0$, (unless $n = 0$, which we won't be considering) so we have $R_n - R_{n-1} = 0 - \frac{1}{n}\binom{n}{0} - R_{n-1}$, which implies that $R_n = -\frac{1}{n}$. Applying this result to our special case, we see that it proves the claim. □

□

If we have a situation in which we have a non-zero contribution to $-L_{P,j,r}$ of the term $c\hbar^{-\nu}z^{\hat{n}}$ (with $d > 0$), we can assemble a balanced tropical h curve from this data. Begin by gluing the disks $h_1, \ldots, h_\nu$ together at their outgoing vertices, add on $d-n_j$ unbounded edges in the direction $\hat{\rho}_j$ for $0 \leq j \leq 2$ and two additional edges E_x and E_{p_l} that will be collapsed to mark x and p_l. This procedure yields a graph Γ whose valency at the new vertex V is given by $Val(V) = \nu + 3d - |\hat{n}| + 2$. Thus we have a parametrized $h : \Gamma \to M_{\mathbb{R}}$ with $h(E_x) = h(p_l) = h(V) = P_l$. This lemma allows us to easily describe the contribution to $-L_{P,j,r}$ of the term $c\hbar^{-\nu}z^{\hat{n}}$ upon crossing the corresponding rays emanating from P_l as

$$\binom{Val(V) - 3 - ((d-n_0) + (d-n_1))}{r - ((d-n_0) + (d-n_1))} Mult_x^0(h) \left(\prod_{k=1}^{\nu} Mult(h_k)\right) u_{I(h)} u_l \cdot \hbar^{-(\nu - r - 1 + 3d - |\hat{n}|)}$$

Recalling our original monomial $c\hbar^{-\nu}z^{\hat{n}}$ and the definition of $B(c)$, suppose that $I(h) = \{i_1, \ldots, i_{3d-\nu'-B(c)-r}\}$ for some $\nu' \geq 0$. Because h is obtained by gluing $Val(V) - 2$ semirigid disks, we have

$$\begin{aligned} Val(V) - 2 &= \sum_{i=1}^{\nu} (|\Delta(h_i)| - |I(h_i)| - overvalency(h_i)) + 3d - |\hat{n}| \\ &= |\hat{n}| - (3d - \nu' - 1 - B(c) - r) - B(c) + 3d - |\hat{n}| \\ &= \nu' + r + 1 \end{aligned}$$

where $overvalency(h_i) := \sum_{V \in \mathfrak{V}} (Val(V) - 3)$ where $\mathfrak{V}$ is the set of all non-univalent vertices of h_i. Therefore, the contribution to $-L_{P,j,r}$ from γ_P crossing rays associated to this term is precisely the contribution of h to

$$\langle \psi^{r_1} P_{i_1}, \ldots, \psi^{r_{3d-\nu'}} P_{3d-\nu'}, \psi^r P_l, \psi^{\nu'} M_{\mathbb{R}} \rangle_{d, \sigma_{j,j+1}}^{trop} u_{I(h)} \hbar^{-\nu'}$$

Conversely, it is easy to see that any such curve h contributing to the invariant will be accounted for by the integral by deconstructing it into its marked semirigid disks and applying the above procedure to create the relevant set of rays for γ_P to cross.

We still need to cover the case in which $d = 0$, the situation that accounts for the odd term in the statement of the result. Our lemma shows that there is no contribution to $-L_{P,j,r}$ with positive power of $\hbar$ if $d \neq 0$, but the same does not hold if $d = 0$.

Suppose $d = 0$. Then an examination of Expression 3.3.1 quickly shows that any non-zero contribution must occur when $n_0 = n_2 = 0$. In this case, $m_2 = 0$, which forces $m_1 = k - s = r + 1 - s$ so we get

$$\hbar^{-(\nu-r-1-(n_1))} \sum_{s=0}^{r+1} \frac{(-1)^{r-s-n_1}(n_1+r-s)!}{(r+1-s)!} \left(\binom{\nu-1}{s-1}(-n_1) + \binom{\nu}{s}(-r-1+s) \right).$$

If $n_1 > d = 0$, then the same argument as was applied in the first case of Lemma 3.3.17 shows that the above quantity is equal to 0. In fact, one can easily see that the first part of that Lemma does not rely on the fact that $d > 0$. If $n_1 = 0$, we know that $\nu = 0$, so the above simplifies to

$$\hbar^{-(-r-1))} \sum_{s=0}^{r+1} \frac{(-1)^{r-s-n_1}(n_1+r-s)!}{(r+1-s)!} \left(\binom{0}{s}(-r-1+s) \right)$$
$$\hbar^{r+1} \frac{(-1)^r(r)!}{(r+1)!} \left(\binom{0}{0}(-r-1) \right).$$

In this case the contribution to $-L_{P,j,r}$ from γ_P is equal to $(-1)^r \hbar^{r+1} t_l^r$. □

Lemma 3.3.19.

$$-L^d_{i,\gamma_j,\rho_{j+1}\to\sigma_{j,j+1}} = \hbar \sum_{l \text{ s.t. } P_l \in Q+\sigma_{j,j+1}} \delta_{d,0}\delta_{2,i} u_l (1 - t_l\hbar + \ldots + (-t_l\hbar)^{n-1}) +$$
$$\sum_{\nu \geq 0} \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ I=\{i_1,\ldots,I_l\} \\ i_1<\ldots<i_l \\ r_1+\ldots+r_l=3d-2+i-\nu-l}} \langle \psi^{r_1} P_{i_1}, \ldots, \psi^{r_l} P_{i_l}, \psi^\nu S \rangle^{\text{trop}}_{d,\sigma_{j,j+1}j} u_{I,r} h^{-(\nu+2-i)}$$

for $S = Q$, L, or $M_{\mathbb{R}}$ and $i = 0$, 1, and 2, respectively.

Proof. This follow from the previous lemmas. Note, in particular, the first sum that results from the previous remark as r is varied from 0 to n. □

Chapter 4

Conclusion

The upshot of the analysis in the previous chapters is that the integrals recover conjectured tropical methods for computing Gromov-Witten invariants. In an attempt to further plumb the structure of these integrals, we assemble the terms we've computed to form an expression for the whole integral.

Theorem 4.0.1. We can write

$$\sum_{i=0}^{2} \alpha^i \int_{\Xi_i} e^{W_{k,n}(Q)/\hbar}\Omega = \hbar^{-3\alpha} \sum_{i=0}^{2} \phi_i(\alpha\hbar)^i$$

where

$$\phi_i(y_0, y_1, u_1, \ldots, u_k, t_1, \ldots, t_k, \hbar^{-1}) := \delta_{0,i} + \sum_{j=1}^{\infty} \phi_{i,j}(y_0, y_1, , u_1, \ldots, u_k, t_1, \ldots, t_k)\hbar^{-j},$$

for $i = 0, 1$ and

$$\begin{aligned}\phi_2(y_0, y_1, u_1, \ldots, u_k, t_1, \ldots, t_k, \hbar^{-1}) :=& \delta_{0,2} + \sum_{l=1}^{k} u_l(1 + (-t_l\hbar)^1 + \ldots + (-t_l\hbar)^m) + \\ & \sum_{j=2}^{\infty} \phi_{2,j}(y_0, y_1, , u_1, \ldots, u_k, t_1, \ldots, t_k)\hbar^{-j}\end{aligned}$$

with

$$\phi_{0,1} = y_0$$

$$\phi_{1,1} = y_1$$

Proof. Follows from assembling the terms of the integral. □

Theorem 4.0.2. Making the identification $u_1 + u_2 + \ldots + u_k = y_n$ and under the forgetful map $t_i \mapsto t$, we can write the functions ϕ in terms of our conjectured tropical Gromov-Witten invariants. Noting that the values of the integrals are independent of the position of the points as long as they are general, we write T_2 for the point conditions P_i and T_0, T_1, and T_2 for the cases in which $S = M_{\mathbb{R}}$, L, and Q, respectively. Then, using the potential $W_{k,n}$ defined over $R_{k,n}$, we see that:

$$L_0^d = \delta_{d,0} + \sum_{\nu \geq 0} \sum_{\substack{m \leq k \\ r_1+\ldots+r_m=3d-2-\nu-m \\ r_i \geq 0}} \langle \psi^{r_1} T_2, \ldots, \psi^{r_m} T_2, \psi^{\nu} T_2 \rangle_{0,d}^{\text{trop}} \frac{y_2^m t^{3d-2+\nu-m}}{m!} \hbar^{-(\nu+2)}$$

$$L_1^d = \sum_{\nu \geq 0} \sum_{\substack{m \leq k \\ r_1+\ldots+r_m=3d-1-\nu-m \\ r_i \geq 0}} \langle \psi^{r_1} T_2, \ldots, \psi^{r_m} T_2, \psi^{\nu} T_1 \rangle_{0,d}^{\text{trop}} \frac{y_2^m t^{3d-1+\nu-m}}{m!} \hbar^{-(\nu+1)}$$

$$L_2^d = \delta_{d,0} \hbar y_2 (1 - t\hbar + \ldots + (-t\hbar)^{n-1}) + \sum_{\nu \geq 0} \sum_{\substack{m \leq k \\ r_1+\ldots+r_m=3d-\nu-m \\ r_i \geq 0}} \langle \psi^{r_1} T_2, \ldots, \psi^{r_m} T_2, \psi^{\nu} T_0 \rangle_{0,d}^{\text{trop}} \frac{y_2^m t^{3d+\nu-m} s}{m!} \hbar^{-\nu}.$$

With this definition,

$$\phi_0 = e^{y_0/\hbar} \sum_{d=0}^{\infty} e^{dy_1} (L_0^d)$$

$$\phi_1 = e^{y_0/\hbar} \sum_{d=0}^{\infty} e^{dy_1} \hbar^{-1} (y_1 L_0^d + L_1^d)$$

$$\phi_2 = e^{y_0/\hbar} \sum_{d=0}^{\infty} e^{dy_1} \hbar^{-2} (\frac{y_1^2}{2} L_0^d + y_1 L_1^d + L_2^d)$$

Proof. Follows from above. □

Because of Gross' work and Theorem 1.2.6, the tropical invariants of the type $\langle \psi^{r_1} T_2 \ldots, \psi^{r_m} T_2, \psi^{\nu} T_i \rangle_{0,d}^{trop}$ are known to coincide with their classical invariants. On the other hand, Markwig and Rau have shown that the tropical invariants of the type $\langle \psi_1^r T_2 \ldots, \psi^{r_m} T_2, T_i \rangle_{0,d}^{trop}$ also agree with their classical counterparts. The rest of this thesis will deal with a discussion of how one may able show that the

other tropical invariants appearing in our functions L_i^d correspond to classical invariants. Note that we cannot immediately apply Theorem 1.2.6 as the result of our integral, as seen in Theorem 4.0.1, fails to conform to the necessary shape in terms of its expansion in $\hbar$.

4.1 Directions forward

4.1.1 Topological Recursion

Descendent Gromov-Witten invariants satisfy a set of relationships such that those in genus 0 are completely determined in terms of their non-descendent counterparts [11].

Definition 4.1.1 (Descendent Gromov-Witten invariant axioms)**.** Some of the more important rules are listed below.

- *Divisor axiom.* If $n + 2g \geq 4$ or $\beta \neq 0$ and $n \geq 1$, and $\alpha_n \in H^2(X, \mathbb{Q})$, then

$$\begin{aligned}&\langle \psi^{r_1}\alpha_1, \ldots, \psi^{r_{n-1}}\alpha_{n-1}, \alpha_n\rangle_{g,\beta} = \\ &\left(\int_\beta \alpha_n\right) \langle \psi^{r_1}\alpha_1, \ldots, \psi^{r_{n-1}}\alpha_{n-1}\rangle_{g,\beta} \\ &+ \sum_{i=1}^{n-1} \left(\int_\beta \alpha_n\right) \langle \psi^{r_1}\alpha_1, \ldots, \psi^{r_i-1}\alpha_i \cup \alpha_n, \ldots, \psi^{r_{n-1}}\alpha_{n-1}\rangle_{g,\beta}.\end{aligned}$$

- *Point mapping axiom* If $n \leq 3$, then

$$\langle \psi^{r_1}\alpha_1, \ldots, \psi^{r_n}\alpha_n\rangle_{0,0} = 0$$

  unless $n = 3$ and $r_1 = r_2 = r_3 = 0$.

- *Fundamental class axiom.* If $[X]$ is the fundamental class of X, then

$$\begin{aligned}&\langle \psi^{r_1}\alpha_1, \ldots, \psi^{r_{n-1}}\alpha_{n-1}, [X]\rangle_{g,\beta} = \\ &= \sum_{i=1}^{n-1} \left(\int_\beta \alpha_n\right) \langle \psi^{r_1}\alpha_1, \ldots, \psi^{r_i-1}\alpha_i \cup \alpha_n, \ldots, \psi^{r_{n-1}}\alpha_{n-1}\rangle_{g,\beta}\end{aligned}$$

- *Dilaton Axiom*

$$\langle \psi^{r_1}\alpha_1, \ldots, \psi^{r_{n-1}}\alpha_{n-1}, \psi[X]\rangle_{g,\beta} = (2g-3+n)\langle \psi^{r_1}\alpha_1, \ldots, \psi^{r_{n-1}}\alpha_{n-1}\rangle_{g,\beta}$$

- *Topological recursion relationship (TRR)* This particular result is the most important for the following discussion and unique to genus 0. Let T_i generate $H^{2i}(X,\mathbb{Z})$, and $T^0, \ldots, T^m$ be the Poincaré dual basis.

$$\langle \psi^{r_1}\alpha_1, \ldots, \psi^{r_n}\alpha_n\rangle_{g,\beta} =$$
$$\sum \langle \psi^{r_1-1}\alpha_1, \prod_{i\in S_1} \psi^{r_i}\alpha_i, T_e\rangle_{0,\beta_1} \sum \langle T^e, \psi^{r_2}\alpha_2, \psi^{r_3}\alpha_3, \prod_{i\in S_2} \psi^{r_i}\alpha_i, T_e\rangle_{0,\beta_2}$$

  where the sum is over all $0 \leq e \leq m$, all splittings $\beta_1 + \beta_2 = \beta$, and $S_1 \cup S_2 = \{4, \ldots, n\}$.

It is relatively easy to see that these axioms allow us to reduce the invariants of concern in the previous results to those for which tropical counting methods (in the case of $\mathbb{P}^2$) are firmly established. Thus, if one is able to show that the conjectural methods of tropical curve counting satisfy these invariants, in particular the TRR in combination with the simpler effects of the fundamental class and divisor axioms, one can show that they coincide with the actual values of the Gromov-Witten invariants associated to $\mathbb{P}^2$. It is reasonable to expect that a fairly straightforward combinatorial argument of the type seen in this thesis could show that the TRR is satisfied.

4.1.2 Normalization

While the bulk of this thesis can be seen as a combinatorial effort to show some results about a conjectural curve counting method, it is more interesting if seen as an alternate construction of an unfolding of the Landau-Ginzburg potential. If one revisits the discussion of Barannikov's oscillatory integrals and Givental's J-function in the first chapter and compares the result of the integrals in Theorem 1.2.6 with those discussed in Theorem 4.0.2, it is immediately observed that ϕ_2 fails to only contain terms whose power on $\hbar$ is less than or equal to -1. However, given the properties of our unfolding $W_{k,n}$ of W, Barannikov's result shows that

we should be able to construct an integral that has the desired form. In order to achieve this, one must alter the function f discussed in Section 1.2.3 (we set $f = 1$ in our calculations) so that the *normalization* condition occurs, that is, the result of the integrals satisfies the form set out at the end of the section.

If such a function f can be found, Theorem 1.2.6 (Mirror symmetry for $\mathbb{P}^2$) may give a novel relationship between the tropically counted descendent Gromov-Witten invariants and the classical invariants used in the definition of Givental's J-function and may give the coincidence of our conjectured tropical invariants with their classical counterparts.

Bibliography

[1] K. Behrend and B. Fantechi. "The intrinsic normal cone". In: *Invent. Math.* 128.1 (1997), pp. 45–88. ISSN: 0020-9910. DOI: 10.1007/s002220050136. URL: http://dx.doi.org/10.1007/s002220050136.

[2] P. Candelas, M. Lynker, and R. Schimmrigk. "Calabi-Yau manifolds in weighted $\mathbf{P}_4$". In: *Nuclear Phys. B* 341.2 (1990), pp. 383–402. ISSN: 0550-3213. DOI: 10.1016/0550-3213(90)90185-G. URL: http://dx.doi.org/10.1016/0550-3213(90)90185-G.

[3] Philip Candelas et al. "An exactly soluble superconformal theory from a mirror pair of Calabi-Yau manifolds". In: *Particles, strings and cosmology (Boston, MA, 1991)*. World Sci. Publ., River Edge, NJ, 1992, pp. 667–680.

[4] David A. Cox and Sheldon Katz. *Mirror symmetry and algebraic geometry*. Vol. 68. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society, 1999, pp. xxii+469. ISBN: 0-8218-1059-6.

[5] Andreas Gathmann. *Growi 1.02*. URL: http://www.mathematik.uni-kl.de/~gathmann/growi.php?lang=en.

[6] Andreas Gathmann. *Tropical algebraic geometry*. eprint: math/0601322. URL: http://arxiv.org/abs/math/0601322.

[7] Andreas Gathmann, Michael Kerber, and Hannah Markwig. *Tropical fans and the moduli spaces of tropical curves*. Aug. 2007. eprint: 0708.2268. URL: http://arxiv.org/abs/0708.2268.

[8] Angela Gibney and Diane Maclagan. *Equations for Chow and Hilbert Quotients*. July 2007. eprint: 0707.1801. URL: http://arxiv.org/abs/0707.1801.

[9] Alexander B. Givental. "Semisimple Frobenius structures at higher genus". In: *Internat. Math. Res. Notices* 23 (2001), pp. 1265–1286. ISSN: 1073-7928. DOI: 10.1155/S1073792801000605. URL: http://dx.doi.org/10.1155/S1073792801000605.

[10] Mark Gross. "Mirror symmetry for P^2 and tropical geometry". In: (Mar. 2009). eprint: 0903.1378. URL: http://arxiv.org/abs/0903.1378.

[11] Mark Gross. *Tropical geometry and mirror symmetry*. Vol. 114. CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2011, pp. xvi+317. ISBN: 978-0-8218-5232-3.

[12] Y.P. Lee and Rahul Pandharipande. *Frobenius manifolds, Gromov-Witten theory, and Virasoro constraints Part II*. URL: http://www.math.princeton.edu/~rahulp/Part2.ps.

[13] Jun Li and Gang Tian. "Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties". In: *J. Amer. Math. Soc.* 11.1 (1998), pp. 119–174. ISSN: 0894-0347. DOI: 10.1090/S0894-0347-98-00250-1. URL: http://dx.doi.org/10.1090/S0894-0347-98-00250-1.

[14] Yuri I. Manin. *Frobenius manifolds, quantum cohomology, and moduli spaces*. Vol. 47. American Mathematical Society Colloquium Publications. Providence, RI: American Mathematical Society, 1999, pp. xiv+303. ISBN: 0-8218-1917-8.

[15] Hannah Markwig and Johannes Rau. *Tropical descendant Gromov-Witten invariants*. Sept. 2008. eprint: 0809.1102. URL: http://arxiv.org/abs/0809.1102.

[16] Grigory Mikhalkin. *Enumerative tropical algebraic geometry in R2*. eprint: math/0312530. URL: http://arxiv.org/abs/math/0312530.

[17] Rahul Pandharipande and Y.P. Lee. *Frobenius manifolds, Gromov-Witten theory, and Virasoro constraints Part I*. URL: http://www.math.princeton.edu/~rahulp/Part1.ps.

[18] S.Barannikov. *Semi-infinite Hodge structures and mirror symmetry for projective spaces*. eprint: math/0010157. URL: http://arxiv.org/abs/math/0010157.