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A SUFFICIENT CONDITION FOR ULTRAFILTERS ON A UNIFORM SPACE TO BE CAUCHY FILTERS

by

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INTRODUCTION

The concept of a filter is defined on an arbitrary nonempty set X, the filter being a generalization of a sequence. If the set X is equipped with a topology, then the notion of convergence of a filter is defined within the structure of the topology. Furthermore, if the topological space is uniformizable, then the concept of a Cauchy filter is defined within the uniform structure. This is, of course, analogous to a Cauchy sequence in a metric space.

It can be shown by using Zorn's lemma that the partially ordered set of filters on a set X contains certain maximal elements which are called ultrafilters. The most obvious ultrafilters are trivial ones. In a uniform space the trivial ultrafilters are always Cauchy filters.

It is natural to ask if non-trivial ultrafilters exist. The question is answered affirmatively, but another is posed. In which uniform space is every ultrafilter a Cauchy filter? It is this question with which we are concerned in this paper.

The following form of Zorn's lemma is used: "Every inductive partially ordered set contains at least one maximal element." A partially ordered set is said to be inductive if every linearly ordered subset has an upper bound.

Much of the background material presented is adapted from Bourbaki [1]. Details of the proof of Theorem 4. 1 are adapted from notes of lectures delivered by the late Dr. Hisahiro Tamano.

Numbers in brackets refer to the bibliography.

CHAPTER I

FILTERS AND ULTRAFILTERS

Definition 1.1. A filter on a nonempty set X is a collection $\mathcal{F}$ of subsets of X which satisfies the following conditions:

(F.1) Every subset of X which contains a member of $\mathcal{F}$ is also a member of $\mathcal{F}$.

(F.2) Every finite intersection of members of $\mathcal{F}$ is also a member of $\mathcal{F}$.

(F.3) The empty set is not a member of $\mathcal{F}$.

As an example, consider a subset A of a topological space X. The collection of all neighborhoods of A is a filter on X called the neighborhood filter of A. (A neighborhood of a set A is defined to be a set which contains an open set which contains A.)

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two filters on a set X, then $\mathcal{F}_1$ is said to be coarser than $\mathcal{F}_2$ (or $\mathcal{F}_2$ is finer than $\mathcal{F}_1$) if $\mathcal{F}_1$ is a subset of $\mathcal{F}_2$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are also unequal, then $\mathcal{F}_1$ is strictly coarser than $\mathcal{F}_2$ (or $\mathcal{F}_2$ is strictly finer than $\mathcal{F}_1$).

Definition 1.2. A point x in a topological space X is said to be a limit of a filter $\mathcal{F}$ on X (or $\mathcal{F}$ is said to converge to x) if $\mathcal{F}$ is finer than the neighborhood filter of x.

For a given collection $\mathcal{A}$ of subsets of a set X, there may or may not exist a filter on X which contains $\mathcal{A}$. For such a filter to exist, it is obviously necessary for $\mathcal{A}$ to have the finite intersection property. The following theorem shows that this condition is also sufficient for the existence of such a filter.

Theorem 1.1. Let $\mathcal{A}$ be a collection of subsets of X. There exists a filter on X which contains $\mathcal{A}$ if and only if every finite intersection of members of $\mathcal{A}$ is nonempty.

Proof: The necessity of the finite intersection property is obvious. For the sufficiency, suppose that $\mathcal{A}$ has the finite intersection property, and let $\mathcal{B}$ be the collection of all finite intersections of members of $\mathcal{A}$. Now let $\mathcal{F}$ be the collection of all subsets of X which contain some member of $\mathcal{B}$. Then $\mathcal{F}$ clearly satisfies F.1, F.2, and F.3 and is, therefore, a filter on X which contains $\mathcal{A}$. This completes the proof of the theorem.

Note that $\mathcal{F}$ is also the coarsest filter on X containing $\mathcal{A}$. For if $\mathcal{F}'$ is another such filter coarser than $\mathcal{F}$ and F is a member of $\mathcal{F} - \mathcal{F}'$, then F contains a member of $\mathcal{F}'$ but does not belong to $\mathcal{F}'$. Therefore, $\mathcal{F}'$ is not a filter.

The filter $\mathcal{F}$ in Theorem 1.1 is said to be generated by $\mathcal{A}$, and $\mathcal{A}$ is called a subbase of the filter $\mathcal{F}$.

If $\mathcal{A}$ is a subbase of a filter $\mathcal{F}$ on X, then $\mathcal{F}$ is not necessarily the collection of subsets of X which contain some member of $\mathcal{A}$. For this to be true, it is necessary and sufficient that $\mathcal{A}$ satisfy the additional condition of being a filter base.

Definition 1.3. Let $\mathcal{B}$ be a collection of subsets of X satisfying the following:

(B. 1) The intersection of two members of $\mathcal{B}$ contains a member of $\mathcal{B}$.

(B. 2) $\mathcal{B}$ is not empty, and the empty subset of X does not belong to $\mathcal{B}$.

Then $\mathcal{B}$ is called a base of the filter which it generates.

Two filter bases are said to be equivalent if they generate the same filter.

Theorem 1.2. Let $\mathcal{B}$ be a collection of subsets of X. Then the collection $\mathcal{F}$ of subsets of X each of which contains a member of $\mathcal{B}$ is a filter if and only if $\mathcal{B}$ satisfies B. 1 and B. 2.

Proof: Suppose that $\mathcal{F}$ is a filter, and let B_1 and B_2 be members of $\mathcal{B}$. Then B_1 and B_2 also belong to $\mathcal{F}$; therefore, their intersection belongs to $\mathcal{F}$. Therefore, $B_1 \cap B_2$ must contain a member of $\mathcal{B}$, and B. 1 is satisfied.

If $\mathcal{B}$ is empty, then it is vacuously true that a member of $\mathcal{B}$ is contained in each subset of X, hence $\mathcal{F}$ contains all subsets of X including the empty set $\emptyset$. But this implies that $\mathcal{F}$ is not a filter. Therefore, $\mathcal{B}$ is nonempty.

If $\emptyset$ is a member of $\mathcal{B}$, then $\emptyset$ is a subset of X which contains a member of $\mathcal{B}$; hence $\emptyset$ belongs to $\mathcal{F}$, and $\mathcal{F}$ is not a filter. Therefore, $\emptyset$ is not a member of $\mathcal{B}$, and B.2 is satisfied.

Conversely, suppose that $\mathcal{B}$ satisfies B.1 and B.2. Since $\emptyset$ does not belong to $\mathcal{B}$, then F.3 is satisfied.

If $F_1, \ldots, F_n$ are members of $\mathcal{F}$ and contain members $B_1, \ldots, B_n$ of $\mathcal{B}$ respectively, then $F_1 \cap \ldots \cap F_n$ contains $B_1 \cap \ldots \cap B_n$ which contains a member of $\mathcal{B}$. Hence F.2 is satisfied.

A subset F of X containing a member G of $\mathcal{F}$ also contains a member of $\mathcal{B}$ which is contained in G. Hence F belongs to $\mathcal{F}$, and F.1 is satisfied. Therefore, $\mathcal{F}$ is a filter, and the theorem is proved.

Theorem 1.3. A subset $\mathcal{B}$ of a filter $\mathcal{F}$ on X is a base of $\mathcal{F}$ if and only if every member of $\mathcal{F}$ contains a member of $\mathcal{B}$.

Proof: If $\mathcal{B}$ is a base of $\mathcal{F}$, then every member of $\mathcal{F}$ contains a member of $\mathcal{B}$ by Definition 1.3 and Theorem 1.2. Conversely, if each member of $\mathcal{F}$ contains a member of $\mathcal{B}$, then by F.1 the filter generated by $\mathcal{B}$ is precisely $\mathcal{F}$. This completes the proof.

Let $\{x_n\}$ be an arbitrary sequence of points in a set X, and let $B_n = \{x_i : i \geq n\}$ and $\mathcal{B} = \{B_n : n \in N\}$. Then $\mathcal{B}$ clearly satisfies B. 1 and B. 2, hence it is a base of a filter $\mathcal{F}$. $\mathcal{F}$ is called the <u>elementary filter</u> associated with the sequence $\{x_n\}$.

We now consider some properties of the collection of all filters on a set X partially ordered by set inclusion.

<u>Theorem 1.4.</u> A set Φ of filters on X has a least upper bound if and only if, for all finite sequences $\{\mathcal{F}_i\}_{i=1}^n$ of elements of Φ and all A_i belonging to $\mathcal{F}_i$ $(1 \leq i \leq n)$, the intersection $A_1 \cap \ldots \cap A_n$ is nonempty.

Proof: The intersection $A_1 \cap \ldots \cap A_n$ is nonempty if and only if the union of all filters in Φ has the finite intersection property. By Theorem 1. 1, this is necessary and sufficient for the existence of a filter containing each member of Φ. The construction of Theorem 1. 1 produces the coarsest such filter, hence it is the least upper bound, and the proof is complete.

<u>Corollary 1.4.1.</u> The partially ordered set of all filters on X is inductive.

Proof: Let Φ be a linearly ordered subset of filters on X, and let $\{\mathcal{F}_i\}_{i=1}^n$ be any finite sequence of members of Φ. Choose a member A_i of $\mathcal{F}_i$ for $1 \leq i \leq n$. Since Φ is linearly ordered, we have

$A_1 \cap \ldots \cap A_n \neq \emptyset$. Hence, by Theorem 1.4, Φ has an upper bound. Therefore, the set of all filters on X is inductive.

Definition 1.4. An ultrafilter on a set X is a filter which is a maximal element in the partially ordered set of all filters on X.

The existence of ultrafilters is a consequence of Zorn's lemma. In fact, Zorn's lemma implies an even stronger result.

Theorem 1.5. If $\mathcal{F}$ is a filter on X, then there exists an ultrafilter on X finer than $\mathcal{F}$.

Proof: Let S be the set of all filters on X finer than $\mathcal{F}$. Note that S is nonempty, for $\mathcal{F}$ belongs to S. Now let C be a linearly ordered subset of S. Then C has an upper bound in the set of all filters on X. But this upper bound belongs to S since S contains all filters which contain $\mathcal{F}$. Hence every linearly ordered subset of S has an upper bound in S. By Zorn's lemma, S contains a maximal element, which is an ultrafilter finer than $\mathcal{F}$, and this is the desired conclusion.

As an example of an ultrafilter, consider the collection $\mathcal{F}$ of all subsets of X which contain the point x_o in X. Clearly, this is a filter. $\mathcal{F}$ is easily shown to be an ultrafilter. If $\mathcal{G}$ is an ultrafilter strictly finer than $\mathcal{F}$, then $\mathcal{G}$ contains a set A such that $A \cap \{x_o\} = \emptyset$.

But $\{x_o\}$ is in $\mathcal{L}$, hence $\mathcal{L}$ is not a filter. This contradiction proves that $\mathcal{F}$ is an ultrafilter. Ultrafilters of this type are called <u>trivial</u> ultrafilters.

Examples of ultrafilters other than trivial ones are not immediately obvious. However, their existence may be implied by the following argument. Let $\{x_n\}$ be a sequence of distinct points in an infinite set X, and let $A_n = \{x_i : i \geq n\}$. Now A_n is a base of a filter $\mathcal{F}$ on X, and Theorem 1.5 implies the existence of an ultrafilter $\mathcal{F}^*$ finer than $\mathcal{F}$. Clearly, the intersection of the A_n's is empty. But the intersection of the members of $\mathcal{F}^*$ is a subset of the intersection of the members of $\mathcal{F}$, which is in turn a subset of the intersection of all the A_n's. Hence the ultrafilter $\mathcal{F}^*$ is not of the trivial type.

It is interesting to note that the intersection of all members of any ultrafilter contains at most a single point. To show this, note that if two distinct points x and y both belong to the intersection of the members of a filter $\mathcal{F}$, then the trivial ultrafilter generated by $\{x\}$ and that generated by $\{y\}$ are both strictly finer than $\mathcal{F}$. Therefore, $\mathcal{F}$ is not an ultrafilter.

We shall later make use of the following results, which characterize ultrafilters.

<u>Theorem 1.6.</u> If the union of two subsets A_1 and A_2 of X belongs to an ultrafilter $\mathcal{F}$ on X, then either A_1 or A_2 belongs to $\mathcal{F}$.

Proof: Suppose that neither A_1 nor A_2 belongs to $\mathcal{F}$. Let $\mathcal{G}$ be the collection of subsets of X whose union with A_1 belongs to $\mathcal{F}$.

If M is a member of $\mathcal{G}$, then $A_1 \cup M$ belongs to $\mathcal{F}$. If M is also a subset of some set N, then $M \cup A_1 \subset N \cup A_1$. Therefore, $A_1 \cup N \in \mathcal{F}$, and $N \in \mathcal{G}$. Hence F. 1 is satisfied. If M and N are members of $\mathcal{G}$, then $M \cup A_1$ and $N \cup A_1$ are members of $\mathcal{F}$. Then $(M \cap N) \cup A_1 = (M \cup A_1) \cap (N \cup A_1) \in \mathcal{F}$; hence $M \cap N \in \mathcal{G}$, and F. 2 is satisfied. Also, $\emptyset \cup A_1 = A_1 \notin \mathcal{F}$ so that $\emptyset \notin \mathcal{G}$, satisfying F. 3. Therefore, $\mathcal{G}$ is a filter.

However, $\mathcal{G}$ is strictly finer than $\mathcal{F}$ since $A_2 \in \mathcal{G}$, hence $\mathcal{F}$ is not an ultrafilter. This contradiction proves the theorem.

Corollary 1.6.1. If the union of a finite collection of subsets of X belongs to an ultrafilter $\mathcal{F}$ on X, then some member of the collection belongs to $\mathcal{F}$.

The proof follows from Theorem 1.6 by induction on the number of members of the collection.

In particular, if $\{A_i: i = 1, \ldots, n\}$ is a covering of X, then some A_i belongs to $\mathcal{F}$.

Theorem 1.7. Let $\mathcal{G}$ be a collection of subsets of X satisfying the finite intersection property ($\mathcal{G}$ is a filter subbase). Then $\mathcal{G}$ is an ultrafilter if and only if for every subset A of X, either A or its complement belongs to $\mathcal{G}$.

Proof: If $\mathcal{U}$ is an ultrafilter, then the desired conclusion follows from Theorem 1.6 since X belongs to each filter on X.

Conversely, suppose that $\mathcal{U}$ has the finite intersection property and that either A or its complement belongs to $\mathcal{U}$ for each subset A of X. Now let $\mathcal{F}$ be an ultrafilter which contains $\mathcal{U}$ (the existence of $\mathcal{F}$ is a consequence of Theorems 1.1 and 1.5). If B belongs to $\mathcal{F}$, then B^c does not, hence $B^c \notin \mathcal{U}$. But then B belongs to $\mathcal{U}$, so that $\mathcal{F} \subset \mathcal{U}$. But $\mathcal{U} \subset \mathcal{F}$; therefore, $\mathcal{U} = \mathcal{F}$. Hence $\mathcal{U}$ is an ultrafilter, and the proof is complete.

Definition 1.5. A point x in a topological space X is called a cluster point of a filter base $\mathcal{B}$ on X if x lies in the closure of each member of $\mathcal{B}$.

In particular, if x is a cluster point of a filter base $\mathcal{B}$ of a filter $\mathcal{Z}$, then x is also a cluster point of $\mathcal{F}$.

Theorem 1.8. A point x is a cluster point of a filter $\mathcal{Z}$ if and only if there exists a filter finer than $\mathcal{Z}$ which converges to x.

Proof: Let x be a cluster point of a filter $\mathcal{Z}$. Then x lies in the closure of each member of $\mathcal{Z}$. Hence if x belongs to an open set V, then $V \cap F \neq \emptyset$ for each F in $\mathcal{Z}$. Let $\mathcal{N}(x)$ be the neighborhood filter of x, and let $\mathcal{Z}^* = \mathcal{Z} \cup \mathcal{N}(x)$. Clearly, $\mathcal{Z}^*$ is a filter. By definition, $\mathcal{Z}^*$ is finer than both $\mathcal{Z}$ and $\mathcal{N}(x)$; hence it converges to x.

Conversely, suppose $\mathcal{F}^*$ is a filter convergent to x and $\mathcal{F}$ is a filter coarser than $\mathcal{F}^*$. Then $\mathcal{F}^*$ is finer than the neighborhood filter $\mathcal{N}(x)$ of x. If N is an open member of $\mathcal{N}(x)$ and F is a member of $\mathcal{F}$, then the intersection of N and F is nonempty since both belong to $\mathcal{F}^*$. Hence x belongs to the closure of each member of $\mathcal{F}$; therefore, it is a cluster point of $\mathcal{F}$. This completes the proof of the theorem.

In particular, if $\mathcal{F}$ converges to x, then x is a cluster point of $\mathcal{F}$.

Corollary 1.8.1. An ultrafilter $\mathcal{U}$ converges to a point x if and only if x is a cluster point of $\mathcal{U}$.

Proof: The result follows from the theorem and the fact that $\mathcal{U}$ is the only filter finer than itself.

Since compact spaces play an important role in topology, it is significant that they can be characterized in terms of their filters. We recall that a topological space is said to be compact if every open cover has a finite subcover.

Theorem 1.9. For a topological space X, the following are equivalent:

(a) X is compact.

(b) Every filter on X has at least one cluster point.

(c) Every ultrafilter on X is convergent.

(d) Every family of closed subsets of X having empty

intersection has a finite subfamily with empty intersection.

Proof: We shall show that (a) implies (d), (d) implies (a), (d) implies (b), (b) implies (d), (b) implies (c), and (c) implies (b).

Suppose that X is compact, and let $\mathcal{G}$ be a family of closed sets with empty intersection. If $\mathcal{V}$ is the family of complements of members of $\mathcal{G}$, then $\mathcal{V}$ is an open cover of X, by DeMorgan's law. Since X is compact, there exists a finite subfamily $\{V_1, \ldots, V_n\}$ of $\mathcal{V}$ which covers X. Let $G_1, \ldots, G_n$ be the complements of $V_1, \ldots, V_n$. Then, by DeMorgan's law, $V_1 \cup \ldots \cup V_n = X$ implies that $G_1 \cap \ldots \cap G_n$ is empty. Therefore, (a) implies (d).

Suppose that (d) is true, and let $\mathcal{V}$ be an open cover of X. If $\mathcal{G}$ is the collection of complements of members of $\mathcal{V}$, then, by DeMorgan's law, the members of $\mathcal{G}$ have empty intersection. By hypothesis, there exists a subfamily $\{G_1, \ldots, G_n\}$ of $\mathcal{G}$ with empty intersection. Then, by DeMorgan's law, the complements $V_1, \ldots, V_n$ of $G_1, \ldots, G_n$ cover X and form a subfamily of $\mathcal{V}$. Therefore, X is compact, and (d) implies (a).

Suppose that (d) is true, and let $\mathcal{F}$ be a filter with no cluster point. Then the closures of the members of $\mathcal{F}$ form a family of closed subsets of X with empty intersection. By hypothesis, there exists a finite subfamily of this family which also has empty

intersection. But this contradicts the fact that a filter cannot contain the empty set. Therefore, $\mathcal{Z}$ must have a cluster point, and (d) implies (b).

If every filter on X has a cluster point, then let $\mathcal{G}$ be a family of closed subsets of X with empty intersection, and suppose that every finite subfamily of $\mathcal{G}$ has nonempty intersection. Then $\mathcal{G}$ is a subbase of a filter $\mathcal{Z}$ which has a cluster point x. Since the members of $\mathcal{G}$ are closed, each contains x. But this contradicts the fact that $\mathcal{G}$ has empty intersection. Therefore, (b) implies (d).

Suppose that every filter on X has a cluster point, and let $\mathcal{U}$ be an ultrafilter on X with a cluster point x. By Corollary 1.8.1, $\mathcal{U}$ converges to x; therefore, (b) implies (c).

Suppose that every ultrafilter on X is convergent. If $\mathcal{F}$ is a filter on X, then there exists an ultrafilter $\mathcal{Z}^*$ finer than $\mathcal{F}$. By hypothesis, $\mathcal{F}^*$ converges to a point x. Then x belongs to the closure of each member of $\mathcal{Z}^*$. But each member of $\mathcal{F}$ is also a member of $\mathcal{F}^*$, therefore, x belongs to the closure of each member of $\mathcal{F}$. Hence x is a cluster point of $\mathcal{F}$, and (c) implies (b).

This completes the proof of the theorem.

We close the chapter with some observations concerning the requirement that a filter must not contain the empty set.

Suppose that a filter is defined by F.1 and F.2; i.e., the empty set may belong to a filter. Suppose, also that $\mathcal{A}$ is a filter on a set

X to which $\emptyset$ belongs. Now let A be any subset of X. Since $\emptyset \subset A$, then by F. 1, A belongs to $\mathcal{Z}$. Thus $\mathcal{Z}$ contains all subsets of X and hence is finer than any other filter on X. In particular, $\mathcal{Z}$ is finer than the neighborhood filter of any point in X (if X is a topological space); therefore, it converges to every point in X. Note, also, that $\mathcal{Z}$ is the unique ultrafilter on X, for no collection of subsets of X can contain it other than itself.

If property F. 3 is valid, then a filter in a Hausdorff space can converge to at most a single point. For if a filter converges to two distinct points, the points can be separated by disjoint neighborhoods which must belong to the filter.

We see then that the removal of F. 3 from the defining properties of a filter leads to a rather trivial situation.

CHAPTER II

THE SET PRODUCT

A uniform space (to be defined in Chapter III) is described in terms of subsets of the Cartesian product X x X, where X is the set on which the uniform structure is being defined. These subsets are nothing more than relations on X. This chapter is devoted to a discussion of a set product of these subsets (a composition of relations) which is used in describing uniform spaces.

Definition 2.1. Let V and W be subsets of X x X (relations on X). Then the set product (composition) of V and W, denoted by V o W, is defined by $V \circ W = \{(w, y) \in X \times X: (x, z) \in W, (z, y) \in V \text{ for some } z \in X\}$. We shall denote V o V by V^2, and $V^{n+1} = V^n \circ V$ for each positive integer n.

Definition 2.2. If V is a subset of X x X and A is a subset of X, then

$$V(A) = \{y \in X: (x, y) \in V \text{ for } x \in A\}.$$

In particular, if $x_o \in X$, then $V(x_o) = \{y \in X: (x_o, y) \in V\}$.

The set product may be visualized geometrically. Choose x in X, and consider the set $W(x) = \{z \in X: (x, z) \in W\}$ as a subset of X.

Now consider the set $V(W(x)) = \{y \in X: (z, y) \in V \text{ for some } z \in W(x)\}$. If y belongs to $V(W(x))$, then (x, y) belongs to $V \circ W$. In fact, the set $V \circ W$ is precisely the set of all pairs (x, y) such that y belongs to $V(W(x))$ as x varies over X.

Theorem 2.1. Let V and W be subsets of X x X, and let $S_x = \{(x, y) \in X \times X: y \in V(W(x))\}$ for each x in X. Then $V \circ W = \bigcup_{x \in X} S_x$.

Proof: Let $(x_o, y_o) \in V \circ W$. Then there exists $z_o \in X$ such that $(x_o, z_o) \in W$ and $(z_o, y_o) \in V$. Then $z_o \in W(x_o)$ and $y_o \in V(W(x_o))$. Therefore, $(x_o, y_o) \in S_{x_o}$, hence (x_o, y_o) belongs to $\bigcup_{x \in X} S_x$, and $V \circ W \subset \bigcup_{x \in X} S_x$.

Conversely, if (x_o, y_o) belongs to the union of the sets S_x (x in X), then (x_o, y_o) belongs to S_{x_1} for some x_1 in X. But $S_{x_1} = \{(x_1, y): y \in V(W(x_1))\}$. Then $(x_o, y_o) \in S_{x_1}$ implies that $x_o = x_1$ and $y_o \in V(W(x_1))$. So we may refer to S_{x_1} as S_{x_o}. Now $(x_o, y_o) \in S_{x_o}$ implies that $y_o \in V(W(x_o))$. Therefore, there exists $z_o \in W(x_o)$ such that $(z_o, y_o) \in V$. But $z_o \in W(x_o)$ implies that $(x_o, z_o) \in W$. Then $(x_o, y_o) \in V \circ W$, and $\bigcup_{x \in X} S_x$ is a subset of $V \circ W$.

The double set inclusion implies equality, and the theorem is proved.

Definition 2.3. If V is a subset of X x X, then the inverse V^{-1} of V is given by $V^{-1} = \{(x, y): (y, x) \in V\}$.

Definition 2.4. A subset V of X x X is said to be symmetric if

$$V = V^{-1}.$$

Theorem 2.2. If $V \subset X \times X$, then $(V^{-1})^{-1} = V$.

Proof: If $(x, y) \in V$, then $(y, x) \in V^{-1}$ and $(x, y) \in (V^{-1})^{-1}$. Conversely, if $(x, y) \in (V^{-1})^{-1}$, then $(y, x) \in V^{-1}$ and $(x, y) \in V$. Therefore, $V = (V^{-1})^{-1}$.

An example of the set product is illustrated geometrically in Figure 1. If X is the closed interval $[0, 1]$ and V and W are as shown, then V o W is as shown in the figure. Figure 2 illustrates the product W o V for the same sets V and W. It is obvious from the figures that in this case $W \circ V = (V \circ W)^{-1}$. A sufficient condition for this symmetry is given in Corollary 2.4.1.

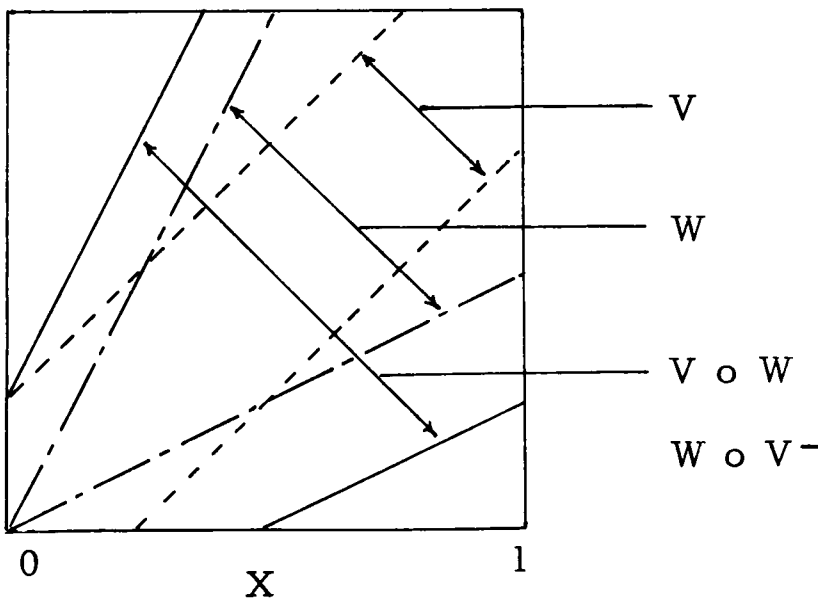

Fig. 1.

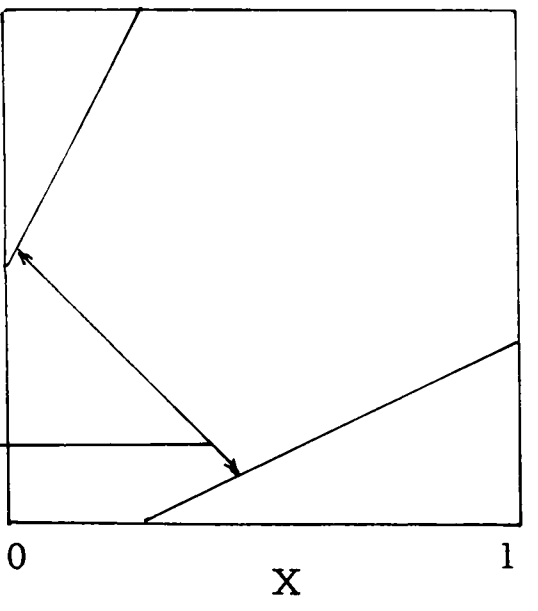

Fig. 2.

We now mention several properties of the set product.

Theorem 2.3. If U, V, and W are subsets of X x X, then (U o V) o W = U o (V o W), that is, the set product is associative.

Proof: Let $(w, z) \in (U \circ V) \circ W$. Then there exists x in X such that $(w, x) \in W$ and $(x, z) \in (U \circ V)$. But this implies that there exists a point y in X such that $(x, y) \in V$ and $(y, z) \in U$. Now $(w, x) \in W$ and $(x, y) \in V$ imply that $(w, y) \in (V \circ W)$. Similarly, $(w, y) \in (V \circ W)$ and $(y, z) \in U$ imply that $(w, z) \in U \circ (V \circ W)$. Therefore, $(U \circ V) \circ W$ is a subset of $U \circ (V \circ W)$.

Conversely, let (w, z) be a member of $U \circ (V \circ W)$. Then there exists a point y in X such that $(w, z) \in V \circ W$ and $(y, z) \in V$. Now there exists x in X such that $(w, x) \in W$ and $(x, y) \in V$. But $(x, y) \in V$ and $(y, z) \in U$ imply that $(x, z) \in (U \circ V)$. Also, $(x, z) \in U \circ V$ and $(w, x) \in W$ imply that $(w, z) \in (U \circ V) \circ W$. Therefore, $U \circ (V \circ W)$ is a subset of $(U \circ V) \circ W$. The double inclusion implies equality, and the proof is complete.

Theorem 2.4. If W and V are subsets of X x X, then $(V \circ W)^{-1} = W^{-1} \circ V^{-1}$.

Proof: Let (x, y) be a point in $(V \circ W)^{-1}$. Then (y, x) belongs to $V \circ W$, and there exists a point z in X such that $(y, z) \in W$ and $(z, x) \in V$. Then $(z, y) \in W^{-1}$ and $(x, z) \in V^{-1}$, so that $(x, y) \in W^{-1} \circ V^{-1}$.

Conversely, if $(x, y) \in W^{-1} \circ V^{-1}$, then there exists z in X such that $(x, z) \in V^{-1}$ and $(z, y) \in W^{-1}$. But this implies that $(y, z) \in W$ and

$(z, x) \in V$, so that $(y, x) \in V \circ W$. Therefore, $(V \circ W)^{-1} = W^{-1} \circ V^{-1}$, and the theorem is proved.

Corollary 2.4.1. If two subsets V and W of X x X are symmetric, then $(V \circ W)^{-1} = W \circ V$.

Proof: Replace W^{-1} and V^{-1} in the theorem by W and V respectively, and the conclusion follows.

Figure 3 illustrates two subsets V and W of X x X for which $V \circ W = V \cap W \neq \emptyset$ while $W \circ V = \emptyset$. One might hastily conjecture that one condition might imply the other. Neither implication is true, however. For if X is any nonempty set and V = W = X x X, then $V \circ W = W \circ V = V \cap W = X \times X \neq \emptyset$. On the other hand, if X = [0, 3], and if V and W are defined by $V = \{(x, y): 0 \leq x \leq 1, \ 0 \leq y \leq 1\}$ and $W = \{(x, y): 2 \leq x \leq 3, \ 2 \leq y \leq 3\}$, then $V \circ W = W \circ V = V \cap W = \emptyset$.

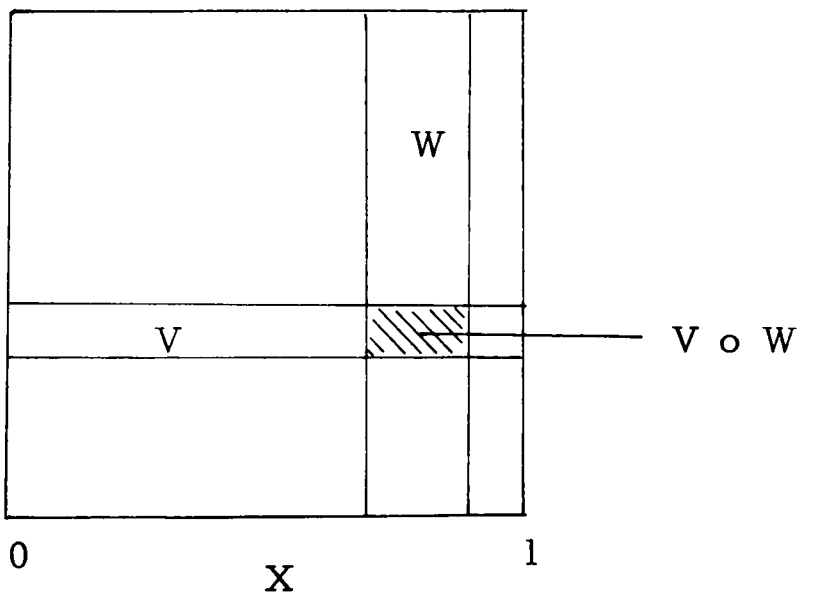

Fig. 3.

The following theorem gives a sufficient condition for V o W to be nonempty.

Theorem 2.5. If V and W are two subsets of X x X such that $V \cap W \cap \Delta \neq \emptyset$, where $\Delta = \{(x, x) \in X \times X: x \in X\}$, then V o W $\neq \emptyset$ and W o V $\neq \emptyset$.

Proof: Since $V \cap W \cap \Delta \neq \emptyset$, then there exists a point x in X such that $(x, x) \in V$, $(x, x) \in W$, and $(x, x) \in \Delta$. Set x = y = z. Now $(x, z) \in W$ and $(z, y) \in V$, so that $(x, y) \in V$ o W. Also, $(x, z) \in V$ and $(z, y) \in W$, so that $(x, y) \in W$ o V. Thus the sufficiency is proved.

The condition is not necessary, for choose $V = V^{-1} = W$ such that $V \cap \Delta = \emptyset$ as in Figure 4. The condition is not met, yet the set product is nonempty.

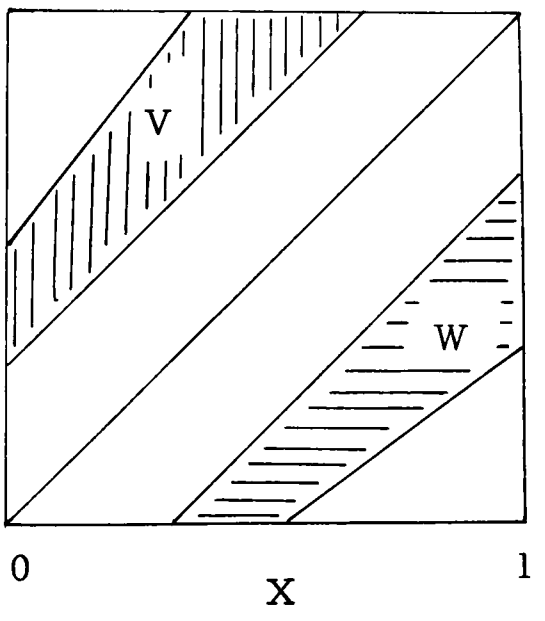

Fig. 4.

CHAPTER III

UNIFORM SPACES AND CAUCHY FILTERS

Definition 3.1. A uniformity on a set X is a collection $\mathcal{U}$ of subsets of X x X which satisfy the following:

(U.1) Every subset of X x X which contains a member of $\mathcal{U}$ belongs to $\mathcal{U}$.

(U.2) Every finite intersection of members of $\mathcal{U}$ belongs to $\mathcal{U}$.

(U.3) Every member of $\mathcal{U}$ contains the diagonal Δ of X x X.

(U.4) If $V \in \mathcal{U}$. then $V^{-1} \in \mathcal{U}$.

(U.5) For each $V \in \mathcal{U}$ there exists $W \in \mathcal{U}$ such that $W \circ W \subset V$.

The members of $\mathcal{U}$ are called the entourages of the uniform structure $\mathcal{U}$ on X. The pair $(X, \mathcal{U})$ is called a uniform space.

Note that if the set X is nonempty, then a uniformity $\mathcal{U}$ on X is a filter on X x X.

Theorem 3.1. If $(X, \mathcal{U})$ is a uniform space and X' is a subset of X, then the trace on X' x X' of the entourages of $\mathcal{U}$ forms a uniformity on X'.

Proof: If U is an entourage of $\mathcal{U}$, then let U' be the trace of U on X' x X', that is, $U' = U \cap (X' \times X')$; and let $\mathcal{U}' = \{U' : U \in \mathcal{U}\}$. To verify U. 1, let U' be a member of $\mathcal{U}'$ and let V' be a subset of X' x X' which contains U'. Set $V = U \cup V'$. Now $U \subset V$, so that $V \in \mathcal{U}$. By construction, $V' = V \cap (X' \times X')$, and $V' \in \mathcal{U}'$.

If U_1' and U_2' are members of $\mathcal{U}'$. with U_1 and U_2 in $\mathcal{U}$, then $U_1' \cap U_2' = (U_1 \cap (X' \times X')) \cap (U_2 \cap (X' \times X')) = (U_1 \cap U_2) \cap (X' \times X')$. Therefore, $U_1' \cap U_2'$ is a member of $\mathcal{U}'$. By induction, $U_1 \cap \ldots \cap U_n' \in \mathcal{U}$, and U. 2 is valid.

Clearly, each U' contains the diagonal Δ' of X' x X', and U. 3 is satisfied.

If $U' \in \mathcal{U}'$, then $U \in \mathcal{U}$ implies that $U^{-1} \in \mathcal{U}$. But it is obvious that $(U^{-1})' = (U')^{-1}$. Therefore, $(U')^{-1}$ is a member of $\mathcal{U}'$, and U. 4 is satisfied.

Let U' be a member of $\mathcal{U}'$ and U a member of $\mathcal{U}$. Then there exists a member W of $\mathcal{U}$ such that W o W is a subset of U. Let $W' = W \cap (X' \times X')$. If (x, y) is any point in W' o W', then there exists a point z in X' such that (x, z) and (z, y) are in W'. Since W' is a subset of W, then (x, y) belongs to W o W, which is a subset of U, and $(x, y) \in U'$. Therefore, W' o W' is a subset of U', U. 5 is verified, and the proof is complete.

Definition 3.2. If $(X, \mathcal{U})$ is a uniform space, X' a subset of X, and $\mathcal{U}'$ the trace on $X' \times X'$ of the entourages of $\mathcal{U}$, then the uniform space $(X', \mathcal{U}')$ is called a uniform subspace of $(X, \mathcal{U})$.

In Chapter II we saw an example of two subsets V and W of $X \times X$ such that $V \circ W = V \cap W$ while $W \circ V = \emptyset$. A more interesting conclusion can be drawn if V and W are entourages of a uniformity on X.

Theorem 3.2. If V and W are entourages of a uniform space $(X, \mathcal{U})$, then $V \cup W \subseteq V \circ W$.

Proof: Let (x, y) be a point in V. Then $(x, x) \in W$, and $(x, y) \in V \circ W$. Hence $V \subset V \circ W$. Similarly, if (w, z) is a point in W, then $(z, z) \in V$, and $(w, z) \in V \circ W$. Hence $W \subset V \circ W$. Therefore, $V \cup W \subset V \circ W$, and the proof is complete.

Corollary 3.2.1. Let V and W be entourages of a uniform space $(X, \mathcal{U})$. Then

(1) $V \cup W \subset V \circ W \cap W \circ V$,

(2) $V \cap W \subset V \circ W \cap W \circ V$,

(3) $\Delta \subset V \circ W$, and

(4) $X \neq \emptyset$ implies $V \circ W \neq \emptyset$.

The proof of each statement is obvious. In particular, $W \circ V$, $V \circ W$, and $(W \circ V) \cap (V \circ W)$ are entourages.

Definition 3.3. A fundamental system of entourages of a uniformity $\mathcal{U}$ on X is a collection $\mathcal{B}$ of entourages such that every entourage of $\mathcal{U}$ contains a member of $\mathcal{B}$.

Theorem 3.4. Let $\mathcal{U}$ be a uniformity on a set X, and let $\mathcal{B}$ be the collection of symmetric entourages of $\mathcal{U}$. Then $\mathcal{B}$ is a fundamental system of entourages.

Proof: Let V be any entourage of $\mathcal{U}$. Then V^{-1} is also an entourage. By U.2, $V \cap V^{-1}$ is an entourage. But $V \cap V^{-1}$ is symmetric, hence, it belongs to $\mathcal{B}$. Since V contains $V \cap V^{-1}$, the theorem is proved.

Theorem 3.5. A collection $\mathcal{B}$ of subsets of X x X forms a fundamental system of entourages of a uniformity on X if and only if $\mathcal{B}$ satisfies the following:

(E.1) The intersection of two members of $\mathcal{B}$ contains a member of $\mathcal{B}$.

(E.2) Every member of $\mathcal{B}$ contains the diagonal of X x X.

(E.3) For each V in $\mathcal{B}$ there exists W in $\mathcal{B}$ such that $W \subset V^{-1}$.

(E.4) For each V in $\mathcal{B}$ there exists W in $\mathcal{B}$ such that $W \circ W \subset V$.

The proof of the theorem is based on the following lemma.

Lemma 3.6. Let V be an entourage of the uniform space $(X, \mathcal{U})$.

Then V^n is a subset of V^{n+1} for each positive integer n.

Proof: Let (x, y) be a member of V. Then $(y, y) \in V$ and $(x, y) \in V \circ V$, so that $V \subset V \circ V$. If $(w, z) \in V^n$, then $(z, z) \in V$ implies that $(w, z) \in V^{n+1}$, so that $V^n \subset V^{n+1}$ for each positive integer n.

Proof of Theorem 3.5: First, let $\mathcal{B}$ be a fundamental system of entourages of a uniformity $\mathcal{U}$ on a set X. If B_1 and B_2 are members of $\mathcal{B}$, then they are also members of $\mathcal{U}$. Then $B_1 \cap B_2$ is an entourage, hence it contains a member of $\mathcal{B}$, and E. 1 is satisfied.

Let V be a member of $\mathcal{B}$. Then V, V^{-1}, and $V \cap V^{-1}$ belong to $\mathcal{U}$. Therefore, $V \cap V^{-1}$ contains a member W of $\mathcal{B}$, so that $W \subset V^{-1}$, satisfying E. 3.

If B is a member of $\mathcal{B}$, then it is also a member of $\mathcal{U}$. Hence B contains the diagonal of X x X, and E. 2 is satisfied.

Again, let V be a member of $\mathcal{B}$. Then V belongs to $\mathcal{U}$, and there exists $U \in \mathcal{U}$ such that $U \circ U \subset V$. But if $U \in \mathcal{U}$, then there exists W in $\mathcal{B}$ such that $W \subset U$. Then $W \circ W \subset U \circ U \subset V$, and E. 4 is satisfied.

Conversely, let $\mathcal{B} = \{B_\alpha \subset X \times X: \alpha \in A\}$ be a collection of subsets of X x X satisfying E. 1 through E. 4. Let $\mathcal{U}$ be the collection of subsets of X x X each of which contains a member of $\mathcal{B}$. We shall show that $\mathcal{U}$ is a uniformity on X.

Let S be a subset of X x X such that S contains some member U of $\mathcal{U}$. Now U contains some member B of $\mathcal{B}$, hence S contains B. But then S belongs to $\mathcal{U}$, and U. 1 is satisfied.

Let $U_1, \ldots, U_n$ be members of $\mathcal{U}$. Then there exist members $B_1, \ldots, B_n$ of $\mathcal{B}$ such that U_i contains B_i for $1 \leq i \leq n$. By applying mathematical induction to E. 1, we see that $B_1 \cap \ldots \cap B_n$ is a member of $\mathcal{B}$. But this intersection is a subset of each B_i, $1 \leq i \leq n$, hence of each U_i. Therefore, $B_1 \cap \ldots \cap B_n \subset U_1 \cap \ldots \cap U_n$, so that $U_1 \cap \ldots \cap U_n$ belongs to $\mathcal{U}$. Hence U. 2 is satisfied.

If U is a member of $\mathcal{U}$, then there exists B in $\mathcal{B}$ such that $\Delta \subset B \subset U$. Hence U. 3 is satisfied.

Let U be a member of $\mathcal{U}$. Then there exists V in $\mathcal{B}$ such that $V \subset U$. Then by E. 3 there exists W in $\mathcal{B}$ such that $W \subset V^{-1} \subset U^{-1}$. Hence U^{-1} is in $\mathcal{U}$, and U. 4 is satisfied.

Finally, let U be a member of $\mathcal{U}$ and B a member of $\mathcal{B}$ such that $B \subset U$. Then by E. 4 there exists V in $\mathcal{B}$ such that $V \circ V \subset B$. By Lemma 3. 6, $V \subset V \circ V \subset B \subset U$. Since $V \subset V$, then $V \subset \mathcal{U}$, and U. 5 is satisfied.

Since U. 1 through U. 5 are satisfied, then $\mathcal{U}$ is a uniform structure on X, which is the desired conclusion.

We see then that a uniformity can be completely determined in a canonical manner by a given fundamental system of entourages. We use this fact to give an example of a uniformity.

Let (X, d) be any metric space. For each positive integer n, let $V_n = \{(x, y) \in X \times X: d(x, y) < 1/n\}$. Then $\mathcal{V} = \{V_n: n \in N\}$ is a fundamental system of entourages of a uniformity $\mathcal{U}$ on X. The uniformity is the collection of subsets of X x X each of which contains a member of $\mathcal{V}$. We see then that any metric space induces a uniform structure with a countable fundamental system of entourages.

Note that if X is nonempty and if $\mathcal{B}$ is a fundamental system of entourages of a uniformity $\mathcal{U}$ on X, then $\mathcal{B}$ is a filter base of the filter on X x X formed by the entourages of $\mathcal{U}$.

Definition 3.4. An isomorphism (uniform isomorphism) of the uniform space $(X, \mathcal{U})$ onto the uniform space $(X', \mathcal{U}')$ is a one-to-one mapping f of X onto X' such that the images under g (where g = f x f) of the entourages of $\mathcal{U}$ are precisely the entourages of $\mathcal{U}'$.

Definition 3.5. A mapping f of a uniform space X into a uniform space Y is said to be uniformly continuous if for each entourage V of Y there exists an entourage U of X such that $(x, y) \in U$ implies that $(f(y)) \in V$.

Theorem 3.7. A one-to-one mapping f of a uniform space X onto a uniform space Y is an isomorphism if and only if f and f^{-1} are uniformly continuous.

The result is an immediate consequence of the definitions.

As is shown by the following theorem, a uniformity on a set X induces a topology on X. This topology will be referred to as the topology of the uniform space $(X, \mathcal{U})$.

Theorem 3.8. Let $(X, \mathcal{U})$ be a uniform space. For each x in X, let $\mathcal{V}(x) = \{V(x) \subset X: V \in \mathcal{U}\}$. Then there exists a unique topology on X such that for each x in X, $\mathcal{V}(x)$ is the neighborhood filter of x in this topology.

Proof: We recall that a topology is uniquely determined by a postulated system of neighborhoods. If to each x in X there corresponds a collection $\mathcal{N}(x)$ of subsets of X which satisfies the following four postulates, then there exists a unique topology on X such that $\mathcal{N}(x)$ is the collection of neighborhoods of x in that topology:

(N. 1) Every subset of X which contains a member of $\mathcal{N}(x)$ is a member of $\mathcal{N}(x)$.

(N. 2) Every finite intersection of members of $\mathcal{N}(x)$ is a member of $\mathcal{N}(x)$.

(N. 3) The point x belongs to each member of $\mathcal{N}(x)$.

(N. 4) If N belongs to $\mathcal{N}(x)$, then there exists a set M in $\mathcal{N}(x)$ such that for each y in M, N belongs to $\mathcal{N}(y)$.

It suffices to show that the collections $\mathcal{V}(x)$, $(x \in X)$ satisfy N. 1 through N. 4.

N. 1, N. 2, and N. 3 are immediate consequences of the properties U. 1, U. 2, and U. 3, respectively. It remains to show that $\mathcal{V}(x)$ satisfies N. 4.

Let x be an arbitrary member of X, V an arbitrary member of $\mathcal{U}$, and V(x) the corresponding member of $\mathcal{V}(x)$. We must exhibit a member W(x) of $\mathcal{V}(x)$ such that $V(x) \in \mathcal{V}(y)$ for each y in W(x).

By U. 5 there exists an entourage W such that $W \circ W \subset V$. Let y be a point in W(x); then (x, y) belongs to W. Let z be a point in W(y); then (y, z) belongs to W. Then (x, z) belongs to $W \circ W$, which is a subset of V. Therefore, $z \in V(x)$ so that $W(y) \subset V(x)$ for each y in W(x). But $y \in W(y) \subset V(x)$ implies that V(x) belongs to $\mathcal{V}(y)$. W(x) is the required set, and the theorem is proved.

Definition 3. 6. The topology on X defined by the neighborhood system in Theorem 3. 4 is called the topology induced by the uniformity $\mathcal{U}$.

We often find it convenient to refer to the topological properties of the topology induced by a uniformity without specifically mentioning the topology. For example, the expression "a uniform space is Hausdorff" means that the topology induced by the uniformity is Hausdorff.

Since Hausdorff spaces play an important role in topology, it is natural to investigate the uniform properties of Hausdorff

spaces. The following results show that Hausdorff uniform spaces can be characterized in terms of their entourages.

Theorem 3.9. Let $(X, \mathcal{U})$ be a uniform space. Then for every symmetric entourage V and every subset M of X x X, the set V o M o V is a neighborhood of M in the product space X x X; and the closure of M in this space is given by $\overline{M} = \bigcap_{V \in \mathcal{A}} (V \circ M \circ V)$, where $\mathcal{A}$ denotes the collection of symmetric entourages of $\mathcal{U}$.

Proof: Let (p, q) be an arbitrary point of M. Then V(p) x V(q) is a neighborhood of (p, q) in X x X. Choose a point (x, y) in this neighborhood. Then (x, p) and (q, y) belong to V, hence $(x, y) \in (V \circ M \circ V)$. Therefore, (V o M o V) is a neighborhood of M.

Now let (x, y) be an arbitrary point in $\overline{M}$. Then either (x, y) belongs to M or it is a limit point of M. If $(x, y) \in M$, then $(x, y) \in (V \circ M \circ V)$ for each V in $\mathcal{A}$, so that (x, y) belongs to $\cap(V \circ M \circ V)$. On the other hand, if (x, y) is a limit point of M, then every neighborhood of (x, y) meets M. Since V(x) x V(y) is a neighborhood of (x, y), then there exists a point (p, q) in M such that $(p, q) \in V(x) \times V(y)$. But then $p \in V(x)$ and $q \in V(y)$ imply that $(x, p) \in V$ and $(q, y) \in V$. Therefore, $(x, y) \in (V \circ M \circ V)$ for each V in $\mathcal{A}$, so that (x, y) is a member of $\cap(V \circ M \circ V)$. As a result, we see that $\overline{M}$ is a subset of $\cap(V \circ M \circ V)$.

Conversely, let (x, y) be a point in $\cap(V \circ M \circ V)$. Then (x, y) belongs to $V \circ M \circ V$ for each V in $\mathcal{L}$. For a given member V of $\mathcal{L}$ there exists a point (p, q) in M such that $(x, p) \in V$, $(p, q) \in M$, and $(q, y) \in V$. But then $p \in V(x)$ and $q \in V(y)$ imply that (p, q) belongs to $V(x) \times V(y)$, which is a neighborhood of (x, y) in $X \times X$. Hence every neighborhood of (x, y) meets M, so that $(x, y) \in \overline{M}$. Therefore, $\cap(V \circ M \circ V)$ is a subset of $\overline{M}$. This inclusion and the previous one show that equality holds, and the proof is complete.

Corollary 3.9.1. If $(X, \mathcal{U})$ is a uniform space, then the interiors (alternatively, the closures) of the entourages of $\mathcal{U}$ in the product space $X \times X$ form a fundamental system of entourages.

Proof: If V is any entourage, then there exists a symmetric entourage U such that $U^2 \subset V$. Further, there exists a symmetric entourage W such that $W \circ W \subset U$. Thus we have $W^3 \subset W^4 \subset U^2 \subset V$, or $W^3 \subset V$. By the previous theorem, W^3 is a neighborhood of W. Therefore, $\mathrm{Int}(V)$ contains W, and $\mathrm{Int}(V)$ is an entourage. Since V contains $\mathrm{Int}(V)$ for each V in $\mathcal{U}$, then the interiors of the entourages form a fundamental system of entourages.

Also, $W \subset \overline{W} \subset W^3 \subset V$ by the previous theorem, so that V contains the closure of an entourage. Therefore, the closures of the entourages form a fundamental system of entourages.

Theorem 3.10. A uniform space X is Hausdorff if and only if the intersection of all its entourages is the diagonal Δ of $X \times X$.

Proof: Suppose that the intersection of the entourages of X is Δ. The entourages which are closed in $X \times X$ form a fundamental system of entourages whose intersection is also Δ, hence Δ is closed. Let x and y be distinct members of X so that (x, y) does not belong to the diagonal. Then there exist entourages V and W such that $(V(x) \times W(y)) \cap \Delta$ is empty, which implies that the neighborhoods V(x) and W(y) of x and y are disjoint. Therefore, X is Hausdorff.

Conversely, if X is Hausdorff and if x and y are distinct points in X, then there exists an entourage V such that y does not belong to V(x). But this implies that (x, y) does not belong to V. Since x and y were arbitrary, then the intersection of all entourages consists of the diagonal. This completes the proof of the theorem.

We saw in Chapter I that an arbitrary sequence $\{x_n\}$ in a set X has associated with it an elementary filter on X. We wish now to generalize the concept of a Cauchy sequence in a metric space to that of a Cauchy filter on a uniform space.

Definition 3.7. Let $(X, \mathcal{U})$ be a uniform space and V an entourage of $\mathcal{U}$. A subset A of X is said to be V-small if A x A is a subset of V.

Theorem 3.11. Let $(X, \mathcal{U})$ be a uniform space and V a member of $\mathcal{U}$. If two subsets A and B of X are V-small and have nonempty intersection, then $A \cup B$ is V^2-small.

Proof: Choose arbitrary points x and y in $A \cup B$ and z in $A \cap B$. Then (x, z) and (z, y) belong to V. Thus (x, y) belongs to $V \circ V$ so that $A \cup B$ is V^2-small.

Definition 3.8. A filter $\mathcal{Z}$ on a uniform space X is called a Cauchy filter if for every entourage V of the uniformity there exists a V-small member of $\mathcal{Z}$.

Theorem 3.12. Let (X, d) be a metric space, $\mathcal{U}$ its uniformity, and $\{x_i\}$ a sequence in X. Then the elementary filter $\mathcal{Z} = \{F_i\}$ associated with $\{x_i\}$ is a Cauchy filter if and only if $\{x_i\}$ is a Cauchy sequence.

Proof: Let $V_n = \{(x, y): d(x, y) < 1/n\}$ and let $\mathcal{V} = \{V_n: n \in N\}$. Then $\mathcal{V}$ is a fundamental system of entourages for the uniformity $\mathcal{U}$. Let V be an entourage of $(X, \mathcal{U})$. Then there exists a positive integer k such that $V_k \subset V$. Choose $\epsilon < 1/k$. Then there exists a positive integer M such that if m and n are greater than M, then $d(x_m, x_n)$

is less than ϵ. Now if n is greater than M, then x_n belongs to the member A_M of the elementary filter $\mathcal{F}$ associated with $\{x_i\}$, and $A_M \times A_M$ is a subset of V_k, which is a subset of V. Hence $\mathcal{F}$ is a Cauchy filter.

Conversely, suppose $\mathcal{F}$ is a Cauchy filter, and let $\epsilon > 0$ be given. Choose N such that $1/N$ is less than ϵ. Then for the corresponding entourage V_N there exists a member F_N of $\mathcal{F}$ such that F_N is V-small. Clearly, if m and n are greater than N, then x_m and x_n belong to F_N, hence $d(x_m, x_n) < 1/N < \epsilon$. Therefore, $\{x_i\}$ is a Cauchy sequence. This completes the proof of the theorem.

As another example of a Cauchy filter, let x_o be a point in a uniform space $(X, \mathcal{U})$, and consider the collection $\mathcal{F}$ of all subsets of X which contain x_o. $\mathcal{F}$ is a trivial ultrafilter as described in Chapter I. $\mathcal{F}$ is also a Cauchy filter. To show this, let V be an entourage of $\mathcal{U}$. Now $\{x_o\}$ is a subset of X which is V-small and belongs to $\mathcal{F}$. Hence $\mathcal{F}$ is a Cauchy filter.

We now present a number of properties of Cauchy filters of which we shall have occasion to make use in the succeeding chapter.

Theorem 3.13. Let f be a uniformly continuous mapping of a uniform space X into a uniform space X'. Then the image under f of a Cauchy filter base on X is a Cauchy filter base on X'.

Proof: Let $g = f \times f$, and let V' be an entourage of X'. Since f is uniformly continuous, then $g^{-1}(V')$ is an entourage of X. If A is $g^{-1}(V')$-small, then f(A) is V'-small. If $\mathcal{B}$ is a Cauchy filter base on X, then it is obvious that $f(\mathcal{B})$ is a filter base on X'. If B is a $g^{-1}(V')$-small member of $\mathcal{B}$, then f(B) is V'-small, and $f(\mathcal{B})$ is therefore a Cauchy filter base on X'.

Definition 3.9. A minimal Cauchy filter on a uniform space X is a minimal element with respect to set inclusion in the set of all Cauchy filters in X.

Theorem 3.14. Let $\mathcal{F}$ be a Cauchy filter on a uniform space $(X, \mathcal{U})$. Then there exists a unique minimal Cauchy filter $\mathcal{F}_o$ coarser than $\mathcal{F}$.

Proof: Let $\mathcal{A}$ be the set of symmetric entourages of $\mathcal{U}$, and let $\mathcal{B}$ be a base of the Cauchy filter $\mathcal{F}$ on X. If M and N are members of $\mathcal{B}$ and V and W are members of $\mathcal{A}$, then there exist members B of $\mathcal{B}$ and S of $\mathcal{A}$ such that $B \subset (M \cap N)$ and $S \subset (V \cap W)$. Therefore, the collection $\mathcal{B}_o = \{V(M): V \in \mathcal{A}, M \in \mathcal{B}\}$ satisfies property B.1. Since this $\mathcal{B}_o$ is obviously nonempty and does not contain $\emptyset$, it also satisfies B.2. Hence it is a base of a filter $\mathcal{F}_o$ on X.

If M is a V-small member of $\mathcal{B}$, then V(M) is V^3-small. For if (x, y) is a point in V(M) x V(M), then there exist points w

and z in M such that (w, x) and (z, y) belong to V. But since V is symmetric and M is V-small, then (x, w) $\in$ V, (w, z) $\in$ V, and (z, y) $\in$ V, so that (x, y) $\in V^3$. Therefore, $\mathcal{Z}_o$ is a Cauchy filter which is clearly coarser than the filter $\mathcal{Z}$.

Let $\mathcal{Z}'$ be a Cauchy filter coarser than $\mathcal{Z}$. It is sufficient to show that $\mathcal{Z}_o$ is finer than $\mathcal{Z}'$. If **F** is a member of $\mathcal{Z}_o$, then it contains a member V(M) of $\mathcal{B}_o$. Since $\mathcal{Z}'$ is a Cauchy filter, it has a V-small member N. N also belongs to the finer filter $\mathcal{Z}$, hence it has nonempty intersection with M.

We claim that N is a subset of V(M). For if y is a point in N and x is chosen from M $\cap$ N, then (x, y) belongs to V, hence y belongs to V(M). Therefore, V(M) belongs to $\mathcal{Z}'$, so that **F** is a member of $\mathcal{Z}'$. $\mathcal{Z}_o$ is then coarser than $\mathcal{Z}'$, and the theorem is proved.

Corollary 3.14.1. Let x be a point in a uniform space X. Then the neighborhood filter of X (in the topology induced by the uniformity) is a minimal Cauchy filter.

Proof: Let $\mathcal{B} = \{x\}$ and $\mathcal{Z}$ the filter generated by $\mathcal{B}$. Following the construction of the previous theorem, $\mathcal{Z} = \mathcal{Z}_o$. Hence $\mathcal{Z}$ is a minimal Cauchy filter.

Corollary 3.14.2. If $\mathcal{Z}$ is a minimal Cauchy filter on the uniform space $(X, \mathcal{U})$, then every member of $\mathcal{Z}$ has a nonempty

interior which also belongs to $\mathcal{F}$, that is, $\mathcal{F}$ has a base consisting of open sets.

Proof: Let V be an entourage of $\mathcal{U}$. Then by Corollary 3.9.1 there exists an entourage U contained in V and open in the product topology of X x X. For each subset M of X, U(M) is open from the construction of the induced topology and is contained in V(M). The construction of the filter base in Theorem 3.14 gives the desired result.

Just as every convergent sequence in a metric space is a Cauchy sequence, the analogous statement is true for filters.

Theorem 3.15. Every convergent filter on a uniform space is a Cauchy filter.

Proof: Let the filter $\mathcal{F}$ converge to the point x in the uniform space $(X, \mathcal{U})$. Let $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ be the symmetric entourages of X. Then $\mathcal{V}$ is a fundamental system of entourages. Since $\mathcal{F}$ converges to x, then $\mathcal{F}$ is finer than the neighborhood filter of x. Let W be any entourage. Then there exists $V_\beta \in \mathcal{V}$ such that $V_\beta \circ V_\beta \subset W$ (V_β is W-small). Since $V_\beta(x)$ is a neighborhood of x, then $V_\beta(x)$ belongs to $\mathcal{F}$. Therefore, $\mathcal{F}$ is a Cauchy filter, and the proof is complete.

It is well known that a Cauchy sequence in a metric space need not be convergent to a point in the space. From the analogies

which have been drawn between sequences and filters, one might suspect that a Cauchy filter in a uniform space need not converge to a point in the space. The following shows that this is true.

Let X be any infinite set. A partition of X is a collection of disjoint subsets of X whose union is X. For each finite partition $P = \{A_i: 1 \leq i \leq n\}$ of X, let $V_P = \bigcup_{i=1}^{n} A_i \times A_i$. The sets V_P form a fundamental system of entourages for a uniformity $\mathcal{U}$ on X called the uniformity of finite partitions. To show this, we must verify that properties E. 1 through E. 4 are satisfied.

To show E. 1, let V_P and V_Q be two sets corresponding to finite partitions $P = \{B_j: 1 \leq j \leq m\}$ and $Q = \{C_k: 1 \leq k \leq n\}$. Then the sets $B_j \cap C_k$, which are nonempty, form a finite partition R of X. Therefore, we have $V_R \subset V_P \cap V_Q$, and E. 1 is satisfied.

Clearly, the diagonal of X x X is contained in each V_P, so that E. 2 is satisfied. Also, $V_P = V_P^{-1}$ for each finite partition P. Setting $W = V_P$, then $W \subset V_P^{-1}$, and E. 3 is satisfied.

It is easy to see that $V_P = V_P \circ V_P$ for each P. Setting $U = V_P$ gives $U \subset V_P \circ V_P$, satisfying E. 4.

We now note the unusual property that every ultrafilter is a Cauchy filter with respect to this uniformity. For if $\mathcal{F}$ is any ultrafilter on X, then let V_P be any member of the fundamental system of entourages, and let $P = \{A_i: 1 \leq i \leq n\}$ be the corresponding

finite partition of X. Then, by Corollary 1.6.1, some A_i belongs to $\mathcal{F}$, and A_i is V_P-small. Therefore, the ultrafilter $\mathcal{F}$ is a Cauchy filter.

We note also that the topology induced by the uniformity of finite partitions is discrete. For if x is a member of X, then $\{x\}$ and $\{x\}^c$ form a finite partition of X. Let $V = (\{x\} \times \{x\}) \cup (\{x\}^c \times \{x\}^c)$. Then $V(x) = \{x\}$ is open.

Since X is an infinite discrete topological space, then X cannot be compact. Now by Theorem 1.9, some ultrafilter on X does not converge.

Definition 3.10. A complete uniform space is a uniform space in which every Cauchy filter is convergent.

At this point, two questions are raised. Can a uniform space which is not complete be "completed" in the same sense as a metric space? What conditions are sufficient for the property that every ultrafilter is a Cauchy filter? The answer to the first question provides a foundation for answering the second. Both are considered in the next chapter.

CHAPTER IV

THE COMPLETION OF A UNIFORM SPACE

We have seen in Chapter I that every ultrafilter on a compact topological space is convergent. If the compact space is also uniformizable, then each convergent filter, hence each ultrafilter, is a Cauchy filter by Theorem 3.15. Since the uniformity of finite partitions is not compact, then compactness is obviously not necessary for each ultrafilter to be a Cauchy filter. In order to find a sufficient condition weaker than compactness, we shall explore further the concept of completeness of a uniform space.

We recall that an incomplete metric space can be imbedded in (is isomorphic to a dense subspace of) a complete metric space. Our next theorem answers affirmatively for Hausdorff uniform spaces the first question raised at the close of the preceding chapter.

Theorem 4.1. Let $(X, \mathcal{U})$ be a Hausdorff uniform space. There exists a complete Hausdorff uniform space $(\hat{X}, \tilde{\mathcal{U}})$ such that $(X, \mathcal{U})$ is isomorphic to a dense subspace of $(\hat{X}, \tilde{\mathcal{U}})$.

Proof: We begin by defining the set $\hat{X}$ to be the set of all minimal Cauchy filters on X. Let $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ be the

symmetric entourages of $\mathcal{U}$. For each V in $\mathcal{V}$, let $\tilde{V}$ be the set of all pairs $(\mathcal{X}, \mathcal{Y})$ of minimal Cauchy filters in $\tilde{X}$ which have in common a V-small set. Let $\tilde{\mathcal{V}} = \{\tilde{V}_\alpha : \alpha \in A\}$.

We shall show that $\tilde{\mathcal{V}}$ forms a fundamental system of entourages for a uniform structure $\tilde{\mathcal{U}}$ on $\tilde{X}$. In particular, the following must be verified:

(E. 1) The intersection of two members of $\tilde{\mathcal{V}}$ contains a member of $\tilde{\mathcal{V}}$.

(E. 2) Every member of $\tilde{\mathcal{V}}$ contains the diagonal of $\tilde{X} \times \tilde{X}$.

(E. 3) For each $\tilde{V}$ in $\tilde{\mathcal{V}}$ there exists $\tilde{V}'$ in $\tilde{\mathcal{V}}$ such that $\tilde{V}'$ is a subset of $\tilde{V}^{-1}$.

(E. 4) For each $\tilde{V}$ in $\tilde{\mathcal{V}}$ there exists $\tilde{W}$ in $\tilde{\mathcal{V}}$ such that $(\tilde{W})^2$ is a subset of $\tilde{V}$.

Let $\tilde{V}$ and $\tilde{V}'$ be two members of $\tilde{\mathcal{V}}$, and let V and V' be the corresponding symmetric entourages of $\mathcal{U}$. Then $W = V \cap V'$ is a symmetric entourage. Since every W-small set is also V-small and V'-small, then $\tilde{W} \subset \tilde{V} \cap \tilde{V}'$. Thus E. 1 is satisfied.

Since each $\mathcal{X}$ in $\tilde{X}$ is a Cauchy filter, then $(\mathcal{X}, \mathcal{X})$ belongs to $\tilde{V}$ for every symmetric entourage V. Therefore, E. 2 is satisfied.

By definition, each $\tilde{V}$ is symmetric. By setting $\tilde{V}' = \tilde{V}$, E. 3 is satisfied.

Let $\tilde{V}$ be an arbitrary member of $\tilde{\mathcal{V}}$ and V the corresponding member of $\mathcal{V}$. Since $\mathcal{V}$ is a fundamental system of entourages, there exists a symmetric entourage W such that W o W is a subset of V.

Now let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be three minimal Cauchy filters on X such that $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ belong to $\tilde{W}$. In order to show that $\tilde{W}$ o $\tilde{W}$ is contained in $\tilde{V}$, we must show that $(\mathcal{X}, \mathcal{Z})$ belongs to $\tilde{V}$. Since $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ belong to $\tilde{W}$, then there exist W-small sets M and N such that M belongs to $\mathcal{X} \cap \mathcal{Y}$ and N belongs to $\mathcal{Y} \cap \mathcal{Z}$. But then M and N belong to $\mathcal{Y}$, so that $M \cap N$ is nonempty. By Theorem 3.11, $M \cup N$ is W^2-small, hence V-small.

Since $M \cup N$ is a subset of X containing $M \in \mathcal{X}$, then $M \cup N$ belongs to $\mathcal{X}$. Similarly, $M \cup N \in \mathcal{Z}$. Now $\mathcal{X}$ and $\mathcal{Z}$ have in common the V-small set $M \cup N$. Therefore, $(\mathcal{X}, \mathcal{Z})$ is a member of $\tilde{V}$; hence $\tilde{W}$ o $\tilde{W} \subset \tilde{V}$, and E.4 is satisfied.

If $\tilde{\mathcal{U}}$ is the collection of subsets of $\hat{X} \times \hat{X}$, each of which contains a member of $\tilde{\mathcal{V}}$, then $\tilde{\mathcal{U}}$ is a uniform structure on $\hat{X}$, and $\tilde{\mathcal{V}}$ is a fundamental system of entourages.

In order to show that $\hat{X}$ is Hausdorff, it is necessary and sufficient, by Theorem 3.10, to show that the intersection of all the entourages of its uniformity is the diagonal of $\hat{X} \times \hat{X}$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two minimal Cauchy filters on X such that $(\mathcal{X}, \mathcal{Y}) \in \tilde{V}$ for each symmetric entourage V of $\mathcal{U}$. Such a pair

exists, for each V contains the diagonal. Now let $\mathcal{B} = \{M \cup N: M \in \mathcal{X}, N \in \mathcal{Y}\}$. We claim that $\mathcal{B}$ is a base of a filter $\mathcal{Z}$, that $\mathcal{Z}$ is a Cauchy filter, and that $\mathcal{Z}$ is coarser than $\mathcal{X}$ and $\mathcal{Y}$.

Let $M \cup N$ and $M' \cup N'$ be two members of $\mathcal{B}$ with M and M' in $\mathcal{X}$ and N and N' in $\mathcal{Y}$. Then $(M \cup N) \cap (M' \cup N') = ((M \cup N) \cap M') \cup ((M \cup N) \cap N') = (M \cap M') \cup (N \cap M') \cup (M \cap N') \cup (N \cap N')$. (1)

Now $M^* = M \cap M'$ is a member of $\mathcal{X}$ and $N^* = N \cap N'$ is a member of $\mathcal{Y}$. Hence $M^* \cup N^*$ is a member of $\mathcal{B}$. But equation (1) above shows that $M^* \cup N^* \subset (M \cup N) \cap (M' \cup N')$, so that B. 1 is satisfied.

Since $\mathcal{X}$ and $\mathcal{Y}$ are filters, then $\mathcal{X}$, $\mathcal{Y}$, M, and N are all nonempty, so that $M \cup N$ is nonempty. Hence $\emptyset$ is not a member of $\mathcal{B}$, and $\mathcal{B}$ is nonempty. Therefore, B. 2 holds, and $\mathcal{B}$ is a base for a filter $\mathcal{Z}$.

Let F be a member of $\mathcal{Z}$. Then F contains some member of $\mathcal{B}$, that is, $M \cup N \subset F$ with M in $\mathcal{X}$ and N in $\mathcal{Y}$. But since F contains both M and N, then F belongs to both $\mathcal{X}$ and $\mathcal{Y}$. Hence $\mathcal{Z}$ is coarser than both $\mathcal{X}$ and $\mathcal{Y}$. Since $\mathcal{X}$ and $\mathcal{Y}$ are minimal Cauchy filters, we have $\mathcal{X} = \mathcal{Y} = \mathcal{Z}$. Thus the only pairs $(\mathcal{X}, \mathcal{Y})$ belonging to each V are the pairs $(\mathcal{X}, \mathcal{X})$, which are precisely the elements of the diagonal of $\hat{X} \times \hat{X}$. Therefore, X is Hausdorff.

We have seen that for each point x in X, the neighborhood filter $\mathcal{N}(x)$ of x is a minimal Cauchy filter on X, hence it belongs

to $\hat{X}$. We define the mapping i of X into $\hat{X}$ in the following manner: for each x in X, $i(X) = \mathcal{N}(x)$. It is obvious that i is a mapping of X onto i(X). By Theorem 3.1, i(X) is a uniform subspace of X. We must verify that i is an isomorphism of X onto i(X), i(X) is dense in $\hat{X}$, and that $\hat{X}$ is complete.

Let x and y be elements of X; let f = i x i; and suppose that $i(x) = i(y)$ with $x \neq y$. From the definition of i, we can see that $\mathcal{N}(x) = \mathcal{N}(y)$. If X is Hausdorff, then x and y can be separated by disjoint neighborhoods whose intersection must belong to the neighborhood filter. But this violates F.3, hence i is a one-to-one mapping.

If $(i(x), i(y))$ is a point in an arbitrary member $\tilde{V}$ of $\tilde{\mathcal{V}}$, then there exists a V-small set M which belongs to both i(x) and i(y), and M is a neighborhood of both x and y. Therefore, $(x, y) \in V$, so that $f^{-1}(\tilde{V}) \subset V$. This implies that i^{-1} is uniformly continuous.

Conversely, let (x, y) be an element of V, and choose two arbitrary elements u and v from the set $M = V(x) \cup V(y)$. Then either (x, u) or (y, u) belongs to V, and either (x, v) or (y, v) belongs to V. Since V is symmetric, then (u, v) belongs to V^3. Therefore, $(V(x) \cup V(y)) \times (V(x) \cup V(y)) \subset V^3$, and M is V^3-small. Since M is a neighborhood of both x and y, then $M \in i(x) \cap i(y)$ and $(i(x), i(y)) \in \tilde{V}^3$, or equivalently, $f(x, y) \in \tilde{V}^3$. But then $(x, y) \in f^{-1}(\tilde{V}^3)$, and V is a

subset of $f^{-1}(\tilde{V}^3)$. Now let a symmetric entourage W be given. It is always possible to choose V such that $V^3 \subset W$. Then, by the above, V is a subset of $f^{-1}(\tilde{V}^3)$, which is a subset of $f^{-1}(\tilde{W})$, so that i is uniformly continuous.

The above results with Theorem 3.7 show that the uniform space X is isomorphic to the uniform space i(X).

In order to show that i(X) is dense in $\hat{X}$, let $\mathfrak{x}$ be any member of $\hat{X}$ ($\mathfrak{x}$ is a minimal Cauchy filter on X), and choose $\tilde{V}$ in $\tilde{\mathcal{U}}$. Then $\tilde{V}(\mathfrak{x})$ is a neighborhood of $\mathfrak{x}$.

The trace of $\tilde{V}(\mathfrak{x})$ on i(X) is the set of all points i(x) in $\hat{X}$ ($x \in X$) such that ($\mathfrak{x}$, i(x)) belongs to $\tilde{V}$. Now if i(x) belongs to the trace of $\tilde{V}(\mathfrak{x})$ on i(X), then i(x) and $\mathfrak{x}$ have in common a V-small neighborhood M of x. In other words, x is an interior point of a V-small set M which belongs to $\mathfrak{x}$.

Let M^* be the union of the interiors of all V-small members of $\mathfrak{x}$. Then by Corollary 3.14.2, we see that $\tilde{V}(\mathfrak{x}) \cap i(X) = i(M^*) \neq \emptyset$, so that i(X) is dense in $\hat{X}$.

Also, the trace of $\tilde{V}(\mathfrak{x})$ on i(X) belongs to the filter base $i(\mathfrak{x})$ on $\hat{X}$, so that $i(\mathfrak{x})$ converges to $\mathfrak{x}$ in $\hat{X}$.

To show that $\hat{X}$ is complete, let $\mathfrak{Z}^*$ be a Cauchy filter in $\hat{X}$, and let $\mathfrak{Z}$ be the restriction of $\mathfrak{Z}^*$ to i(X). Then, since i^{-1} is a uniformly continuous mapping, Theorem 3.13 shows that $i^{-1}(\mathfrak{Z})$ is a base of a Cauchy filter $\mathfrak{F}$ in X.

Let $\mathfrak{X}$ be the minimal Cauchy filter on X coarser than $\mathfrak{F}$. We shall show that $\mathfrak{Z}^*$ converges to $\mathfrak{X}$.

Now $\mathfrak{Y} = \{H_\alpha : \alpha \in A,\ H \subset X\}$ is a minimal Cauchy filter on X. Since i is uniformly continuous, then $i(\mathfrak{X}) = \{i(H_\alpha) : H_\alpha \in \mathfrak{X}\}$ is a Cauchy filter base on i(X), and $\mathfrak{Z} = i(i^{-1}(\mathfrak{Z}))$ is finer than the filter whose base is $i(\mathfrak{X})$. We saw above that the latter converges to $\mathfrak{Y}$ in $\hat{X}$; therefore, $\mathfrak{Z}$ converges to $\mathfrak{X}$. Similarly, since $\mathfrak{Z}^*$ is finer than $\mathfrak{Z}$, then $\mathfrak{Z}^*$ converges to $\mathfrak{X}$, and $\hat{X}$ is a complete uniform space.

This completes the proof of the theorem.

In the above proof, the fact that X is Hausdorff was used only to establish that the mapping i is a one-to-one mapping. Therefore, the statement of the theorem can be generalized for arbitrary uniform spaces.

Corollary 4.1.1. If $(X, \mathcal{U})$ is a uniform space, then there exists a complete Hausdorff uniform space $(\hat{X}, \tilde{\mathcal{U}})$ and a uniformly continuous mapping i of X into $\hat{X}$ such that i(X) is dense in $\hat{X}$.

Definition 4.1. The complete Hausdorff uniform space defined by Corollary 4.1.1 is called the Hausdorff completion of X, and the mapping i is called the canonical mapping of X into its Hausdorff completion.

Definition 4.2. A uniform space is said to be precompact if its Hausdorff completion is compact.

Theorem 4.2. A uniform space $(X, \mathcal{U})$ is precompact if and only if for each entourage V, there exists a finite covering of X by V-small sets.

Proof: Let i be the canonical mapping of X into $\hat{X}$. Then the entourages of X are the inverse images under i x i of the entourages of $\hat{X}$.

Let X be precompact. If U is any entourage of $\hat{X}$, then there is a symmetric entourage W of $\hat{X}$ such that $W \circ W \subset U$. Since $\hat{X}$ is compact, then there exists a finite set of points $\{x_1, \ldots, x_n\}$ in $\hat{X}$ such that the sets $W(x_i)$ cover $\hat{X}$.

Note also that each $W(x_i)$ is U-small. For if (x, y) is an element of $W(x_i) \times W(x_i)$, then (x_i, x) and (x_i, y) belong to W. Since W is symmetric, then (x, x_i) also belongs to W, so that $(x, y) \in W \circ W \subset U$.

Let V be the inverse image of U under i x i. Then V is an entourage of X, and the sets $i^{-1}(W(x_i))$ are V-small and cover X.

Conversely, suppose that for each entourage V of X there exists a finite covering of X by V-small sets. In order to show that $\hat{X}$ is compact, we need only to show that each ultrafilter on $\hat{X}$ is convergent. Since $\hat{X}$ is complete, it is sufficient to show

that each ultrafilter on $\hat{X}$ is a Cauchy filter. We must show that for each entourage W of $\tilde{\mathcal{U}}$ and for an arbitrary ultrafilter $\mathcal{Z}$ on $\hat{X}$, some W-small set belongs to $\mathcal{Z}$.

Since the closures of the entourages form a fundamental system of entourages, then W contains a closed entourage U. Let V be the inverse image under i x i of U. Let $\{A_1, \ldots, A_n\}$ be a finite collection of V-small sets which cover X. The sets $B_k = i(A_k)$ are U-small and cover i(X). Since U is closed, then the closures $\overline{B_k}$ of the sets B_k are U-small. Since i(X) is dense in $\hat{X}$, then the sets $\overline{B_k}$ cover $\hat{X}$.

By corollary 1.6.1, some $\overline{B_i}$ belongs to the ultrafilter $\mathcal{Z}$, and the proof is complete.

We recall that the construction of the uniformity of finite partitions was based on a finite cover consisting of V-small sets. In fact, as our next theorem shows, the existence of this finite cover is sufficient for the phenomenon observed in that uniformity that every ultrafilter is a Cauchy filter.

Theorem 4.3. If $(X, \mathcal{U})$ is a precompact uniform space, then each ultrafilter on X is a Cauchy filter with respect to the uniformity $\mathcal{U}$.

Proof: Let $\mathcal{Z}$ be an ultrafilter on X and V an arbitrary entourage of $\mathcal{U}$. We must demonstrate the existence of a V-small set which belongs to $\mathcal{Z}$.

By hypothesis, there exists a finite cover of X consisting of V-small sets $A_1, \ldots, A_n$. By Corollary 1.6.1, some A_i belongs to the ultrafilter $\mathcal{Z}$. $\mathcal{Z}$ is therefore a Cauchy filter, and the theorem is proved.

We have thus answered the second question posed at the end of Chapter III. In order that every ultrafilter on a uniform space be a Cauchy filter, it is sufficient that the space be precompact or have a compact Hausdorff completion.

CONCLUSION

We have seen that every ultrafilter in a compact uniform space is a Cauchy filter. It is not necessary, however, that the space be compact. The weaker condition that the space have a compact Hausdorff completion is also sufficient. It is not known to this author whether the weaker condition is necessary.

Weil [5] has shown that the topology induced by a uniformity is completely regular. If the topology is also Hausdorff, then it is meaningful to consider the Stone-Čech compactification of the topological space. It may then be reasonably expected that the Stone-Čech compactification plays an important role in the theory of uniform spaces. Tamano [4] has investigated some of the topological and uniform properties of Tychanoff spaces (completely regular T_1-spaces) in connection with the properties of the Stone-Čech compactification.

It has not been possible to explore here all the ramifications of filters and uniform spaces. An attempt has been made to mention some of the more interesting analogies between filters and sequences in addition to the development of the theory necessary to support the result mentioned above.

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