Mathematical Appendix

In this short appendix we provide the full derivation of some expressions in the paper. To derive eq. (2.5) as a solution of the ME problem (2.4), we must solve for p_{ij} in the first order conditions $\frac{\partial \mathcal{L}}{\partial p_{ij}} = 0$ with

$$\mathcal{L} = \sum_i \sum_j p_{ij} \log p_{ij} - \sum_i \left(\lambda_i \sum_j p_{ij} - r_i \right) - \sum_j \left(\mu_j \sum_i p_{ij} - c_j \right) \quad (1)$$

which yields

$$p_{ij} = \exp(\lambda_i + \mu_j - 1) \quad (2)$$

where λ_i and μ_j are Lagrange multipliers respectively of the first and second set of constraints. By substituting the solution into the constraints and the normalization condition we obtain

$$\begin{aligned}
\sum_{j=1}^{m} \exp(\lambda_i + \mu_j - 1) &= \exp(\lambda_i - 1/2) \sum_{j=1}^{m} \exp(\mu_j - 1/2) = r_i \\
\sum_{i=1}^{n} \exp(\lambda_i + \mu_j - 1) &= \exp(\mu_j - 1/2) \sum_{i=1}^{n} \exp(\lambda_i - 1/2) = c_j \\
\sum_{i=1}^{n} \sum_{j=1}^{m} \exp(\lambda_i + \mu_j - 1) &= \sum_{j=1}^{m} \exp(\mu_j - 1/2) \sum_{i=1}^{n} \exp(\lambda_i - 1/2) = 1
\end{aligned}$$

Combining the latter expression with the former two we get

$$\begin{aligned}
\exp(\lambda_i - 1/2) &= \sum_{i=1}^{n} \exp(\lambda_i - 1/2) r_i \\
\exp(\mu_j - 1/2) &= \sum_{j=1}^{m} \exp(\mu_j - 1/2) c_j
\end{aligned}$$

If we now substitute the latter expressions into eq. (2), we get the desired result

$$p_{ij} = r_i c_j \sum_{i=1}^{n} \exp(\lambda_i - 1/2) \sum_{j=1}^{m} \exp(\mu_j - 1/2) = r_i c_j = \frac{w_i^{out} w_j^{in}}{v^2} \quad (3)$$

If we add the constrain (3.1) to the problem (2.4) we obtain the following Lagrangean:

$$\mathcal{L} = \sum_i \sum_j p_{ij} \log p_{ij} - \sum_i \left(\lambda_i \sum_j p_{ij} - r_i \right) - \sum_j \left(\mu_j \sum_i p_{ij} - c_j \right) - \\ - \left(\theta \sum_{ij} (1 - \delta_{ij}) p_{ij} - s \right)$$

Solving for p_{ij} in the FOC, we obtain eq. (3.2).

The upper bound provided by eq. (3.4) is based on Markov inequality. In order to prove it, we need to show that $E[\Sigma^2(K)] = \dfrac{nm}{v}$. In fact, since the w_{ij} are binomially distributed, we know that

$$E\left[(w_{ij} - E[w_{ij}])^2\right] = \sigma^2(w_{ij}) = p_{ij}(1 - p_{ij})v \tag{4}$$

If we take the normalization factors as constants [1], we obtain for the k_{ij}

$$E\left[(k_{ij} - E[k_{ij}])^2\right] = \sigma^2(k_{ij}) = \frac{\sigma^2(w_{ij})}{w_i^{out} w_j^{in}} = \frac{1 - p_{ij}}{v} \leqslant \frac{1}{v} \tag{5}$$

which confirms our claim.

[1] This assumption is justified whenever the null hypothesis is tested for a network whose strength distribution is used to build the binomial ensemble. In this case, the network eventually belongs to the restricted ensemble $\mathcal{G}^*$, composed of those networks which follow exactly the expected strength distribution. In $\mathcal{G}^*$, thus, the strength distributions are constants. Then, it is legitimate to test the null hypothesis $G \in \mathcal{G}^*$ with the aid of this assumption.