# Applying the spacetime geometric algebra in magnetized plasma electromagnetism: Is it only a matter of beauty?

K. Hizanidis, E. Koukoutsis, P.C. Papagiannis, *National Technical University of Athens, Greece*,
A.K. Ram, *Plasma Science and Fusion Center-Massachusetts Institute of Technology, USA,*
G. Vahala, *William and Mary, USA*

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## Extended Summary

Geometric algebras (GAs) offer an elegant way to describe rotations on Bloch hyperspheres. The latter are synonymous to the so-called qubits in the realm of Quantum Information and their implementation via Quantum Computing techniques [1]. The question that naturally arises is if elegance is a high price to pay in adapting the geometric approach in a problem that cubits are involved. However, it is generally accepted in the scientific community that GAs are, by far, the easiest way to describe rotations in space and in higher dimensions [2]. This is become more obvious if one deals with such transformations in the realm of special relativity and the associated Lorentz transformations in four dimensional Minkowskian spacetime.

On the other hand, in the realm of the geometric representation of a field theory, in particular the electromagnetic field theory (EM), there exist two geometric approaches [3]: Either representing the field as a mixture of a vector and a bivector [frequently called biparavector, similar to the widely used Riemann-Silberstein-Weber vector (RSW)], or as a genuine bivector, $\boldsymbol{\mathcal{F}}$. The geometric algebra behind the first approach is the Clifford Geometric Algebra $\boldsymbol{\mathcal{C}\ell}\,(\mathbb{R}^3)$ in the three-dimensional space, $\mathbb{R}^3$ , which is called Pauli Algebra ($\boldsymbol{\mathcal{PA}}$). The respective geometric algebra behind the second approach is, instead, the Clifford Geometric Algebra $\boldsymbol{\mathcal{C}\ell}(\mathbb{R}^{1,3})$ which is called Dirac Algebra, $\boldsymbol{\mathcal{DA}}$, or, alternatively, spacetime geometric algebra (STGA)], which refers to the four dimensional Minkowskian spacetime, $\mathbb{R}^{1,3}$, with the metric (*1,-1,-1,-1*).

In this work we adopt the second approach as far more suitable since it brings in naturally Lorentz invariance and symmetries of the EM. Note that $\boldsymbol{\mathcal{DA}}$ is isomorphic to the algebra of *4X4* real matrices (while $\boldsymbol{\mathcal{PA}}$ is isomorphic to the algebra of *2X2* complex matrices). The EM bivector $\boldsymbol{\mathcal{F}}$ consists of six bivectorial components: The first three correspond to the Gibbsian electric field intensity, **E**, and they are time-like, that is, there are defined by the three blades (geometric product of mutually orthogonal directions in spacetime) that possess time-like directions, i.e., the geometric products $\gamma_m\gamma_0,\ (m = 1,2,3)$ with $\gamma_0$ being the unit vector along the temporal direction (0). The latter three components are purely spatial left-handed (*32, 13, 21*) unit blades, $\gamma_m\gamma_n,\ (m,n = 1,2,3 = x,y,z)$ and correspond to the Gibbsian magnetic field induction, **B**. Thus (the double arrow in the symbols signifies the spacetime ($\gamma_m\gamma_0$) bivectorial nature of the symbols involved):

$$\mathcal{F} = \overleftrightarrow{\mathbf{E}} + c\overleftrightarrow{\mathbf{B}}\gamma_{0123},\ \gamma_{0123} \equiv \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 = \gamma_0\gamma_1\gamma_2\gamma_3 \quad (1)$$

where the exterior or wedge ($\wedge$) product coincides with the geometric product of all four unit directions in spacetime since all are mutually orthogonal. That is the RHS multiplication of the spacetime bivector $\overleftrightarrow{\mathbf{B}}$ with the (called pseudoscalar) $\gamma_{0123}$ yields the aforementioned



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purely special bivector for the magnetic field induction. The Clifford conjugate bivector (generalization of the Hermitian conjugate for Clifford Algebras) for $\mathcal{F}$ is:

$$\mathcal{F}^{\dagger} = \overleftrightarrow{\mathbf{E}} - c\overleftrightarrow{\mathbf{B}}I \quad (2)$$

This conjugate bivector is important in material media where the induced polarization and/or magnetization depends on $\overleftrightarrow{\mathbf{E}}$ and/or $\overleftrightarrow{\mathbf{B}}$. Therefore, in a material medium the EM field is described with the columnal bivector $\begin{pmatrix}\mathcal{F}\\ \mathcal{F}^{\dagger}\end{pmatrix}$ which is our geometrical object at hand. The geometric perspective in our approach is equipped with a differential operator in Minkowskian space-time (the Dirac operator), $\mathfrak{O} = \gamma_0\partial^0 + \gamma_m\partial^m$ (differentiation with respect to *ct*, "0", and space, "*m*") which brings in two distinct aspects: (1) the contraction-grade reduction of the differential properties of our geometrical object and (2) the expansion-grade increase of these differential properties. Both properties will be driven by proper "sources". Note that (1) is the so-called **convergence** (Maxwell's himself definition, opposite to divergence) of the geometrical object, while (2) is the **vorticity (space-time handiness)** of our geometrical object. The basic differential law will be (1), while the existence of vorticity (2) will be only under special circumstances. In field-theoretical tensorial language, (2) will be rather the validity (or the "violation") of the Bianchi mathematical identity. The differential law will be therefore a law equivalent to the Maxwell's equations. In a material medium in the presence of (total) 4-dimensional currents (scalar and bivectorial elements of $\boldsymbol{\mathcal{DA}}$), $\mathcal{J}_t$:

$$\mathfrak{O}\mathcal{F} = \mathfrak{O}\circ\mathcal{F} + \mathfrak{O}\wedge\mathcal{F} = \mu_0 c\mathcal{J}_t,\ \mathfrak{O}^{\dagger}\mathcal{F}^{\dagger} = \mathfrak{O}^{\dagger}\circ\mathcal{F}^{\dagger} + \mathfrak{O}^{\dagger}\wedge\mathcal{F}^{\dagger} = \mu_0 c\mathcal{J}_t^{\dagger} \quad (3)$$

The driving source term for cold lossless magnetized plasmas, are the generalized currents of the polarization space-time bivector $\overleftrightarrow{\mathbf{P}} = P_m\left(\overleftrightarrow{\mathbf{E}}\right)\gamma_m\gamma_0$ for each participating species $\alpha$

$$P_m\left(\overleftrightarrow{\mathbf{E}}\right) = P_m(\{\chi_m\},\chi_0) = \frac{\varepsilon_0}{c}\int_0^{\infty} d\xi_0\ \kappa_{mn}\left(\{\chi_m\},\xi_0\right)E_n(\{\chi_m\},\chi_0-\xi_0) \quad (4)$$

In the frequency (ω) domain the susceptibility kernel $\kappa_{mn}\left(\{\chi_m\},\xi_0\right)$ (Stix's [4]) is:

$$\kappa_{mn}^{(\alpha)} \equiv \begin{pmatrix} \frac{L_\alpha+R_\alpha}{2} & i\frac{L_\alpha-R_\alpha}{2} & 0 \\ i\frac{R_\alpha-L_\alpha}{2} & \frac{L_\alpha+R_\alpha}{2} & 0 \\ 0 & 0 & P_\alpha \end{pmatrix}, L_\alpha = \frac{P_\alpha}{1-\varepsilon_\alpha C_\alpha}, R_\alpha = \frac{P_\alpha}{1+\varepsilon_\alpha C_\alpha},\ P_\alpha = -\left(\frac{\omega_{p\alpha}}{\omega}\right)^2,\ C_\alpha = \frac{\omega_{c\alpha}}{\omega} \quad (5)$$

with $\omega_{p\alpha}$ and $\omega_{c\alpha}$ being the plasma and gyrofrequency of the species involved respectively, and $\varepsilon_\alpha$ the respective sign of its charge.


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**Results**

Implementing (3-5) leads to the following compact Dirac type evolution equation:

$$\partial^0 \begin{bmatrix} \begin{pmatrix} \mathcal{F} \\ \mathcal{F}^\dagger \end{pmatrix} \\ \begin{pmatrix} A^{(\alpha,L)} \\ A^{\dagger(\alpha,L)} \end{pmatrix} \\ \begin{pmatrix} A^{(\alpha,R)} \\ A^{\dagger(\alpha,R)} \end{pmatrix} \\ \begin{pmatrix} A^{(\alpha,\|)} \\ A^{\dagger(\alpha,\|)} \end{pmatrix} \\ \begin{pmatrix} I^{(\alpha,L)} \\ I^{\dagger(\alpha,L)} \end{pmatrix} \\ \begin{pmatrix} I^{(\alpha,R)} \\ I^{\dagger(\alpha,R)} \end{pmatrix} \\ \begin{pmatrix} I^{(\alpha,\|)} \\ I^{\dagger(\alpha,\|)} \end{pmatrix} \end{bmatrix} = \begin{bmatrix} -\partial^r \begin{pmatrix} \gamma_0\gamma_r & 0 \\ 0 & \gamma_r\gamma_0 \end{pmatrix} & \left\{\partial^r \begin{pmatrix} \gamma_0\gamma_r & 0 \\ 0 & \gamma_r\gamma_0 \end{pmatrix} \circ, \alpha\right\} & \left\{\partial^r \begin{pmatrix} \gamma_0\gamma_r & 0 \\ 0 & \gamma_r\gamma_0 \end{pmatrix} \cdot, \alpha\right\} & \left\{\partial^r \begin{pmatrix} \gamma_0\gamma_r & 0 \\ 0 & \gamma_r\gamma_0 \end{pmatrix} \cdot, \alpha\right\} & \{0,\alpha\} & \{0,\alpha\} & \{0,\alpha\} \\ 0 & \{0,\alpha\} & \{0,\alpha\} & \{0,\alpha\} & \{1,\alpha\} & \{0,\alpha\} & \{0,\alpha\} \\ 0 & \{0,\alpha\} & \{0,\alpha\} & \{0,\alpha\} & \{0,\alpha\} & \{1,\alpha\} & \{0,\alpha\} \\ 0 & \{0,\alpha\} & \{0,\alpha\} & \{0,\alpha\} & \{0,\alpha\} & \{0,\alpha\} & \{1,\alpha\} \\ -\varepsilon_0 \frac{\omega_{pa}^2}{4c} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} (\gamma_m \wedge \gamma_n) & \{0,\alpha\} & \{0,\alpha\} & \{0,\alpha\} & \{\varepsilon_\alpha \omega_{c\alpha}, \alpha\} & \{0,\alpha\} & \{0,\alpha\} \\ -\varepsilon_0 \frac{\omega_{pa}^2}{4c} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} (\gamma_m \wedge \gamma_n) & \{0,\alpha\} & \{0,\alpha\} & \{0,\alpha\} & \{0,\alpha\} & \{-\varepsilon_\alpha \omega_{c\alpha}, \alpha\} & \{0,\alpha\} \\ -\varepsilon_0 \frac{\omega_{pa}^2}{2c} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} (\gamma_m \wedge \gamma_n) & \{0,\alpha\} & \{0,\alpha\} & \{0,\alpha\} & \{0,\alpha\} & \{0,\alpha\} & \{0,\alpha\} \end{bmatrix} \begin{bmatrix} \begin{pmatrix} \mathcal{F} \\ \mathcal{F}^\dagger \end{pmatrix} \\ \begin{pmatrix} A^{(\alpha,L)} \\ A^{\dagger(\alpha,L)} \end{pmatrix} \\ \begin{pmatrix} A^{(\alpha,R)} \\ A^{\dagger(\alpha,R)} \end{pmatrix} \\ \begin{pmatrix} A^{(\alpha,\|)} \\ A^{\dagger(\alpha,\|)} \end{pmatrix} \\ \begin{pmatrix} I^{(\alpha,L)} \\ I^{\dagger(\alpha,L)} \end{pmatrix} \\ \begin{pmatrix} I^{(\alpha,R)} \\ I^{\dagger(\alpha,R)} \end{pmatrix} \\ \begin{pmatrix} I^{(\alpha,\|)} \\ I^{\dagger(\alpha,\|)} \end{pmatrix} \end{bmatrix} \quad (6)$$

where $\{A, \alpha\}$ represents a row segment with $\alpha$ elements (species indexing) of the operator $A$ (number or otherwise). The differential operators involved are $\partial^r \begin{pmatrix} \gamma_0\gamma_r & 0 \\ 0 & \gamma_r\gamma_0 \end{pmatrix}$ [the full (exterior and dot) geometric one] and $\partial^r \begin{pmatrix} \gamma_0\gamma_r & 0 \\ 0 & \gamma_r\gamma_0 \end{pmatrix} \circ$ (the "dot" one). Also, all 3 by 3 matrices involved are considered as *mn* matrices (*m,n=1,2,3=x,y,z*) and double summation convention is assumed for *m* and *n*. Under proper boundary conditions the operator in the RHS of (6) is leads to unitary evolution and, thus, the "generalized EM state" [containing the bivectorial entities $\mathcal{F}, A^{(\alpha;L,R,\|)}, I^{(\alpha;L,R,\|)}, \mathcal{F}^\dagger, A^{\dagger(\alpha;L,R,\|)}$ and $I^{\dagger(\alpha;L,R,\|)}$] is amenable to qubit (Bloch Sphere) encoding and processing in a quantum computing simulator or a quantum computer of suitable capability. In these encodings, a suitable spatial discretization technique are the so-called quantum lattice algorithms (QLAs) [5].

**References**


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