THE CONTINUUM COLLABORATION

THE ARCHITECTURE OF SPACE-TIME SEMINARS

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SEMINAR-IV: SINGULARITIES AND COSMOLOGY

Cosmology is a very interesting field concerning the global aspects of the Universe using classical and quantum mechanics.

Singularities in Cosmology are quite important, and we will be discussing the idea of Singularity theorems, and eventually a small outline of the Hawking-Penrose theorem and its meaning in Cosmology.

A SKETCH OF SINGULARITY THEOREMS

- In order to understand the construction of Singularity theorems, we need to first understand the physical modelling of them.

- We will start by understanding some energy conditions, or conditions on the physical aspects of matter in a model, and then about some important conditions on the geometric features of Space-time.

ENERGY CONDITIONS

- Energy conditions are certain constraints on the matter content of Space-time.

- The form of relations we will be looking at are independent of the exact value of the energy-momentum tensor – instead, these would just be inequalities which would suffice to understand the possibility of singularities.

- The Weak energy condition says that for a time-like vector X^a at a point $p \in M$, we have the following for all points $p_i \in M$:

$$T_{ab}X^aX^b \geq 0$$

- This is say that the local energy density for an observer is always positive. When taken into effect, quantum mechanics provides a form of violation of such condition (due to the Casimir effect, where the energy density is negative.)

- We will only consider a classical form of this theory.

- The Strong energy condition is to say that gravity is always attractive. In general, this places the conditions on the energy-momentum tensor as

$$\left(T_{ab} - \frac{1}{2} T g_{ab}\right) \geq 0$$

- The field equations $R_{ab} - \frac{1}{2} g_{ab} R = k T_{ab}$ can be written in terms of the trace of the energy-momentum tensor as

$$R_{ab} = k(T_{ab} - \frac{1}{2} T g_{ab})$$

- Therefore, the strong energy condition places a constraint on the nature of gravity, which would be that since the energy-momentum tensor is always non-negative, the effect of curvature on two nearby time-like curves would be attractive.

- The sketch of singularity theorems can be summarized as follows:

"In order for a Space-time to house singularities, it must contain a set of conditions (on matter and on the geometry of Space-time)"

- As we will be seeing, the energy conditions together with the idea of Closed Trapped Surfaces (CTSs) model the Penrose's 1965 theorem.
- CTSs can be understood by understanding the behavior of light rays going radially outwards and inwards of a Sphere – based on the nature of Space-time, the Spheres of light thus formed will either be greater than in volume with each other or less.

- The extension of the Penrose's theorem into Cosmology was seen in the Hawking-Penrose theorem, which can be used to show that the Universe, under suitable conditions can have an "initial singularity".

A QUICK VIEW OF SINGULARITIES

- Singularities are a blow-up of the geometry of the manifold (M, g).
- A coordinate singularity is one that arises merely due to a "bad" coordinate system, and by a suitable extension of coordinates can be covered (e.g. $r = 2GM$ in the Schwarzschild metric – the event horizon and the interior can be understood by the Kruskal coordinates).

- A coordinate invariant curvature singularity is one that is a blow-up of the Kretschmann scalar $\mathcal{R} = R_{abcd}R^{abcd}$. As this is coordinate invariant, no coordinate system under any extensions can cover this singularity.

A QUICK VIEW OF SINGULARITIES

- The energy conditions we looked at previously can be used to model a form of the Penrose singularity theorem, which we can state as follows:

- If the energy conditions together with conditions on the nature of (M, g) are such that they show the following meaning, we can say that singularities are possible:

- *1. Gravitational interaction felt due to non-zero energy-momentum tensor is always attractive,*
- *2. Light rays at some 2 –surface going normally outward and inwards would have decreasing areas.*

A QUICK VIEW OF SINGULARITIES

- The complete set of conditions are more restrictive – one of the key conditions being that we can understand the physical nature of the manifold from an initial data set on a space-like hypersurface (the condition of global hyperbolicity).

- The Penrose theorem can now be written as:

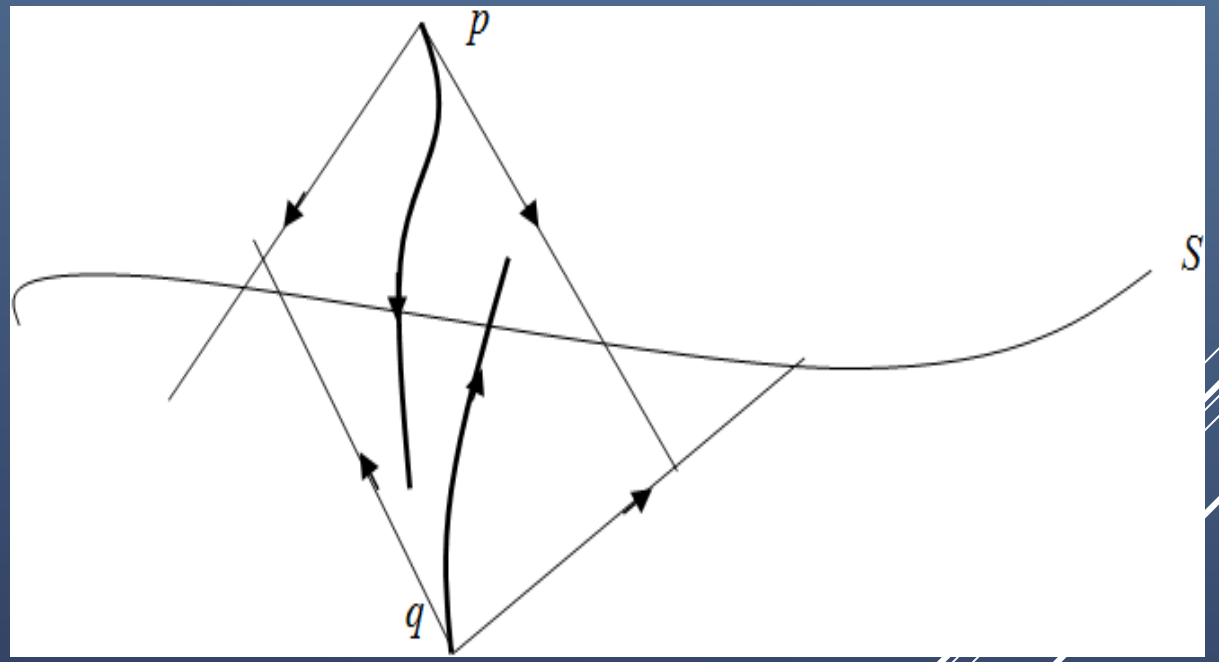
If a pair (M, g) is globally hyperbolic, satisfies a form of null energy condition and if there exists a CTS for a 2 –surface in M, we can say that there is a form of geodesic incompleteness for some null vector.

A QUICK VIEW OF SINGULARITIES

- Global hyperbolicity has to do with a very nice thing, the idea of chronological precedence. We define $I^{\pm}(p)$ as the set of all points that can be reached by a future (or past) directed causal curve (non-space-like curve).
- For $p, q \in U$, the set $I^{+}(p) \cap I^{-}(q)$ would represent a "diamond" of causal curves. If in our model there are no Closed time-like curves (CTCs), or time-like curves that start and end at a point x, and pass through a point y, where $x \neq y$, and there exists a compact $I^{+}(p) \cap I^{-}(q)$ for all p, q, then we say that U is globally hyperbolic.

- Global hyperbolicity has an important meaning to it. If there are "slices" of non-time-like hypersurfaces Σ such that they intersect every point of time-like curves in U only once, then such Σ are called Cauchy surfaces.

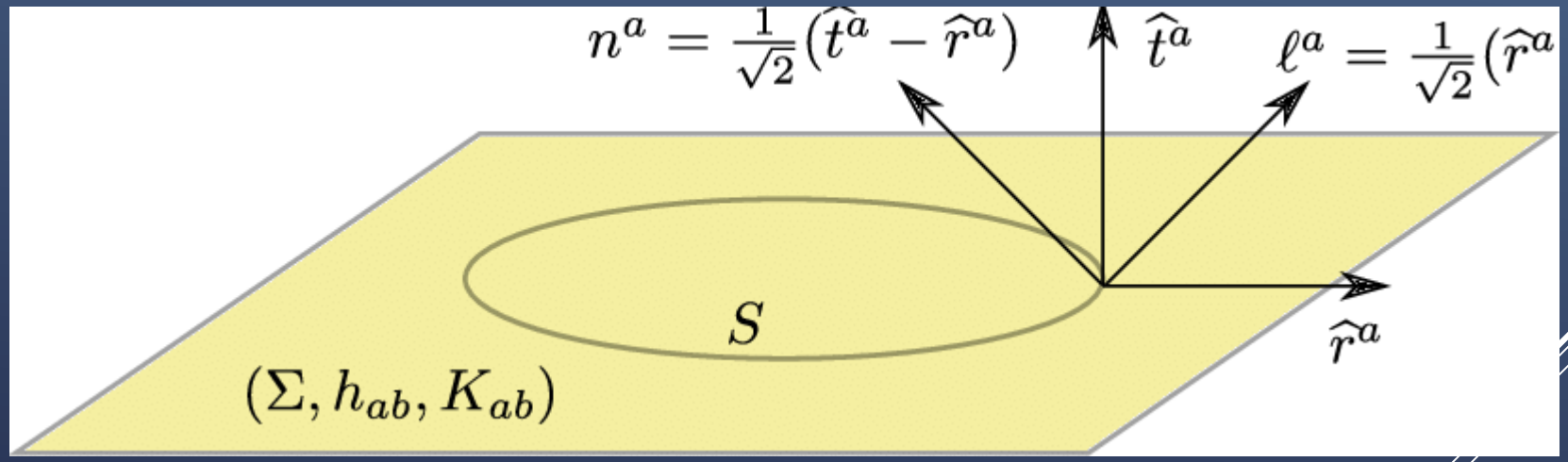
- Global hyperbolicity allows us to understand the physical aspects of U from such $\Sigma(t)$.



- This diagram represents a Cauchy surface S, which intersects points of time-like curves only once.

- This is, in general, an idea of the global hyperbolicity condition we saw in the construction of Penrose's theorem.

- This is an example of a space-like Σ with a metric h_{ab} and a symmetric covariant rank two tensor:



CLOSED TRAPPED SURFACES

- CTSs have to do with the reconverging effect of light rays travelling outward and inward.
- Consider a Sphere emitting light radially outwards and inwards. You would get two Spheres of light, one composed of light rays travelling outwards (Γ_a) and one of light rays travelling inwards (Γ_b).
- In stationary Space-time, the area of Γ_a would be greater than that of Γ_b.
- In contracting Space-time, this is not so.

CLOSED TRAPPED SURFACES

- Trapped surface on one of the first pages of The Large scale structure of Space-time (Hawking and Ellis, 1973)

- We can see that if the Universe was contracting, both the Spheres of light would be less in area.

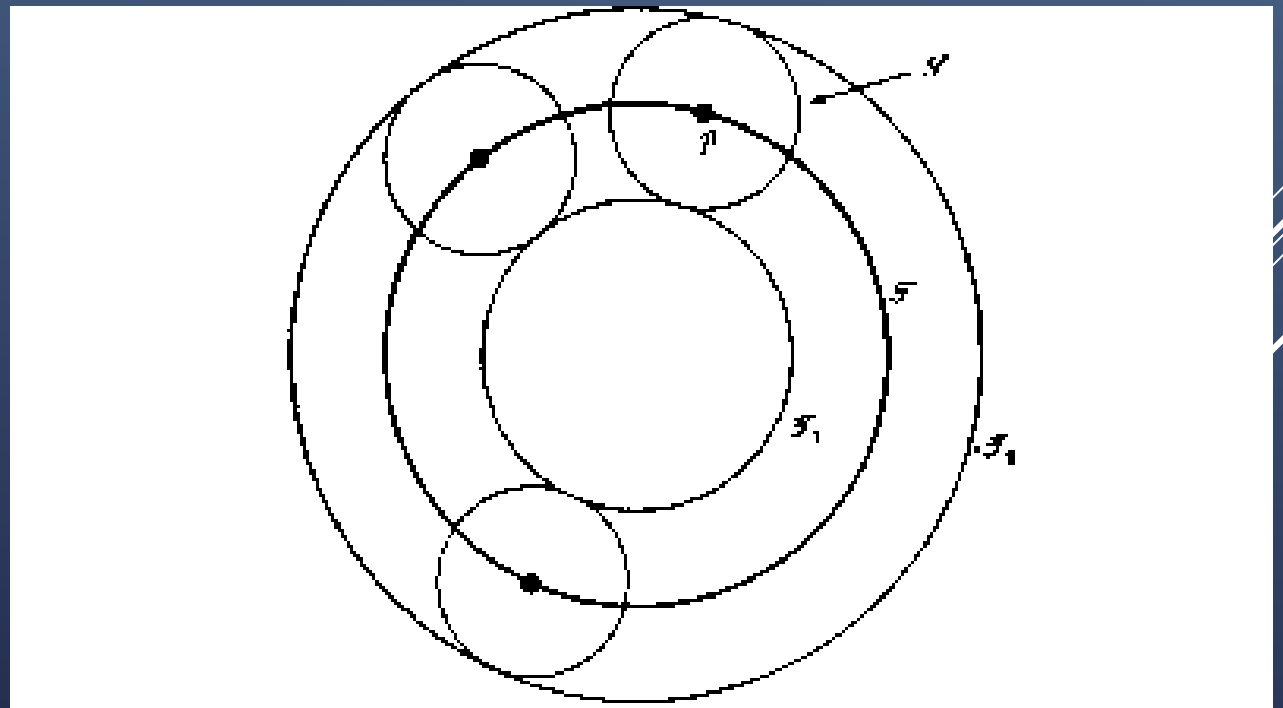


FIGURE 1. At some instant, the sphere $\mathcal{F}$ emits a flash of light. At a later time, the light from a point p forms a sphere $\mathcal{S}$ around p, and the envelopes $\mathcal{F}_1$ and

THE HAWKING-PENROSE THEOREM

- The Hawking-Penrose theorem can be summarized as the following:

- If a pair (M, g) is such that it follows the following, then we can conclude there exists some form of geodesic incompleteness:
- 1. No CTCs exist,
- 2. A CTS,
- 3. A negative expansion of null geodesics,
- 4. Or a point with a reconverging light cone.

THE HAWKING-PENROSE THEOREM

- This can be used to show that if a Universe has sufficient matter content, then the CMB observations can be used to show that it has a time-reversal singularity.

COSMIC CENSORSHIP

- Roger Penrose conjectured that *(weakly)* singularities cannot be found without an event horizon.
- An observer at infinity would not be able to see the singularity.
- GR is a "deterministic" theory – knowing the initial data at a slice of time, we can predict the evolution of the model completely (omitting certain points).
- If the Cosmic Censorship conjecture is false, this would mean that this nature of GR is not so.

COSMIC CENSORSHIP

- We will talk about the idea of the initial value problem in GR tomorrow.
- We will discuss the aspects of a static Universe, and the idea of evolution and constraint equations.
- We will also talk of the famous 1952 Choquet-Bruhat theorem, and some observations on it.

THANK YOU

- References:

- [1] The Large scale structure of Space-time, Hawking and Ellis
- [2] Singularities of gravitational collapse and Cosmology, S. Hawking and R. Penrose
- [3] The 1965 Penrose theorem, J. Senovilla