# Helicon and lower hybrid current drive comparisons in tokamak geometry

**Peter J. Catto[1] and Muni Zhou[1,2,3]**

[1]Plasma Science and Fusion Center, Massachusetts Institute of Technology, Cambridge, MA

[2]Department of Astrophysical Sciences, Princeton University, Princeton, NJ

[3]School of Natural Science, Institute for Advanced Study, Princeton, NJ

## Abstract

The parallel current driven by applied helicon waves is evaluated in tokamak geometry along with the radio frequency (rf) power absorbed by the passing electrons. The results are compared to the corresponding expressions for lower hybrid current drive. The efficiency of both current drive schemes is found to be the same in the single wave frequency, single mode number limit. The evaluation of the parallel currents is performed using an adjoint technique and tokamak geometry is retained by using an eigenfunction expansion appropriate for a transit averaged long mean free path treatment of electrons making correlated poloidal passes through the applied rf fields.

## 1. Introduction

Investigating helicon (or whistler) waves as a means to drive a parallel current in a tokamak is a less studied area of current drive (Pinsker 2015) than the more common mechanism of lower hybrid waves (Fisch 1978; Karney & Fisch 1979 & 1985; Fisch & Boozer 1980; Fisch & Karney 1981; Cordey *et al.* 1982; Taguchi 1983; Cohen 1987; Giruzzi 1987; Chiu *et al.* 1989; Ehst & Karney 1991) which is extensively reviewed by Bonoli (2014). Nonetheless, helicon current drive (HCD) remains of considerable interest (Prater *et al.* 2014; Pinsker 2015; Pinsker *et al.* 2018) because of its many similarities to lower hybrid current drive (LHCD). Both HCD and LHCD rely on a Landau resonance and the preferential heating of electrons, although HCD uses a perpendicular component of the applied electric field rather than the parallel component used for LHCD. An early attempt at evaluating the parallel current that can be driven by helicon waves and the associated efficiency appears in de Assis & Busnardo-Neto (1988) who give rough estimates for a model collision operator in a constant magnetic field with some inadequately defined notation. Little else in the way of estimates seems to be available, even though a great deal of effort has been expended planning experiments on the DIII-D tokamak (Prater *et al.* 2014; Pinsker *et al.* 2018). The purpose of the investigation here is to derive expressions for both the parallel current that can be driven by helicon waves and the associated HCD efficiency in a tokamak, and to compare them to the recently derived expressions for LHCD (Catto 2021). It turns out that the adjoint procedure (Antonsen & Chu 1982) used for these evaluations have many similarities so the comparisons can be made in a meaningful way. Moreover, these results are derived in tokamak geometry by using the Cordey (1976) eigenfunctions and associated results (Hsu *et al.* 1990; Xiao *et al.* 2007; and Parker & Catto 2012). In addition, since the electrons are only weakly collisional, successive poloidal passes through the applied rf fields are correlated and the quasilinear (QL) description employed (Catto & Tolman 2021) accounts for this feature.

The next section introduces the transit quasilinear operator to be employed for the adjoint evaluations of the parallel current as well as notation. The adjoint technique is briefly

summarized in section 3, where the unlike collision operator is also presented. Section 4 summarizes the solution for the adjoint equation in a tokamak and also gives the like particle collision operator employed. The parallel current driven by helicon waves is derived in section 5, which also presents an improved evaluation of the lower hybrid results. The rf power absorbed by the passing electrons and the current drive efficiencies associated with HCD and LHCD are presented in section 6. Section 7 gives results when HCD and LHCD are both operative. The Appendix presents some cold plasma material that suggests that HCD and LHCD are able to drive comparable currents.

## 2. Background

In a tokamak the transit averaged QL operator for electrons when the applied wave field is at a frequency $\omega$ much smaller than the electron cyclotron frequency $\Omega_e = eB/m_e c$ and the unperturbed electron distribution function is nearly the Maxwellian $f_0$ , is

$$\overline{Q}\{f_0\} = \sum_{\vec{k}} \frac{1}{\tau_f} \frac{\partial}{\partial E}(\tau_f v^2 D \frac{\partial f_0}{\partial E}), \tag{2.1}$$

with $\tau_f = \oint_f d\tau$ the time for a full (f) poloidal passing ($\sigma = v_{||}/|v_{||}| = \pm 1$) or trapped ($\sigma = 0$) poloidal circuit, the sum over the applied rf wave vectors $\vec{k}$, and D the velocity space diffusivity (Catto & Tolman 2021). Here e is the charge on a proton, c is the speed of light, $m_e$ is the electron mass, and $E = v^2/2 - e\Phi/m_e$ is the total energy, with $\Phi$ the electrostatic potential, $v = |\vec{v}|$ the electron speed, and $v_{||} = \vec{n} \cdot \vec{v}$ the parallel electron velocity along the tokamak magnetic field $\vec{B} = B\vec{n} = I\nabla\zeta + \nabla\zeta \times \nabla\psi$ . The unit vector along the magnetic field is $\vec{n}$, $\zeta$ is the toroidal angle variable, $\psi$ is the poloidal flux function, and the flux function $I(\psi)$ is $I = RB_t$, where $B_t$ is the toroidal magnetic field, R is the major radius, and $B_p$ is the poloidal magnetic field in $|\nabla\psi| = RB_p$. The poloidal angle $\vartheta$ satisfies $\nabla\zeta \cdot \nabla\psi = 0 = \nabla\zeta \cdot \nabla\vartheta$ and is chosen such that $\vec{B} \cdot \nabla\vartheta = |I|/qR^2 = q^{-1}|\vec{B} \cdot \nabla\zeta|$, making the safety factor a flux function, $q = q(\psi)$. Taking the toroidal current to be in the $\nabla\zeta$ direction makes $B_p > 0$ , $\vec{B} \cdot \nabla\vartheta > 0$, and $\vartheta$ increases in the $\vec{B}_p = \nabla\zeta \times \nabla\psi$ direction. Then the incremental time along a trajectory is $d\tau = d\vartheta/v_{||}\vec{n} \cdot \nabla\vartheta > 0$, with $d\vartheta$ and $v_{||}$ reversing signs together when a trapped electron reflects, while $d\vartheta < 0$ for a passing electron with $v_{||} < 0$.

The QL diffusivity D in tokamak geometry for successive correlated interactions with the applied rf (Catto & Tolman 2021) is

$$D = \frac{\pi e^2}{2m_e^2 v^2 \tau_f} \sum_\ell \delta(\oint_f d\tau\, \Lambda - 2\pi\ell) \left| \oint_f d\tau \vec{e}_k \cdot [\vec{n} v_{||} J_0(\eta) - i\vec{n} \times \vec{k} \frac{v_\perp}{k_\perp} J_1(\eta)] e^{-i\int_{\tau_0}^{\tau} d\tau' \Lambda(\tau')} \right|^2, \tag{2.2}$$

with

$$\oint_f d\tau\, \Lambda = \omega\tau_f - 2\pi\sigma(qn - m), \tag{2.3}$$

the transit averaged resonance condition, and the exponential phase factor is

$$\int_{\tau_0}^{\tau} d\tau' \Lambda(\tau') = \omega(\tau - \tau_0) - \sigma(qn - m)\vartheta(\tau), \tag{2.4}$$

where $\tau_0$ is taken to be the trajectory time at the equatorial plane crossing where B is a minimum. In the preceding and what follows $\vec{e}_k$ is the Fourier amplitude of the applied electric field of wave vector $\vec{k} = \vec{k}_\perp + k_{||}\vec{n}$ having $\vec{k}_\perp = k_\perp(\vec{\psi}\cos\varsigma + \vec{p}\sin\varsigma)$ , $k_{||} = (qn - m)/qR$ and $\eta = k_\perp v_\perp/\Omega_e$ , with $\vec{\psi} = \nabla\psi/|\nabla\psi| = \nabla\psi/RB_p$, $\vec{p} = \nabla\zeta \times \nabla\psi/|\nabla\zeta \times \nabla\psi| = \vec{B}_p/B_p$ and $\vec{n}$ orthonormal unit vectors satisfying $\vec{\psi} \times \vec{p} = \vec{n}$ , n and m toroidal and poloidal mode numbers, respectively, and large aspect ratio assumed to write $\vec{n} \cdot \nabla\vartheta \approx 1/qR$. The integer $\ell$

denotes the resonant path in velocity space as electrons can experience other (usually less important) resonances besides $\ell = 0$ in toroidal geometry as their velocity changes along a field line so they cannot remain in resonance indefinitely. Instead, they can have other resonant interactions by crossing the various resonant paths or curves in velocity space defined by $\oint_f d\tau\, \Lambda = 2\pi\ell$.

For lower hybrid current drive (LHCD) the applied rf fields are tailored to make the first or $v_{||}J_0$ term dominate. For helicon current drive (HCD) the rf fields must be applied in a manner that makes the second or $v_\perp J_1$ term dominate (Prater *et al.* 2014; Pinsker 2015; Pinsker *et al.* 2018). Of course, both HCD and LHCD rely on there being a Landau resonance. Here the goal is to evaluate HCD by an adjoint technique (Antonsen & Chu 1982) very similar to the one used recently to evaluate LHCD in full toroidal geometry to find the aspect ratio modifications to the driven current and efficiency (Catto 2021). Unlike for LHCD, HCD does not seem to have an accepted expression for the driven current or efficiency even without toroidal effects retained.

## 3. Adjoint technique summary

For an adjoint evaluation of HCD the preceding QL operator is all that is required along with the perturbed electron kinetic equation,

$$v_{||}\vec{n}\cdot\nabla f_1 = C\{f_1\} + Q\{f_0\}, \tag{3.1}$$

and its adjoint equation for the adjoint function h associated with $f_1$,

$$v_{||}\vec{n}\cdot\nabla h + C\{h\} = -(\frac{B}{I} - \frac{m_e\nu_u}{T_e x^3}V_{||})v_{||}f_0, \tag{3.2}$$

where the term with the parallel mean ion velocity $V_{||}$ is needed to account for the non-self-adjoint term in the electron-ion collision operator and $f_0$ is the Maxwellian

$$f_0 = f_0(\psi, E) = n_e(\psi)[\frac{m_e}{2\pi T_e(\psi)}]^{3/2}e^{-m_e v^2/2T_e(\psi)} = n_e(\frac{m_e}{2\pi T_e})^{3/2}e^{-[m_eE+e\Phi(\psi)]/T_e(\psi)}. \tag{3.3}$$

The potential may be assumed to be a flux function to lowest order. In the electron kinetic equation $f_1$ is the perturbed electron distribution function with $f = f_0 + f_1$, $Q\{f_0\}$ is the QL operator prior to transit averaging, and $C\{f_1\} = C_{ee}\{f_1\} + C_{ei}\{f_1\}$ is the sum of the full self-adjoint electron-electron collision operator plus the electron-ion collision operator

$$C_{ei}\{f_1\} = \frac{\nu_u}{x^3}L\{f_1 - \frac{m_e}{T_e}V_{||}v_{||}f_0\} = \frac{\nu_u}{x^3}[L\{f_1\} + \frac{m_e}{T_e}V_{||}v_{||}f_0], \tag{3.4}$$

where $L\{h\}$ is the self-adjoint Lorentz operator

$$L\{h\} = \frac{1}{2}\nabla_v\cdot[(v^2\overleftrightarrow{I} - \vec{v}\vec{v})\cdot\nabla_v f_1] = \frac{2B_0}{B}\xi\frac{\partial}{\partial\lambda}(\lambda\xi\frac{\partial h}{\partial\lambda}), \tag{3.5}$$

with $x = v/v_e$ and $v_e = (2T_e/m_e)^{1/2}$ the electron thermal speed. The pitch angle is defined as $\lambda = 2\mu B_0/Bv^2 = B_0 v_\perp^2/Bv^2$ and $\xi = v_{||}/v$, with $B_0$ a normalizing flux function to be defined shortly, and $\nu_u$ the unlike collision frequency defined as

$$\nu_u = \sqrt{2}\pi Z^2 e^4 n_i \ell n\Lambda_C/m_e^{1/2}T_e^{3/2} \to 3\sqrt{\pi}\nu_{ei}/4, \tag{3.6}$$

where $\nu_{ei} = 4\sqrt{2\pi}Z^2e^4n_i\ell n\Lambda_C/3m_e^{1/2}T_e^{3/2} = Z\nu_{ee}$ for a quasineutral plasma with the ion, $n_i$, and electron, $n_e$, densities satisfying $Zn_i = n_e$, Z the ion charge number, $\nu_{ee}$ the electron-electron collision frequency and $\ell n\Lambda_C$ the Coulomb logarithm.

Defining the flux surface average of any quantity A by

$$\langle A\rangle = (\oint d\vartheta\, A/\vec{B}\cdot\nabla\vartheta)/(\oint d\vartheta\, /\vec{B}\cdot\nabla\vartheta), \tag{3.7}$$

using $\langle\vec{B}\cdot\nabla(B^{-1}\int d^3v\, v_{||}hf_1/f_0\rangle = 0$ and the self-adjointness of $C_{ee}$ (that requires $f_0$ to be Maxwellian) and L to combine the equations leads to the convenient adjoint relation

$$\langle B\int d^3v\, v_{||} f_1\rangle = I\langle\int d^3v\, h\left(\frac{Q\{f_0\}}{f_0} - \frac{m_e\nu_u}{T_e x^3}V_{||}v_{||}\right)\rangle \approx I\left(\int d^3v \frac{v_{||}\bar{h}}{Bf_0}\tau_f\bar{Q}\{f_0\}\right)/\left(\oint d\vartheta\,/\vec{B}\cdot\nabla\vartheta\right) \tag{3.8}$$

As $V_{||}\sim v_i\rho_{pi}/a$, with $v_i$ the ion thermal speed, $\rho_{pi}$ the poloidal ion gyroradius and a the minor radius, the $V_{||}$ term is expected to be negligible for the rf amplitudes of interest since

$$\frac{m_e\nu_u\langle V_{||}\int d^3v h x^{-3}v_{||}\rangle}{T_e\langle\int d^3v h f_0^{-1}Q\{f_0\}\rangle} \sim \frac{V_{||}\nu_{ee}f_0}{v_e Q\{f_0\}} \sim \frac{v_i}{v_e}\frac{\rho_{pi}}{a}\frac{\nu_{ee}f_0}{Q\{f_0\}} \ll 1. \tag{3.9}$$

To write the final form of $\langle B\int d^3v\, v_{||}f_1\rangle$ the lowest order result $\vec{n}\cdot\nabla\bar{h} = 0$ is employed along with the transit average definition $\bar{A} = \oint_f d\tau\, A/\tau_f$. It is convenient to let $B_0^2 = \langle B^2\rangle$.

Recall the Ohmic current is in the positive toroidal direction. Therefore, the helicon waves must drive the current in the same direction, requiring $\langle B\int d^3v\, v_{||}f_1\rangle < 0$.

## 4. Solution of the adjoint equation

The advantage of the adjoint method is that only the simpler adjoint equation

$$v_{||}\vec{n}\cdot\nabla h + C_{ee}\{h\} + \nu_u x^{-3}L\{h\} = -I^{-1}Bv_{||}f_0, \tag{4.1}$$

need be solved instead of the more complicated electron kinetic equation. To solve the preceding equation, it is adequate to approximate the electron-electron collision operator by its standard high speed ($v^2 \gg v_e^2$) expansion and self-adjoint form

$$C_{ee}\{h\} = \nu_\ell\{x^{-3}L\{h\} + \nabla_v\cdot\left[\frac{T_e f_0}{m_e x^3}\nabla_v\left(\frac{h}{f_0}\right)\right\} = \frac{2\nu_\ell B_0\xi}{B}\frac{\partial}{\partial\lambda}\left(\lambda\xi\frac{\partial h}{\partial\lambda}\right) + \frac{\nu_\ell T_e}{m_e v^2}\frac{\partial}{\partial v}\left[\frac{v^2 f_0}{x^3}\frac{\partial}{\partial v}\left(\frac{h}{f_0}\right)\right], \tag{4.2}$$

where

$$\nu_\ell = \sqrt{2}\pi e^4 n_e \ell n\Lambda_C/m_e^{1/2}T_e^{3/2} = 3\sqrt{\pi}\nu_{ee}/4, \tag{4.3}$$

with $\nu_{ee} = 4\sqrt{2\pi}e^4 n_e \ell n\Lambda_C/3m_e^{1/2}T_e^{3/2}$. This like particle operator is the usual non-momentum conserving high speed expansion of the Rosenbluth potentials for collisions with a Maxwellian used by Karney & Fisch (1979, 1985). Recent estimates (Catto 2021; Catto & Tolman 2021a&b) indicate $f_0$ must be nearly Maxwellian for QL theory to remain valid.

The adjoint equation is solved using the Cordey eigenfunctions (Cordey 1976, Hsu *et al.* 1990, Xiao *et al.* 2007, Parker & Catto 2012) by writing $h = \bar{h} + \tilde{h}$ with $\partial\bar{h}/\partial\vartheta = 0$ to obtain, upon annihilating $v_{||}\vec{n}\cdot\nabla\tilde{h}$ the term, the transit average equation

$$\bar{C}_{ee}\{\bar{h}\} + \nu_u x^{-3}L\{\bar{h}\} = -I^{-1}\overline{Bv_{||}}f_0. \tag{4.4}$$

Integration over a full trapped (t) bounce gives $\overline{Bv_{||}} = 0$ implying that $\bar{h}_t = 0$. For the passing (p) electrons flux surface averages are used to rewrite the adjoint equation as

$$2(Z+1)\frac{\partial}{\partial\lambda}\left[\lambda\langle\xi\rangle\frac{\partial}{\partial\lambda}\left(\frac{\bar{h}_p}{f_0}\right)\right] + \frac{T_e x^3\langle B/v_{||}\rangle}{m_e B_0 v^2 f_0}\frac{\partial}{\partial v}\left[\frac{v^2 f_0}{x^3}\frac{\partial}{\partial v}\left(\frac{\bar{h}_p}{f_0}\right)\right] = -\frac{\langle B^2\rangle v x^3}{IB_0\nu_\ell}, \tag{4.5}$$

where $\overline{Bv_{||}}\oint_f d\tau = \langle B^2\rangle\oint d\vartheta\,/\vec{B}\cdot\nabla\vartheta$.

For the purposes here the recent solution (4.5) (Catto 2021) is adequate and convenient. Ignoring order $\epsilon = r/R \ll 1$ terms it is given by

$$\frac{\bar{h}_p}{f_0} \approx \frac{vx^3}{R\nu_\ell}\left\{\frac{(1+0.62\sqrt{\epsilon})\Lambda_1(\lambda) - 1.02[(Z+5)/(7Z+11)]\sqrt{\epsilon}\Lambda_2(\lambda)}{[(Z+1)(1+1.46\sqrt{2\epsilon})+4]}\right\} \equiv \frac{vx^3}{R\nu_\ell}\frac{\Lambda_{1+2}(\sqrt{\epsilon},Z,\lambda)}{[(Z+1)(1+1.46\sqrt{2\epsilon})+4]}, \tag{4.6}$$

where the Cordey (1976) eigenfunctions $\Lambda_j$ with eigenvalues $\kappa_j$ satisfy the Sturm-Liouville differential equation

$$\frac{\partial}{\partial\lambda}\left(\lambda\langle\xi\rangle\frac{\partial\Lambda_j}{\partial\lambda}\right) = \kappa_j\frac{\partial\langle\xi\rangle}{\partial\lambda}\Lambda_j, \tag{4.7}$$

with $\partial\langle\xi\rangle/\partial\lambda = -\langle B/2B_0\xi\rangle$, $\Lambda_j(\lambda = 0) = 1$,

$$\Lambda_{1+2}(\sqrt{\epsilon}, Z, \lambda = 0) = 1 + [0.62 - 1.02(Z+5)/(7Z+11)]\sqrt{\epsilon}\,, \tag{4.8}$$

$$\langle \xi \rangle = 2\sqrt{2\epsilon}E(k)/\pi\sqrt{(1-\epsilon)k^2 + 2\epsilon}\,, \tag{4.9}$$

E(k) is the elliptic integral of the second kind, $k^2 = 2\epsilon\lambda/[1-(1-\epsilon)\lambda]$, and $\Lambda_j = 0$ at $k = 1$. The response $\bar{h}_p$ increases with speed because of the v dependence of C. For the passing

$$\tau_p = \oint_p d\tau = \oint_p d\vartheta/v_{||}\vec{n}\cdot\nabla\vartheta \approx 4qR\sqrt{(1-\epsilon)k^2+2\epsilon}K(k)/v\sqrt{2\epsilon}, \tag{4.10}$$

with K(k) the elliptic integral of the first kind. The $\Lambda_2$ term has a somewhat small numerical coefficient. It was ignored in Catto (2021), but is retained here. More eigenfunction details are available in Catto (2021), Hsu *et al.* (1990), Xiao *et al.* (2007), and Parker & Catto (2012).

## 5. Helicon driven current in a tokamak compared to LHCD

Only the passing drive current giving

$$\langle B\int d^3v\, v_{||} f_1\rangle = \frac{\langle B^2\rangle}{2\pi q}\int d^3v \frac{v_{||}\bar{h}_p}{Bf_0}\tau_p\bar{Q}\{f_0\} = \frac{\langle B^2\rangle}{2\pi q}\sum_{\vec{k}}\int d^3v \frac{v_{||}\bar{h}_p}{Bvf_0}\frac{\partial}{\partial v}\Big|_\mu(\tau_p vD\frac{\partial f_0}{\partial v}\Big|_\mu), \tag{5.1}$$

at large aspect ratio, with the QL diffusivity simplifying to

$$D = \frac{\pi e^2|\vec{e}_k\cdot\vec{n}\times\vec{k}|^2\tau_p v_\perp^2 J_1^2(\eta)}{8m_e^2k_\perp^2v^2}\sum_\ell \delta[\omega\tau_p - 2\pi\sigma(qn-m) - 2\pi\ell]\,\Theta(v,\lambda,n,m), \tag{5.2}$$

where

$$\Theta = \frac{1}{\tau_p^2}\left|\oint_p d\tau\, e^{-i[\omega(\tau-\tau_0)-\sigma(qn-m)\vartheta(\tau)]}\right|^2 = \frac{q^2R^2}{\tau_p^2v^2}\left|\int_{-\pi}^{\pi}\frac{d\vartheta}{\xi}e^{-i[\omega qR\int_0^\vartheta d\vartheta/v_{||}-\sigma(qn-m)\vartheta]}\right|^2 \le 1 \tag{5.3}$$

and $d^3v \to 2\pi B dE d\mu/v_{||} \to 2\pi B v^3 dv d\lambda/B_0 v_{||}$. Integrating by parts in E, μ variables yields

$$\langle B\int d^3v\, v_{||} f_1\rangle = \frac{m_e\langle B^2\rangle}{2\pi q T_e}\sum_{\vec{k}}\int d^3v\frac{v_{||}}{B}\tau_p vDf_0\frac{\partial}{\partial v}\Big|_\mu(\frac{\bar{h}_p}{f_0})\,. \tag{5.4}$$

Catto (2021) has an extra $v^2$ multiplying $\bar{h}_p$ in his (4.7), implying the factors of v in his (4.27) for the lower hybrid driven parallel current need to be corrected as will be found shortly.

To sustain or enhance the poloidal magnetic field the parallel current driven by the helicon waves must be positive, requiring $\langle B\int d^3v\, v_{||} f_1\rangle < 0$. Therefore, passing electrons with $v_{||} < 0$ must drive the current implying $\sigma = -1$ and $\ell + m - qn > 0$ in the argument of the delta function (requiring $k_{||} < 0$ for $\ell = 0$). As a result, writing it in a form allowing the speed integral to be performed gives

$$\delta[\omega\tau_p - 2\pi(\ell+m-qn)] = \frac{\delta(v-v_{\omega/k_{||}})}{\omega|\partial\tau_p/\partial v|_{\omega/k_{||}}} = \frac{v_{\omega/k_{||}}^2\delta(v-v_{\omega/k_{||}})}{\omega v\tau_p}\,, \tag{5.5}$$

with

$$v_{\omega/k_{||}} = \omega v\tau_p/2\pi(\ell+m-qn) > 0\,, \tag{5.6}$$

and the exponential factor

$$x_{\omega/k_{||}}^2 = m_e v_{\omega/k_{||}}^2/2T_e = m_e(\omega v\tau_p)^2/8\pi^2(\ell+m-qn)^2T_e\,, \tag{5.7}$$

where recall a high speed expansion of the like collision operator is used so $x_{\omega/k_{||}}^2 \gg 1$ in $f_0$. In addition, the barely passing do not contribute significantly to the current since $v\tau_p/4qR \underset{k\to1}{\longrightarrow} (2\epsilon)^{-1/2}\ell n(4/\sqrt{1-k^2}) \gg 1$. Therefore, because of the exponential from $f_0$ the freely passing ($k^2 \lesssim \epsilon \ll 1$) electrons make the dominant contribution. As a result, only $\ell = 0$ need be retained in

$$D \approx \frac{\pi e^2|\vec{e}_k\cdot\vec{n}\times\vec{k}|^2\lambda J_1^2(\eta)}{8m_e^2k_\perp^2|k_{||}|}\delta(v-v_{\omega/k_{||}}) \equiv D_{\vec{k}}\,\lambda J_1^2(\eta)\delta(v-v_{\omega/k_{||}})\,, \tag{5.8}$$

with the exponential factor of $f_0$ for $\ell = 0$ defined as $X^2(\lambda) \equiv x^2_{\omega/k_{||}}(\ell = 0)$. Therefore,

$$X^2(\lambda) = (\omega^2/k_{||}^2 v_e^2)\{[2\sqrt{(1-\epsilon)k^2 + 2\epsilon}K(k)/\pi\sqrt{2\epsilon}]^2 - 1\} \approx (\omega^2/k_{||}^2 v_e^2)[\lambda/(1-\lambda)], \quad (5.9)$$

as $v\tau_p \xrightarrow[k^2 \ll 1]{} 2\pi q R\sqrt{1 + k^2/2\epsilon}$, $v_{\omega/k_{||}} \to \omega q R\sqrt{1 + k^2/2\epsilon}/(\ell + m - qn) \xrightarrow[\ell=0]{} \omega\sqrt{1 + k^2/2\epsilon}/k_{||}$, $X^2(\lambda) \to \omega^2(1 + k^2/2\epsilon)/k_{||}^2 v_e^2$ and

$$\Theta \xrightarrow[\epsilon \ll 1]{} \left|\int_{-\pi}^{\pi} \frac{d\vartheta}{2\pi} e^{-i\ell\vartheta}\right|^2 = \delta_{0\ell} = \begin{cases} 1 & \ell = 0 \\ 0 & \ell \neq 0 \end{cases}. \quad (5.10)$$

Consequently, integrating over $v_{||} < 0$ as D vanishes for $v_{||} > 0$ leads to

$$\langle \frac{B}{B_0} \int d^3v\, v_{||} f_1 \rangle = -4\pi R \sum_{\vec{k}}^{k_{||}<0} \frac{\omega^2}{k_{||}^2 v_e^2} D_{\vec{k}} \int_0^{\lambda<1-\epsilon} d\lambda\, \lambda J_1^2(\eta) \int_0^{\infty} dv \delta(v - v_{\omega/k_{||}})\, v\, f_0 \frac{\partial}{\partial v}\Big|_{\mu} (\frac{\bar{h}_p}{f_0}), \quad (5.11)$$

where in $v, \lambda$ variables

$$v\, f_0 \frac{\partial}{\partial v}\Big|_{\mu} (\frac{\bar{h}_p}{f_0}) = v f_0 [\frac{\partial}{\partial v}\Big|_{\lambda} (\frac{\bar{h}_p}{f_0}) + \frac{\partial \lambda}{\partial v}\Big|_{\mu} \frac{\partial}{\partial \lambda}\Big|_{v} (\frac{\bar{h}_p}{f_0})] = (4 - \frac{2\lambda}{\Lambda_{1+2}} \frac{\partial \Lambda_{1+2}}{\partial \lambda}) \bar{h}_p\,, \quad (5.12)$$

since the lowest order solution is $\bar{h}_p/f_0 \propto v^4 \Lambda_{1+2}(\lambda)$. The exponential in the Maxwellian makes the evaluation of the $\lambda$ integral insensitive to its upper limit ($\lambda < 1 - \epsilon$) as will be shown shortly. The pitch angle derivative of $\Lambda_{1+2}$ is finite at $\lambda = 0$ making $\Lambda_{1+1}^{-1}\, \partial \Lambda_{1+2}/\, \partial \lambda \sim 1$ leaving the lowest order result

$$\langle \frac{B}{B_0} \int d^3v\, v_{||} f_1 \rangle = -16\pi R \sum_{\vec{k}}^{k_{||}<0} \frac{\omega^2}{k_{||}^2 v_e^2} D_{\vec{k}} \int_0^{\lambda<1-\epsilon} d\lambda\, \lambda J_1^2(\eta) \bar{h}_p\,, \quad (5.13)$$

where at resonance $\eta = \sqrt{\lambda} k_\perp \omega/k_{||}\Omega_e$ and

$$\bar{h}_p = \frac{4 n_e \omega^4 e^{-\omega^2/k_{||}^2 v_e^2} \Lambda_{1+2}(\sqrt{\epsilon}, Z, \lambda) e^{-X^2(\lambda)}}{3\pi^2 R \nu_{ee} k_{||}^4 v_e^6 [(Z+1)(1+1.46\sqrt{2\epsilon})+4]}\,. \quad (5.14)$$

Inserting $\bar{h}_p$ leads to the expression for the parallel current driven by helicon waves to be

$$J_{||}^H = -e \langle \frac{B}{B_0} \int d^3v\, v_{||} f_1 \rangle =$$

$$\frac{8 e n_e}{(Z+1)(1+1.46\sqrt{2\epsilon})+4} \sum_{\vec{k}}^{k_{||}<0} \frac{e^2 |\vec{e}_k \cdot \vec{n} \times \vec{k}|^2}{3 m_e^2 k_\perp^2} \frac{\omega^6 e^{-\omega^2/k_{||}^2 v_e^2}}{\nu_{ee} |k_{||}^7| v_e^8} \int_0^{\lambda<1-\epsilon} d\lambda\, \lambda J_1^2(\eta) \Lambda_{1+2}(\lambda) e^{-X^2(\lambda)}\,, \quad (5.15)$$

where $\eta \sim \sqrt{\lambda} k_\perp v_e/\Omega_e = \sqrt{\lambda} k_\perp \rho_e$. As a result, $X^2(\lambda)$ provides a cut-off before $\lambda \to 1 - \epsilon$. However, the procedure is more complex in the HCD case than in the LHCD case.

To compare helicon and lower hybrid driven current evaluations the preceding steps are repeated starting with the replacement $k_\perp^{-2} |\vec{e}_k \cdot \vec{n} \times \vec{k}|^2 v_\perp^2 J_1^2(\eta) \to |\vec{e}_k \cdot \vec{n}|^2 v_{||}^2 J_0^2(\eta)$ and then using $v_\perp^2 \approx \lambda v^2 \approx \lambda \omega^2/k_{||}^2 \to v_{||}^2 \approx \omega^2/k_{||}^2$. As a result, the driven parallel LH current is

$$J_{||}^{LH} = \frac{8 e n_e}{(Z+1)(1+1.46\sqrt{2\epsilon})+4} \sum_{\vec{k}}^{k_{||}<0} \frac{e^2 |\vec{e}_k \cdot \vec{n}|^2}{3 m_e^2} \frac{\omega^6 e^{-\omega^2/k_{||}^2 v_e^2}}{\nu_{ee} |k_{||}^7| v_e^8} \int_0^{\lambda<1-\epsilon} d\lambda\, J_0^2(\eta) \Lambda_{1+2}(\lambda) e^{-X^2(\lambda)}\,, \quad (5.16)$$

where again $\eta = \sqrt{\lambda} k_\perp \omega/k_{||}\Omega_e$. However, an integration by parts is performed by using $e^{-X^2(\lambda)} = -[(1-\lambda)^2/z]\, \partial e^{-X^2(\lambda)}/\, \partial \lambda$ to obtain an asymptotic expansion in inverse powers of $z = \omega^2/k_{||}^2 v_e^2 \gg 1$ to find

$$J_{||}^{LH} \approx \frac{8 e n_e \Lambda_{1+2}(\sqrt{\epsilon}, Z, 0)}{(Z+1)(1+1.46\sqrt{2\epsilon})+4} \sum_{\vec{k}}^{k_{||}<0} \frac{e^2 |\vec{e}_k \cdot \vec{n}|^2}{3 m_e^2} \frac{\omega^4 e^{-\omega^2/k_{||}^2 v_e^2}}{\nu_{ee} |k_{||}^5| v_e^6}\,, \quad (5.17)$$

as $\Lambda_j(0) = 1 = J_0^2(0)$ and $\int_0^{\lambda<1-\epsilon} d\lambda\, \lambda J_0^2 \Lambda_{1+2} e^{-X^2} \approx \Lambda_{1+2}(\lambda = 0)/z$. This result is 2/3's the value given by Catto (2021) when $\Lambda_2$ is neglected, which seems to be due to the extra power of $v^2$ multiplying $\bar{h}_p$ inside the v derivative beginning at (4.7).

In the helicon case three integrations by parts are required. Ignoring exponentially small terms

$$\int_0^{\lambda<1-\epsilon} d\lambda\,\lambda J_1^2\Lambda_{1+2}e^{-X^2} \approx \frac{1}{z^2}\int_0^{\lambda<1-\epsilon} d\lambda\, e^{-X^2}\frac{\partial}{\partial\lambda}\{(1-\lambda)^2\frac{\partial}{\partial\lambda}[(1-\lambda)^2\lambda J_1^2\Lambda_{1+2}]\}$$
$$\approx \frac{1}{z^3}\frac{\partial}{\partial\lambda}\{(1-\lambda)^2\frac{\partial}{\partial\lambda}[(1-\lambda)^2\lambda J_1^2\Lambda_{1+2}]\}|_{\lambda=0} = \frac{1}{z^3}\frac{\partial J_1^2}{\partial\lambda}\Big|_{\lambda=0} = \frac{k_\perp^2\rho_e^2}{2z^2}. \quad (5.18)$$

Therefore, for HCD

$$J_{||}^H = \frac{4en_e\Lambda_{1+2}(\sqrt{\epsilon},Z,0)}{(Z+1)(1+1.46\sqrt{2\epsilon})+4}\sum_{\vec{k}}^{k_{||}<0}\frac{e^2|\vec{e}_k\cdot\vec{n}\times\vec{k}|^2}{3m_e^2k_\perp^2}\frac{\omega^2 e^{-\omega^2/k_{||}^2v_e^2}}{\nu_{ee}|k_{||}^3|v_e^4}k_\perp^2\rho_e^2\,. \quad (5.19)$$

Letting $z = \omega^2/k_{||}^2v_e^2$, then HCD and LHCD depend on $z^{3/2}e^{-z}$ and $z^{5/2}e^{-z}$, respectively, and are maximized at $z = 3/2$ and $z = 5/2$, consistent with assuming $v^2 > v_e^2$ in $C_{ee}$.

## 6. RF power and current drive efficiency

To form helicon current drive efficiency requires evaluating the rf power absorbed by the passing electrons. Integrating over $v_{||} < 0$ and ignoring the power into the barely passing leads to

$$P_{cd}^H = \frac{m_e}{2}\langle\int d^3v\, v^2Q\rangle = \frac{m_e}{2}\int d^3v\frac{v_{||}}{B}v^2\langle\frac{B}{v_{||}}Q\rangle \approx \frac{m_eB_0}{4\pi qR}\int d^3v\frac{v_{||}}{B}v^2\tau_p\overline{Q} =$$
$$\frac{m_eB_0}{4\pi qR}\int d^3v\frac{v_{||}}{B}v\frac{\partial}{\partial v}\Big|_\mu(\tau_pvD\frac{\partial f_0}{\partial v}\Big|_\mu) = \frac{m_e^2}{qRT_e}\int_0^{\lambda<1-\epsilon}d\lambda\int_0^\infty dv\,\tau_pv^5Df_0 =$$
$$\frac{\pi^{1/2}}{2}m_en_e\sum_{\vec{k}}^{k_{||}<0}\frac{e^2|\vec{e}_k\cdot\vec{n}\times\vec{k}|^2\omega^4}{m_e^2k_\perp^2\,|k_{||}^5|v_e^5}e^{-\omega^2/k_{||}^2v_e^2}\int_0^{\lambda<1-\epsilon}d\lambda\,\lambda J_1^2(\eta)e^{-X^2(\lambda)}\,. \quad (6.1)$$

Similarly, fixing $v^2$ and the numerical factor in Catto (2021) for LHCD gives

$$P_{cd}^{LH} = \frac{\pi^{1/2}}{2}m_en_e\sum_{\vec{k}}^{k_{||}<0}\frac{e^2|\vec{e}_k\cdot\vec{n}|^2\omega^4}{m_e^2|k_{||}^5|v_e^5}e^{-\omega^2/k_{||}^2v_e^2}\int_0^{1-\epsilon\to1/2}d\lambda\,J_0^2(\eta)e^{-X^2(\lambda)}. \quad (6.2)$$

Again using $e^{-X^2(\lambda)}$ to integrate by parts gives

$$P_{cd}^H = \frac{\pi^{1/2}}{4}m_en_e\sum_{\vec{k}}^{k_{||}<0}\frac{e^2|\vec{e}_k\cdot\vec{n}\times\vec{k}|^2}{m_e^2k_\perp^2\,|k_{||}|v_e}k_\perp^2\rho_e^2e^{-\omega^2/k_{||}^2v_e^2}, \quad (6.3)$$

and

$$P_{cd}^{LH} = \frac{\pi^{1/2}}{2}m_en_e\sum_{\vec{k}}^{k_{||}<0}\frac{e^2|\vec{e}_k\cdot\vec{n}|^2\omega^2}{m_e^2|k_{||}^3|v_e^3}e^{-\omega^2/k_{||}^2v_e^2}. \quad (6.4)$$

The numerical coefficient of $P_{cd}^{LH}$ is half that of (5.7) in Catto (2021).

The current drive efficiency is defined by the ratio $J_{||}/P_{cd}$. Consequently,

$$\frac{J_{||}^H}{P_{cd}^H} = \frac{16e\{1+[0.62-1.02(Z+5)/(7Z+11)]\sqrt{\epsilon}\}\sum_{\vec{k}}^{k_{||}<0}\frac{e^2|\vec{e}_k\cdot\vec{n}\times\vec{k}|^2\omega^2}{\nu_{ee}m_e^2k_\perp^2\,|k_{||}^3|v_e^4}k_\perp^2\rho_e^2e^{-\omega^2/k_{||}^2v_e^2}}{3\pi^{1/2}m_e[(Z+1)(1+1.46\sqrt{2\epsilon})+4]\sum_{\vec{k}}^{k_{||}<0}\frac{e^2|\vec{e}_k\cdot\vec{n}\times\vec{k}|^2}{m_e^2k_\perp^2\,|k_{||}|v_e}k_\perp^2\rho_e^2e^{-\omega^2/k_{||}^2v_e^2}}, \quad (6.5)$$

while

$$\frac{J_{||}^{LH}}{P_{cd}^{LH}} = \frac{16e\{1+[0.62-1.02(Z+5)/(7Z+11)]\sqrt{\epsilon}\}\sum_{\vec{k}}^{k_{||}<0}\frac{e^2|\vec{e}_k\cdot\vec{n}|^2\omega^4}{\nu_{ee}m_e^2|k_{||}^5|v_e^6}e^{-\omega^2/k_{||}^2v_e^2}}{3\pi^{1/2}m_e[(Z+1)(1+1.46\sqrt{2\epsilon})+4]\sum_{\vec{k}}^{k_{||}<0}\frac{e^2|\vec{e}_k\cdot\vec{n}|^2\omega^2}{m_e^2|k_{||}^3|v_e^3}e^{-\omega^2/k_{||}^2v_e^2}}. \quad (6.6)$$

Remarkably, for a single $\omega$ and $\vec{k}$ the preceding forms are identical

$$\frac{J_{||}^H/en_ev_e}{P_{cd}^H/n_em_ev_e^2\nu_{ee}} = \frac{16\{1+[0.62-1.02(Z+5)/(7Z+11)]\sqrt{\epsilon}\}\omega^2}{3\pi^{1/2}[(Z+1)(1+1.46\sqrt{2\epsilon})+4]k_{||}^2v_e^2} = \frac{J_{||}^{LH}/en_ev_e}{P_{cd}^{LH}/n_em_ev_e^2\nu_{ee}}. \quad (6.7)$$

Approximation (6.7) suggests that comparable current drive efficiencies are possible with helicon and lower hybrid waves. Moreover, based on the cold plasma estimate (A.15) of the Appendix it seems possible to drive comparable parallel currents with helicon and lower hybrid waves. The $\epsilon = 0$ form of (6.7) is nearly the same as the non-relativistic, large $z = v_p^2 \gg 1$ limit (31) in Karney & Fisch (1985).

## 7. Combined helicon and lower hybrid

The preceding evaluations of the parallel currents driven and the form of the QL operator suggest it might be possible to combine HCD with LHCD for the same applied wave frequency. Indeed, an antenna used for LHCD could be driving some level of helicon waves and vice versa. Retaining only $\ell = 0$, using the approximation $e^{-i\int_{\tau_0}^{\tau} d\tau' \Lambda(\tau')} \approx 1$, and inserting the delta function, the combined QL diffusivity for the passing electrons becomes

$$D = \frac{\pi e^2}{2m_e^2 v^2 \tau_p} \left| \oint_p d\tau \vec{e}_k \cdot [\vec{n} v_{||} J_0(\eta) - i\vec{n} \times \vec{k} \frac{v_\perp}{k_\perp} J_1(\eta)] \right|^2 \frac{v_{\omega/k_{||}}^2 \delta(v - v_{\omega/k_{||}})}{\omega v \tau_p}, \quad (7.1)$$

with

$$v_{\omega/k_{||}} = \omega v \tau_p / 2\pi(m - qn) > 0 . \quad (7.2)$$

The combined parallel current that can be driven is therefore

$$J_{||}^{H+LH} = \frac{8e^3 n_e \{1 + [0.62 - 1.02(Z+5)/(7Z+11)]\sqrt{\epsilon}\}}{3m_e^2 \nu_{ee}[(Z+1)(1+1.46\sqrt{2\epsilon})+4]} \sum_{\vec{k}}^{k_{||}<0} \frac{\omega^6 e^{-\omega^2/k_{||}^2 v_e^2}}{|k_{||}^7| v_e^8} \{|\vec{e}_k \cdot \vec{n}|^2 +$$

$$i \frac{|k_{||}| v_e \rho_e}{2\omega} [(\vec{e}_k \cdot \vec{n})(\vec{e}_k^* \cdot \vec{n} \times \vec{k}) - (\vec{e}_k^* \cdot \vec{n})(\vec{e}_k \cdot \vec{n} \times \vec{k})] + \frac{k_{||}^2 v_e^2 \rho_e^2}{2\omega^2} |\vec{e}_k \cdot \vec{n} \times \vec{k}|^2 \}, \quad (7.3)$$

where the integrals are performed by integration by parts as before but with one new integral appearing in the cross terms, namely

$$\int_0^{\lambda<1-\epsilon} d\lambda \lambda^{1/2} J_1(\eta) J_0(\eta) \Lambda_{1+2}(\lambda) e^{-X^2(\lambda)} \approx \frac{1}{z} \frac{\partial(\lambda^{1/2} J_1)}{\partial \lambda}\Big|_{\lambda=0} = \frac{k_\perp \rho_e}{2z^{3/2}} \Lambda_{1+2}(\sqrt{\epsilon}, Z, \lambda = 0) . \quad (7.4)$$

The rf power absorbed by passing electrons is

$$P_{cd}^{H+LH} = \frac{\pi^{1/2} e^2 n_e}{2m_e} \sum_{\vec{k}}^{k_{||}<0} \frac{\omega^2}{|k_{||}^3| v_e^3} e^{-\omega^2/k_{||}^2 v_e^2} \{|\vec{e}_k \cdot \vec{n}|^2 +$$

$$i \frac{|k_{||}| v_e \rho_e}{2\omega} [(\vec{e}_k \cdot \vec{n})(\vec{e}_k^* \cdot \vec{n} \times \vec{k}) - (\vec{e}_k^* \cdot \vec{n})(\vec{e}_k \cdot \vec{n} \times \vec{k})] + \frac{k_{||}^2 v_e^2 \rho_e^2}{2\omega^2} |\vec{e}_k \cdot \vec{n} \times \vec{k}|^2 \} . \quad (7.5)$$

As a result, the efficiency is the ratio $J_{||}^{H+LH}/P_{cd}^{H+LH}$, which for a single frequency and wavenumber recovers the same expression as given at the end of section 6. Moreover, the cross term may allow more current to be driven and more power to be absorbed.

## 8. Summary

The new results are the expressions for the parallel current that can be driven in a tokamak by a helicon wave, (5.19), and the associated efficiency of HCD, (6.5) and (6.7). In addition, the numerical coefficients of corresponding tokamak expressions for LHCD (Catto 2021) are corrected, (5.17), (6.4), (6.6), and (6.7), using a more systematic derivation. Interestingly, for a single applied frequency and wave vector the efficiency of HCD and LHCD are shown to be the same, (6.7). In addition, HCD and LHCD can be combined for the same applied frequency as shown in (7.3) and (7.5) of section 7, and, of course, recover the same single wave vector efficiency, (6.7). If a combination of HCD and LHCD is possible from the same antenna and rf source, more current might be driven and steady state operation might be slightly more feasible.

## Appendix: helicon and lower hybrid waves in a cold plasma

To highlight the differences between helicon and lower hybrid waves in the simplest fashion, cold plasma theory can be employed. Helicon current drive (HCD) employs a perpendicular component of the applied electric field, while the parallel component is used for lower hybrid current drive (LHCD).

In a cold plasma, Maxwell's equations lead to the need for the Fourier transform of the applied electric field, $\vec{e}_k$, to satisfy

$$[\overleftrightarrow{\varepsilon} - n^2(\overleftrightarrow{I} - \vec{k}\vec{k})] \cdot \vec{e}_k = 0\,, \tag{A.1}$$

with $\vec{k}$ a wave vector, $k = |\vec{k}|$, $\overleftrightarrow{I}$ the unit dyad, and $n = kc/\omega$ the index of refraction, and the dielectric tensor written as

$$\overleftrightarrow{\varepsilon} = \varepsilon_\perp(\overleftrightarrow{I} - \vec{n}\vec{n}) + \varepsilon_{||}\vec{n}\vec{n} - i\varepsilon_\times \vec{n} \times \overleftrightarrow{I}\,, \tag{A.2}$$

where the magnetic field is $\vec{B} = B\vec{n}$. The components of the dielectric tensor are defined as

$$\varepsilon_\perp = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \Omega_s^2} \approx 1 + \frac{\omega_{pe}^2}{\Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2} = 1 + \frac{\omega_{pe}^2}{\Omega_e^2}\left(1 - \frac{\Omega_i\Omega_e}{\omega^2}\right), \tag{A.3}$$

$$\varepsilon_{||} = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \approx -\frac{\omega_{pe}^2}{\omega^2}, \tag{A.4}$$

$$\varepsilon_\times = -\sum_s \frac{\omega_{ps}^2\Omega_s}{\omega(\omega^2 - \Omega_s^2)} \approx -\frac{\omega_{pe}^2}{\omega\Omega_e} - \frac{\omega_{pi}^2\Omega_i}{\omega^3} \approx -\frac{\omega_{pe}^2}{\omega\Omega_e}, \tag{A.5}$$

with $\omega_{pe}^2\Omega_i = \omega_{pi}^2\Omega_e$, where the approximate forms are valid for $\Omega_e \gg \omega \gg \Omega_i = Z_i eB/m_i c$, which is the frequency range of interest here. The species (s) plasma frequency is defined as $\omega_{ps}^2 = 4\pi Z_s^2 e^2 n_s/m_s$ The orderings give $\varepsilon_\times^2/\varepsilon_{||} \sim \omega_{pe}^2/\Omega_e^2 \sim 1$ and $\varepsilon_{||} \gg \varepsilon_\times \gg \varepsilon_\perp \sim 1$. Moreover, for a resonant Landau interaction $\omega/k_{||}v_e > 1$ suggesting the ordering $n_{||} = k_{||}c/\omega \gtrsim 1$.

Letting

$$\vec{e}_k = e_{||}\vec{n} + \frac{e_\perp}{k_\perp\sqrt{1+|\Upsilon|^2}}(\vec{k}_\perp + \Upsilon\vec{n} \times \vec{k})\,, \tag{A.6}$$

the three components of the cold plasma equation are related by

$$e_{||}/e_\perp = n_{||}n_\perp/[(n_\perp^2 - \varepsilon_{||})\sqrt{1 + |\Upsilon|^2}]\,, \tag{A.7}$$

and

$$\Upsilon = i\varepsilon_\times/(\varepsilon_\perp - n^2)\,, \tag{A.8}$$

and the cold plasma dispersion relation is

$$n_{||}^2\varepsilon_{||}/(\varepsilon_{||} - n_\perp^2) = \varepsilon_\perp - \varepsilon_\times^2/(\varepsilon_\perp - n^2)\,, \tag{A.9}$$

which when rewritten in powers of $n_\perp^2$ yields

$$\varepsilon_\perp n_\perp^4 + [(\varepsilon_{||} + \varepsilon_\perp)(n_{||}^2 - \varepsilon_\perp) + \varepsilon_\times^2]n_\perp^2 + \varepsilon_{||}[(n_{||}^2 - \varepsilon_\perp)^2 - \varepsilon_\times^2] = 0\,. \tag{A.10}$$

The two distinct branches satisfying accessibility ($n_\perp^2 > 0$) are found by assuming $[(\varepsilon_{||} + \varepsilon_\perp)(n_{||}^2 - \varepsilon_\perp) + \varepsilon_\times^2]^2 \gg 4\varepsilon_\perp\varepsilon_{||}[(n_{||}^2 - \varepsilon_\perp)^2 - \varepsilon_\times^2]$ to find

$$n_\perp^2 \approx \begin{cases} -[(\varepsilon_{||} + \varepsilon_\perp)(n_{||}^2 - \varepsilon_\perp) + \varepsilon_\times^2]/\varepsilon_\perp & \text{lower hybrid/slow} \\ -\varepsilon_{||}[(n_{||}^2 - \varepsilon_\perp)^2 - \varepsilon_\times^2]/[(\varepsilon_{||} + \varepsilon_\perp)(n_{||}^2 - \varepsilon_\perp) + \varepsilon_\times^2] & \text{helicon/fast} \end{cases}. \tag{A.11}$$

For the lower hybrid branch $\varepsilon_\times^2 + \varepsilon_\perp n_\perp^2 + \varepsilon_{||}(n_{||}^2 - \varepsilon_\perp) \approx 0$. Keeping $|\Upsilon|^2 \ll 1$, requires $n^2 \gg \varepsilon_\times \gg \varepsilon_\perp$, leading to

$$\omega^2 = \frac{\Omega_i\Omega_e(1 + \omega_{pe}^2/k_\perp^2c^2 + k_{||}^2\Omega_e/k_\perp^2\Omega_i)}{(1 + \Omega_e^2/\omega_{pe}^2)(1 + \omega_{pe}^2/k_\perp^2c^2) + \omega_{pe}^2/k_\perp^2c^2} \approx \frac{\Omega_i\Omega_e(1 + k_{||}^2\Omega_e/k_\perp^2\Omega_i)}{1 + \Omega_e^2/\omega_{pe}^2}, \tag{A.12}$$

where the last form assumes $k_\perp^2c^2 \gg \omega_{pe}^2$ or $n_\perp^2 \gg \varepsilon_{||}$ to further increase the size $e_{||}$ to allow $e_{||}/e_\perp \approx k_{||}/k_\perp$.

The helicon or whistler branch can be approximated by

$$n_\perp^2 = \frac{\varepsilon_\times^2 - (n_{||}^2 - \varepsilon_\perp)^2}{(n_{||}^2 - \varepsilon_\perp) + \varepsilon_\times^2/\varepsilon_{||}} . \tag{A.13}$$

Substituting in the components of the dielectric tensor and rewriting gives

$$\omega^2 = \frac{\Omega_i \Omega_e (1 + k_{||}^2 c^2/\omega_{pi}^2)}{1 + k_\perp^2/k^2 + \Omega_e^2/\omega_{pe}^2 + \omega_{pe}^2/k^2 c^2} \rightarrow \Omega_i \Omega_e \left(\frac{k_{||}^2 c^2}{\omega_{pi}^2}\right)\left(\frac{k^2 c^2}{\omega_{pe}^2}\right) , \tag{A.14}$$

where the first form allows $k_\perp^2 c^2 \sim \omega_{pe}^2$, while the last assumes $\omega_{pe}^2/k^2 c^2 \gg 2 + \Omega_e^2/\omega_{pe}^2$ and $k_{||}^2 c^2/\omega_{pi}^2 \gg 1$ to recover the form in Preinhaelter & Vaclavik (1967) and used in de Assis & Busnardo-Neto (1988). The last form is sometimes referred to as an oblique whistler wave.

For the same value of $\omega^2/k_{||}^2 v_e^2$ the ratio of the parallel currents driven by helicon and lower hybrid waves (or the power absorbed) is roughly

$$\frac{J_{||}^H}{J_{||}^{LH}} \sim \frac{\rho_e^2 |\vec{e}_k \cdot \vec{n} \times \vec{k}|^2}{|\vec{e}_k \cdot \vec{n}|^2} \sim \frac{k_\perp^2 \rho_e^2 |Y|\, e_\perp^2}{(1 + |Y|^2) e_{||}^2} \sim k_\perp^2 \rho_e^2 \frac{\varepsilon_\times (n_\perp^2 - \varepsilon_{||})^2}{n_\perp^2 n_{||}^2 (n^2 - \varepsilon_\perp)} \sim k_\perp^2 \rho_e^2 \frac{\omega \omega_{pe}^2 (\omega_{pe}^2 + k_\perp^2 c^2)^2}{\Omega_e k_{||}^2\, c^2 k_\perp^2 c^2 k^2 c^2} , \tag{A.15}$$

which can be order unity if $\omega_{pe}^2 \gtrsim k_\perp^2 c^2$.


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