Additional documentation for "Exotic ribbon disks and symplectic surfaces"

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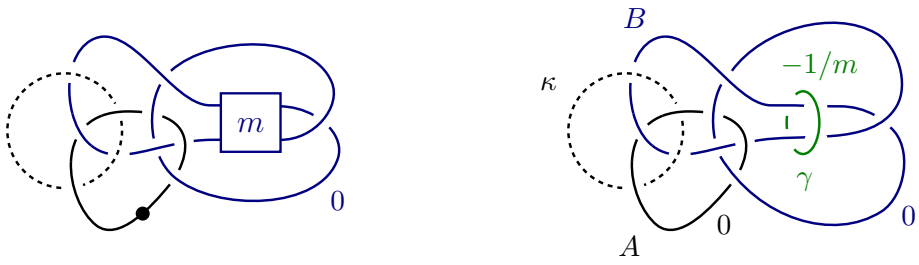
This is a guide to the computer-assisted proofs in [4]. All files referenced below are located on the site [3]. The calculations described rely on the software SnapPy [1] and the knot Floer homology calculator [5]. The SnapPy calculations can be verified by running SnapPy inside Sage [6]; see the official SnapPy documentation regarding verified computations.

Notation. For a link $\mathcal{L}$ with ordered components $L_1, \dots, L_n$ and a choice of Dehn filling slopes $r_1, \dots, r_n \in \mathbb{Q} \cup \infty$, let $\mathcal{L}(r_1, \dots, r_n)$ denote the 3-manifold obtained by Dehn filling $S^3 \setminus \mathcal{L}$ along the component L_i with slope r_i. If L_i is not filled, we write $r_i = *$.

A. Proof of Theorem 1.2

SnapPy

We follow the notation from [4] wherever possible. The first claim to verify is that $S^3 \setminus K$ admits a hyperbolic structure with trivial isometry group for $m = 0$ and $m \ll 0$, where K is the knot in ∂B^4 corresponding to the dashed curve in Figure 1(a) below. In Figure 1(b), we have recast K using another surgery description of S^3. To describe this precisely, let $\mathcal{L}$ denote the link with (ordered) components A, B, γ, and κ depicted in Figure 1(b). Then $S^3 \setminus K$ is diffeomorphic to $\mathcal{L}(0, 0, -1/m, *)$, i.e. the 3-manifold obtained from Dehn filling $S^3 \setminus \mathcal{L}$ along the link components A, B, and γ with framings zero, zero, and $-1/m$, respectively.



(a) (b)

Figure 1

We claim that $\mathcal{L}(0,0,*,*)$ admits a hyperbolic structure with trivial isometry group. The diagram for $\mathcal{L}$ is saved as the SnapPy projection file `L2.lnk`. Using SnapPy's link editor, we extracted a Dowker-Thistlethwaite code for this link:

$$\text{DT: [(6,8),(14,22,16,28,18,24,26,-30),(4,2,10),(12,-20)]} = [\kappa, A, B, \gamma].$$

To verify that $\mathcal{L}(0,0,*,*)$ admits a hyperbolic structure with trivial isometry group using SnapPy, we ran the commands in the file `L2-input.py`. (These commands may be entered manually or by opening the file `L2-input.py` in SnapPy.) The activity log is reproduced below; see the file `L2-output.py` for a copy saved by SnapPy.

```
In:  L=Manifold('DT:[(6,8),(14,22,16,28,18,24,26,-30),(4,2,10),(12,-20)]')
In:  L.dehn_fill((0,1),1)
In:  L.solution_type()
Out:  'all tetrahedra positively oriented'
In:  L.dehn_fill((0,1),2)
In:  Y=L.filled_triangulation()
In:  Y.solution_type()
Out:  'all tetrahedra positively oriented'                    (A1)
In:  Y.symmetry_group()
Out:  0                                                       (A2)
In:  L.dehn_fill((1,0),3)
In:  Y0=L.filled_triangulation()
In:  Y0.solution_type()
Out:  'all tetrahedra positively oriented'                    (A3)
In:  Y0.symmetry_group()
Out:  0                                                       (A4)
```

In the log above, `Y` corresponds to $\mathcal{L}(0,0,*,*)$, and lines A1 and A2 imply that $\mathcal{L}(0,0,*,*)$ admits a hyperbolic structure with trivial isometry group. By [2, Lemma 2.2], the same is true of $\mathcal{L}(0,0,-1/m,*)$ for $|m| \gg 0$, including $m \ll 0$, as claimed.

Similarly, `Y0` corresponds to $\mathcal{L}(0,0,\infty,*)$, which admits a hyperbolic structure with trivial isometry group by lines A3 and A4. The slope ∞ corresponds to $-1/m$ for $m = 0$. This implies that $S^3 \setminus K$ admits a hyperbolic structure with trivial isometry group for $m = 0$, as claimed.

HFK Calculator

The next claim to verify is that the knot L_0 shown in [4, Figure 4(f)] has $\tau(L_0) = 2$. A diagram of L_0 is saved as the SnapPy projection file `L0knot.lnk`. From these diagrams, we used SnapPy's link editor to extract Planar Diagram codes for L_0. This PD code is saved as `L0knot.txt`, formatted for use with [5]. We then used this file to complete the desired calculation, following the documentation for [5]; the resulting activity log is copied into the summary file `L0knot-summary.txt`. In particular, the output tells us $\tau(L_0) = 2$.

B. Proof of Theorem 1.1

As above, we follow the notation in [4]. The claim to verify is that $S^3 \setminus (K_0 \cup \gamma)$ admits a hyperbolic structure with trivial isometry group for $m = 0$ and $m \ll 0$. The diagram for the link $K_0 \cup \gamma$ is saved as the SnapPy projection file `L1.lnk`. Using SnapPy's link editor, we extracted a Dowker-Thistlethwaite code for this link:

$$\begin{aligned}\text{DT: } [(&-88,-34,-78,-42,66,-24,-76,4,92,-30,54,-70,38,\\ &-8,80,-90,46,-14,-58,-68,26,-56,-16,94,62,-72,\\ &-82,-28,6,12,-86,52,22,-40,10,84,-50,-60,36,44,\\ &-20,-64,74),(-2,48,18,-32)]\\ &= [K_0, \gamma]\end{aligned}$$

To verify that $S^3 \setminus (K_0 \cup \gamma)$ admits a hyperbolic structure with trivial isometry group using SnapPy, we ran the commands in the file `L1-input.py`. (These commands may be entered manually or by opening the file `L1-input.py` in SnapPy.) The activity log is reproduced below; see the file `L1-output.py` for a copy saved by SnapPy.

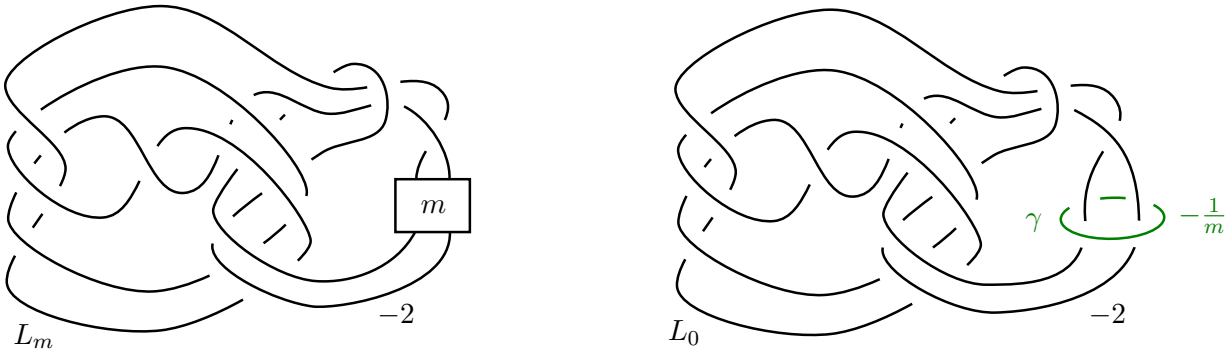
```
In:  L=Manifold('DT:[(-88,-34,-78,-42,66,-24,-76,4,92,-30,54,-70,38,-8,80,
-90,46,-14,-58,-68,26,-56,-16,94,62,-72,-82,-28,6,12,-86,52,22,-40,10,84,
-50,-60,36,44,-20,-64,74),(-2,48,18,-32)]')
In:  L.solution_type()
Out:  'all tetrahedra positively oriented'
In:  L.symmetry_group()
Out:  0
```

C. Proof of Theorem 1.1

We continue to follow the notation in [4]. The claim to verify is that $S^3_{-2}(L_m)$ admits a hyperbolic structure with trivial isometry group for $m \ll 0$, where L_m is the Legendrian link shown in Figure 2(a) below (and [4, Figure 16(a)]). Letting $\mathcal{L} = L_0 \cup \gamma$ be the link shown in Figure 2(b), we have

$$S^3_{-2}(L) = S^3_{-2,-1/m}(L_0 \cup \gamma) = \mathcal{L}(-2, -1/m).$$

We claim that $\mathcal{L}(-2, *)$ admits a hyperbolic structure with trivial isometry group. The diagram for the link $\mathcal{L} = L_0 \cup \gamma$ is saved as the SnapPy projection file `L3.lnk`. In our next calculation, we will be performing Dehn filling with nonzero slopes, so it is important that the link is entered into SnapPy with the correct chirality; that is, we must take care to work with $\mathcal{L}$ and not its mirror. For this reason, we encode the diagram of $\mathcal{L}$ using a



(a) (b)

Figure 2

Planar Diagram code instead of a Dowker-Thistlethwaite code, extracted from `L3.lnk` using SnapPy's link editor:

PD: $[(47,39,48,38),(36,50,37,49),(32,44,33,43),(46,30,47,29),(27,49,28,48),$
$(28,37,29,38),(39,26,40,27),(25,40,26,41),(24,36,25,35),(45,18,46,19),$
$(30,18,31,17),(15,42,16,43),(14,34,15,33),(22,13,23,14),(10,23,11,24),$
$(8,42,9,41),(9,34,10,35),(44,6,45,5),(31,6,32,7),(16,8,17,7),(2,22,3,21),$
$(1,13,2,12),(11,1,12,50),(51,20,52,21),(54,4,51,3),(19,52,20,53),(4,54,5,53)]$

When fed to SnapPy, this encodes $\mathcal{L}$ as an (ordered) two-component link whose first component is L_0 and whose second component is γ. To verify that $\mathcal{L}(-2,*)$ admits a hyperbolic structure with trivial isometry group using SnapPy, we ran the commands in the file `L3-input.py`. (These commands may be entered manually or by opening the file `L3-input.py` in SnapPy.) The activity log is reproduced below; see the file `L3-output.py` for a copy saved by SnapPy.

```
In:  TheLink=Link([[47,39,48,38],[36,50,37,49],[32,44,33,43],[46,30,47,29],
[27,49,28,48],[28,37,29,38],[39,26,40,27],[25,40,26,41],[24,36,25,35],
[45,18,46,19],[30,18,31,17],[15,42,16,43],[14,34,15,33],[22,13,23,14],
[10,23,11,24],[8,42,9,41],[9,34,10,35],[44,6,45,5],[31,6,32,7],
[16,8,17,7],[2,22,3,21],[1,13,2,12],[11,1,12,50],[51,20,52,21],
[54,4,51,3],[19,52,20,53],[4,54,5,53]])
In:  L=TheLink.exterior()
In:  L.dehn_fill((-2,1),0)
In:  L.solution_type()
Out:  'all tetrahedra positively oriented'
In:  L.symmetry_group()
Out:  0
```

This implies that $\mathcal{L}(-2,*)$ admits a hyperbolic structure with trivial isometry group. By [2,

Lemma 2.2], the same is true of $\mathcal{L}(-2,-1/m)$ for $|m| \gg 0$, including $m \ll 0$, as claimed.

References

[1] **M Culler**, **N M Dunfield**, **M Goerner**, **J R Weeks**, *SnapPy, a computer program for studying the geometry and topology of 3-manifolds*, Available at http://snappy.computop.org

[2] **N M Dunfield**, **N R Hoffman**, **J E Licata**, *Asymmetric hyperbolic L-spaces, Heegaard genus, and Dehn filling*, Math. Res. Lett. 22 (2015) 1679–1698

[3] **K Hayden**, Data and code to accompany 'Exotic ribbon disks and symplectic surfaces', https://doi.org/10.7910/DVN/JWE5AX (March 2020)

[4] **K Hayden**, *Exotic ribbon disks and symplectic surfaces*, Preprint (March 2020)

[5] **Z Szabó**, *Knot Floer homology calculator*, Available at https://web.math.princeton.edu/~szabo/HFKcalc.html (2019)

[6] **The Sage Developers**, *SageMath, the Sage mathematics software system*, Available at https://www.sagemath.org (2019)

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