Appendix of the paper "Loyalty discounts and price-cost tests," by Giacomo Calzolari and Vincenzo Denicolò

```
(*December 2019*)
```

Setup of the model

In[121]:=
```
(*Buyer's utility*)
v = (q1 + q2) - 1 / 2 (q1^2 + q2^2) - γ q1 q2;
ve1 = v /. q2 → 0;
ve2 = v /. q1 → 0;
```

In[124]:=
```
(*Firms' costs and
 profits: first we will study the case where the only asymmetry is on capacity
   (i.e. c=0) and then we will introduce back also cost asymmetry (c>0).*)
c1 = 0;
c2 = c;

P1 = (p1 - c1) q1;
P2 = (p2 - c2 ) q2;
```

Equilibrium with exclusivity

In[131]:=
```
(*Reserve utility  is obtaind by maximizing the
 buyer' surplus under the constraint that firm 2 does
 not make negative profits, given capacity constraint q2≤
 k. This entails maximization of bilateral surplus ve2-c2 q2. Since the
   constraint is always binding (by assumption), then the surplus is:*)
```

In[132]:=
```
Ve2 = FullSimplify[ve2 - c2 q2 /. q2 → k]
```

Out[132]=
$$-\frac{1}{2} k \left(-2 + 2 c + k\right)$$

In[133]:=
```
(*This solution could be implemented for example,
by setting p2=D[ve2,q2]/.q2→k and then F2=-(p2-c2)k.*)
```

In[134]:=
```
(*Utility in exclusivity with firm 1*)
qe1 = FullSimplify[Solve[D[ve1 - p1 q1, q1] == 0, q1]][[1, 1, 2]]
Ve1 = FullSimplify[ve1 - p1 q1 /. q1 → qe1]
```

Out[134]= $1 - p1$

Out[135]= $\frac{1}{2} (-1 + p1)^2$

In[136]:=
```
(*Monopoly price in exclusivity firm 1*)
P1 /. q1 → qe1;
p1mon = FullSimplify[Solve[D[%, p1] == 0, p1]][[1, 1, 2]]
```

Out[137]= $\frac{1}{2}$

In[138]:=
```
(*This solution applies as long as the buyer's utility
 is greater than ve2 so the implied fixed fees is strictly
 positive. If instead the participation constraint is binding,
then since it is never optimal to set a negative fixed fee,
the marginal price must be set so as to guarantee participation.*)
```

In[139]:=
```
(*The optimal exclusive price when firm 1 is thus constrained and is: *)
p1econstr = FullSimplify[Solve[Ve2 == Ve1, p1]][[1, 1, 2]]
FullSimplify[% /. c → 0]
```

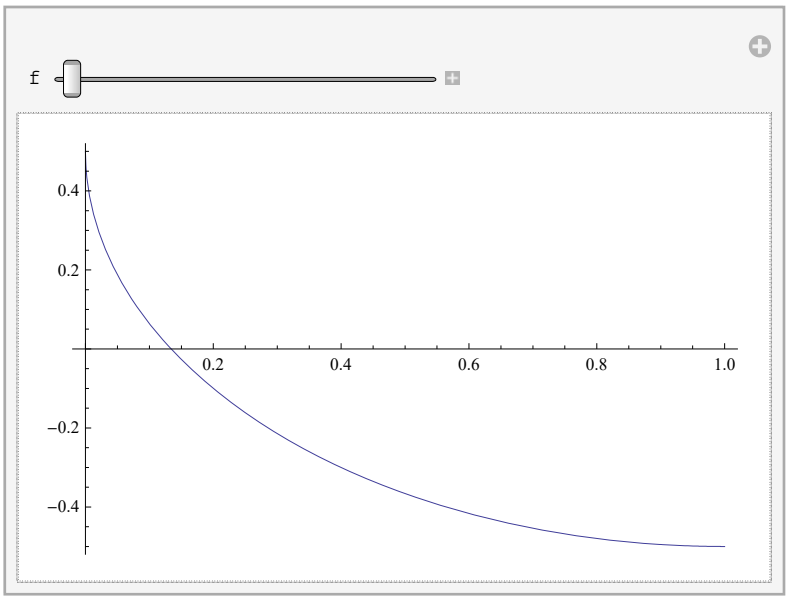
Out[139]= $1 - \sqrt{-k (-2 + 2 c + k)}$

Out[140]= $1 - \sqrt{-(-2 + k) k}$

In[141]:=
```
(*In this constrained case is exactly F1=0.*)
```

In[142]:=
```
(*Where the constrained exclusive
 price prevails: for k larger than the following
   threshold the exclusive price is the constrained one.*)
p1mon;
p1econstr;
Manipulate[Plot[% - %%, {k, 0, 1}], {c, 0, 1}]
```

Out[144]=



In[145]:=
```
FullSimplify[Solve[p1econstr == p1mon, k]]
(*The relevant root is the small one:*)
kpe1constr = %[[1, 1, 2]];
```

Out[145]= $\left\{\left\{k \to 1 - c - \frac{1}{2}\sqrt{3 + 4\ (-2 + c)\ c}\right\}, \left\{k \to \frac{1}{2}\left(2 - 2\,c + \sqrt{3 + 4\ (-2 + c)\ c}\right)\right\}\right\}$

In[147]:=
```
(*Summarizing, in exclusivity the buyer's payoff gross of the fixed fees is,
respectively when k<(>)kpe1constr:*)
Ve1mon = Ve1 /. p1 → p1mon
Ve1constr = Ve1 /. p1 → p1econstr
```

Out[147]= $\frac{1}{8}$

Out[148]= $-\frac{1}{2}\,k\ (-2 + 2\,c + k)$

In[234]:=
```
(*Minimal c for having firm 1 more efficient
 than firm 2 for any combination of c and K:*)
FullSimplify[v /. q1 -> q1mon /. q2 → 0 /. p1 → 0]
FullSimplify[v - p2 q2 /. q2 → k /. q1 → 0 /. p2 → c]
cmin = FullSimplify[Solve[%% == %, c]][[1, 1, 2]]
```

Out[234]= $\frac{1}{2}$

Out[235]= $-\frac{1}{2} k \left(-2 + 2 c + k\right)$

Out[236]= $-\frac{(-1 + k)^2}{2 k}$

Non-Exclusive equilibrium and figures 1 and 2

Regime with k large

In[152]:=
```
(*We first determine the equilibrium for k suffiently large,
that firms prices at "Bertrand" prices.*)
```

In[153]:=
```
qNE = FullSimplify[
    Solve[{D[v - p1 q1 - p2 q2, q1] == 0, D[v - p1 q1 - p2 q2, q2] == 0}, {q1, q2}]][[1]]
```

Out[153]= $\left\{q1 \to \frac{-1 + p1 + \gamma - p2 \gamma}{-1 + \gamma^2}, q2 \to \frac{-1 + p2 + \gamma - p1 \gamma}{-1 + \gamma^2}\right\}$

In[154]:=
```
(*"Bertrand" equilibrium*)
P2 /. qNE;
P1 /. qNE;
pNE = FullSimplify[Solve[{D[%, p1] == 0, D[%%, p2] == 0}, {p1, p2}]][[1]]
VCRNE = FullSimplify[v - q1 p1 - q2 p2 /. qNE /. pNE]
```

Out[156]= $\left\{p1 \to \frac{-2 + \gamma - c \gamma + \gamma^2}{-4 + \gamma^2}, p2 \to \frac{-2 - 2 c + \gamma + \gamma^2}{-4 + \gamma^2}\right\}$

Out[157]= $\frac{-4 \left(2 + \left(-2 + c\right) c\right) + 3 \left(2 + \left(-2 + c\right) c\right) \gamma^2 - 2 \left(-1 + c\right) \gamma^3}{2 \left(-4 + \gamma^2\right)^2 \left(-1 + \gamma^2\right)}$

In[158]:=
```
(*Firm 2 is constrained when k is smaller than the following:*)
FullSimplify[qNE[[2, 2]] /. pNE /. c → 0]
```

Out[158]= $\frac{1}{2 + \gamma - \gamma^2}$

In[159]:=
```
(*For future reference: efficient quantitites*)
qeff = FullSimplify[Solve[{D[v - (c) q2, q1] == 0, D[v - (c) q2, q2] == 0}, {q1, q2}]]
```

Out[159]= $\left\{\left\{q1 \to \frac{-1+\gamma-c\,\gamma}{-1+\gamma^2},\ q2 \to \frac{-1+c+\gamma}{-1+\gamma^2}\right\}\right\}$

In[160]:=
```
(*For completeness we also caracterize the equilibrium with
 limit pricing: for c sufficiently large, firm 2 cannot produce.*)
FullSimplify[qNE[[2, 2]] /. pNE];
clim = FullSimplify[Solve[% == 0, c]][[1, 1, 2]]

(*And the equilibrium limit price of firm 1 is:*)
D[v, q2] /. q2 → 0;
FullSimplify[Solve[% == c, q1]];
v - p1 q1 /. q2 → 0;
q1mon = FullSimplify[Solve[D[%, q1] == 0, q1]][[1, 1, 2]]
p1lim = FullSimplify[Solve[q1mon == %%%[[1, 1, 2]], p1]][[1, 1, 2]]
Limit[%, γ → 1]
```

Out[161]= $1+\frac{\gamma}{-2+\gamma^2}$

Out[165]= $1-p1$

Out[166]= $\frac{-1+c+\gamma}{\gamma}$

Out[167]= c

In[168]:=
```
(*Summarizing: for c<clim, the equilibrium is with "Bertrand" prices,
for larger c, it is with limit price.*)
```

Regime with k small and figure 1

In[169]:=
```
(*As explained in the text, when the additional marginal cost for firm 2,
θ, is sufficiently large, p2 guarantees that the demand for product
 2 is exactly equal to k. This implies the following best reponse:*)

BR2k = FullSimplify[Solve[qNE[[2, 2]] == k, p2]][[1, 1, 2]]
```

Out[169]= $1+(-1+p1)\,\gamma+k\left(-1+\gamma^2\right)$

```
(*Now, the BR of firm 1:*)
```

In[170]:=
```
P1 /. qNE;
BR1 = FullSimplify[Solve[D[%, p1] == 0, p2]][[1, 1, 2]]
```

Out[171]= $\frac{-1+2\,p1+\gamma}{\gamma}$

In[172]:=
```
(*Equilibrium in this regime:*)
p1eqrat = FullSimplify[Solve[{BR1 == BR2k}, {p1}]][[1, 1, 2]]
p2eqrat = FullSimplify[BR2k /. p1 → %]
```

Out[172]= $-\dfrac{(-1+k\,\gamma)\,(-1+\gamma^2)}{-2+\gamma^2}$

Out[173]= $1-2\,k+\dfrac{-2\,k+\gamma}{-2+\gamma^2}$

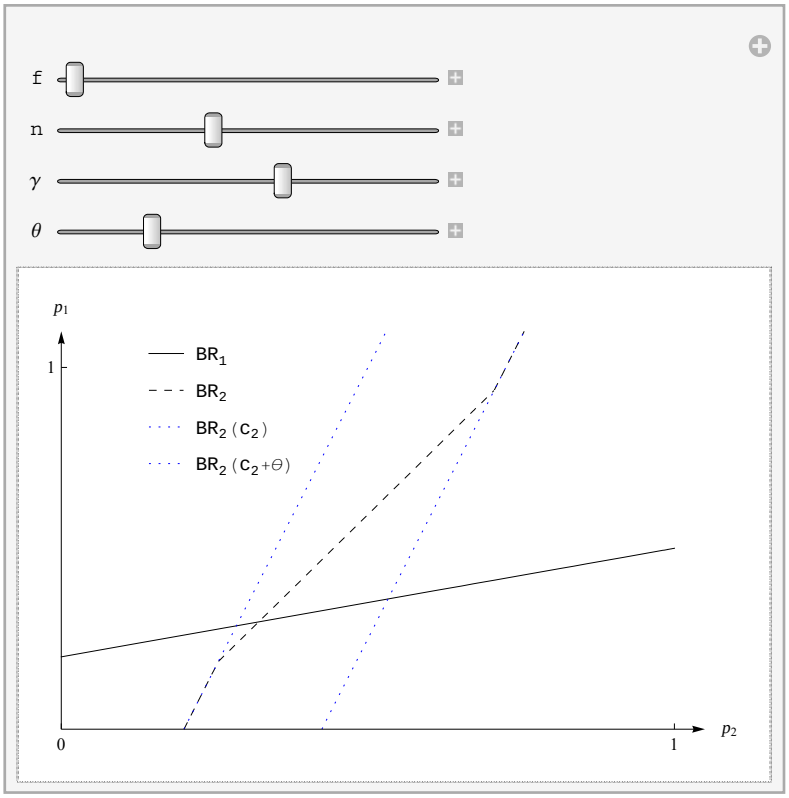
In[177]:=
```
(*Now we derive the BR of firm 2 when facing a cost c+
 θ (not considered in the equilibrium, as we focues on sufficiently large θ)*)
P2 - θ q2 /. qNE;
BR2θ = FullSimplify[Solve[D[%, p2] == 0, p2]][[1, 1, 2]]
(*For completeness also what would be equilibrium prices*)
pNEθ = FullSimplify[Solve[{BR1 == p2, BR2θ == p2}, {p1, p2}]][[1]]
```

Out[178]= $\dfrac{1}{2}\,(1+c+(-1+p1)\,\gamma+\Theta)$

Out[179]= $\left\{p1\to\dfrac{-2+\gamma^2-\gamma\,(-1+c+\Theta)}{-4+\gamma^2},\ p2\to\dfrac{-2\,c+\gamma+\gamma^2-2\,(1+\Theta)}{-4+\gamma^2}\right\}$

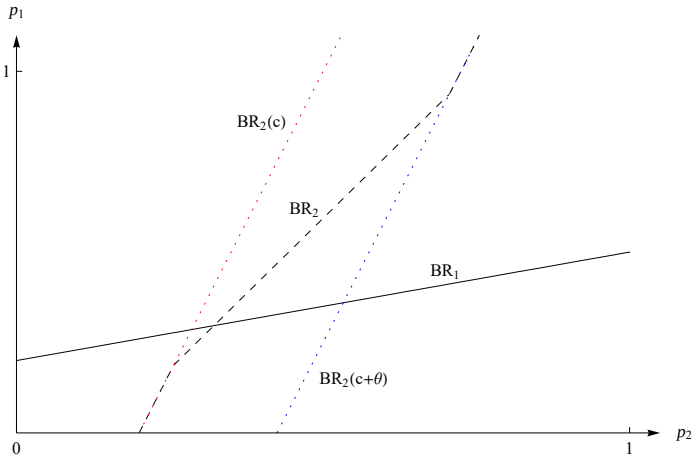
In[180]:=
```
(*Now we derive Figure 1:*)
(*Inverting the BR in order to plot on p2*)
BR1inv = FullSimplify[Solve[BR1 == p2, p1]][[1, 1, 2]];
BR2kinv = FullSimplify[Solve[BR2k == p2, p1]][[1, 1, 2]];
BR2Bertinv = FullSimplify[Solve[BR2θ == p2, p1] /. θ → 0][[1, 1, 2]];
BR2θinv = FullSimplify[Solve[BR2θ == p2, p1]][[1, 1, 2]];
p2low = FullSimplify[Solve[BR2kinv == BR2Bertinv, p2]][[1, 1, 2]];
p2high = FullSimplify[Solve[BR2kinv == BR2θinv, p2]][[1, 1, 2]];
BR2all = Piecewise[{{BR2Bertinv, p2 < p2low},
     {BR2kinv, p2high > p2 ≥ p2low}, {BR2θinv, p2high < p2}}];


{BR1inv, BR2all, BR2Bertinv, BR2θinv};
Manipulate[Plot[%, {p2, 0, 1},
  PlotStyle → {{Black}, {Black, Dashed}, {Blue, Dotted}, {Blue, Dotted}},
  AxesLabel → {"p2", "p1"},
  PlotLegends → Placed[LineLegend[{"BR1", "BR2", "BR2(c2)", "BR2(c2+θ)"},
     LabelStyle → {FontFamily → "Courier"}], {0.25`, 0.8`}],
  Ticks → {{0, 1}, {-1, 1}}, PlotRange → {{0, 1.05`}, {0, 1.1`}},
  AxesStyle → Arrowheads[{0.`, 0.02`}]], {c, 0, 1},
 {{k, 0.4}, 0, 1}, {{γ, 0.6}, 0, 1}, {{θ, 0.45}, 0, 2}]
```

Out[188]=



In[196]:=
```
(*Now we derive Figure 1:*)
parameters = {c → 0, k → 0.4, γ → 0.6, θ → 0.45};
{BR1inv, BR2all, BR2Bertinv, BR2θinv} /. parameters;
Figure1 = Plot[%, {p2, 0, 1},
  PlotStyle → {{Black}, {Black, Dashed}, {Red, Dotted}, {Blue, Dotted}},
  AxesLabel → {"p2", "p1"},
  Epilog → {Text["1", {1.01, -0.04}], Text["0", {-0.01, -0.04}],
    Text["BR2(c)", {0.4, 0.86}],
    Text["BR2(c+θ)", {0.55, 0.15}],
    Text["BR1", {0.7, 0.45}],
    Text["BR2", {0.47, 0.62}]
   }, Ticks → {{0, 1}, {-1, 1}},
  PlotRange → {{0, 1.05`}, {0, 1.1`}}, AxesStyle → Arrowheads[{0.`, 0.02`}]]


SetDirectory[NotebookDirectory[]];
SetDirectory[ParentDirectory[]];
SetDirectory["Figures"];
Export["Figure1.pdf", Figure1];
```

Out[198]=




```
(*Note: setting θ>1 guarantees that the q2≤k.*)
```

Effects of exclusivity on Welfare, Profitability and Figure 2

In[203]:=
```
(*Now, in the space (c,k), we identify the two regions, k small and k large,
when c<clim. This is simply given by the condition q2NE==k*)
cq2k = FullSimplify[Solve[k == qNE[[2, 2]] /. pNE, c]][[1, 1, 2]]
Collect[Solve[% == c, k], c, FullSimplify]
```

Out[203]= $1 - 3\,k + k\,\gamma^2 + \frac{-2\,k + \gamma}{-2 + \gamma^2}$

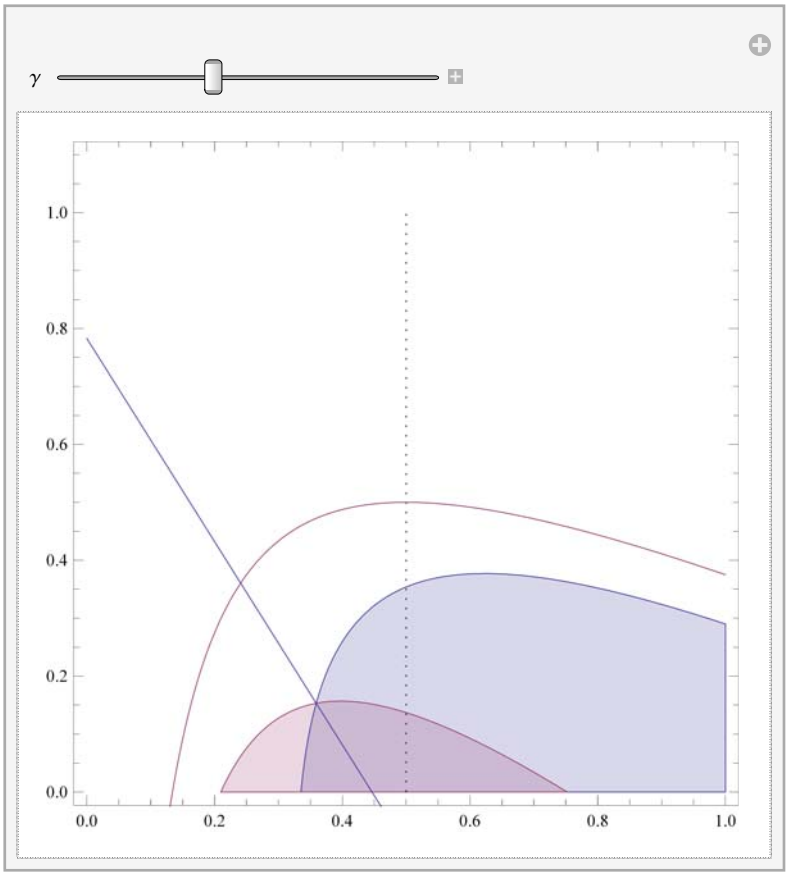
Out[204]= $\left\{\left\{k \to \frac{1}{2 + \gamma - \gamma^2} + \frac{c\,\left(-2 + \gamma^2\right)}{4 - 5\,\gamma^2 + \gamma^4}\right\}\right\}$

In[205]:=
```
(*We also keep track of the two Exclusivity prices:*)
cpe1constr = FullSimplify[Solve[kpe1constr == k, c]][[1, 1, 2]]
```

Out[205]= $1 - \frac{1}{8\,k} - \frac{k}{2}$

In[206]:=
```
(*A first check on welfare:*)
cpe1constr;
cq2k;
vertline = {{0.5, 0}, {0.5, 1}};
FullSimplify[v - p2 q2 - p1 q1 /. qNE /. pNE];
FullSimplify[v - p2 q2 - p1 q1 /. qNE /. {p1 -> p1eqrat, p2 -> p2eqrat}];
FullSimplify[ve1 - 1 / 4 /. q1 → 1 / 2];
FullSimplify[ve1 - p1 q1 /. q1 → 1 - p1 /. p1 -> p1econstr];
Manipulate[Show[RegionPlot[{%%%% - % < 0, %%% - % < 0}, {k, 0, 1}, {c, 0, 1.1}],
  Graphics[{Dotted, Line[%%%%%]}],
  Plot[{%%%%%%, %%%%%%%}, {k, 0, 1}]

 ], {{γ, 0.4}, 0, 1}]
```

Out[213]=



In[214]:=
```
(*Defining the relevan boundaries for welfare*)
FullSimplify[v - p2 q2 - p1 q1 /. qNE /. pNE];
FullSimplify[v - p2 q2 - p1 q1 /. qNE /. {p1 -> p1eqrat, p2 -> p2eqrat}];
FullSimplify[ve1 - 1 / 4 /. q1 → 1 / 2];
FullSimplify[ve1 - p1 q1 /. q1 → 1 - p1 /. p1 -> p1econstr];
cWelfBoundA = FullSimplify[Solve[%%%% - % == 0, c], -4 + γ^2 > 0][[2, 1, 2]]
cWelfBoundB = FullSimplify[Solve[%%%% - %% == 0, c], -4 + γ^2 > 0][[1, 1, 2]]
```

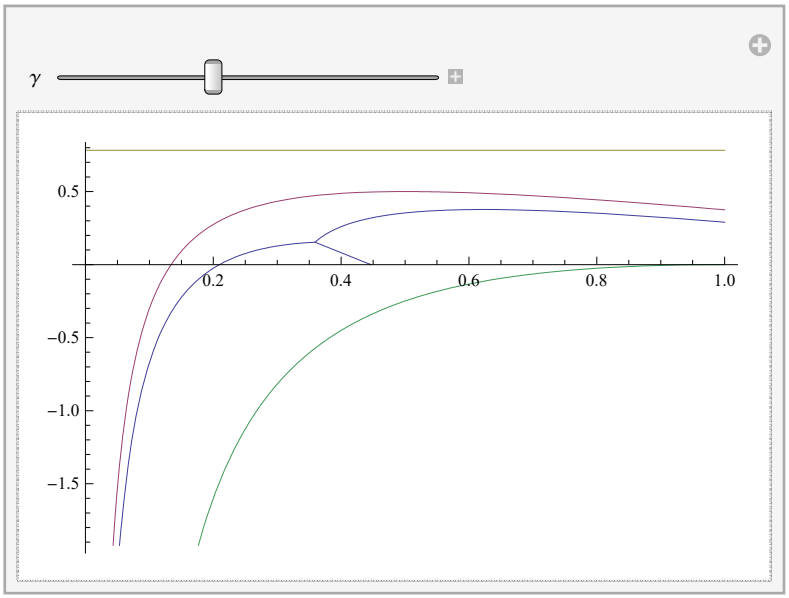
Out[218]=
$$\frac{1}{-4+3\gamma^2}\left(-4+16k+3\gamma^2-24k\gamma^2+\gamma^3+9k\gamma^4-k\gamma^6+\left(-4+\gamma^2\right)\sqrt{\left(-1+\gamma^2\right)\left(1-2k\gamma^3+k^2\left(-12+21\gamma^2-9\gamma^4+\gamma^6\right)\right)}\right)$$

Out[219]=
$$\frac{1}{2}\left(2+\frac{-1+2k\gamma\left(-1+\gamma^2\right)+k^2\left(-8+11\gamma^2-4\gamma^4\right)}{k\left(-2+\gamma^2\right)^2}\right)$$

In[237]:=
```
(*Plotting the boundaries for welfare*)
cmin
clim;
cpe1constr;
FullSimplify[Solve[cWelfBoundB == cq2k, k], -2 + γ^2 > 0][[2, 1, 2]];
FullSimplify[Solve[cq2k == 0, k], -2 + γ^2 > 0][[1, 1, 2]];
Piecewise[{{cWelfBoundB, cWelfBoundB < cq2k}, {cWelfBoundA, cWelfBoundB > cq2k}}];

Manipulate[Show[Plot[{%, %%%%, %%%%%, %%%%%%}, {k, 0, 1}],
  Plot[1 - 3 k + k γ^2 + (-2 k + γ)/(-2 + γ^2), {k, %%%, %%}]
 ], {{γ, 0.4}, 0, 1}]
```

Out[237]=
$$-\frac{(-1+k)^2}{2k}$$

Out[243]=



In[244]:=
```
(*This is the welfare
 effect: above the blue line welfare is reduced by exclusivity.

    Notice that when the change in welfare takes
   place we are always in the region where for exclusivity we
   have the constrained case, that is below the Red line*)

(*Now the analysis of profitability*)
```

```
FullSimplify[P1 /. qNE /. {p1 -> p1eqrat, p2 -> p2eqrat}]
FullSimplify[P1 /. qNE /. pNE]
FullSimplify[P1 /. q1 → q1mon /. p1 → p1econstr]
```

$$-\frac{(-1+k\gamma)^2\,(-1+\gamma^2)}{(-2+\gamma^2)^2}$$

$$-\frac{(-2+\gamma-c\gamma+\gamma^2)^2}{(-4+\gamma^2)^2\,(-1+\gamma^2)}$$

$$k\,(-2+2c+k)+\sqrt{-k\,(-2+2c+k)}$$

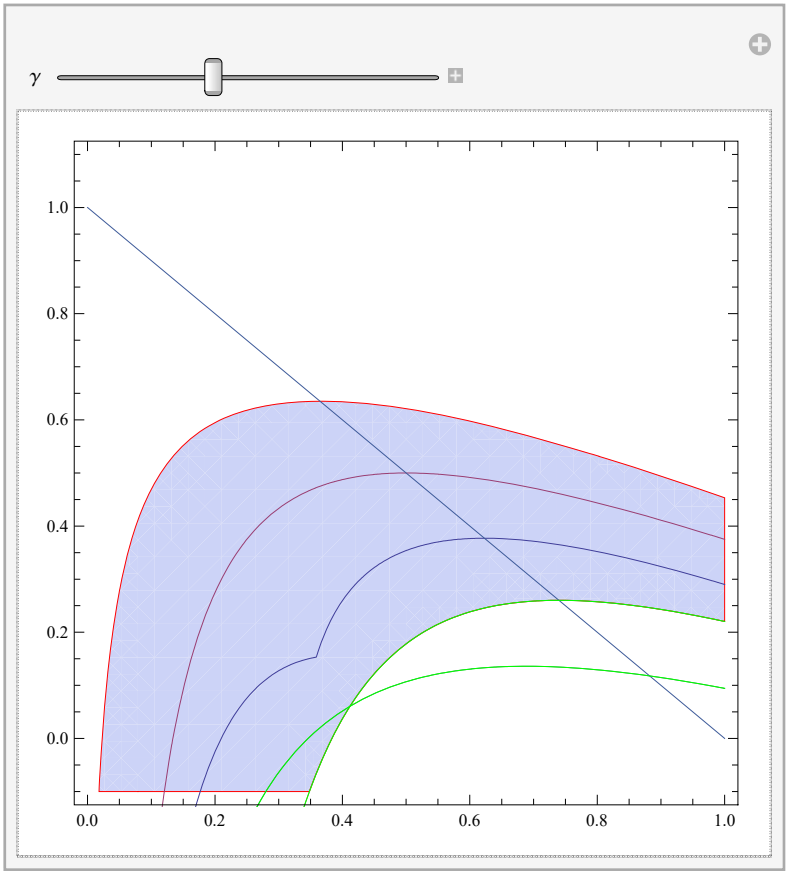
```
(*Since Mathematica does not solve directly the equality between profit
 with exclusivity the constrained case and with CR Bertrand case,
we proceed solving with manual manipulation
  (then we check the solution is the correct one):*)
```

In[249]:=
```
tmpc = Solve[-k (-2 + 2 c + k) == (-(-2 + γ - c γ + γ^2)^2/((-4 + γ^2)^2 (-1 + γ^2)) - k (-2 + 2 c + k))^2, c];
cprofitA = tmpc[[1, 1, 2]];
```

In[251]:=
```
(*Defining the second boundary of profitability*)
FullSimplify[P1 /. qNE /. {p1 -> p1eqrat, p2 -> p2eqrat}];
FullSimplify[P1 /. qNE /. pNE];
FullSimplify[P1 /. q1 → q1mon /. p1 → p1econstr];
tmpc2 = FullSimplify[Solve[% - %%% == 0, c]];

(*PROFITABILITY*)
```

In[255]:=
```
clim;
{cprofitA, tmpc2[[1, 1, 2]]};
FullSimplify[P1 /. qNE /. pNE];
FullSimplify[P1 /. q1 → q1mon /. p1 → p1mon];
FullSimplify[P1 /. q1 → q1mon /. p1 → p1econstr];

cpe1constr;
FullSimplify[Solve[cWelfBoundB == cq2k, k], -2 + γ^2 > 0][[2, 1, 2]];
FullSimplify[Solve[cq2k == 0, k], -2 + γ^2 > 0][[1, 1, 2]];

cWelfare =
  Piecewise[{{cWelfBoundB, cWelfBoundB < cq2k}, {cWelfBoundA, cWelfBoundB > cq2k}}];

Manipulate[Show[
  RegionPlot[{%%%%% > %%%%%%%}, {k, 0, 1}, {c, -0.1, 1.1}, BoundaryStyle → {Red}],
  Plot[{%, %%%%, %%%%%%%%, 1 - k}, {k, 0, 1}],
  Plot[%%%%%%%%, {k, 0, 1}, PlotStyle → {Green}]

 ], {{γ, 0.4}, 0, 1}]
```

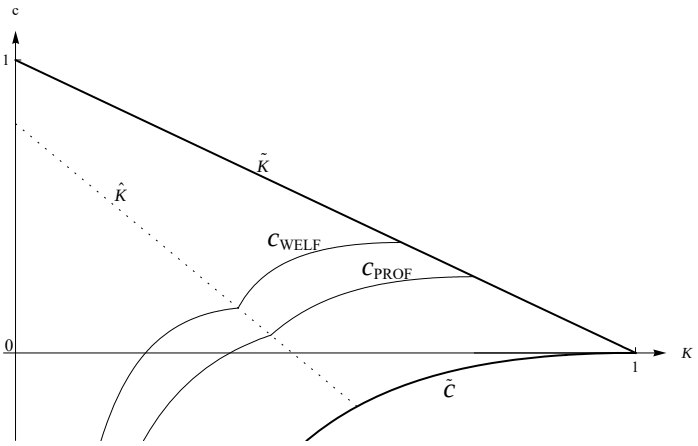
Out[264]=



```
(*Consider first below the red cure
  (i.e. the price with exclusivity is constrained). Then in the Blue area
  Exclusivity is profitable and in the white it is not and will not be used.
   Hence, varying γ one can see that if γ is sufficiently
  large then we have the following patter:
   – for c low, exclusivity is not profitable.
   – increasing c, it is profitable and welfare increasing.
   – further increasing c, it is profitable and welfare reducing
  For γ sufficiently low, instead,
 when exclusivity is profitable it is also uniquely welfare decreasing.

   Above the red curve,
 since the exclusivity pricing is the monopolist' unrestrained price,
 exclusivity is profitable a fortiori and we
  do not need to characterize any other region.)*)
```

```
(*Plotting figure 2*)
γparam = 0.6;
Piecewise[{{cq2k, cq2k ≥ cmin}, {Null, cq2k < cmin}}] /. γ → γparam;
cmin /. γ → γparam;
{Piecewise[{{cprofitA, tmpc2[[1, 1, 2]] < cprofitA && cprofitA < 1 - k},
      {tmpc2[[1, 1, 2]], tmpc2[[1, 1, 2]] ≥ cprofitA && tmpc2[[1, 1, 2]] < 1 - k}}]
  } /. γ -> γparam;
FullSimplify[P1 /. qNE /. pNE] /. γ -> γparam;
FullSimplify[P1 /. q1 → q1mon /. p1 → p1mon] /. γ -> γparam;
FullSimplify[P1 /. q1 → q1mon /. p1 → p1econstr] /. γ -> γparam;

Piecewise[{{cWelfBoundB, cWelfBoundB < cq2k && cWelfBoundB < 1 - k},
     {cWelfBoundA, cWelfBoundB > cq2k && cWelfBoundA < 1 - k},
     {Null, cWelfBoundA > 1 - k}}] /. γ -> γparam;

Figure2 = Show[
  Quiet[Plot[{%, 1 - k, %%%%%, %%%%%%, %%%%%%%}, {k, 0, 1},
    Epilog → {Text["0", {-0.01, 0.025}],
      Text["K̃", {0.4, 0.65}],
      Text[Subscript["C", "WELF"], {0.45, 0.38}],
      Text[Subscript["C", "PROF"], {0.6, 0.28}],
      Text["K̂", {0.17, 0.55}],
      Text["C̃", {0.7, -0.12}]
     },
    PlotRange → {{-0.02, 1.05}, {-0.3, 1.1}}, AxesLabel → {K, "c"},
    PlotStyle → {{Black}, {Black, Thickness[0.003]},
      {Black}, {Black, Thickness[0.003]}, {Thin, Black, Dotted}},
    LabelStyle → Directive[FontSize → 8],
    Ticks → {{0, 1}, {-1, 1}},
    AxesStyle → Arrowheads[{0.0, 0.02}]
   ]
  ]
 ]

SetDirectory[NotebookDirectory[]];
SetDirectory[ParentDirectory[]];
SetDirectory["Figures"];
Export["Figure2.pdf", Figure2];
```

Out[273]=



The Test and figure 3

```
(*We first determine the p1H that is the price firm
 1 would offer if the buyer does not accept exclusivity.
  (Note: if we treated p1H as a parameter since it would be higher
     than the one used here, the test would be failed a fortiori.)*)
```

In[278]:=
```
(*It must be that at this price the
  capacity constraint of 2 is binding (see later why):*)
v - p1 q1 - c q2 /. q2 → k;
q1constr = FullSimplify[Solve[D[%, q1] == 0, q1]][[1, 1, 2]]

(*and the indifference condition for the buyer is:*)
FullSimplify[v - p1 q1 - (c) q2 /. q2 → k /. q1 -> q1constr];
Ve2;
FullSimplify[Solve[%% == %, p1]]
p1HTestconstr = %[[1, 1, 2]]
```

Out[279]= $1 - p1 - k\,\gamma$

Out[282]= $\{\{p1 \to 1 - k\,\gamma\}, \{p1 \to 1 - k\,\gamma\}\}$

Out[283]= $1 - k\,\gamma$

```
(*Note that the buyer would obtain in exclusivity a payoff equal to that
 it would obtain in exclusivity with firm 2, q2=k and same price. Hence,
the only way to make the buyer indifferent is to set a p1 such that q1=0,
which is what p1HTest does.*)
FullSimplify[Solve[q1constr == 0, p1]]
```

$\{\{p1 \to 1 - k\,\gamma\}\}$

```
(*This also shows that,
it must be that the quantity q2 associated with q1=0 must be binding and the
  p1H determined above is the only possible p1H to consider for the test*)
```

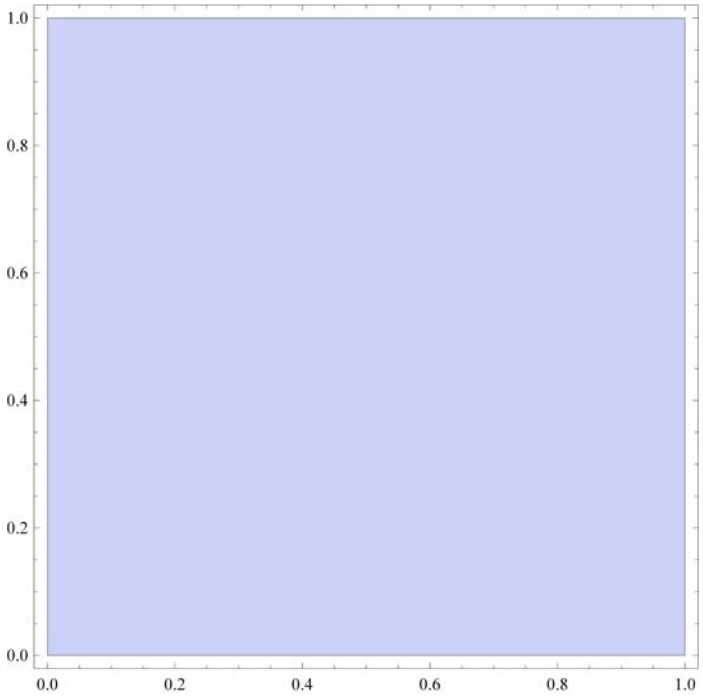
```
(*Now we calculate the "Required K" for the test. We have two case. One in which
 the equilibrium exclusive price is constrained and the other when it is not*)
```

In[284]:=
```
(*The required K with the constrained exclusive price*)
qe1 /. p1 -> p1econstr;
krconstr = FullSimplify[% * (p1HTestconstr - p1econstr) / (p1HTestconstr)]
```

Out[285]=
$$\frac{k\left(-2+2c+k+\sqrt{-k\,(-2+2c+k)}\;\gamma\right)}{-1+k\gamma}$$

In[286]:=
```
(*In the following dark area the test fails*)
krconstr /. {c → 0};
RegionPlot[k < %, {γ, 0, 1}, {k, 0, 1}]
```

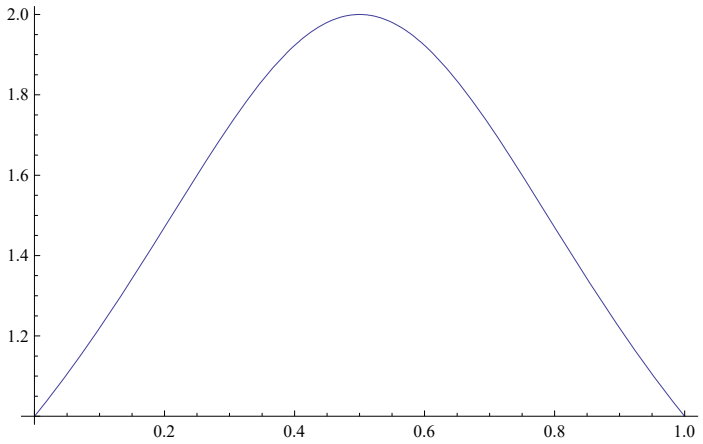
Out[287]=




```
(*As explained in the text:
  in this case (with the cosntrained exclusive price) the test always fails*)
```

In[288]:=
```
(*A double check*)
Solve[krconstr == k, k] /. c → 0;
Plot[%[[3, 1, 2]], {γ, 0, 1}]
```

Out[289]=




```
(*Now we study the other case, when with exclusivity the price is unconstrained,
the price p1H must be recalculated also because
 in exclusivity the buyer obtains a payoff that is higher
 than the reservation payoff with firm 2 used above, Ve2.*)
```

In[290]:=
```
(*First, in common representation (off-equilibrium), q2≤K is binding*)
FullSimplify[ve1 - p1 q1 /. q1 → 1 - p1 /. p1 → p1mon];
FullSimplify[v - p1 q1 - p2 q2 /. q2 → k /. p2 → c /. q1 → q1constr];
FullSimplify[Solve[% == %%, p1]];
N[% /. {c → 0, γ → 0.2, k → 0.1}]
p1HMonkbind = %%[[1, 1, 2]]
N[% /. {c → 0, γ → 0.2, k → 0.1}]
```

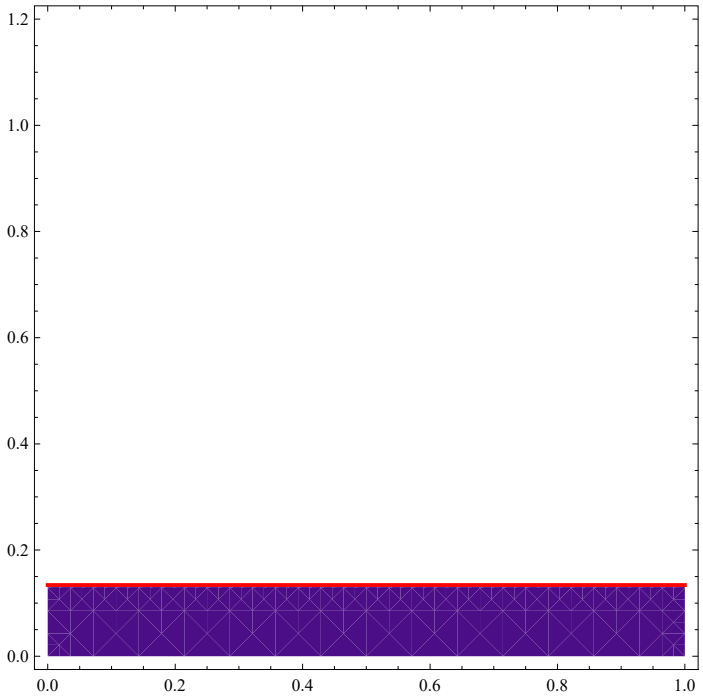
Out[293]= $\{\{p1 \to 0.735051\}, \{p1 \to 1.22495\}\}$

Out[294]= $1 - \frac{1}{2}\sqrt{1 + 4\,k\,(-2 + 2\,c + k)} - k\,\gamma$

Out[295]= 0.735051

In[296]:=
```
(*Now we can calculate the "Required K" for the test and plot it.
   First when in the common representation (off-Equilibrium) q2=K*)
qe1 /. p1 → p1mon;
% * p1mon - (k * c + (% - k) p1HMonkbind + 0) /. {c → 0};
ContourPlot[%, {γ, 0, 1}, {k, 0, 1.2}, Contours → {0}];
kpe1constr /. c → 0;
Plot[%, {γ, 0, 1}, PlotStyle → {Red, Thick}];
Show[%%%, %]
```

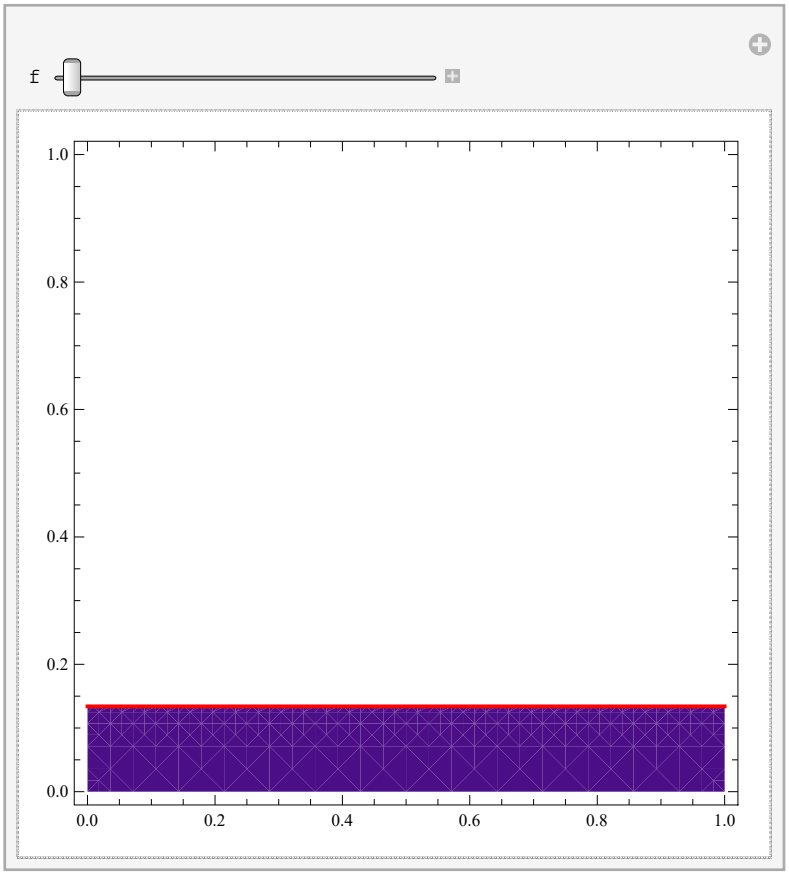
Out[301]=




```
(*Also in this case the test always fails.
  We can conlude, as discussed in the paper that the test always fails.*)
```

In[302]:=
```
(*Now we can calculate the "Required K" for the test and plot it.
   First when in the common representation (off-Equilibrium) q2=K*)
qe1 /. p1 → p1mon;
% * p1mon - (k * c + (% - k) p1HMonkbind + 0);
kpe1constr;

Manipulate[
 Show[ContourPlot[%%, {γ, 0, 1}, {k, 0, 1}, Contours → {0}],
  Plot[%, {γ, 0, 1}, PlotStyle → {Red, Thick}]], {c, 0, 1}]
```

Out[305]=




```
(*Now we generate figure 3 with the second source of asymmetry,
i.e. that referring to marginal cost, c.*)
```

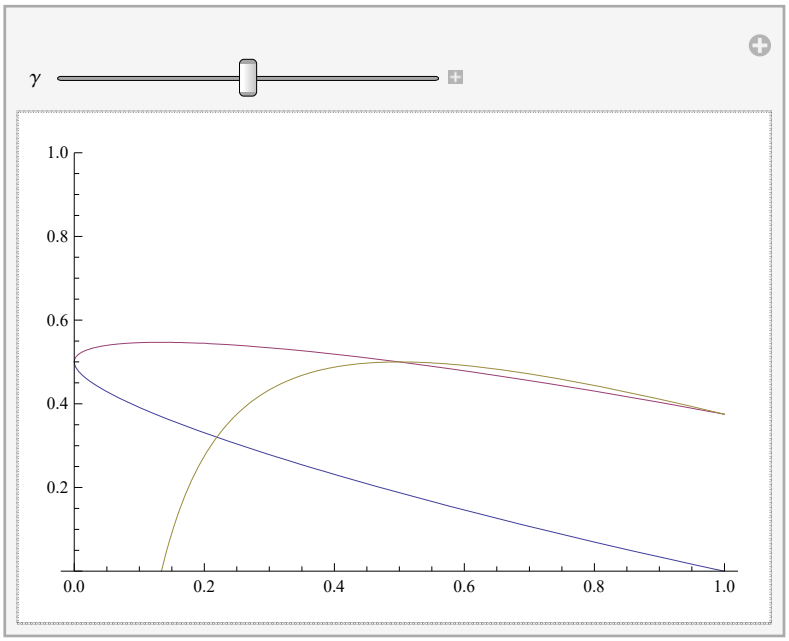
In[306]:=
```
cpe1constr = Solve[kpe1constr == k, c][[1, 1, 2]]
```

Out[306]=
$$\frac{-1 + 8\,k - 4\,k^2}{8\,k}$$

In[307]:=
```
(*First we see the case of exclusive constrained price:*)
tmp = FullSimplify[Solve[krconstr == k, c]]
{tmp[[1, 1, 2]], tmp[[2, 1, 2]], cpe1constr};
Manipulate[Plot[%, {k, 0, 1}, PlotRange → {0, 1}], {{γ, 0.5}, 0, 1}]
cTESTconstr = tmp[[1, 1, 2]]
```

Out[307]=
$$\left\{\left\{c \to \frac{1}{4}\left(2 - \sqrt{k\,\gamma^2\,(4 + k\,(-4+\gamma)\,\gamma)} - k\,(2 + (-2+\gamma)\,\gamma)\right)\right\},\right.$$
$$\left.\left\{c \to \frac{1}{4}\left(2 + \sqrt{k\,\gamma^2\,(4 + k\,(-4+\gamma)\,\gamma)} - k\,(2 + (-2+\gamma)\,\gamma)\right)\right\}\right\}$$

Out[309]=



Out[310]=
$$\frac{1}{4}\left(2 - \sqrt{k\,\gamma^2\,(4 + k\,(-4+\gamma)\,\gamma)} - k\,(2 + (-2+\gamma)\,\gamma)\right)$$

In[311]:=
```
(*Now we see the monopoly price case when in
 common representation (off-equilibrium), q2≤K is binding*)
v - p1 q1 - q2 p2 /. q2 → k;
q1rationed = FullSimplify[Solve[D[%, q1] == 0, q1]][[1, 1, 2]]
FullSimplify[ve1 - p1 q1 /. q1 → q1mon /. p1 → p1mon];
FullSimplify[v - p1 q1 - p2 q2 /. q2 → k /. p2 → c /. q1 → q1rationed];
FullSimplify[Solve[% == %%, p1]];
N[% /. {c → 0, γ → 0.2, k → 0.1}]
p1HMonkbind = %%[[1, 1, 2]]
N[% /. {c → 0, γ → 0.2, k → 0.1}]
```

Out[312]= $1 - p1 - k\,\gamma$

Out[316]= $\{\{p1 \to 0.735051\}, \{p1 \to 1.22495\}\}$

Out[317]= $1 - \frac{1}{2}\sqrt{1 + 4\,k\,(-2 + 2\,c + k)} - k\,\gamma$

Out[318]= 0.735051

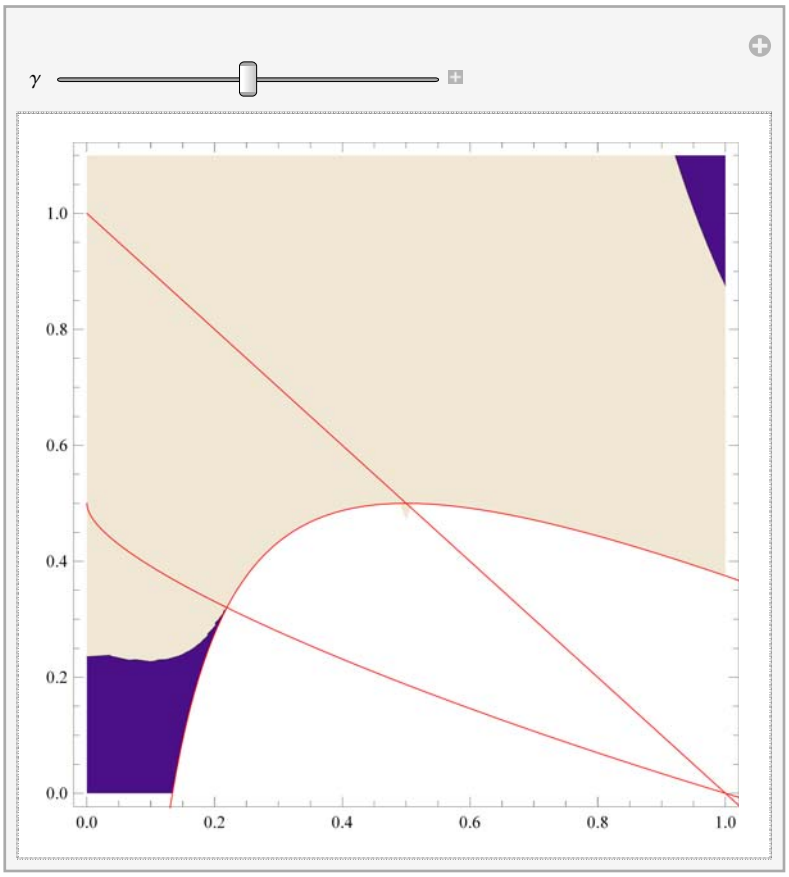
In[319]:=
```
q1mon /. p1 → p1mon;
tmp = Solve[k * 0 + (% - k) p1HMonkbind + 0 - (% * p1mon) == 0, c]
cTestmon = tmp[[1, 1, 2]];
```

Out[320]= $\left\{\left\{c \to \frac{1 - 6\,k + 12\,k^2 - 4\,k^3 - \gamma + 6\,k\,\gamma - 8\,k^2\,\gamma + k\,\gamma^2 - 4\,k^2\,\gamma^2 + 4\,k^3\,\gamma^2}{2\,(-1 + 2\,k)^2}\right\}\right\}$

In[322]:=
```
cTESTconstr
cpe1constr;
q1mon /. p1 → p1mon;
% * p1mon - (k * 0 + (% - k) p1HMonkbind + 0);
Manipulate[Show[ContourPlot[%, {k, 0, 1}, {c, 0, 1.1}, Contours -> {0}],
  Plot[{%%%, 1 - k, %%%%}, {k, 0, 1.1}, PlotStyle → {Red}]], {{γ, 0.5}, 0, 1}]
```

Out[322]=
$$\frac{1}{4}\left(2-\sqrt{k\,\gamma^2\,(4+k\,(-4+\gamma)\,\gamma)}-k\,(2+(-2+\gamma)\,\gamma)\right)$$

Out[326]=



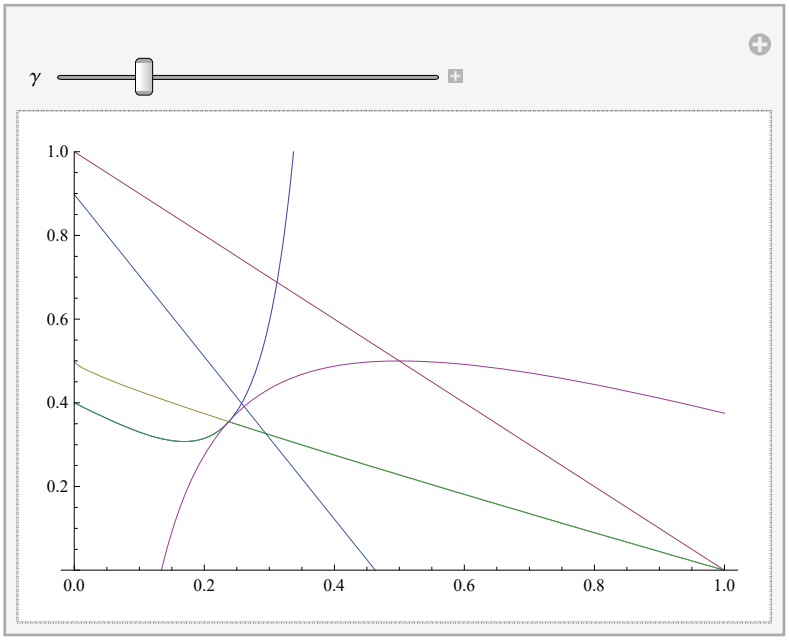

```
(*The white area is the one where the
 unconstrained pricing occurs in exclusivity*)
```

In[327]:=
```
(*The relevant boundaries are:*)
cpe1constr;
cq2k;
cTEST = Min[cTESTconstr, cTestmon];
cTESTconstr
cTestmon
Manipulate[
 Plot[{%, 1 - k, %%, %%%, %%%%, %%%%%}, {k, 0, 1}, PlotRange → {0, 1}], {{γ, 0.2}, 0, 1}]
```

Out[330]=
$$\frac{1}{4}\left(2-\sqrt{k\,\gamma^2\,(4+k\,(-4+\gamma)\,\gamma)}-k\,(2+(-2+\gamma)\,\gamma)\right)$$

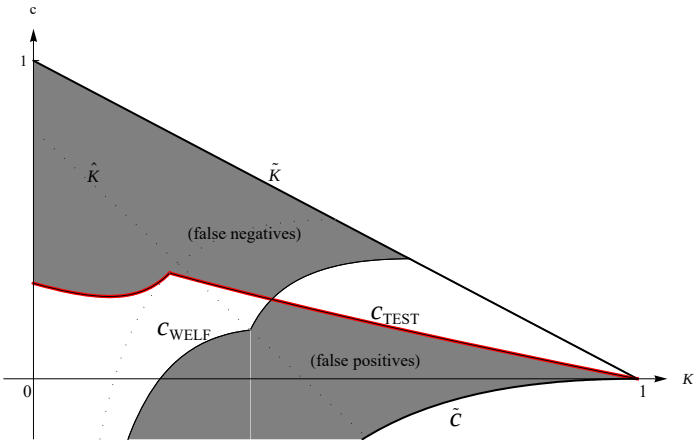
Out[331]=
$$\frac{1-6\,k+12\,k^2-4\,k^3-\gamma+6\,k\,\gamma-8\,k^2\,\gamma+k\,\gamma^2-4\,k^2\,\gamma^2+4\,k^3\,\gamma^2}{2\,(-1+2\,k)^2}$$

Out[332]=



```
(*Figure 3 in the paper, as for the elements on welfare*)

parameters = {γ → 0.6};
cmin /. γ → γparam;
Min[cpe1constr, 1 - k] /. parameters;
cq2k /. parameters;
Max[cTEST, Min[cWelfare, 1 - k]] /. parameters;
Min[cTEST, cWelfare] /. parameters;
cTEST /. parameters;
1 - k;
Piecewise[{{cWelfare, cWelfare ≤ 1 - k}, {Null, cWelfare > 1 - k}}] /. parameters;

Figure3 = Quiet[Plot[{%, %%, %%%, %%%%, %%%%%, %%%%%%, %%%%%%%, %%%%%%%%},
    {k, 0, 1}, Epilog → {Text["1", {1.01, -0.04}], Text["0", {-0.01, -0.04}],
      Text[Subscript[K̃, "WELF"], {0.4, 0.65}],
      Text[Subscript["c", "WELF"], {0.25, 0.15}],
      Text[Subscript[K̂, "TEST"], {0.1, 0.65}],
      Text[Subscript["c", "TEST"], {0.6, 0.2}],
      Text["(false negatives)", {0.35, 0.45}],
      Text["(false positives)", {0.55, 0.05}],
      Text[c̃, {0.7, -0.12}]
     }, AxesLabel → {K, "c"}, PlotStyle →
     {{Black}, {Black, Thickness[0.003]}, {Red, Thickness[0.005]}, {Black}, {Black},
      {Dashing[{0.001, 0.02}], Black, Thin}, {Dashing[{0.001, 0.02}], Black, Thin},
      {Black, Thickness[0.003]}}, LabelStyle → Directive[FontSize → 8],
    Ticks → {{0, 1.01}, {-1, 1}},
    PlotRange → {{-0.05, 1.05}, {-0.19, 1.1}},
    AxesStyle → Arrowheads[{0.0, 0.02}],
    Filling -> {4 → {8}, 5 → {2}},
    FillingStyle → {Gray}
   ]]

SetDirectory[NotebookDirectory[]];
SetDirectory[ParentDirectory[]];
SetDirectory["Figures"];
Export["Figure3.pdf", Figure3];
```

Out[342]=

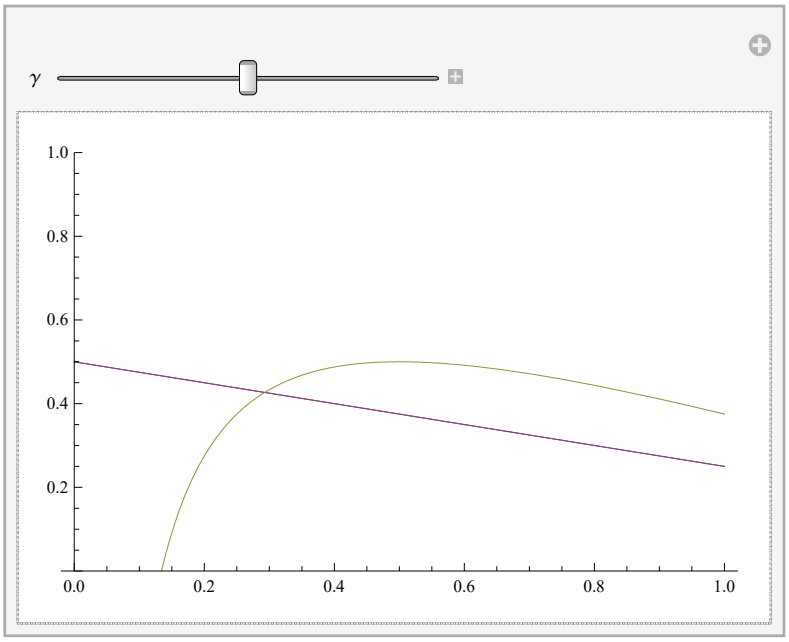

c
1
$\hat{K}$
$\tilde{K}$
(false negatives)
c_{TEST}
c_{WELF}
(false positives)
0
1
K
$\tilde{C}$

The Local-as-efficient Test and figure 4

In[347]:=
```
(*The required K with the constrained exclusive price*)
D[v, q2] /. {q2 → 0, q1 → q1mon} /. p1 → p1econstr;
qe1 /. p1 -> p1econstr;
p1econstr * % - ( k (0 + (%% - p1econstr)) + (% - k) p1HTestconstr );
(*First we see the case of exclusive constrained price:*)
tmp = FullSimplify[Solve[% == 0, c]]
{tmp[[1, 1, 2]], tmp[[2, 1, 2]], cpe1constr};
Manipulate[Plot[%, {k, 0, 1}, PlotRange → {0, 1}], {{γ, 0.5}, 0, 1}]
cTESTconstrLocal = Piecewise[{{tmp[[2, 1, 2]], γ ≤ 0.5}, {tmp[[1, 1, 2]], γ > 0.5}}];
```

Out[350]=
$$\left\{\left\{c \to \frac{1}{4}\left(2 + k\left(-3 + 6\gamma - 4\gamma^2\right) - \sqrt{k\,(1-2\gamma)^2\,(4 + k\,(1 + 4\,(-2+\gamma)\,\gamma))}\right)\right\},\right.$$
$$\left.\left\{c \to \frac{1}{4}\left(2 + k\left(-3 + 6\gamma - 4\gamma^2\right) + \sqrt{k\,(1-2\gamma)^2\,(4 + k\,(1 + 4\,(-2+\gamma)\,\gamma))}\right)\right\}\right\}$$

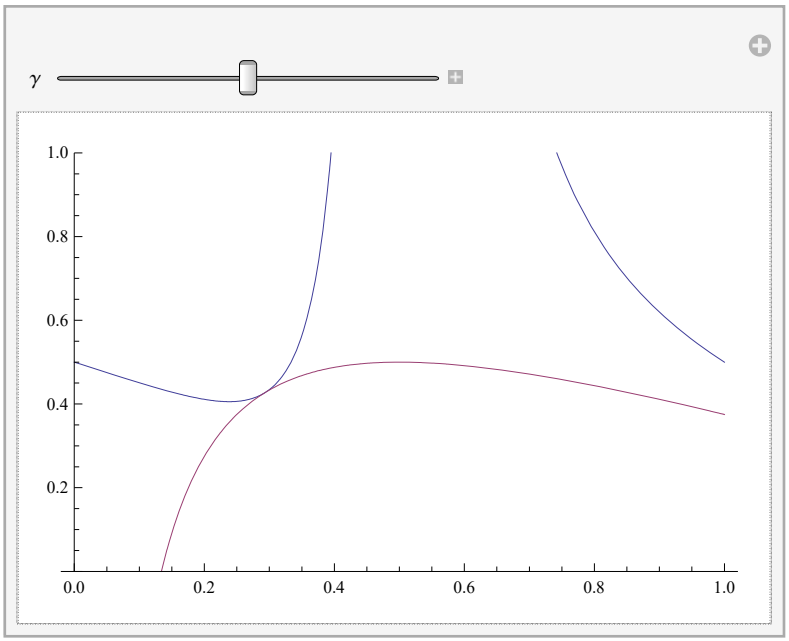
Out[352]=



In[354]:=
```
D[v, q2] /. {q2 → 0, q1 → q1mon} /. p1 → p1mon;
qe1 /. p1 -> p1mon;
p1mon * % - ( k (0 + (%% - p1mon)) + (% - k) p1HMonkbind );
Solve[% == 0, c]
cTestmonLocal = %[[1, 1, 2]];
{cTestmonLocal, cpe1constr};
Manipulate[Plot[%, {k, 0, 1}, PlotRange → {0, 1}], {{γ, 0.5}, 0, 1}]
```

Out[357]=
$$\left\{\left\{c \to \frac{2 - 9\,k + 12\,k^2 - 4\,k^3 - 2\,\gamma + 6\,k\,\gamma - 4\,k^2\,\gamma + 4\,k\,\gamma^2 - 8\,k^2\,\gamma^2 + 4\,k^3\,\gamma^2}{2\,(-1 + 2\,k)^2}\right\}\right\}$$

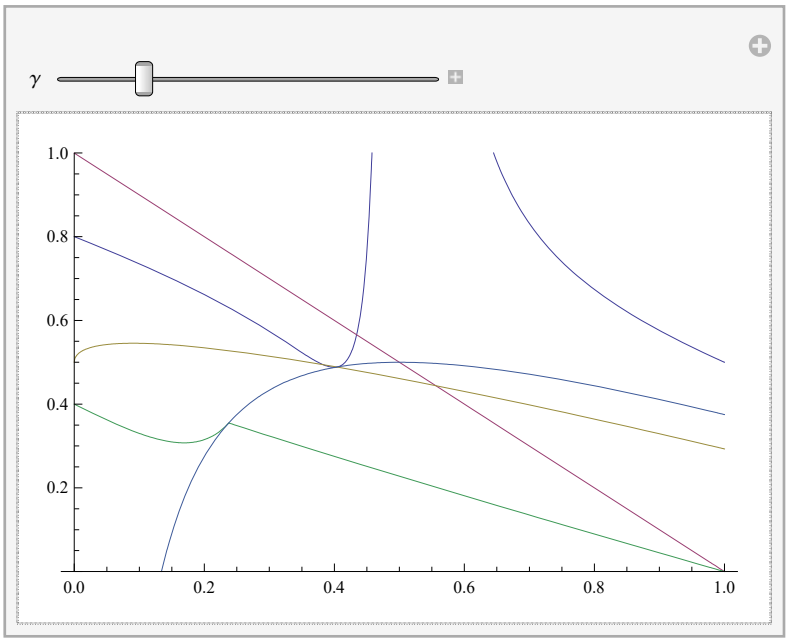
Out[360]=




```
(*The white area is the one where the
 unconstrained pricing occurs in exclusivity*)
```

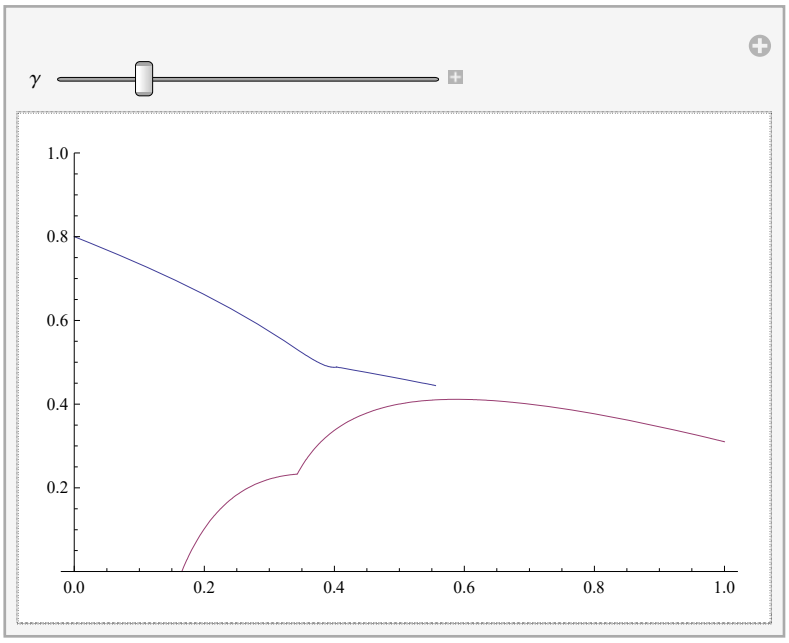
In[361]:=
```
(*Identifying the relevant boundaries are:*)
cpe1constr;
cTEST;
cTESTconstrLocal;
cTestmonLocal;
Manipulate[
 Plot[{%, 1 - k, %%, %%%, %%%%}, {k, 0, 1}, PlotRange → {0, 1}], {{γ, 0.2}, 0, 1}]
```

Out[365]=



In[366]:=
```
(*Defining the test:*)
cWelfare;
cTESTLocal = Piecewise[
    {{cTestmonLocal, cTESTconstrLocal ≥ cpe1constr && cTESTconstrLocal ≤ 1 - k},
     {cTESTconstrLocal, cTESTconstrLocal < cpe1constr && cTESTconstrLocal ≤ 1 - k},
     {Null, cTESTconstrLocal > 1 - k}}];
Manipulate[Quiet[Plot[{%, %%}, {k, 0, 1}, PlotRange → {0, 1}]], {{γ, 0.2}, 0, 1}]
```

Out[368]=



In[369]:=
```
(*Figure 4 in the paper, as for the elements on welfare*)
toplot = Min[Piecewise[{{cTestmonLocal, cTESTconstrLocal ≥ cpe1constr},
      {cTESTconstrLocal, cTESTconstrLocal < cpe1constr}}], 1 - k];

parameters = {γ → 0.6};
cmin /. γ → γparam;
toplot /. parameters;
cTEST /. parameters;
Max[cTESTLocal, Min[cWelfare, 1 - k]] /. parameters;
Min[toplot, cWelfare] /. parameters;
cTESTLocal /. parameters;
1 - k;
Piecewise[{{cWelfare, cWelfare ≤ 1 - k}, {Null, cWelfare > 1 - k}}] /. parameters;

Figure4 = Quiet[Plot[{%, %%, %%%%%%%, %%%%, %%%%%, %%%%%%, %%%, %%%%%%%%}, {k, 0, 1},
   PlotStyle → {{Black}, {Black, Thickness[0.003]}, {Black}, {Black}, {Black},
     {Thin, Dashed, Black}, {Red, Thickness[0.005]}, {Black, Thickness[0.003]}},
   LabelStyle → Directive[FontSize → 8], Ticks → {{0, 1.01}, {-1, 1}},
   Epilog → {Text["1", {1.01, -0.04}], Text["0", {-0.01, -0.04}],
     Text["C_WELF", {0.36, 0.09}],
     Text["C_TEST", {0.32, 0.35}],
     Text["K̃", {0.4, 0.65}],
     Text["(false negatives)", {0.2, 0.5}],
     Text["(false positives)", {0.55, 0.05}],
     Text["C̃", {0.7, -0.12}]
    }, AxesLabel → {K, "c"}, PlotRange → {{-0.05, 1.05}, {-0.19, 1.1}},
   AxesStyle → Arrowheads[{0.0, 0.02}],
   Filling -> {4 → {8}, 5 → {2}}, FillingStyle → {Gray}
  ]]

SetDirectory[NotebookDirectory[]];
SetDirectory[ParentDirectory[]];
SetDirectory["Figures"];
Export["Figure4.pdf", Figure4];
```