THE CONTINUUM COLLABORATION

THE ARCHITECTURE OF SPACE-TIME SEMINARS

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Differential Geometry, General Relativity

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SEMINAR-III: ON BLACK HOLES

We had previously talked about the Penrose diagram for a Schwarzschild black hole. In this seminar, we will try to understand more about the nature of black holes, and how the affect the motion of particles nearby.

SCHWARZSCHILD SOLUTION

- The field equations are of the form

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = kT_{\mu\nu}$$

- One of the most elementary solution to the field equations is the Schwarzschild solution.

- We will talk about the physical nature of this solution, and the construction of the idea of black holes.

SCHWARZSCHILD SOLUTION

- The conditions we wish to impose on the metric solving the field equations for our model are:
- 1. Static metric ($\partial_0 g_{\mu\nu} = 0$),
- 2. Vacuum solution – the field equations are of the following form, $R_{\mu\nu} = 0$,
- 3. Spherical symmetry.

- The metric we wish to solve has the following form:

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2 d\Omega^2$$

We need to determine the functions $f(r)$ and $g(r)$.

- The metric would have the diagonal form as follows:

$$g_{\mu\nu} = diag(-f(r), g(r), r^2, r^2 \sin^2\theta)$$

- The Christoffel symbol can be computed using the following expression:

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}\{\partial_b g_{ce} + \partial_c g_{be} - \partial_e g_{bc}\}$$

With this, the Christoffel symbols can be understood. The non-zero terms of the Christoffel symbols are the following:

$$\Gamma^0_{01} = \Gamma^0_{10}$$
$$\Gamma^1_{00}, \Gamma^1_{ii} \quad (i = 1, 2, 3)$$
$$\Gamma^2_{12} = \Gamma^2_{21}$$
$$\Gamma^3_{13} = \Gamma^3_{31}$$
$$\Gamma^3_{23} = \Gamma^3_{32}$$

These are the non-zero terms.

- We also need to identify the Ricci tensor for this model, which can be found using

$$R_{bc} \equiv R^{a}_{bca} = \partial_c \Gamma^{a}_{bc} - \partial_a \Gamma^{a}_{bc} + \Gamma^{a}_{bd}\Gamma^{d}_{ca} - \Gamma^{a}_{ad}\Gamma^{d}_{bc}$$

- We would then see that we would have non-zero elements of the Ricci tensors along the components $\mu\nu$.

- After getting the Ricci tensors, we would find the Ricci scalar by computing $\mathrm{R} = g^{\mu\nu}R_{\mu\nu}$. With these values, we would need to solve the field equations for this model. We would have the following four equations:

$$R_{00} - \frac{1}{2} g_{00} R = 0$$

$$R_{ii} - \frac{1}{2} g_{ii} R = 0$$

Where again, $i = 1, 2, 3$.

- In the equation comprising of 00 components, we would have a differential equation in only $g(r)$.

$$\frac{g'}{g} + \frac{g-1}{r} = 0$$

Where g' denotes $\frac{dg}{dr}$. We can solve this to find $f(r)$.

- We would see that the value of g as

$$g = \frac{1}{1 - \frac{K}{r}}$$

- Which would be the first part of our solution.
- The field equation in 11 components would be

$$\frac{f'}{rfg} = \frac{1}{r^2} - \frac{1}{r^2 g}$$

We can use the value of g to get f.

- We would then get the value of f as

$$f = 1 - \frac{K}{r}$$

- The metric now would take the form of

$$ds^2 = -\left(1 - \frac{K}{r}\right)dt^2 + \left(1 - \frac{K}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- In the Newtonian limit, we consider a perturbed Minkowski metric of the form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. In the 00 components, we have

$$g_{00} = \eta_{00} + h_{00} = 1 + \frac{2\Phi}{c^2}$$

By comparing this to g_{00} in our metric, we have

$$f = g^{-1} = \left(1 - \frac{K}{r}\right) = (1 - \frac{2GM}{c^2 r})$$

- The metric then would be of the form

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Which is the Schwarzschild metric, a solution to the field equations.

A PROBLEM

- In the Schwarzschild metric, we would observe two problems: one, at $r = \frac{2GM}{c^2}$, and the other at $r = 0$.

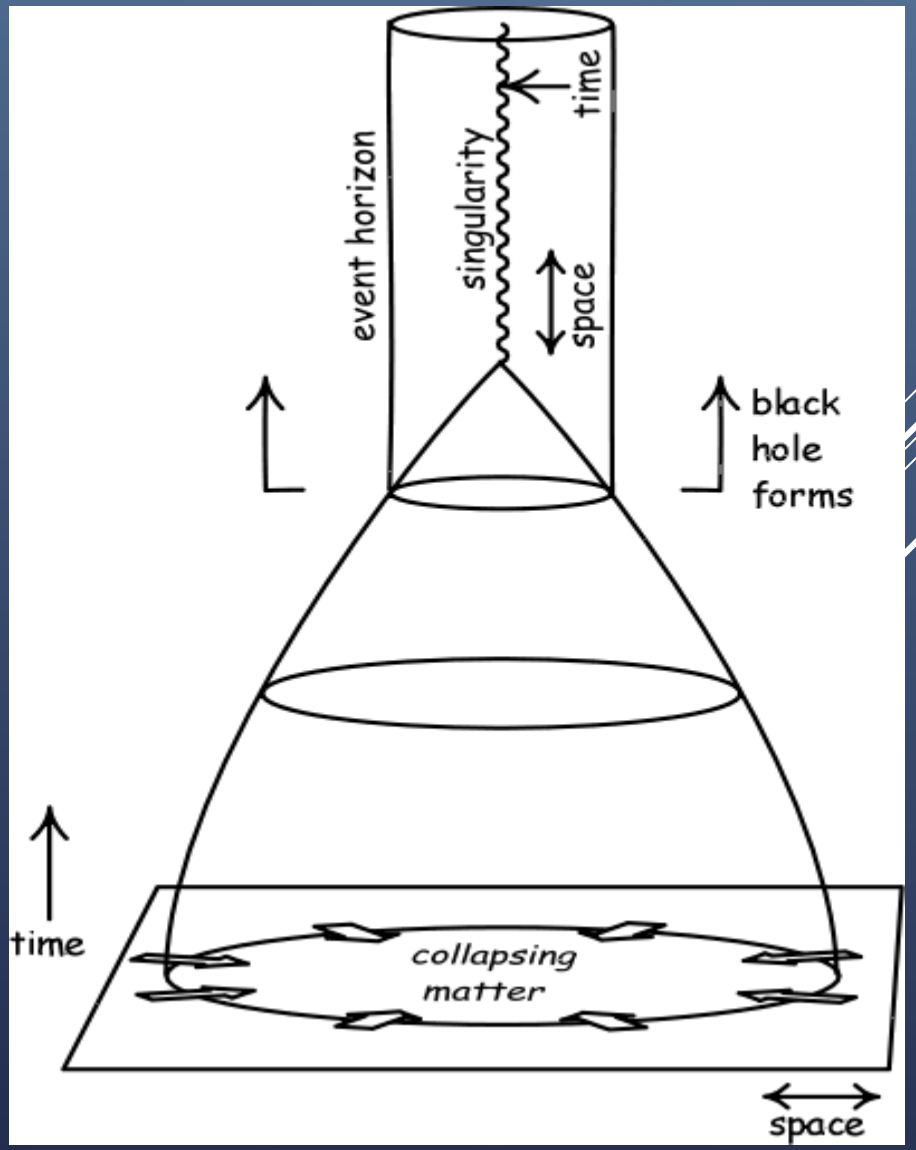
- We need to understand this in a little more detail. To get there, let us understand the structure of gravitational collapse.

- Look at the following diagram:
- Every circle in this diagram represents a 2 –Sphere corresponding to the collapsing star.

- When the star shrinks to $r = 2GM$, an event horizon is formed, as represented by the cylinder.

- When the star shrinks to $r = 0$, a singularity is formed.

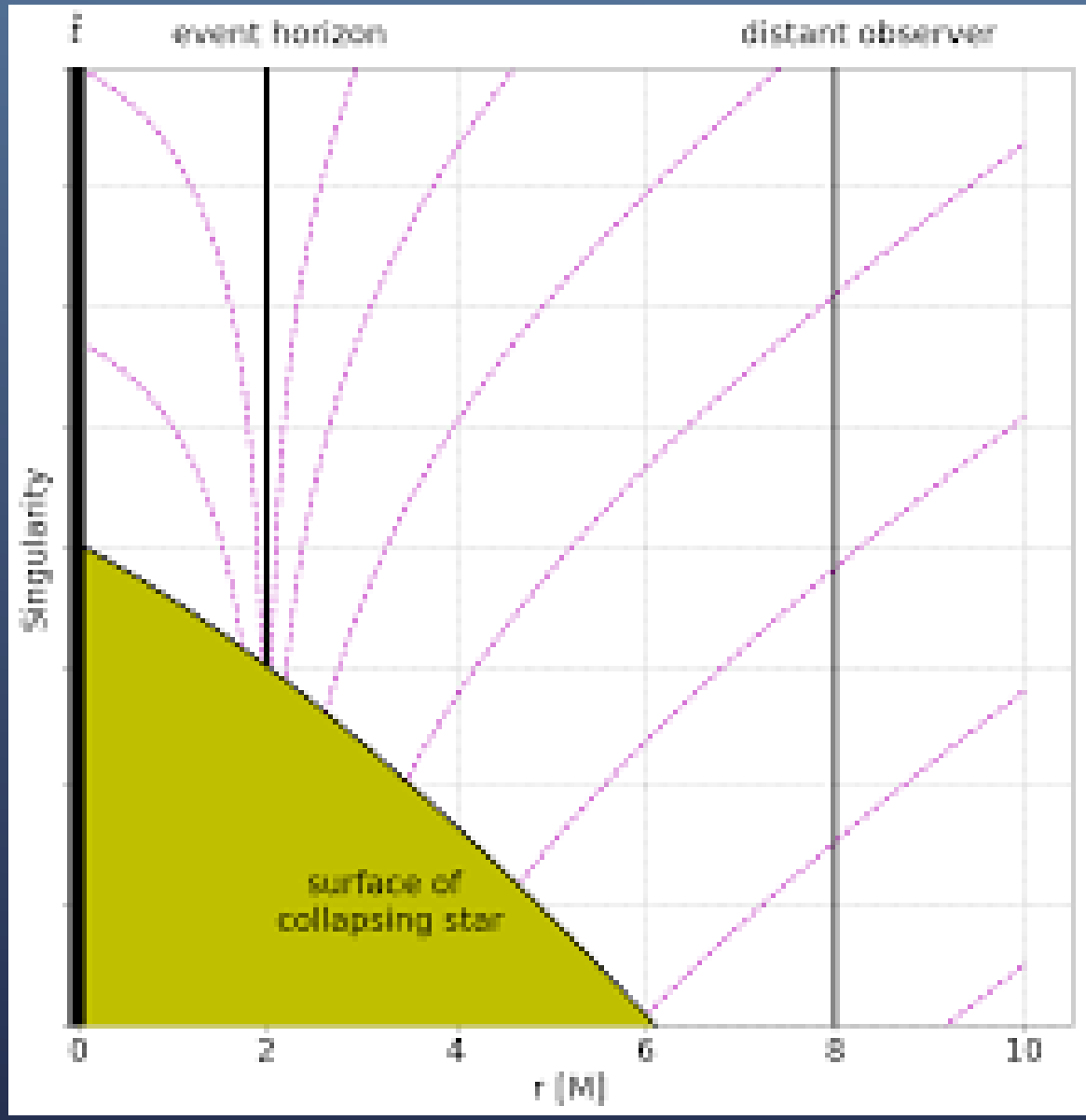
- This also has a deep meaning to the nature of the world-lines of particles emitted before, at and after the event horizon forms.



- In this diagram, we see the pink curves represent radially outgoing light rays.

- Light rays emitted before the event horizon forms get to null infinity. Those emitted close to the time of the formation of the event horizon take longer to reach the observer.

- Those rays emitted at the time the event horizon formed would take forever to reach the observer



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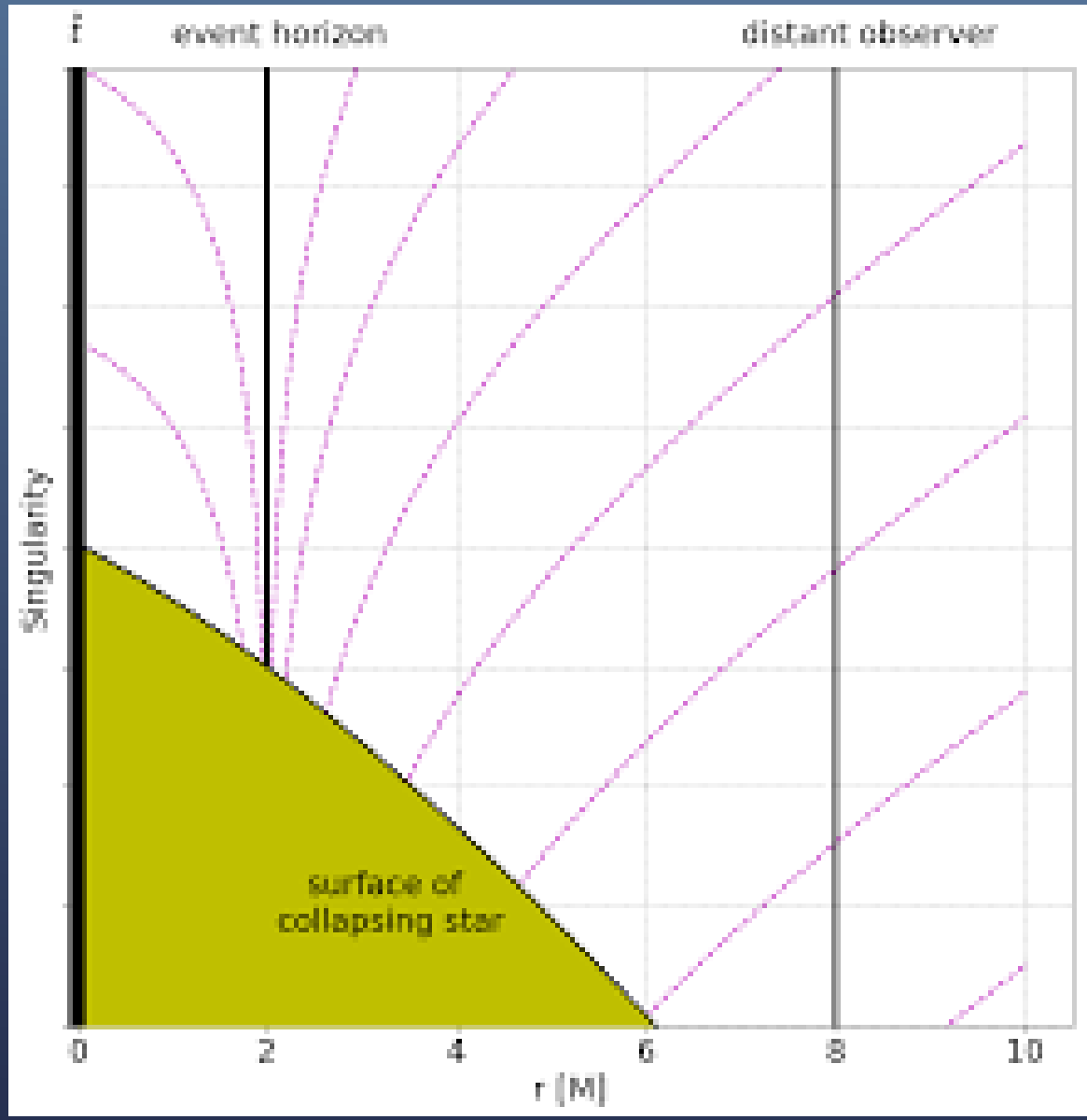
- Light rays emitted after the formation of the event horizon would be pulled into the interior of the black hole and would hit the singularity at $r = 0$.

- The behavior of light cones here is such that the light cones tip inside the black hole.

See the next slide.



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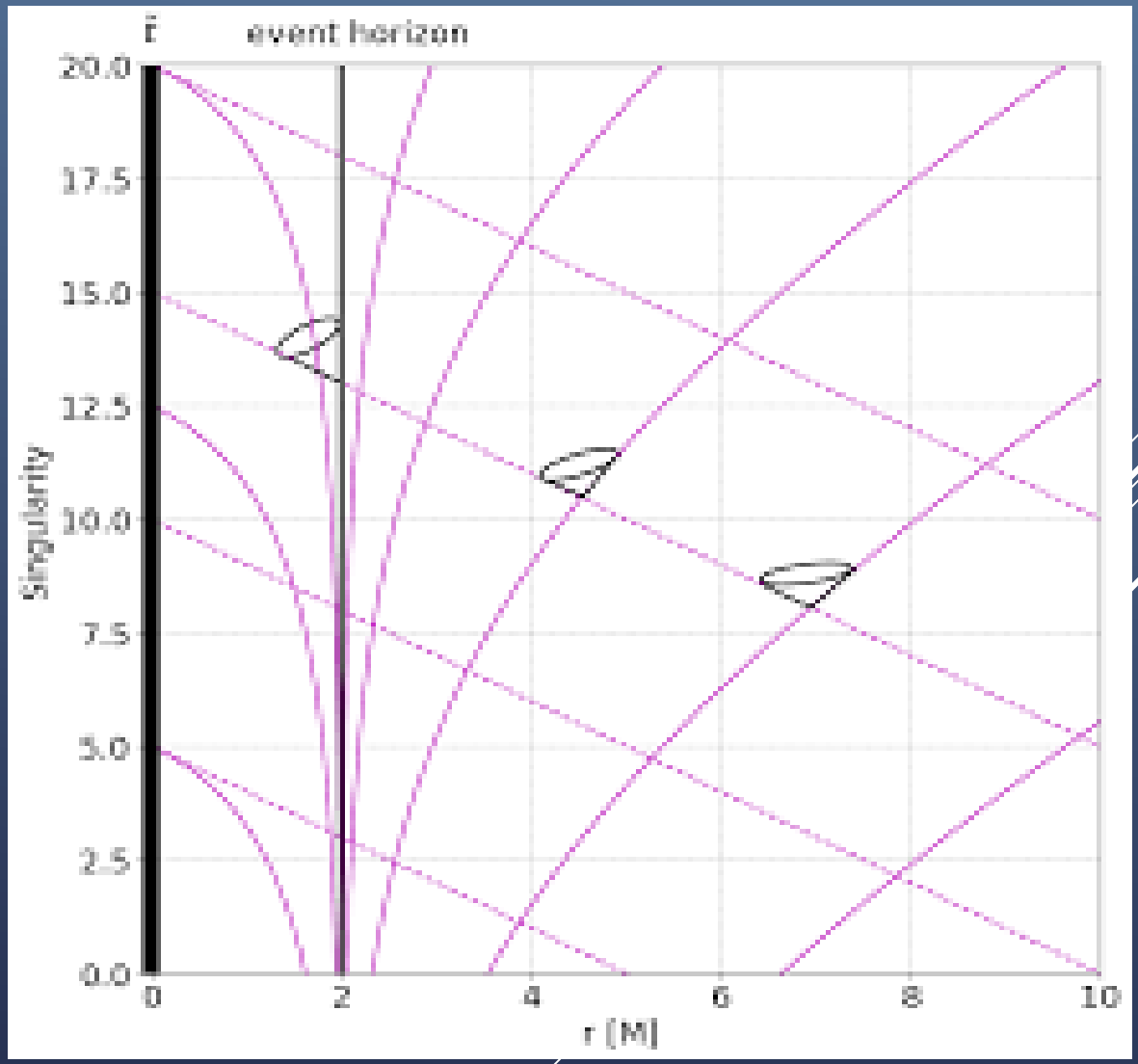
- Here, we see that the light cones emitted after the event horizon forms tip towards the singularity.

- Therefore not even light can escape the region inside $r = 2GM$.



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DESCRIBING BLACK HOLES

- We can describe black holes using three numbers:
- 1. Mass (Schwarzschild)
- 2. Angular momentum (Kerr)
- 3. Charge (Reissner)

- By the no hair theorem, we cannot describe black holes with more than these numbers.

- What about a thermodynamic idea of black holes?

BLACK HOLE ENTROPY

- The entropy of a black hole is quite an interesting thing.
- Since the black hole is formed by the collapse of a star, it is reasonable to ask what happens to the matter content.

- The generalized (second) law of thermodynamics is as follows:

"The sum of entropy outside and that of the black hole can never decreases."

- Hawking radiation formulates a form of violation of the energy condition in our theory – the area of the horizon can decrease.

BLACK HOLE ENTROPY

- The Hawking-Bekenstein equation relates the area of the event horizon to the entropy of the event horizon by

$$S_{BH} = \frac{A}{4}$$

We will talk a little of this in the next slides.

ON THE HAWKING-BEKENSTEIN RELATION

- The entropy of black holes have a very nice feature – they have a certain relation to the area of the event horizon.

- From the idea of Penrose processes, we can see that the entropy is related to the area,

$$S = f(A)$$

- The area of a Kerr-Newmann ($M = J = Q \neq 0$) black hole would be of the form

$$A_{Kerr} = 4\pi r_{\pm} + a^2$$

- Where $r_{\pm}$ is the radius of the first or the second event horizon in our model, $r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2}$

$$a = \frac{J}{M}$$

ON THE HAWKING-BEKENSTEIN RELATION

- By some algebraic manipulations, we can see that we would get two equations:

$$T_{BH} = \frac{1}{8\pi f'(A)} s_g$$

$$S_{BH} = \mu A$$

- Here, s_g represents the surface gravity, $s_g = \frac{2\pi(r_+ - r_-)}{A}$.

- We cannot quite find μ classically, and this would require quantum mechanical observations.

CONCLUSION

- We have talked briefly of some ideas of black holes.
- We talked of the Schwarzschild metric, and the effect of black holes on the motion of particles.
- We talked also of the thermodynamic idea of entropy and area of a black hole.
- We will be talking of Cosmology in the next seminar.

THANK YOU

- References:

- [1] Bekenstein-Hawking entropy, Scholarpedia, Jacob Bekenstein
- [2] Black hole thermodynamics, Physics Today, Jacob Bekenstein
- [3] A classical derivation of the Hawking-Bekenstein relation, *Topology and Physics blog*, Vaibhav Kalvakota