THE CONTINUUM COLLABORATION

THE ARCHITECTURE OF SPACE-TIME SEMINARS

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SEMINAR-II: ON PENROSE DIAGRAMS

In this seminar, we will look at the construction of Penrose diagrams of the Minkowski and the Schwarzschild solutions to the field equations.

We will start by discussing extended coordinates, and then move on to understanding their physical importance.

SIR PENROSE

- Sir Roger Penrose is well known for his works on GR, particularly in the field of differential geometry and their implications in GR.
- He discovered the famous 1965 singularity theorem in terms of closed trapped surfaces.
- He contributed to the idea of singularities at many levels, particularly in understanding their physical nature.

SIR PENROSE

- In this seminar, we will talk about the Carter-Penrose diagrams of the Minkowski and Schwarzschild solutions.
- These are conformal diagrams mapping the causal structure of these models into a compact diagram.

PENROSE DIAGRAMS

- Penrose diagrams are helpful in understanding the causal structure of various models, famously black holes.
- They consider an extension of coordinates to map the entire causal structure into a compact diagram.

- To understand the Penrose diagrams of black holes, we will start by discussing that of the Minkowski Space-time.

FROM THE MINKOWSKI SPACE-TIME

- The metric in the Minkowski solution in polar coordinates is of the form $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$. In order to draw a diagram, we would have to first understand the physical nature of the model.

- The Minkowski diagrams show a 2 –Sphere at every point in the diagram – the points $r \geq 0$ and $-\infty < t < +\infty$ represent a 2 –Sphere at those points.

- In this coordinate system, is it possible to draw the complete set of infinities in a compact diagram? No!

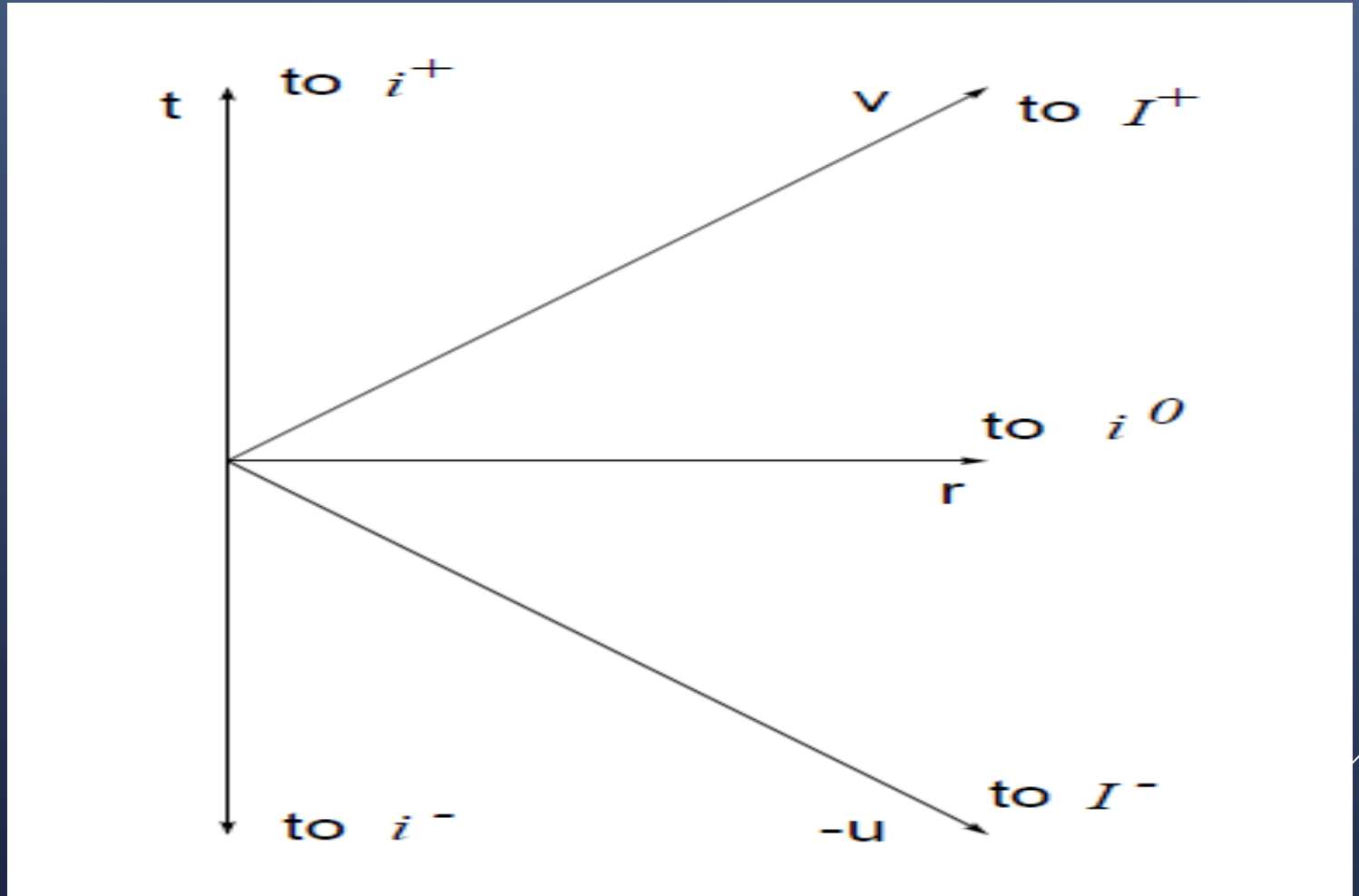
THE u, v COORDINATE SYSTEM

- One thing we can do is to extend the coordinate system.
- Instead of the usual (t, r, θ, ϕ) system, we can draw a new extended version by constructing $(v, u) = (t \pm r)$. This allows us to write the causal structure the same way as the Minkowskian diagrams.

- This way, $-dt^2 + dr^2 = -dudv$. This coordinate system is also called as the light-cone coordinates, composed of advanced and retarded null geodesics.

INFINITIES

- In the u, v coordinate system we observe a set of infinities, three sets distributed into past and future terms.



- We now can see five infinities based on their nature:

- 1. $I^{\pm}$: $v \to \infty$ at fixed u or $u \to -\infty$ at fixed v.
- 2. $i^{\pm}$: $t \to \pm\infty$ at fixed r. (future and past time-like infinity)
- 3. i^{0}: $r \to \infty$ at fixed t. (space-like infinity)

- Since I^{+} is at an infinite value of v, we need to consider a change of coordinates once again.

A FURTHER EXTENSION OF COORDINATES

- With u, v coordinates, we can write the metric as

$$ds^2 = -dudv + \frac{1}{4}(v-u)^2 d\Omega^2$$

- Which after writing in t, r would be the same metric as before.

- We can write a new system of coordinates U, V such that:
- $U = \arctan u$ and
- $V = \arctan v$

- We can define the range as $-\frac{\pi}{2} < U,\ V < \frac{\pi}{2}$. The metric in u, v coordinates now takes the form of

$$ds^2 = (\cos U \cos V)^{-2}[-dUdV + \frac{1}{4}\sin^2(V - U)\, d\Omega^2]$$

- We can define the time-like and the radial coordinates T, R as

$$(T, R) = (V \pm U)$$

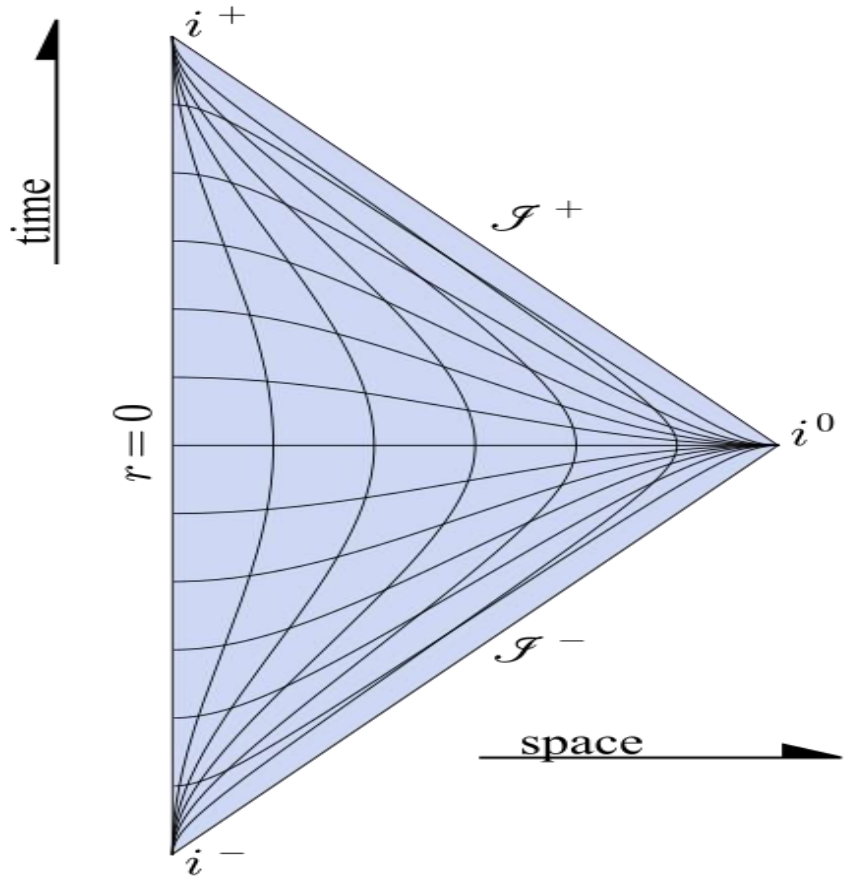
- This would mean that we have a metric of the form $\bar{g}_{\mu\nu}$ that can be written as $\bar{g}_{\mu\nu} = \omega^{-2} g_{\mu\nu}$, which in terms of the metric in T, R would be

$$ds^2 = \omega^{-2}[-dT^2 + dR^2 + \sin^2 R\, d\Omega^2]$$

- In this form, we see that the metric $-dT^2 + dR^2 + \sin^2 R\, d\Omega^2$ is a topology of the form $R \times S^3$. In this way, we see that the Minkowskian metric has a conformal structure as $R \times S^3$.

- This means that the causal structure is the same. That is to say that causal and space-like curves are represented the same way.
- $i^{\pm}$ would be points in a diagram, whereas $I^{\pm}$ would be radially outgoing or ingoing null geodesics. i^0 is the spatial infinity.
- Together, we can draw a conformal diagram of the Minkowskian Space-time as the following diagram.

This is the Penrose diagram for a Minkowski Space-time



WHAT ABOUT BLACK HOLES?

- Black holes can also be depicted in the form of such conformal diagrams.

- We will talk about the case of the Schwarzschild solution, and the Penrose diagram for such a black hole ($J = Q = 0$)

A SMALL REMARK

- Before talking about the Penrose diagrams for the Schwarzschild solution, we need to first have an idea of the structure of such black holes.

- Schwarzschild black holes are models that have a metric such that there exist two singularities, at $R = 2GM$ (coordinate singularity) and $R = 0$ (curvature singularity). The metric for this solution is

$$ds^2 = -dt^2 f(r) + dr^2 g(r) + r^2 d\Omega^2$$

Where $f(r) = g(r)^{-1} = (1 - 2GM)$

- The coordinate singularity at $R = 2GM$ can be (almost) covered by a nice set of coordinates called the Eddington-Finkelstein coordinates.
- The curvature singularity at $R = 0$ cannot be covered by any coordinates, as can be seen by the Kretschmann scalar, and is therefore a coordinate invariant blow-up.

- We can introduce a set of coordinates (analogous to the u, v coordinates in Minkowski Space-time) u, v along with the tortoise coordinate.

- The tortoise coordinate is given by $R^* = R + 2GM\ ln|\frac{R}{2GM} - 1|$, and satisfies the slope condition $\frac{dR}{dR^*} = 1 - \frac{2GM}{R}$.
- The analogue of the Minkowskian null ingoing and outgoing null geodesics in this case would be the following:

$$(v, u) = (t \pm r^*)$$

- In terms of the tortoise coordinate r^* the metric is

$$ds^2 = \left(1 - \frac{2GM}{R}\right)\left(-dt^2 + dr^{*2}\right) + r^* r^2 d\Omega^2$$

- $R = 2GM$ would then be at $r^* \to -\infty$

THE KRUSKAL COORDINATES

- As a further extension to the usual Schwarzschild coordinates (t, r, θ, ϕ), we can write the Kruskal-Szekeres coordinates to completely cover the coordinate singularity.
- Kruskal and Szekeres considered a set of coordinate transformations into a new coordinate system that describes the interior and the exterior of the black hole.
- Note that the point $R = 0$ is omitted (coordinate invariant nature of the blow-up).

THE KRUSKAL COORDINATES

- The metric in the Kruskal coordinates is

$$ds^2 = \frac{32M^3}{R} e^{-\frac{R}{2M}} (d\mathcal{V}^2 - d\mathcal{U}^2) - r^2 d\Omega^2$$

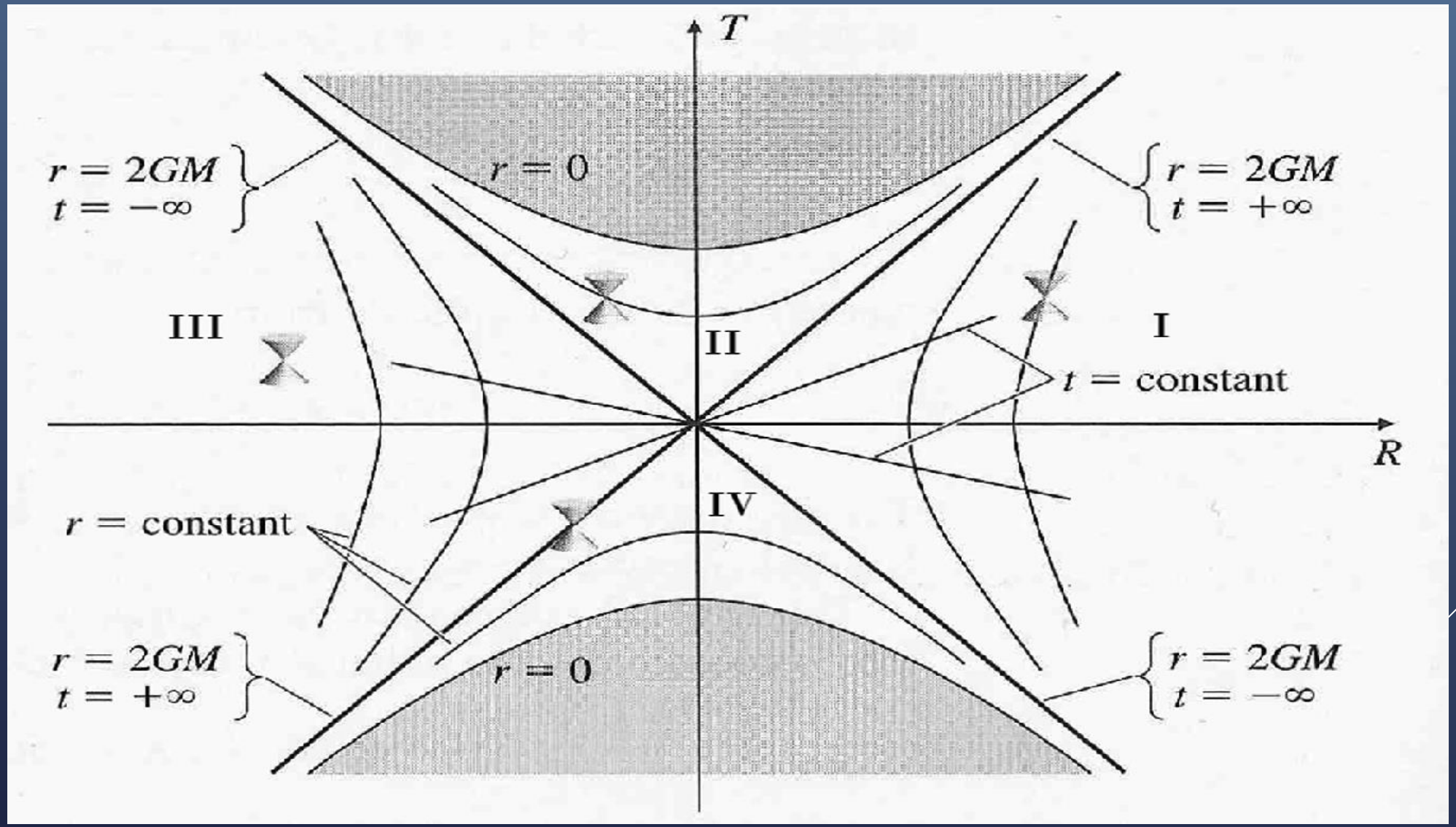
- This can be obtained by choosing the interior and the exterior sectors of the Space-time. These would be
- (Exterior) $\mathcal{V} = A_1(r) \cosh \frac{t}{4M}$ $\mathcal{U} = B_1(r) \sinh \frac{t}{4m}$
- (Interior) $\mathcal{V} = A_2(r) \sinh \frac{t}{4M}$ $\mathcal{U} = B_2(r) \cosh \frac{t}{4M}$

THE KRUSKAL COORDINATES

- The sectors are now in four forms, as depicted in the following diagram.
- The first sector would be when $2GM < R$, the exterior region.
- The second sector would be when $0 < R < 2GM$, avoiding the singularity at $R = 0$ – the interior region.
- Two more regions (parallel region and the interior of the sector with a negative singularity, called a "white hole")

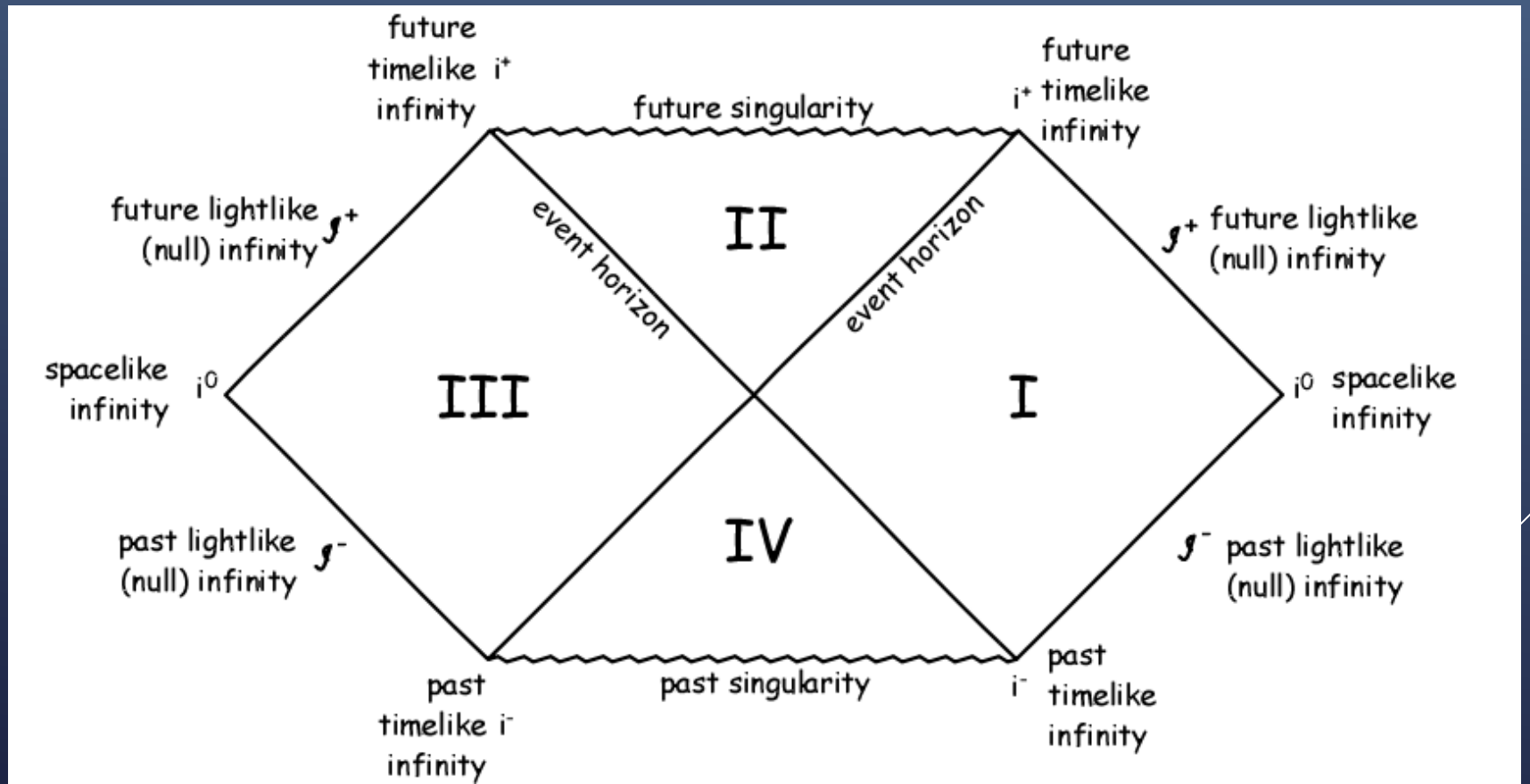
- The Kruskal diagram looks as follows:

- The Kruskal diagram, here we can clearly see the four sectors:



THE PENROSE DIAGRAM

- From the Kruskal diagram, we can construct the Penrose diagram by the same algorithm as we saw for the Penrose diagram for the Minkowskian Space-time. We construct the advanced and retarded null geodesics, and the Penrose diagram would look as:



A REMARK ON TOMORROW

- Tomorrow's seminar would be on black holes, and we will develop the idea of a model housing singularities.
- We would talk about the collapse of a star, and how it affects light and the motion of a particle traveling radially outwards.

- We will talk about the Schwarzschild solution, and the idea of black holes in detail in regards to identifying the nature of black holes.

THANK YOU

- References:

- [1] The large scale structure of Space-time, Hawking and Ellis,
- [2] Les Houches lectures on black holes, A. Strominger,
- [3] Penrose diagrams, Cosmo@NYU 2018, lecture 14 GR.