Appendix for Strong Leader, Fragile Party: How Even a Weak Party Can Protect a Powerful Leader?

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Proof of Lemma 1

Suppose that the leader does not build a party and deduces the elite's optimal strategy with backward induction. Let V_E denote the elite's utility in Period 2;

$V_E(reject) = (\sigma + b)p_2 + k(p_2)(1 - p_2)$,

$V_E(accept) = x_2$.

The leader can prevent the elites from rebelling iff

$$V_E(accept) \geq V_E(reject)$$

$$x_2 > (\sigma + b)p_2 + k(p_2)(1 - p_2) \equiv x_2^*$$

which implies that the incumbent has to offer x_2^* to deter the elite if she deems it preferable. Given the assumption $x_2 < 1$, this is possible if and only if

$$(\sigma + b)p_2 + k(p_2)(1 - p_2) - 1 \leq 0$$

Let $\phi_2\big(p_2, k(p_2)\big)$ denote the left-hand side. Differentiating this function, we have

$$\frac{d\phi_2\big(p_2, k(p_2)\big)}{dp_2} = (\sigma + b) + k'(p_2) - k(p_2) - p_2 k'(p_2) = \sigma + b - k(p_2) + (1 - p_2)k'(p_2)$$

$$\frac{d^2\phi_2\big(p_2, k(p_2)\big)}{dp_2 dp_2} = -k'(p_2) + k''(p_2) - k'(p_2) - p_2 k''(p_2) = -2k'(p_2) + (1 - p_2)k''(p_2) < 0$$

The last inequation is derived from the assumptions $k'(p_2) > 0$ and $k''(p_2) < 0$.

Since $\frac{d\phi_2(p_2, k(p_2))}{dp_2}$ decreases in p_2,

$$\min_{p_2 \in [0,1]} \frac{d\phi_2\big(p_2, k(p_2)\big)}{dp_2} = \frac{d\phi_2\big(p_2, k(p_2)\big)}{dp_2}\Big|_{p_2 = 1} = \sigma + b - \overline{k} > 0 \qquad (1)$$

Inequation (1) is drawn from $\lim_{p_t \to 1} k(p_t) = \overline{k} < 1$, which implies $\phi_2(p_2)$ increases in p_2.

$$\min_{p_2 \in [0,1]} \phi_2\big(p_2, k(p_2)\big) = \phi_2\big(0, k(0)\big) = -1$$

$$\max_{p_2 \in [0,1]} \phi_2\big(p_2, k(p_2)\big) = \phi_2\big(1, k(1)\big) = \sigma + b - 1 > 0$$

This means that the equation $\phi_2\big(p_2, k(p_2)\big) = 0$ has a unique solution in the interval $p_2 \in (0,1)$.

Let's denote the solution as $\hat{p}_2$ so that

$$\phi_2\left(\hat{p}_2, k(\hat{p}_2)\right) = 0.$$

$\hat{p}_2$ is considered the solution of

$$(\sigma + b)p_2 + k(p_2)(1 - p_2) - 1 = 0 \tag{2}$$

Proof of Lemma 2

If $p_2 \geq \hat{p}_2$, then the leader abandons accommodation and offers nothing ($x_2 = 0$). Regardless, the elite accepts the same payoff, $V_E = x_2^* = (b + \sigma)p_2 + \mathrm{k}(\mathrm{p}_2)$.

In the same way, we can specify the condition for peaceful bargaining in period 1.

$U_E(reject) = 2[(b + \sigma)p_1 + \mathrm{k}(p_1)(1 - p_1)]$,

$U_E(accept) = x_1 + (\sigma + b)p_2 + k(p_2)(1 - p_2)$.

$$U_E(accept) \geq U_E(reject)$$

$$x_1 \geq 2[(b + \sigma)p_1 + \mathrm{k}(p_1)(1 - p_1)] - (b + \sigma)p_2 - k(p_2)(1 - p_2) \equiv x_1^*$$

Given $x_1 \leq 1$,

$$2[(b + \sigma)p_1 + \mathrm{k}(p_1)(1 - p_1)] - (b + \sigma)p_2 - k(p_2)(1 - p_2) < 0$$

Denote the left-hand side as $\phi_1\left(p_1, k(p_1)\right)$. Differentiating this function, we have

$$\frac{d\phi_1\left(p_1, k(p_1)\right)}{dp_1} = 2(\sigma + b) + 2k'(p_1) - 2k(p_1) - 2p_1k'(p_1)$$

$$= 2(\sigma + b) + 2(1 - p_1)k'(p_1) - 2k(p_1)$$

$$\frac{d^2\phi_1\left(p_1, k(p_1)\right)}{dp_1 dp_1} = 2k''(p_1) - 2k'(p_1) - 2k'(p_1) - 2p_1k''(p_1) = -4k'(p_1) + 2(1 - p_1)k''(p_1)$$

$$< 0$$

The latter inequation implies that $\frac{d\phi_1\left(p_1, k(p_1)\right)}{dp_1}$ decreases in p_1.

$$\min_{p_1 \in [0,1]} \frac{d\phi_1\left(p_1, k(p_1)\right)}{dp_1} = \frac{d\phi_1\left(p_1, k(p_1)\right)}{dp_1}\Big|_{p_1 = 1} = 2(\sigma + b) - 2\overline{k} > 0,$$

which implies $\frac{d\phi_1\left(p_1, k(p_1)\right)}{dp_1} > 0$ for all $p_1 \in [0,1]$. Since the function $\phi_1\left(p_1, k(p_1)\right)$ strictly increases in $p_1 \in [0,1]$,

$$\min_{p_1 \in [0,1]} \phi_1\left(p_1, k(p_1)\right) = \phi_1\left(0, k(0)\right) = -V^E(p_2) - 1 < 0$$

$$\max_{p_1 \in [0,1]} \phi_1\left(p_1, k(p_1)\right) = \phi_1\left(1, k(1)\right) = 2(\sigma + b) - V^E(p_2) - 1 > 0$$

That means the equation $\phi_1(p_1, k(p_1)) = 0$ has a unique solution in the interval $p_1 \in [0,1]$. Let's denote the solution as $\hat{p}_1$, which means

$$\phi_1(\hat{p}_1, k(\hat{p}_1)) = 0$$

Since the function ϕ_1 contains p_2, the threshold $\hat{p}_1$ can be considered a function of p_2, $\hat{p}_1(p_2)$. Implicit differentiation of $\phi_1\left(\hat{p}_1(p_2), k(\hat{p}_1(p_2))\right) = 0$ shows:

$$\frac{d}{dp_2} 2(\sigma + b)\hat{p}_1(p_2) + 2k(\hat{p}_1(p_2))(1 - \hat{p}_1(p_2)) - V^E(p_2) - 1 = 0$$

$$2(\sigma + b)\frac{\partial \hat{p}_1(p_2)}{\partial p_2} + 2\frac{\partial k(\hat{p}_1(p_2))}{\partial \hat{p}_1(p_2)}\frac{\partial \hat{p}_1(p_2)}{\partial p_2} - 2\left[\hat{p}_1(p_2)\frac{\partial k(\hat{p}_1(p_2))}{\partial \hat{p}_1(p_2)} + k(\hat{p}_1(p_2))\right] - (\sigma + b) - \frac{\partial k(p_2)}{\partial p_2} = 0$$

$$\frac{\partial \hat{p}_1(p_2)}{\partial p_2} = \frac{2\left[\hat{p}_1(p_2)\frac{\partial k(\hat{p}_1(p_2))}{\partial \hat{p}_1(p_2)} + k(\hat{p}_1(p_2))\right] + (\sigma + b) + \frac{\partial k(p_2)}{\partial p_2}}{2[(\sigma + b) + \frac{\partial k(\hat{p}_1(p_2))}{\partial \hat{p}_1(p_2)}]} > 0$$

Proof of Proposition 3

In Period 2, the leader decides whether to buy off the elite with x_2^*, or to discard him by offering nothing. Her payoffs are the following:

$$V_2^R(x_2 = 0) = (1 - p_2)[\sigma + b - k(p_2)],$$
$$V_2^R(x_2 = x_2^*) = 1 + b - x_2^* = 1 + b - (\sigma + b)p_2 - k(p_2)(1 - p_2).$$

In order to derive the condition for the ruler to deter the elite, we have her gain from deterrence:

$$V^R(x_2 = x_2^*) - V^R(x_2 = 0) = 1 + b - (\sigma + b)p_2 - k(p_2)(1 - p_2) - (1 - p_2)[\sigma + b - k(p_2)]$$
$$= 1 - \sigma > 0,$$

which always holds, so that the leader always prefers to deter the elite as long as it is possible.

The leader's utility in Period 1 depends on whether Period 2 allows accommodation between the two players. Let us compare the ruler's payoff in each case. First, if the elite in Period 2 is not deterrable, and the ruler opts for deterrence in Period 1, her payoff is:

$$U^R(x_1 = x_1^*, x_2 = 0) = (1 + b - x_1^*) + (1 - p_2)[(\sigma + b) - k(p_2)] \quad (4)$$

Second, if the ruler is resolved to fight the elite in Period 1, her payoff is:

$$U^R(x_1 = 0) = 2(1 - p_1)[(\sigma + b) - k(p_1)] \quad (5)$$

Third, suppose that the elite in Period 2 is deterrable. If the ruler opts for deterrence in Period 1, her payoff is:

$$U^R(x_1 = x_1^*, x_2 = x_2^*) = (1 + b - x_1^*) + (1 + b - x_2^*) \quad (6)$$

The comparison between (4) and (5) asks whether the ruler prefers to deter the elite in Period 1 when they are destined to fight against each other in Period 2.

$$U^R(x_1 = x_1^*, x_2 = 0) - U^R(x_1 = 0) = 1 - \sigma > 0$$
$$U^R(x_1 = x_1^*, x_2 = x_2^*) - U^R(x_1 = 0) = 2 - 2\sigma > 0$$

Proof of Proposition 4

Given $p_2 + g \in [0,1]$, the threshold is the solution of the following equation:

$$\phi_1(p_1, k(p_1), p_2 = 1) = 2(\sigma + b)p_1 + 2k(p_1)(1 - p_1) - V^E(p_2 = 1) - 1 = 0 \quad (7)$$

It follows from the same logic of Proposition 2 that this equation has a unique solution within the interval $p_1 \in [0,1]$. Denote this solution as $\overline{p}_1$.

The condition for self-destructive party building occurring is

$$\phi_1(p_1, k(p_1), p_2 = \hat{p}_2) = 2(\sigma + b)p_1 + 2k(p_1)(1 - p_1) - V^E(\hat{p}_2) - 1 > 0$$
$$(\sigma + b)p_1 + k(p_1)(1 - p_1) - 1 > 0$$
$$p_1 > \hat{p}_2$$

Proof of Proposition 5

First, recall that $\hat{p}_1$ is the solution of Equation (3).

$$\phi_1(p_1 = \hat{p}_2, k(p_1 = \hat{p}_2), p_2) = 2(\sigma + b)\hat{p}_2 + 2k(\hat{p}_2)(1 - \hat{p}_2) - V^E(p_2) - 1 = 1 - V^E(p_2)$$

The second equation follows from the fact that $\hat{p}_2$ is the solution of Equation (3). Thus, the relationship between $\hat{p}_1$ depends on the sign of $1 - V^E(p_2)$. If $1 - V^E(p_2) > 0$, then

$$\phi_1(p_1 = \hat{p}_2, k(p_1 = \hat{p}_2), p_2) > \phi_1(p_1 = \hat{p}_1, k(p_1 = \hat{p}_1), p_2)$$

Given that the function $\phi_1(p_1, k(p_1), p_2)$ increases in p_1, the above inequation implies

$$\hat{p}_1 < \hat{p}_2$$

In the same way, if $1 - V^E(p_2) \leq 0$,

$$\hat{p}_1 \geq \hat{p}_2$$

Next, recall that $\overline{p}_1$ is the solution of Equation (7) and $\phi_1(p_1, k(p_1), p_2 = 1)$ strictly increases in p_1. Since $V^E(p_2 = 1) = \sigma + b$,

$$\phi_1(p_1 = \hat{p}_2, k(p_1 = \hat{p}_2), p_2 = 1) = 2(\sigma + b)\hat{p}_2 + 2k(\hat{p}_2)(1 - \hat{p}_2) - (\sigma + b) - 1 = 1 - (\sigma + b)$$
$$< 0$$

Thus,

$$\phi_1(p_1 = \hat{p}_2, k(p_1 = \hat{p}_2), p_2 = 1) < \phi_1\left(p_1 = \overline{p}_1, k\left(p_1 = \overline{p}_1\right), p_2 = 1\right)$$

Given that the function $\phi_1(p_1, k(p_1), p_2 = 1)$ increases in p_1, the above inequation implies

$$\hat{p}_2 < \overline{p}_1$$

Proposition 6

Effect of k on thresholds.

Recall that $\overline{p}_1$ is the solution of Equation (7). In the same way of Proposition 5, let's ignore that $k(p_1)$ is the function of p_1 and assume that $k(p_1)$ is determined exogenously.

$$2(\sigma + b)\,\overline{p}_1 + 2k_{\overline{p}_1}\left(1 - \overline{p}_1\right) - (\sigma + b) - 1 = 0$$

By implicit differentiation,

$$\frac{\partial\,\overline{p}_1}{\partial k_{\overline{p}_1}} = \frac{\overline{p}_1 - 1}{b + \sigma - 1} < 0$$

Again, recall that $\hat{p}_2$ is the solution of Equation (2). Suppose

$$(\sigma + b)\hat{p}_2 + k_{\hat{p}_2}(1 - \hat{p}_2) - 1 = 0$$

$$\frac{\partial\,\hat{p}_2}{\partial k_{\hat{p}_2}} = \frac{\hat{p}_2 - 1}{b + \sigma - 1} < 0$$

Thus, $\overline{p}_1$ and $\hat{p}_2$ decreases as k increases.