Theorem 2. Suppose that X is a discrete random variable with the probability mass function $P(X=x_i)=p_i, i=1,2,\cdots,r$ and support in the finite interval $[a,b]$. If $m(x)$ and $n(x)$ are both increasing (or decreasing) in $[a,b]$, then we have $m(X)$ dominates $n(X)$ by ε-AFSD if and only if

$$\sum_{i\in I}[n(x_i)-m(x_i)]p_i \le \varepsilon\sum_{i=1}^{r}|\, n(x_i)-m(x_i)\,|\, p_i\ ,$$

where $I=\{i\,|\,m(x_i)<n(x_i), 1\le i\le r\}$.

Proof of Theorem 2. This proof is very similar to that of Theorem 1 and the concise proof is shown as follows.

Without loss of generality, we first assume that $x_1<x_2<\cdots<x_r$.

(1) "If" part: Suppose $m(x)$ and $n(x)$ are both increasing in $[a,b]$. We only need to prove that if $\sum_{i\in I}[n(x_i)-m(x_i)]p_i > \varepsilon\sum_{i=1}^{r}|\, n(x_i)-m(x_i)\,|\, p_i$, then there exists a utility function $u(x)\in U_1^*(\varepsilon)$, such that $E[u(m(X)]-E[u(n(X)]<0$.

For convenience, we suppose $I=\{s,s+1,\cdots,t\}$. From the known conditions, we deduce that

(i) $m(x_i)\ge n(x_i)$ if $i<s$;

(ii) $m(x_i)<n(x_i)$ if $s\le i\le t$;

(iii) $m(x_i)\ge n(x_i)$ if $i>t$.

Define

$$u(x)=\begin{cases} l_1 x, & x<m(x_s)\\ l_2 x-(l_2-l_1)m(x_s), & m(x_s)\le x\le n(x_t)\\ l_1 x+(l_2-l_1)[n(x_t)-m(x_s)], & x>n(x_t)\end{cases},$$

where $l_2>l_1>0$ and $\dfrac{l_1}{l_1+l_2}=\varepsilon$. It is obvious that $u(x)\in U_1^*(\varepsilon)$ and (i) for any $i<s$, we have $m(x_i)\le m(x_s)$; (ii) for any $s\le i\le t$, we have $m(x_i)\ge m(x_s)$ and $m(x_i)<n(x_i)\le n(x_t)$; (iii) for any $i>t$, we have $m(x_i)\ge n(x_i)\ge n(x_t)$. Hence, we get

$$u(m(x_i)=\begin{cases} l_1 m(x_i), & i<s\\ l_2 m(x_i)-(l_2-l_1)m(x_s), & s\le i\le t\\ l_1 m(x_i)+(l_2-l_1)[n(x_t)-m(x_s)], & i>t\end{cases}.$$

Similarly, we have

$$u(n(x_i)=\begin{cases} l_1 n(x_i), & i<s\\ l_2 n(x_i)-(l_2-l_1)m(x_s), & s\le i\le t\\ l_1 n(x_i)+(l_2-l_1)[n(x_t)-m(x_s)], & i>t\end{cases}.$$

Therefore, we obtain

$$u(m(x_i))-u(n(x_i))=\begin{cases} l_1[m(x_i)-n(x_i)], i<s \\ l_2[m(x_i)-n(x_i)], s\le i\le t \\ l_1[m(x_i)-n(x_i)], i>t \end{cases}.$$

and

$$E(u(m(X))-E(u(n(X))=\sum_{i=1}^{r}[u(m(x_i))-u(n(x_i))]p_i$$

$$=l_1\sum_{i=1}^{s-1}[m(x_i)-n(x_i)]p_i+l_2\sum_{i=s}^{t}[m(x_i)-n(x_i)]p_i+l_1\sum_{i=t+1}^{r}[m(x_i)-n(x_i)]p_i$$

$$=l_1\sum_{i=1}^{r}|\,m(x_i)-n(x_i)\,|\,p_i-(l_1+l_2)\sum_{i=s}^{t}[n(x_i)-m(x_i)]p_i$$

$$=(l_1+l_2)\{\varepsilon\sum_{i=1}^{r}|\,m(x_i)-n(x_i)\,|\,p_i-\sum_{i=s}^{t}[n(x_i)-m(x_i)]p_i\}$$

$$<0.$$

Similarly, if $m(x)$ and $n(x)$ are both decreasing in $[a,b]$, we only need to redefine

$$u(x)=\begin{cases} l_1x, & x<m(x_t) \\ l_2x-(l_2-l_1)m(x_t), & m(x_t)\le x\le n(x_s) \\ l_1x+(l_2-l_1)[n(x_s)-m(x_t)], & x>n(x_s) \end{cases},$$

where $l_2>l_1>0$ and $\dfrac{l_1}{l_1+l_2}=\varepsilon$. By repeating the above process, we can also deduce that $E(u(m(X))-E(u(n(X))<0$.

(2) "Only if" part: Suppose $\sum_{i\in I}[n(x_i)-m(x_i)]p_i\le\varepsilon\sum_{i=1}^{r}|\,n(x_i)-m(x_i)\,|\,p_i$, then for any $u(x)\in U_1^*(\varepsilon)$, if $\inf\limits_{x\in[a,b]}u'(x)=l_1$ and $\sup\limits_{x\in[a,b]}u'(x)=l_2$, we get $\varepsilon\le\dfrac{l_1}{l_1+l_2}$. Thus, we deduce that

$$E(u(m(X))-E(u(n(X))=\sum_{i=1}^{r}[u(m(x_i))-u(n(x_i))]p_i$$

$$=\sum_{i=1}^{r}u'(\xi_{x_i})[m(x_i)-n(x_i)]p_i \text{ (where } \xi_{x_i} \text{ is among } m(x_i) \text{ and } n(x_i)\text{)}$$

$$=\sum_{i=1}^{s-1}u'(\xi_{x_i})[m(x_i)-n(x_i)]p_i+\sum_{i=s}^{t}u'(\xi_{x_i})[m(x_i)-n(x_i)]p_i+\sum_{i=t+1}^{r}u'(\xi_{x_i})[m(x_i)-n(x_i)]p_i$$

$$\ge l_1\sum_{i=1}^{s-1}[m(x_i)-n(x_i)]p_i+l_2\sum_{i=s}^{t}[m(x_i)-n(x_i)]p_i+l_1\sum_{i=t+1}^{r}[m(x_i)-n(x_i)]p_i$$

$$\ge l_1\sum_{i=1}^{r}|\,m(x_i)-n(x_i)\,|\,p_i-(l_1+l_2)\sum_{i=s}^{t}[n(x_i)-m(x_i)]p_i$$

$$=(l_1+l_2)(\varepsilon\sum_{i=1}^{r}|\,m(x_i)-n(x_i)\,|\,p_i-\sum_{i=s}^{t}[n(x_i)-m(x_i)]p_i)$$

$$\ge 0.\square$$