Challenger Entry and Electoral Accountability

Online Appendix

Jacob Morrier

Division of the Humanities and Social Sciences

California Institute of Technology

October 2024

Contents

A Equilibrium Definition **1**

B Equilibrium Analysis Without Endogenous Challenger Entry **4**

C Proof of Lemma 1 **10**

D Proof of Lemma 2 **13**

E Equilibrium Analysis With Endogenous Challenger Entry **15**

F Proof of Proposition 3 **17**

G Proof of Proposition 4 **20**

A Equilibrium Definition

The Incumbent's strategy is characterized by:

(i) the policy enacted by high-ability incumbents in the first period conditional on the state of the world, denoted $y_1(h, \omega_1)$.

(ii) the probability that low-ability incumbents enact policy a in the first period, denoted σ.

(iii) the policy enacted by high-ability politicians in the second period conditional on the state of the world, denoted $y_2(h, \omega_1)$.

(iv) the policy enacted by low-ability politicians in the second period, denoted $y_2(\ell)$.

The Challenger's strategy is characterized by:

(i) The probability that the Challenger runs for office given the Incumbent's first-period policy decision and the posterior probability that she has a high ability, denoted ρ^y.

(ii) the policy enacted by high-ability politicians in the second period conditional on the state of the world, denoted $y_2(h, \omega_1)$.

(iii) the policy enacted by low-ability politicians in the second period, denoted $y_2(\ell)$.

The definition of strategies for both the Incumbent and the Challenger relies on the fact that they encounter identical policymaking decisions in the second period, leading them to adopt the same strategy at this stage of the game.

The Voter's strategy is characterized by the probability that they reelect the Incumbent when the Challenger decides to contest the election, given the Incumbent's first-period policy decision and the posterior probability that she has a high ability, denoted ν^y. For simplicity, consistent with my argument in Section 3, this definition leaves out the Voter's electoral behavior when the Incumbent's type is revealed before the election.

Definition A1. *An equilibrium is a tuple* $\left\langle \sigma, (y_1(h, \omega_1))_{\omega_1 \in \Omega}, (\rho^y)_{y \in Y}, (\nu^y)_{y \in Y}, y_2(\ell), (y_2(h, \omega_2))_{\omega_2 \in \Omega}, (\kappa^y)_{y \in Y} \right\rangle$ *such that:*

(i) Given the state of the world $\omega_2 \in \Omega$, the policy enacted by high-ability politicians in the second period, denoted as $y_2(h, \omega_2)$, maximizes their policy payoffs:

$$y_2(h, \omega_2) \in \arg\max_{y \in Y} u(y, \omega_2);$$

(ii) The policy enacted by low-ability politicians in the second period, denoted as $y_2(\ell)$, maximizes their expected policy payoffs:

$$y_2(\ell) \in \arg\max_{y \in Y} \pi \times u(y, a) + (1 - \pi) \times u(y, b);$$

(iii) When the Challenger opts to contest the election, the Voter reelects the Incumbent with a probability ν^y that maximizes their expected policy payoffs given the Incumbent's first-period policy decision y and the posterior probability that she has a high ability:

$$\nu^y \in \arg\max_{\nu \in [0,1]} \nu \times [\kappa^y \times u_2(h) + (1 - \kappa^y) \times u_2(\ell)] + (1 - \nu) \times [\gamma \times u_2(h) + (1 - \gamma) \times u_2(\ell)],$$

where $u_2(\ell) = \pi \times u(y_2(\ell), a) + (1 - \pi) \times u(y_2(\ell), b)$ and $u_2(h) = \pi \times u(y_2(h, a), a) + (1 - \pi) \times u(y_2(h, b), b)$ are the policy payoffs expected from a low-ability and a high-ability politician holding office in the second period, respectively;

(iv) Given the Incumbent's first-period policy decision y and the posterior probability that she has a high ability, the Challenger runs for office with a probability ρ^y that maximizes his expected payoffs:

$$\rho^y \in \arg\max_{\rho \in [0,1]} \rho \times [q_i \times (1 - \kappa^y) + (1 - q_i) \times (1 - \nu^y) - c] \times [\gamma \times 1 + (1 - \gamma) \times \pi];$$

(v) Given the state of the world $\omega_1 \in \Omega$, the policy enacted by high-ability incumbents in the first period, denoted as $y_1(h, \omega_1)$, maximizes their expected payoffs:

$$y_1(h, \omega_1) \in \arg\max_{y \in Y} u(y, \omega_1) + \delta \times [\rho^y \times (q_i + (1 - q_i) \times \nu^y) + (1 - \rho^y)] \times u_2(h);$$

(vi) Low-ability incumbents enact policy a in the first period with a probability σ that maximizes their

expected payoffs:

$$\sigma \in \underset{s\in[0,1]}{\arg\max} \begin{array}{l} s \times \{\pi \times u(a,a) + (1-\pi) \times u(a,b) + \delta \times [\rho^a \times (1-q_i) \times \nu^a + (1-\rho^a)] \times u_2(\ell)\} \\ + (1-s) \times \left\{\pi \times u(b,a) + (1-\pi) \times u(b,b) + \delta \times \left[\rho^b \times (1-q_i) \times \nu^b + \left(1-\rho^b\right)\right] \times u_2(\ell)\right\}; \end{array}$$

(vii) The posterior probability that the Incumbent has a high ability conditional on having implemented policy $y \in Y$ in the first period is computed using Bayes' Rule:

$$\kappa^y = \frac{\kappa \times [\pi \times \mathbf{1}\{y_1(h,a) = y\} + (1-\pi) \times \mathbf{1}\{y_1(h,b) = y\}]}{\kappa \times [\pi \times \mathbf{1}\{y_1(h,a) = y\} + (1-\pi) \times \mathbf{1}\{y_1(h,b) = y\}] + (1-\kappa) \times [\mathbf{1}\{y = a\} \times \sigma + \mathbf{1}\{y = b\} \times (1-\sigma)]}.$$

B Equilibrium Analysis Without Endogenous Challenger Entry

Proposition A1. *In equilibrium, high-ability incumbents enact the policy corresponding to the state of the world in the first period.*

Low-ability incumbents invariably enact policy a in the first period if the following condition holds:

$$\frac{2 \times \pi - 1}{\delta \times \pi} > 1 - q_i.$$

In this case, when the Voter does not observe the Incumbent's type before the election, they reelect the Incumbent after she has enacted policy a in the first period if $\kappa \geq \frac{\gamma}{\gamma+(1-\gamma)\times\pi}$*, replace her with the Challenger otherwise, and reelect the Incumbent with certainty after she has enacted policy b.*

If the condition above does not hold, the model's equilibria are as follows:

(i) If $\kappa < \gamma$*:*

- *Low-ability incumbents enact policy a in the first period with probability* $\sigma = 1 - \frac{\kappa}{1-\kappa} \times \frac{1-\gamma}{\gamma} \times (1 - \pi)$*;*
- *When the Incumbent's type is not exogenously revealed before the election, they replace the Incumbent with the Challenger with certainty after she has enacted policy a in the first period and reelect the Incumbent with probability* $\nu^b = \frac{2\times\pi-1}{\delta\times\pi\times(1-q_i)}$ *after she has enacted policy b;*

(ii) If $\kappa \in \left(\gamma, \frac{\gamma}{\gamma+(1-\gamma)\times\pi}\right)$*:*

- *Low-ability incumbents enact policy a in the first period with probability* $\sigma = \frac{\kappa}{1-\kappa} \times \frac{1-\gamma}{\gamma} \times \pi$*;*
- *When the Incumbent's type is not exogenously revealed before the election, they reelect the Incumbent with probability* $\nu^a = 1 - \frac{2\times\pi-1}{\delta\times\pi\times(1-q_i)}$ *after she has enacted policy a in the first period and with certainty after she has enacted policy b;*

(iii) If $\kappa > \frac{\gamma}{\gamma+(1-\gamma)\times\pi}$*:*

- *Low-ability incumbents enact policy a in the first period;*
- *When the Incumbent's type is not exogenously revealed before the election, they reelect the Incumbent with certainty, irrespective of her first-period policy decision.*

Proof. I begin by characterizing high-ability incumbents' policy decisions.

Given the state of the world ω_1, it is sequentially rational for high-ability incumbents to enact the policy corresponding to the state of the world in the first period if and only if the following holds:

$$1 + \delta \times \bar{v}_i\left(h, \nu^{\omega_1}\right) \times 1 \geq 0 + \delta \times \bar{v}_i\left(h, \nu^{-\omega_1}\right) \times 1.$$

I use $-\omega_1$ to denote the policy opposite to the state of the world ω_1.

This can be rearranged as follows:

$$\delta \times \left[\bar{v}_i\left(h, \nu^{-\omega_1}\right) - \bar{v}_i\left(h, \nu^{\omega_1}\right)\right] \leq 1.$$

This equation stipulates that when pondering which policy to enact in the first period, high-ability incumbents weigh the cost of enacting the "wrong" policy against the resulting improvement in their reelection prospects.

Since it is the difference between two probabilities, the improvement in high-ability incumbents' reelection prospects associated with implementing the policy that does not correspond to the state of the world is bounded above by one:

$$\bar{v}_i\left(h; \gamma, \nu^{-\omega}\right) - \bar{v}_i\left(h; \gamma, \nu^{\omega}\right) \leq 1.$$

Combined with the assumption that the discount factor δ has a value strictly lower than one, it follows that the previous equation must hold with strict inequality:

$$\delta \times \left[\bar{v}_i\left(h; \gamma, \nu^{-\omega}\right) - \bar{v}_i\left(h; \gamma, \nu^{\omega}\right)\right] \leq \delta \times 1 < 1.$$

Therefore, in equilibrium, high-ability incumbents necessarily enact the policy corresponding to the state of the world in the first period.

Now, I characterize low-ability incumbents' policy decisions.

It is sequentially rational for low-ability incumbents to enact policy a in the first period if and only if the following holds:

$$\pi \times 1 + (1 - \pi) \times 0 + \delta \times \bar{v}_i\left(\ell, \nu^{a}\right) \times \pi \geq \pi \times 0 + (1 - \pi) \times 1 + \delta \times \bar{v}_i\left(\ell, \nu^{b}\right) \times \pi.$$

This can be rearranged as follows:

$$\bar{v}_i\left(\ell, \nu^b\right) - \bar{v}_i\left(\ell, \nu^a\right) \leq \frac{2 \times \pi - 1}{\delta \times \pi}. \tag{A1}$$

This inequality stipulates that, when pondering which policy to enact in the first period, low-ability incumbents weigh the cost of enacting policy b, which is less likely to match the state of the world than policy a, relative to the benefits of holding office in the second period against the resulting improvement in their reelection prospects.

Leveraging Section 3, it can be easily demonstrated that the left-hand side of Equation (A1) equals:

$$\bar{v}_i\left(\ell; \gamma, \nu^b\right) - \bar{v}_i\left(\ell; \gamma, \nu^a\right) = (1 - q_i) \times \left(\nu^b - \nu^a\right).$$

Sequential rationality of the Voter's actions imposes that, absent exogenous information about the Incumbent's type before the election, they elect the candidate who is most likely to have a high ability:

$$\nu^y = 1\,(0) \Rightarrow \kappa^y \geq (\leq)\,\gamma.$$

Since it equals the difference between two probabilities, the potential improvement in low-ability incumbents' reelection probability from enacting policy b, absent exogenous information about the Incumbent's private type before the election, is bounded above by one:

$$\nu^b - \nu^a \leq 1.$$

It follows that if $\frac{2\times\pi-1}{\delta\times\pi} \geq 1 - q_i$, Equation (A1) necessarily holds, reflecting the fact that the cost of enacting policy b in the first period systematically outweighs the potential improvement in low-ability incumbents' reelection prospects. In this case, low-ability incumbents invariably enact policy a in the first period. On the other hand, if $\frac{2\times\pi-1}{\delta\times\pi} < 1 - q_i$, the potential improvement in low-ability incumbents' reelection prospects may be sufficiently valuable for them to distort their policy decisions in equilibrium.

Equation (A1) must hold in equilibrium. Let us assume it did not. In this case, low-ability incumbents would find it sequentially rational to enact policy b invariably in the first period. Accordingly, if she enacted policy a, the Voter would deduce that the Incumbent has a high ability. This would negate the electoral

benefits associated with policy b, thus eliminating the Incumbent's motives for distorting her policy decisions in the first place.

Henceforth, I distinguish two cases: whether Equation (A1) holds with strict inequality or with equality in equilibrium.

In the first case, low-ability incumbents invariably enact policy a in the first period. Accordingly, the Voter infers the Incumbent has a high ability if she enacted policy b in the first period, ensuring her reelection. In contrast, if she enacted policy a, she has a probability $\kappa^a = \frac{\kappa \times \pi}{\kappa \times \pi + (1-\kappa)}$ of having a high ability.

Since I am considering the case wherein low-ability incumbents find the potential improvement in their reelection prospects from enacting policy b sufficiently valuable, it is sequentially rational for them to invariably enact policy a in the first period only if they are guaranteed to be reelected after doing so. For this to occur in equilibrium, the Incumbent must be sufficiently likely to have a high ability after enacting policy a in the first period such that, absent exogenous disclosure of the Incumbent's type before the election, the Voter finds it sequentially rational to reelect the Incumbent rather than replace her with the Challenger:

$$\nu^a = 1 \Rightarrow \kappa^a \geq \gamma.$$

This condition can be reformulated as follows:

$$\kappa^a = \frac{\kappa \times \pi}{\kappa \times \pi + (1-\kappa)} \geq \gamma \Leftrightarrow \kappa \geq \frac{\gamma}{\gamma + (1-\gamma) \times \pi}.$$

In the second case, low-ability incumbents are indifferent between enacting both policies in the first period. Accordingly, they are willing to randomize between enacting each policy in the first period. In equilibrium, the extent to which they do must be set to make the Voter indifferent between reelecting the Incumbent or replacing her with the Challenger after she has enacted one of the two available policies:

$$\kappa^y = \gamma.$$

In turn, after the Incumbent has enacted one of the two available policies, the Voter must randomize between reelecting her and replacing her with the Challenger to the extent that low-ability incumbents are

indifferent between enacting both policies:

$$(1 - q_i) \times \left(\nu^b - \nu^a\right) = \frac{2 \times \pi - 1}{\delta \times \pi}.$$

For this equation to hold, the Incumbent must be more likely to be reelected after enacting policy b than after enacting policy a. Formally, this means that we must have $\nu^b > \nu^a$. In turn, sequential rationality of the Voter's electoral choices imposes that we have $\kappa^b > \kappa^a$.

Given that in equilibrium, the Voter can only randomize between reelecting the Incumbent and replacing her with the Challenger after she has enacted one of the policies, there are two subcases to consider: the one wherein the Voter is indifferent between reelecting the Incumbent and replacing her with the Challenger after she has enacted policy a, and the other wherein they are indifferent after the Incumbent has enacted policy b.

In the first subcase, low-ability incumbents enact policy a with a probability σ, making the posterior probability that the Incumbent has a high ability conditional on having enacted policy a in the first period equal to the probability that the Challenger has a high ability:

$$\kappa^a = \frac{\kappa \times \pi}{\kappa \times \pi + (1 - \kappa) \times \sigma} = \gamma \Leftrightarrow \sigma = \frac{\kappa}{1 - \kappa} \times \frac{1 - \gamma}{\gamma} \times \pi.$$

Demonstrably, this value of σ is strictly positive. This value must also be lower than one, which translates into the following condition:

$$\frac{\kappa}{1 - \kappa} \times \frac{1 - \gamma}{\gamma} \times \pi \leq 1 \Leftrightarrow \kappa \leq \frac{\gamma}{\gamma + (1 - \gamma) \times \pi}.$$

To ensure sequential rationality of the Voter's electoral choices, the Incumbent must have a higher posterior probability of having a high ability conditional on having enacted policy b in the first period than the Challenger:

$$\kappa^b = \frac{\kappa \times (1 - \pi)}{\kappa \times (1 - \pi) + (1 - \kappa) \times (1 - \sigma)} \geq \gamma \Leftrightarrow \kappa \geq \gamma.$$

In the second subcase, low-ability incumbents enact policy b with a probability σ, making the posterior probability that the Incumbent has a high ability conditional on having enacted policy b in the first period

equal to the probability that the Challenger has a high ability:

$$\kappa^b = \frac{\kappa \times (1-\pi)}{\kappa \times (1-\pi) + (1-\kappa) \times (1-\sigma)} = \gamma \Leftrightarrow \sigma = 1 - \frac{\kappa}{1-\kappa} \times \frac{1-\gamma}{\gamma} \times (1-\pi)\,.$$

Demonstrably, this value of σ is strictly lower than one. This value must also be positive, which translates into the following condition:

$$1 - \frac{\kappa}{1-\kappa} \times \frac{1-\gamma}{\gamma} \times (1-\pi) \geq 0 \Leftrightarrow \kappa \leq \frac{\gamma}{\gamma + (1-\gamma) \times (1-\pi)}.$$

To ensure sequential rationality of the Voter's electoral choices, the Incumbent must have a higher posterior probability of having a high ability conditional on having enacted policy a in the first period than the Challenger:

$$\kappa^a = \frac{\kappa \times \pi}{\kappa \times \pi + (1-\kappa) \times \sigma} \leq \gamma \Leftrightarrow \kappa \leq \gamma.$$

Note that this condition is the only one binding since the previous must necessarily hold if this one does. □

C Proof of Lemma 1

Proof. Equation (1) characterizes the conditions under which it is sequentially rational for the Challenger to run for office:

$$\kappa^y \leq \frac{1-(1-q_i)\times \nu^y - c}{q_i}.$$

This equation does not directly describe the Challenger's equilibrium entry strategy. The reason is that both sides depend on κ^y. Indeed, the left-hand side contains it explicitly, whereas the right-hand side contains ν^y, which depends on κ^y through the sequential rationality of the Voter's electoral behavior. Specifically, sequential rationality requires that the Voter elects the candidate who is most likely to have a high ability to hold office in the second period:

$$\nu^y = 1\,(0) \Rightarrow \kappa^y \geq (\leq)\,\gamma.$$

To characterize the Challenger's equilibrium entry strategy, it is necessary to consider three scenarios contingent upon the value of ν^y.

First, if $\nu^y = 1$, the right-hand side of Equation (1) equals $1 - \frac{c}{q_i}$, which I hereafter denote by $\underline{\kappa}$. Sequential rationality of the Voter's electoral choices imposes that we have $\kappa^y \geq \gamma$. Both conditions cannot concurrently hold unless $\gamma < \underline{\kappa}$. In this case, the Challenger runs for office if and only if $\kappa^y \in (\gamma, \underline{\kappa})$. If $\kappa^y = \underline{\kappa}$, the Challenger may randomize between contesting the election and conceding to the Incumbent as he is indifferent between both.

Second, if $\nu^y = 0$, the right-hand side of Equation (1) equals $\frac{1-c}{q_i}$, which I hereafter denote by $\bar{\kappa}$. Sequential rationality of the Voter's electoral choices imposes that we have $\kappa^y \leq \gamma$. Consequently, the Challenger runs for office if and only if $\kappa^y \leq \min\{\gamma, \bar{\kappa}\}$. If $\bar{\kappa} < \gamma$, this means that the Challenger runs for office if and only if $\kappa^y \leq \bar{\kappa}$. Also, if $\kappa^y = \bar{\kappa}$, the Challenger may randomize between contesting the election and conceding to the Incumbent as he is indifferent between both. On the other hand, if $\bar{\kappa} > \gamma$, this means that the Challenger runs for office if and only if $\kappa^y \leq \gamma$.

Third, if $\nu^y \in (0, 1)$, the right-hand side of Equation (1) equals a value between $\underline{\kappa}$ and $\bar{\kappa}$. Sequential rationality of the Voter's electoral choices requires that we have $\kappa^y = \gamma$. Generically, it occurs only if $\sigma \in (0, 1)$. This requires that low-ability incumbents be indifferent between enacting both policies in the first period and that ν^y be defined as such. In this case, the Challenger runs for office if and only if the probability that the Challenger has high ability is lower than or equal to the right-hand side of Equation (1)

induced by this value of ν^y.

The value of ν^y may also be set to make the Challenger indifferent between running for office or not:

$$\nu^y = \frac{q_i \times (\bar{\kappa} - \gamma)}{1 - q_i} =: \hat{\nu}.$$

In this case, the Challenger may randomize between running and not running. He must do so to make the Voter indifferent between reelecting the Incumbent and replacing her with the Challenger, absent any exogenous information about the Incumbent's type before the election. □

Figure A1 illustrates the Challenger's equilibrium entry strategy. The areas over which the Challenger runs for office are crosshatched or shaded. The horizontal axis represents the probability that the Challenger has a high ability. The vertical axis represents the posterior probability that the Incumbent has a high ability given her first-period policy decision. The crosshatched area highlights the cases wherein: (i) the Challenger runs for office, and (ii) the Incumbent is necessarily reelected absent exogenous information disclosure about her type before the election. The shaded area highlights the cases wherein: (i) the Challenger runs for office, and (ii) he necessarily replaces the Incumbent absent exogenous information disclosure about her type before the election.

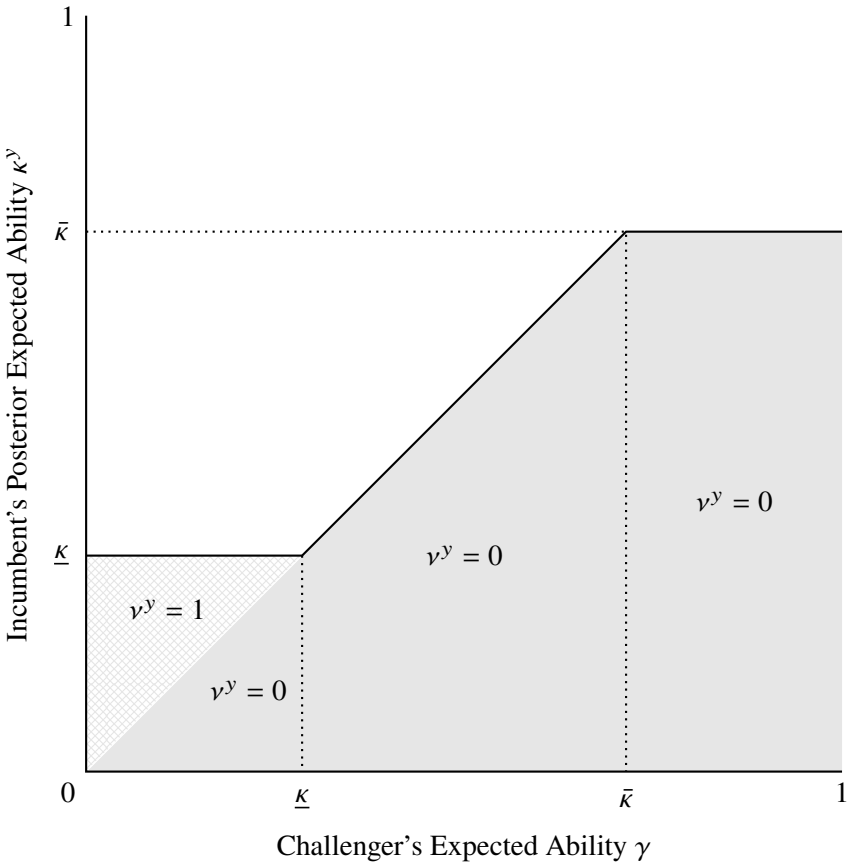

Figure A1: The Challenger's Entry Strategy

D Proof of Lemma 2

Proof. I consider three scenarios contingent on the value of γ relative to the thresholds $\underline{\kappa}$ and $\bar{\kappa}$.

First, I consider the case in which $\gamma < \underline{\kappa}$.

If $\kappa^y < \gamma$, the Voter replaces the Incumbent with the Challenger when no exogenous information about her type is revealed before the election. Since $\gamma < \underline{\kappa} < \bar{\kappa}$, it is sequentially rational for the Challenger to run for office. It follows that low-ability incumbents' reelection probability is null. Note that if $\kappa^y = \gamma$, all values in the interval $[0, 1 - q_i]$ can be sustained in equilibrium since the Voter is indifferent between reelecting the Incumbent and replacing her with the Challenger.

At the other end of the spectrum, if $\kappa^y > \bar{\kappa}$, it is sequentially rational for the Challenger not to run for office. Since $\gamma < \underline{\kappa} < \bar{\kappa}$, it is sequentially rational for the Voter to reelect the Incumbent absent exogenous information about her type before the election. It follows that the Incumbent is reelected with certainty. Note that if $\kappa^y = \bar{\kappa}$, all values in the interval $[1 - q_i, 1]$ can be sustained in equilibrium since the Challenger is indifferent between contesting the election and conceding to the Incumbent.

If $\kappa^y \in (\gamma, \underline{\kappa})$, it is sequentially rational for the Challenger to run for office and for the Voter to reelect the Incumbent absent exogenous information about her type before the election. It follows that low-ability incumbents' reelection probability is $1 - q_i$.

Second, I consider the case in which $\gamma \in (\underline{\kappa}, \bar{\kappa})$.

If $\kappa^y < \gamma$, it is sequentially rational for the Challenger to run for office and for the Voter to replace the Incumbent with the Challenger absent exogenous information about her type before the election. In this case, low-ability incumbents' reelection probability is null.

On the other hand, if $\kappa^y > \gamma$, it is sequentially rational for the Challenger to concede to the Incumbent and for the Voter to reelect the Incumbent absent exogenous information about her type before the election. In this case, low-ability incumbents' reelection is guaranteed.

I now consider the case when $\kappa^y = \gamma$. In this case, I show that there are equilibrium values of ν^y and ρ^y such that low-ability incumbents' reelection probability can take any value in the interval $[0, 1]$ in equilibrium. Note that since $\gamma \in (\underline{\kappa}, \bar{\kappa})$, there is a value of $\nu^y \in (0, 1)$ such that the right-hand side of Equation (1) equals γ. I denote this value as $\hat{\nu}$:

$$\hat{\nu} = \frac{q_i \times (\bar{\kappa} - \gamma)}{1 - q_i}.$$

If $\nu^y < \hat{\nu}$, the Challenger necessarily runs for office. In contrast, if $\nu^y > \hat{\nu}$, the Challenger concedes to the Incumbent.

Given that $\kappa^y = \gamma$, the Voter is indifferent between reelecting the Incumbent and replacing her with the Challenger. Thus, all values of $\nu^y \in (0, 1)$ can be sustained in equilibrium. Further, if $\nu^y \leq \bar{\nu}$, it is sequentially rational for the Challenger to run for office. It follows that all values in the interval $[0, (1 - q_i) \times \bar{\nu}]$ can be sustained as equilibrium values of low-ability incumbents' reelection probability.

If $\kappa^y = \gamma$ and $\nu^y = \bar{\nu}$, the Challenger is indifferent between contesting the election and conceding to the Incumbent. In this case, all values of $\rho^y \in (0, 1)$ are sustainable in equilibrium. This implies that all values in the interval $[(1 - q_i) \times \bar{\nu}, 1]$ can be sustained as equilibrium values of low-ability incumbents' reelection probability.

Third, I consider the case in which $\gamma > \bar{\kappa}$.

If $\kappa^y < \bar{\kappa}$, it is sequentially rational for the Challenger to run for office. Also, since $\gamma > \bar{\kappa}$, the Voter replaces the Incumbent absent exogenous information about her type before the election. It follows that low-ability incumbents' reelection probability is null.

Next, if $\kappa^y = \bar{\kappa}$, the Challenger is indifferent between contesting the election and conceding to the Incumbent, hence all values in the interval $[0, 1]$ can be sustained as equilibrium values of low-ability incumbents' reelection probability.

Finally, if $\kappa^y > \bar{\kappa}$, it is sequentially rational for the Challenger not to run for office. This is true if $\kappa^y < \gamma$ and the Voter replaces the Incumbent with the Challenger absent exogenous information about her type before the election and even more if $\kappa^y > \gamma$ and the Voter reelects the Incumbent absent exogenous information about her type before the election. It follows that low-ability incumbents are reelected with certainty. □

E Equilibrium Analysis With Endogenous Challenger Entry

Proposition A2. *In equilibrium, low-ability incumbents invariably enact policy a in the first period if the following condition is met:*

$$\frac{2\pi - 1}{\delta \times \pi} > 1.$$

If this condition does not hold, the probability that low-ability incumbents enact policy a in the first period is as follows:

(a) If $\gamma < \underline{\kappa}$*:*

(i) If $\kappa < \gamma$*:*

- *If* $\frac{2\pi-1}{\delta\times\pi} < 1 - q_i$*:*

$$\sigma = 1 - \frac{\kappa}{1-\kappa} \times \frac{1-\gamma}{\gamma} \times (1-\pi)\,;$$

- *If* $\frac{2\pi-1}{\delta\times\pi} \in (1 - q_i, 1)$*:*

$$\sigma = 1 - \frac{\kappa}{1-\kappa} \times \frac{1-\underline{\kappa}}{\underline{\kappa}} \times (1-\pi)\,;$$

(ii) If $\kappa \in \left(\gamma, \frac{\underline{\kappa}\times\gamma}{\pi\times\underline{\kappa}+(1-\pi)\times\gamma}\right)$*:*

- *If* $\frac{2\pi-1}{\delta\times\pi} < 1 - q_i$*:*

$$\sigma = \frac{\kappa}{1-\kappa} \times \frac{1-\gamma}{\gamma} \times \pi;$$

- *If* $\frac{2\pi-1}{\delta\times\pi} \in (1 - q_i, 1)$*:*

$$\sigma = 1 - \frac{\kappa}{1-\kappa} \times \frac{1-\underline{\kappa}}{\underline{\kappa}} \times (1-\pi)\,;$$

(iii) If $\kappa \in \left(\frac{\underline{\kappa}\times\gamma}{\pi\underline{\kappa}+(1-\pi)\times\gamma}, \underline{\kappa}\right)$*:*

- *If* $\frac{2\pi-1}{\delta\times\pi} < q_i$*:*

$$\sigma = 1 - \frac{\kappa}{1-\kappa} \times \frac{1-\underline{\kappa}}{\underline{\kappa}} \times (1-\pi)\,;$$

- *If* $\frac{2\pi-1}{\delta\times\pi} \in (q_i, 1)$*:*

$$\sigma = \frac{\kappa}{1-\kappa} \times \frac{1-\gamma}{\gamma} \times \pi;$$

(iv) If $\kappa > \underline{\kappa}$*:*

- *If* $\frac{2\pi-1}{\delta\times\pi} < q_i$*:*

– *If* $\kappa \in \left(\underline{\kappa}, \frac{\underline{\kappa}}{\underline{\kappa}+\left(1-\underline{\kappa}\right)\times\pi}\right)$*:*

$$\sigma = \frac{\kappa}{1-\kappa} \times \frac{1-\underline{\kappa}}{\underline{\kappa}} \times \pi;$$

– *If* $\kappa > \frac{\underline{\kappa}}{\underline{\kappa}+\left(1-\underline{\kappa}\right)\times\pi}$*:*

$$\sigma = 1;$$

• *If* $\frac{2\pi-1}{\delta\times\pi} \in (q_i, 1)$*:*

– *If* $\underline{\kappa} < \frac{\gamma}{\gamma+(1-\gamma)\times\pi}$ *and* $\kappa \in \left(\underline{\kappa}, \frac{\gamma}{\gamma+(1-\gamma)\times\pi}\right)$*:*

$$\sigma = \frac{\kappa}{1-\kappa} \times \frac{1-\gamma}{\gamma} \times \pi;$$

– *If* $\kappa > \max\left\{\underline{\kappa}, \frac{\gamma}{\gamma+(1-\gamma)\times\pi}\right\}$*:*

$$\sigma = 1;$$

(b) If $\gamma \in \left(\underline{\kappa}, \bar{\kappa}\right)$*:*

(i) If $\kappa < \gamma$*:*

$$\sigma = 1 - \frac{\kappa}{1-\kappa} \times \frac{1-\gamma}{\gamma} \times (1-\pi);$$

(ii) If $\kappa \in \left(\gamma, \frac{\gamma}{\gamma+(1-\gamma)\times\pi}\right)$*:*

$$\sigma = \frac{\kappa}{1-\kappa} \times \frac{1-\gamma}{\gamma} \times \pi;$$

(iii) If $\kappa > \frac{\gamma}{\gamma+(1-\gamma)\times\pi}$*:*

$$\sigma = 1;$$

(c) If $\gamma > \bar{\kappa}$*:*

(i) If $\kappa < \bar{\kappa}$*:*

$$\sigma = 1 - \frac{\kappa}{1-\kappa} \times \frac{1-\bar{\kappa}}{\bar{\kappa}} \times (1-\pi);$$

(ii) If $\kappa \in \left(\bar{\kappa}, \frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi}\right)$*:*

$$\sigma = \frac{\kappa}{1-\kappa} \times \frac{1-\bar{\kappa}}{\bar{\kappa}} \times \pi;$$

(iii) If $\kappa > \frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi}$*:*

$$\sigma = 1.$$

F Proof of Proposition 3

Proof. As a preamble, note that the thresholds $\underline{\kappa}$ and $\bar{\kappa}$ have an identical partial derivative with respect to parameter c and, in particular, marginally decrease with its value:

$$\frac{\partial \underline{\kappa}}{\partial c} = \frac{\partial \bar{\kappa}}{\partial c} = \frac{-1}{q_i}.$$

There are two sets of conditions under which the equilibrium probability that low-ability incumbents enact policy a in the first period depends positively on the value of c:

(i) $\gamma < \underline{\kappa}$, $\frac{2\pi-1}{\delta\times\pi} < q_i$, and $\kappa \in \left(\underline{\kappa}, \frac{\underline{\kappa}}{\underline{\kappa}+\left(1-\underline{\kappa}\right)\times\pi}\right)$; or

(ii) $\gamma > \bar{\kappa}$ and $\kappa \in \left(\bar{\kappa}, \frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi}\right)$.

I consider each case in turn.

In the first case, the probability that low-ability incumbents enact policy a in the first period equals:

$$\sigma = \frac{\kappa}{1-\kappa} \times \frac{1-\underline{\kappa}}{\underline{\kappa}} \times \pi.$$

The partial derivative of this probability with respect to parameter c equals:

$$\frac{\partial \sigma}{\partial c} = \frac{\kappa}{1-\kappa} \times \pi \times \frac{\partial\left(\frac{1-\underline{\kappa}}{\underline{\kappa}}\right)}{\partial c} = \frac{\kappa}{1-\kappa} \times \pi \times \frac{-\frac{\partial \underline{\kappa}}{\partial c}}{\underline{\kappa}^2} = \frac{\kappa}{1-\kappa} \times \pi \times \frac{-1 \times \frac{-1}{q_i}}{\underline{\kappa}^2} = \frac{\kappa}{1-\kappa} \times \pi \times \frac{1}{q_i \times \underline{\kappa}^2} > 0.$$

Accordingly, the equilibrium probability that low-ability incumbents enact policy a in the first period marginally increases with the value of c.

I now rearrange the conditions outlined in point (i) so that they are expressed as bounds on the value of the parameter c.

First, $\gamma < \underline{\kappa}$ can be rearranged as follows:

$$\gamma < \underline{\kappa} = 1 - \frac{c}{q_i} \Leftrightarrow c < q_i \times (1-\gamma).$$

Second, $\kappa > \underline{\kappa}$ can be rearranged as follows:

$$\kappa > \underline{\kappa} = 1 - \frac{c}{q_i} \Leftrightarrow c > q_i \times (1-\kappa).$$

Third, $\kappa < \frac{\underline{\kappa}}{\underline{\kappa}+(1-\underline{\kappa})\times\pi}$ can be rearranged as follows:

$$\kappa < \frac{\underline{\kappa}}{\underline{\kappa}+(1-\underline{\kappa})\times\pi} = \frac{\underline{\kappa}}{\pi+(1-\pi)\times\underline{\kappa}} \Leftrightarrow \frac{\pi\times\kappa}{1-(1-\pi)\times\kappa} < \underline{\kappa} = 1-\frac{c}{q_i} \Leftrightarrow c < q_i\times\left(1-\frac{\pi\times\kappa}{1-(1-\pi)\times\kappa}\right).$$

Combined, these conditions can be expressed as follows:

$$q_i\times(1-\kappa) < c < \min\left\{q_i\times\left(1-\frac{\pi\times\kappa}{1-(1-\pi)\times\kappa}\right), q_i\times(1-\gamma)\right\} = q_i\times\left(1-\max\left\{\frac{\pi\times\kappa}{1-(1-\pi)\times\kappa}, \gamma\right\}\right).$$

These conditions may hold only if $\kappa > \gamma$.

In the second case, the probability that low-ability incumbents enact policy a in the first period equals:

$$\sigma = \frac{\kappa}{1-\kappa}\times\frac{1-\bar{\kappa}}{\bar{\kappa}}\times\pi.$$

The partial derivative of this probability with respect to parameter c equals:

$$\frac{\partial\sigma}{\partial c} = \frac{\kappa}{1-\kappa}\times\pi\times\frac{\partial\left(\frac{1-\bar{\kappa}}{\bar{\kappa}}\right)}{\partial c} = \frac{\kappa}{1-\kappa}\times\pi\times\frac{-\frac{\partial\bar{\kappa}}{\partial c}}{\bar{\kappa}^2} = \frac{\kappa}{1-\kappa}\times\pi\times\frac{-1\times\frac{-1}{q_i}}{\bar{\kappa}^2} = \frac{\kappa}{1-\kappa}\times\pi\times\frac{1}{q_i\times\bar{\kappa}^2} > 0.$$

Accordingly, the equilibrium probability that low-ability incumbents enact policy a in the first period marginally increases with the value of c.

I now rearrange the conditions outlined in point (ii) so that they are expressed as bounds on the value of the parameter c.

First, $\gamma > \bar{\kappa}$ can be rearranged as follows:

$$\gamma > \bar{\kappa} = \frac{1-c}{q_i} \Leftrightarrow c > 1-q_i\times\gamma.$$

Second, $\kappa > \bar{\kappa}$ can be rearranged as follows:

$$\kappa > \bar{\kappa} = \frac{1-c}{q_i} \Leftrightarrow c > 1-q_i\times\kappa.$$

Third, $\kappa < \frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi}$ can be rearranged as follows:

$$\kappa < \frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi} = \frac{\bar{\kappa}}{\pi+(1-\pi)\times\bar{\kappa}} \Leftrightarrow \frac{1-c}{q_i} = \bar{\kappa} > \frac{\pi\times\kappa}{1-(1-\pi)\times\kappa} \Leftrightarrow c < 1 - q_i \times \frac{\pi\times\kappa}{1-(1-\pi)\times\kappa}.$$

Combined, these conditions can be expressed as follows:

$$1 - q_i \times \min\{\kappa, \gamma\} = \max\{1 - q_i \times \kappa, 1 - q_i \times \gamma\} < c < 1 - q_i \times \frac{\pi\times\kappa}{1-(1-\pi)\times\kappa}.$$

These conditions may hold only if $\gamma > \frac{\pi\times\kappa}{1-(1-\pi)\times\kappa}$. □

G Proof of Proposition 4

Proof. When the Challenger always runs for office, the Voter's welfare equals:

$$\kappa \times [\pi \times u(y_1(h,a),a) + (1-\pi) \times u(y_1(h,b),b)]$$
$$+ (1-\kappa) \times [\sigma \times (\pi \times u(a,a) + (1-\pi) \times u(a,b)) + (1-\sigma) \times (\pi \times u(b,a) + (1-\pi) \times u(b,b))]$$
$$+ \hat{\kappa} \times u_2(h) + (1-\hat{\kappa}) \times u_2(\ell),$$

where $\hat{\kappa}$ denotes the second-period officeholder's expected ability.

In this case, the second-period officeholder's expected ability equals:

$$\hat{\kappa} = q_i \times [\kappa + (1-\kappa) \times \gamma] + (1-q_i) \times \max\{\kappa, \gamma\}.$$

In general, when the Incumbent's type is not revealed before the election, the second-period officeholder's expected ability is defined as follows:

$$[\kappa \times \pi + (1-\kappa) \times \sigma] \times \max\{\kappa^a, \gamma\} + [\kappa \times (1-\pi) \times (1-\kappa) \times (1-\sigma)] \times \max\left\{\kappa^b, \gamma\right\}$$

The definition of $\hat{\kappa}$ found above takes advantage of the fact that in equilibrium, without endogenous challenger entry, $\kappa^a \leq \gamma$ and $\kappa^b \leq \gamma$ if $\kappa \leq \gamma$, and $\kappa^a \geq \gamma$ and $\kappa^b \geq \gamma$ otherwise. Therefore, the second-period officeholder's expected ability when the Incumbent's type is not revealed before the election simplifies to $\max\{\kappa, \gamma\}$.

With endogenous challenger entry, the Voter's welfare equals:

$$\kappa \times [\pi \times u(y_1(h,a),a) + (1-\pi) \times u(y_1(h,b),b)]$$
$$+ (1-\kappa) \times [\sigma' \times (\pi \times u(a,a) + (1-\pi) \times u(a,b)) + (1-\sigma') \times (\pi \times u(b,a) + (1-\pi) \times u(b,b))]$$
$$+ \hat{\kappa}' \times u_2(h) + (1-\hat{\kappa}') \times u_2(\ell),$$

where σ' denotes the probability that low-ability incumbents enact policy a in the first period and $\hat{\kappa}'$ the second-period officeholder's expected ability.

In this case, the second-period officeholder's expected ability equals:

$$\hat{\kappa}' = [\kappa \times \pi + (1-\kappa) \times \sigma'] \times \{\rho^a \times [q_i \times (\kappa^a + (1-\kappa^a) \times \gamma) + (1-q_i) \times \max\{\kappa^a, \gamma\}] + (1-\rho^a) \times \kappa^a\}$$
$$+ [\kappa \times (1-\pi) + (1-\kappa) \times (1-\sigma')] \times \left\{\rho^b \times \left[q_i \times \left(\kappa^b + \left(1-\kappa^b\right) \times \gamma\right) + (1-q_i) \times \max\left\{\kappa^b, \gamma\right\}\right] + \left(1-\rho^b\right) \times \kappa^b\right\}.$$

The difference between the Voter's welfare with and without endogenous challenger entry equals:

$$(1-\kappa) \times (\sigma' - \sigma) \times \underbrace{[\pi \times (u(a,a) - u(b,a)) + (1-\pi) \times (u(a,b) - u(b,b))]}_{=\pi \times 1 + (1-\pi) \times -1 = 2\pi - 1} + (\hat{\kappa}' - \hat{\kappa}) \times \underbrace{[u_2(h) - u_2(\ell)]}_{=1-\pi}.$$

For endogenous challenger entry to improve the Voter's welfare, this difference must be positive, which turns into the following condition:

$$(1-\kappa) \times (\sigma' - \sigma) \times (2 \times \pi - 1) \geq (\hat{\kappa} - \hat{\kappa}') \times (1-\pi).$$

The left-hand side of this inequality reflects the benefits of endogenous challenger entry in terms of fewer policy distortions. On the other hand, the right-hand side represents the cost of endogenous challenger entry in terms of weaker electoral selection.

Henceforth, I consider the case wherein endogenous challenger entry mitigates policy distortions, that is, when $\gamma > \bar{\kappa}$ and $\kappa \in \left(\frac{\gamma \times \bar{\kappa}}{\pi \times \gamma + (1-\pi) \times \bar{\kappa}}, \frac{\gamma}{\gamma + (1-\gamma) \times \pi}\right)$. In general, the latter interval is divided into three parts partitioned by the values $\frac{\bar{\kappa}}{\bar{\kappa} + (1-\bar{\kappa}) \times \pi}$ and γ.

I successively consider the left-hand and right-hand sides of the inequality under which endogenous challenger entry improves the Voter's welfare.

I begin by considering the left-hand side of the inequality.

If $\kappa < \min\left\{\frac{\bar{\kappa}}{\bar{\kappa} + (1-\bar{\kappa}) \times \pi}, \gamma\right\}$, the difference between the probability that low-ability incumbents enact policy a in the first period with and without endogenous challenger entry equals:

$$\sigma' - \sigma = \frac{\kappa}{1-\kappa} \times \frac{1-\bar{\kappa}}{\bar{\kappa}} \times \pi - \left[1 - \frac{\kappa}{1-\kappa} \times \frac{1-\gamma}{\gamma} \times (1-\pi)\right] = \frac{\kappa}{1-\kappa} \times \left[\pi \times \frac{1-\bar{\kappa}}{\bar{\kappa}} + (1-\pi) \times \frac{1-\gamma}{\gamma}\right] - 1.$$

If $\kappa > \max\left\{\frac{\bar{\kappa}}{\bar{\kappa} + (1-\bar{\kappa}) \times \pi}, \gamma\right\}$, the difference between the probability that low-ability incumbents enact

policy a in the first period with and without endogenous challenger entry equals:

$$\sigma' - \sigma = 1 - \frac{\kappa}{1-\kappa} \times \frac{1-\gamma}{\gamma} \times \pi.$$

There remain two cases to consider, depending on which of $\frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi}$ and γ is greater.

When $\gamma > \frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi}$, the difference between the probability that low-ability incumbents enact policy a in the first period with and without endogenous challenger entry if $\kappa \in \left(\frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi}, \gamma\right)$ equals:

$$\sigma' - \sigma = 1 - \left[1 - \frac{\kappa}{1-\kappa} \times \frac{1-\gamma}{\gamma} \times (1-\pi)\right] = \frac{\kappa}{1-\kappa} \times \frac{1-\gamma}{\gamma} \times (1-\pi)\,.$$

When $\gamma < \frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi}$, the difference between the probability that low-ability incumbents enact policy a in the first period with and without endogenous challenger entry if $\kappa \in \left(\gamma, \frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi}\right)$ equals:

$$\sigma' - \sigma = \frac{\kappa}{1-\kappa} \times \frac{1-\bar{\kappa}}{\bar{\kappa}} \times \pi - \frac{\kappa}{1-\kappa} \times \frac{1-\gamma}{\gamma} \times \pi = \frac{\kappa}{1-\kappa} \times \left(\frac{1-\bar{\kappa}}{\bar{\kappa}} - \frac{1-\gamma}{\gamma}\right) \times \pi.$$

On the whole, the benefits of endogenous challenger entry in terms of fewer policy distortions are maximized when $\kappa = \max\left\{\frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi}, \gamma\right\}$.

Next, I consider the right-hand side of the inequality.

Note that if $\kappa \in \left(\bar{\kappa}, \frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi}\right)$, the second-period officeholder's expected ability with endogenous challenger entry equals:

$$\begin{aligned}
&[\kappa \times \pi + (1-\kappa) \times \sigma] \times \{\rho^a \times [q_i \times (\kappa^a + (1-\kappa^a) \times \gamma) + (1-q_i) \times \gamma] + (1-\rho^a) \times \kappa^a\} \\
&+ [\kappa \times (1-\pi) + (1-\kappa) \times (1-\sigma)] \times \kappa^b \\
=&\kappa + [\kappa \times \pi + (1-\kappa) \times \sigma] \times \rho^a \times [q_i \times (\kappa^a + (1-\kappa^a) \times \gamma) + (1-q_i) \times \gamma - \kappa^a] \\
=&\kappa + \left[\kappa \times \pi + \kappa \times \frac{1-\bar{\kappa}}{\bar{\kappa}} \times \pi\right] \times \rho^a \times [q_i \times (1-\bar{\kappa}) \times \gamma + (1-q_i) \times (\gamma - \bar{\kappa})] \\
=&\kappa \times \left\{1 + \frac{\pi}{\bar{\kappa}} \times \rho^a \times [q_i \times (1-\bar{\kappa}) \times \gamma + (1-q_i) \times (\gamma - \bar{\kappa})]\right\}.
\end{aligned}$$

If $\kappa < \min\left\{\frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi}, \gamma\right\}$, the difference between the second-period officeholder's expected ability with

and without endogenous challenger entry equals:

$$\hat{\kappa} - \hat{\kappa}' = \underbrace{q_i \times [\kappa + (1-\kappa) \times \gamma] + (1-q_i) \times \gamma}_{=\gamma + q_i \times (1-\gamma) \times \kappa} - \kappa \times \left\{ 1 + \frac{\pi}{\bar{\kappa}} \times \rho^a \times [q_i \times (1-\bar{\kappa}) \times \gamma + (1-q_i) \times (\gamma - \bar{\kappa})] \right\}.$$

If $\kappa > \max\left\{ \frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi}, \gamma \right\}$, the difference between the second-period officeholder's expected ability with and without endogenous challenger entry equals:

$$\hat{\kappa} - \hat{\kappa}' = q_i \times [\kappa + (1-\kappa) \times \gamma] + (1-q_i) \times \kappa - \kappa = q_i \times (1-\kappa) \times \gamma.$$

When $\gamma > \frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi}$, the difference between the second-period officeholder's expected ability with and without endogenous challenger entry if $\kappa \in \left(\frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi}, \gamma \right)$ equals:

$$\hat{\kappa} - \hat{\kappa}' = q_i \times [\kappa + (1-\kappa) \times \gamma] + (1-q_i) \times \gamma - \kappa = \gamma - [1 - q_i \times (1-\gamma)] \times \kappa.$$

When $\gamma < \frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi}$, the difference between the second-period officeholder's expected ability with and without endogenous challenger entry if $\kappa \in \left(\gamma, \frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi} \right)$ equals:

$$\begin{aligned}\hat{\kappa} - \hat{\kappa}' &= q_i \times [\kappa + (1-\kappa) \times \gamma] + (1-q_i) \times \kappa - \kappa \times \left\{ 1 + \frac{\pi}{\bar{\kappa}} \times \rho^a \times [q_i \times (1-\bar{\kappa}) \times \gamma + (1-q_i) \times (\gamma - \bar{\kappa})] \right\} \\ &= q_i \times (1-\kappa) \times \gamma - \kappa \times \frac{\pi}{\bar{\kappa}} \times \rho^a \times [q_i \times (1-\bar{\kappa}) \times \gamma + (1-q_i) \times (\gamma - \bar{\kappa})].\end{aligned}$$

In all cases, the cost of weaker electoral selection induced by endogenous challenger entry decreases monotonically with κ.

Combined, these results imply that if endogenous challenger entry improves the Voter's welfare for some value of the Incumbent's expected ability, it must necessarily do so when the latter equals $\max\left\{ \gamma, \frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi} \right\}$. In other words, for endogenous challenger entry to improve the Voter's welfare, it must improve it when $\kappa = \max\left\{ \gamma, \frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi} \right\}$. Formally, this is reflected in the following inequality:

$$\left(1 - \frac{\kappa}{1-\kappa} \times \frac{1-\gamma}{\gamma} \times \pi \right) \times (2 \times \pi - 1) \geq q_i \times \gamma \times (1-\pi).$$

Evaluated at $\kappa = \gamma$, this condition simplifies to:

$$(1-\pi) \times (2 \times \pi - 1) \geq q_i \times \gamma \times (1-\pi) \Leftrightarrow 2 \times \pi - 1 \geq q_i \times \gamma.$$

Evaluated at $\kappa = \frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi}$, this condition simplifies to:

$$\left(1 - \frac{\bar{\kappa}}{1-\bar{\kappa}} \times \frac{1-\gamma}{\gamma}\right) \times (2 \times \pi - 1) \geq q_i \times \gamma \times (1-\pi) \Leftrightarrow (\gamma - \bar{\kappa}) \times (2\pi - 1) \geq q_i \times (1-\bar{\kappa}) \times \gamma^2 \times (1-\pi)$$

$$\Leftrightarrow [q_i \times \gamma - (1-c)] \times (2 \times \pi - 1) \geq q_i \times \gamma^2 \times [q_i - (1-c)] \times (1-\pi)\,.$$

When either of these necessary conditions is met, endogenous challenger entry improves the Voter's welfare across a range of values of the Incumbent's expected ability that contains the value $\max\left\{\gamma, \frac{\bar{\kappa}}{\bar{\kappa}+(1-\bar{\kappa})\times\pi}\right\}$ and is encompassed by the interval over which endogenous challenger entry improves policy decisions. □