A simple model of the Monetary Circuit

Functions to derive a system of coupled Ordinary Differential Equations

$$\text{System}(x) := \left| \begin{array}{l} \text{Functions} \leftarrow \text{submatrix}(x, 2, 2, 1, \text{cols}(x) - 1) \\ \text{Equations} \leftarrow \text{submatrix}(x, 3, \text{rows}(x) - 1, 1, \text{cols}(x) - 1) \\ \text{for } i \in 0 .. \text{cols}(\text{Functions}) - 1 \\ \quad E_i \leftarrow \frac{d}{dt}\text{Functions}_i = \sum \text{Equations}^{\langle i \rangle} \\ \text{return } E \end{array} \right.$$

$$\text{Vars}(x) := \left| \begin{array}{l} \text{vars}_{raw} \leftarrow \text{submatrix}(x, 2, 2, 1, \text{cols}(x) - 1) \\ \text{vars} \leftarrow \text{vars}_{raw}^{T} \\ \text{return vars} \end{array} \right.$$

$$\text{Audit}(x) := \left| \begin{array}{l} \text{Equations} \leftarrow \text{submatrix}(x, 3, \text{rows}(x) - 1, 1, \text{cols}(x) - 1) \\ \text{for } i \in 0 .. \text{rows}(\text{Equations}) - 1 \\ \quad \text{Sum}_i \leftarrow \sum_{j=0}^{\text{cols}(\text{Equations})-1} \text{Equations}_{i,j} \\ \text{return Sum} \end{array} \right.$$

Basic Flow Table

$$S_1 := \begin{pmatrix} \text{"Type"} & 1 & -1 & -1 & 0 \\ \text{"Name"} & \text{"Firm Loan"} & \text{"Firm Deposit"} & \text{"Worker Deposit"} & \text{"Bank Income"} \\ \text{"Symbol"} & F_L(t) & F_D(t) & W_D(t) & B_I(t) \\ \text{"Loan Transactions"} & A - A & -A & 0 & A \\ \text{"Deposit Interest Firms"} & 0 & B & 0 & -B \\ \text{"Wages"} & 0 & -C & C & 0 \\ \text{"Deposit Interest Workers"} & 0 & 0 & D & -D \\ \text{"Consumption"} & 0 & E + F & -E & -F \end{pmatrix}$$

$$\text{Audit}(S_1) \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Symbolic substitutions for alphabetic placeholders

$A := r_L \cdot F_L(t)$ $B := r_D \cdot F_D(t)$ $C := w \cdot F_D(t)$ $D := r_D \cdot W_D(t)$ $E := \omega \cdot W_D(t)$ $F := \beta \cdot B_I(t)$

$$S_1 := \begin{pmatrix} \text{"Type"} & 1 & -1 & -1 & 0 \\ \text{"Name"} & \text{"Firm Loan"} & \text{"Firm Deposit"} & \text{"Worker Deposit"} & \text{"Bank Income"} \\ \text{"Symbol"} & F_L(t) & F_D(t) & W_D(t) & B_I(t) \\ \text{"Loan Transactions"} & A - A & -A & 0 & A \\ \text{"Deposit Interest Firms"} & 0 & B & 0 & -B \\ \text{"Wages"} & 0 & -C & C & 0 \\ \text{"Deposit Interest Workers"} & 0 & 0 & D & -D \\ \text{"Consumption"} & 0 & E + F & -E & -F \end{pmatrix}$$

Resulting system of ODEs

$$\text{System}(S_1) \rightarrow \begin{pmatrix} \frac{d}{dt} F_L(t) = 0 \\ \frac{d}{dt} F_D(t) = \beta \cdot B_I(t) + r_D \cdot F_D(t) - r_L \cdot F_L(t) + \omega \cdot W_D(t) - w \cdot F_D(t) \\ \frac{d}{dt} W_D(t) = w \cdot F_D(t) - \omega \cdot W_D(t) + r_D \cdot W_D(t) \\ \frac{d}{dt} B_I(t) = r_L \cdot F_L(t) - r_D \cdot F_D(t) - \beta \cdot B_I(t) - r_D \cdot W_D(t) \end{pmatrix}$$

Derivation of equilibrium conditions

Given

$$F_L = \Lambda$$

$$0 = \beta \cdot B_I + r_D \cdot F_D - r_L \cdot F_L + \omega \cdot W_D - w \cdot F_D$$

$$0 = w \cdot F_D - \omega \cdot W_D + r_D \cdot W_D$$

$$0 = r_L \cdot F_L - r_D \cdot F_D - \beta \cdot B_I - r_D \cdot W_D$$

$$F_D + W_D + B_I = F_L$$

$$\begin{pmatrix} F_{L_eq} \\ F_{D_eq} \\ W_{D_eq} \\ B_{I_eq} \end{pmatrix} := \text{Find}\left(\begin{pmatrix} F_L \\ F_D \\ W_D \\ B_I \end{pmatrix}\right) \left|\begin{array}{l}\text{simplify} \\ \text{collect}, \Lambda\end{array}\right. \rightarrow \begin{bmatrix} \Lambda \\ \frac{(\beta - r_L)\cdot(\omega - r_D)}{(\beta - r_D)\cdot(\omega - r_D + w)}\cdot\Lambda \\ \frac{w\cdot(\beta - r_L)}{(\beta - r_D)\cdot(\omega - r_D + w)}\cdot\Lambda \\ \left(-\frac{r_D - r_L}{\beta - r_D}\right)\cdot\Lambda \end{bmatrix}$$

Rewriting w in terms of rate of surplus ***s*** and rate of turnover τ_S.

$$\begin{bmatrix} \Lambda \\ \frac{(\beta - r_L)\cdot(\omega - r_D)}{(\beta - r_D)\cdot(\omega - r_D + w)}\cdot\Lambda \\ \frac{w\cdot(\beta - r_L)}{(\beta - r_D)\cdot(\omega - r_D + w)}\cdot\Lambda \\ \left(-\frac{r_D - r_L}{\beta - r_D}\right)\cdot\Lambda \end{bmatrix} \left|\begin{array}{l}\text{substitute}, w = \frac{1 - s}{\tau_S} \\ \\ \text{simplify}\end{array}\right. \rightarrow \begin{bmatrix} \Lambda \\ -\frac{\Lambda\cdot\tau_S\cdot(\beta - r_L)\cdot(\omega - r_D)}{(\beta - r_D)\cdot(s - \omega\cdot\tau_S + r_D\cdot\tau_S - 1)} \\ \frac{\Lambda\cdot(\beta - r_L)\cdot(s - 1)}{(\beta - r_D)\cdot(s - \omega\cdot\tau_S + r_D\cdot\tau_S - 1)} \\ -\frac{\Lambda\cdot(r_D - r_L)}{\beta - r_D} \end{bmatrix}$$

Showing equilibrium bank account balances as a function of key parameters

$\beta := 1$ $\omega := 26$ $r_L := 5\%$ $r_D := 1\%$ $\Lambda := 100$ $w := 4$

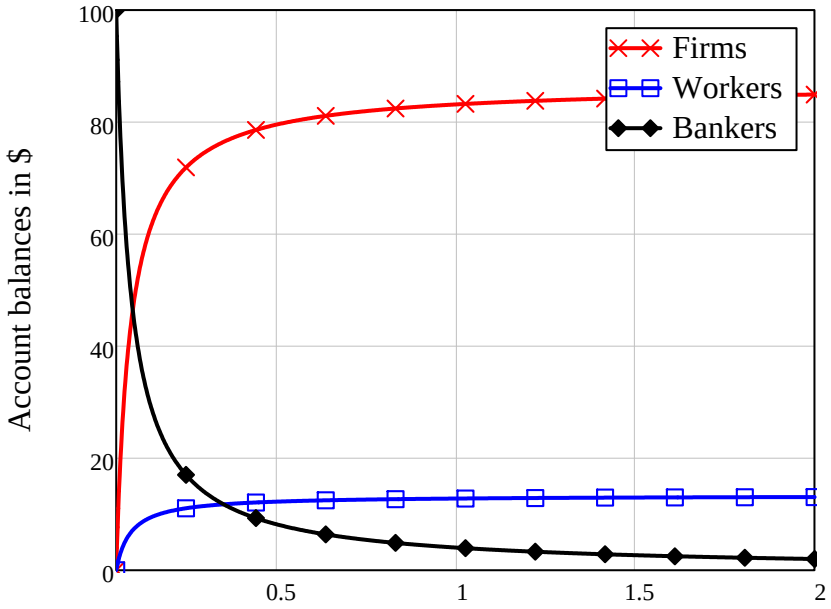
$$F_{Le}(\beta, \omega, w) := \Lambda$$

$$F_{De}(\beta, \omega, w) := \frac{(\beta - r_L)\cdot(\omega - r_D)}{(\beta - r_D)\cdot(\omega - r_D + w)}\cdot\Lambda$$

$$W_{De}(\beta, \omega, w) := \frac{w\cdot(\beta - r_L)}{(\beta - r_D)\cdot(\omega - r_D + w)}\cdot\Lambda$$

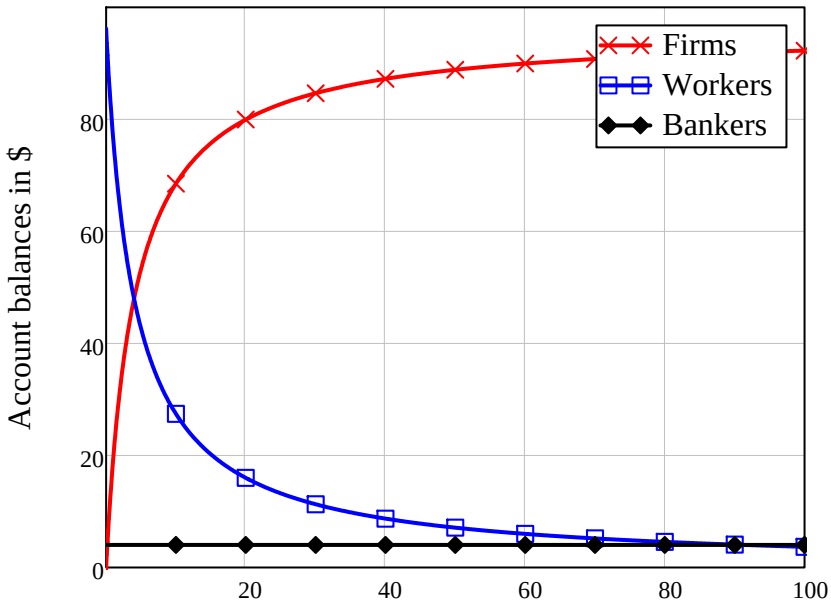
$$B_{Ie}(\beta, \omega, w) := \left(-\frac{r_D - r_L}{\beta - r_D}\right)\cdot\Lambda$$

Account Balances as a function of β

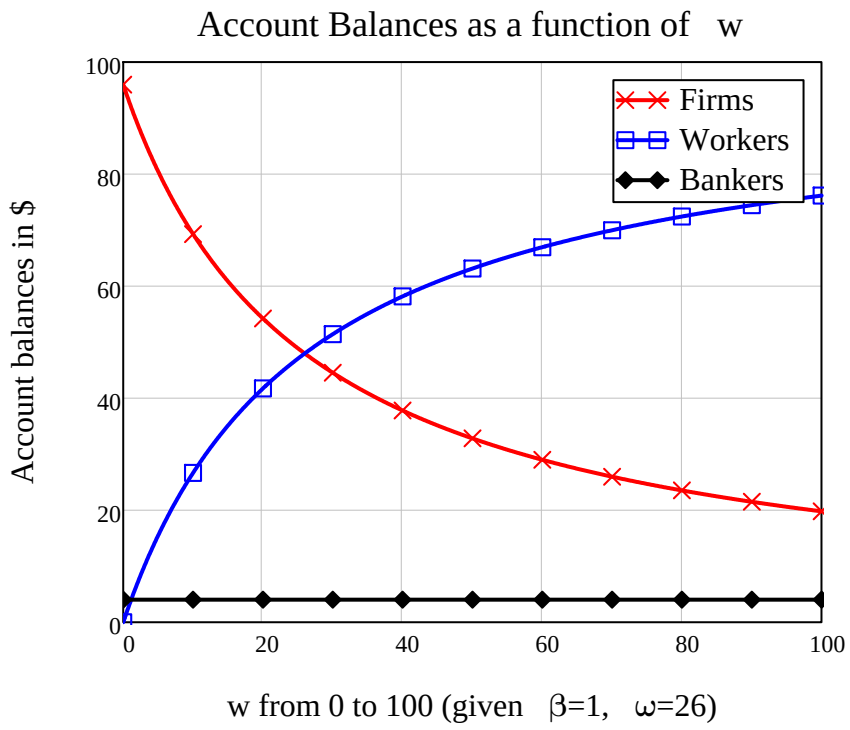


β from rL to 1 (given ω=26, w=4)

Account Balances as a function of ω



ω from rd to 100 (given β=1, w=4)



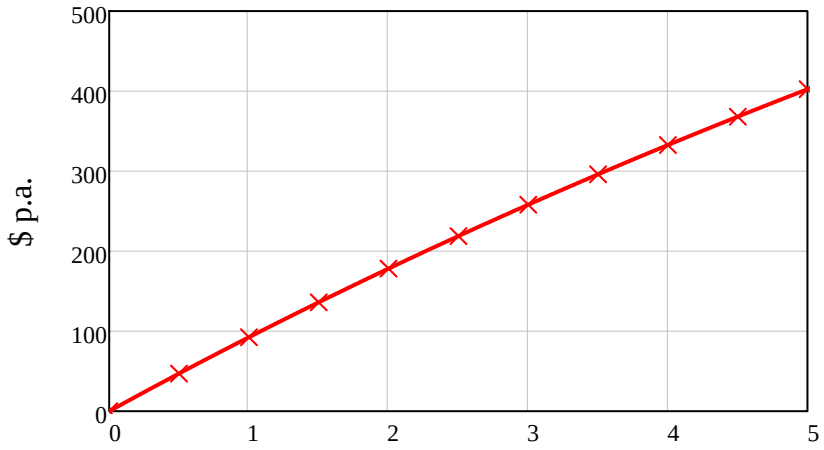
Equilibrium account values given parameter values

$$\begin{pmatrix} \beta \\ \omega \\ r_L \\ r_D \\ w \\ \Lambda \end{pmatrix} = \begin{pmatrix} 1 \\ 26 \\ 0.05 \\ 0.01 \\ 4 \\ 100 \end{pmatrix} \qquad \begin{bmatrix} \Lambda \\ \dfrac{(\beta - r_L)\cdot(\omega - r_D)}{(\beta - r_D)\cdot(\omega - r_D + w)}\cdot\Lambda \\ \dfrac{w\cdot(\beta - r_L)}{(\beta - r_D)\cdot(\omega - r_D + w)}\cdot\Lambda \\ \left(-\dfrac{r_D - r_L}{\beta - r_D}\right)\cdot\Lambda \end{bmatrix} = \begin{pmatrix} 100 \\ 83.161 \\ 12.799 \\ 4.04 \end{pmatrix}$$

Equilibrium incomes as a function of parameter values

$$\text{Wages}(\beta, \omega, w) := w\cdot F_{De}(\beta, \omega, w)$$

Annual Wage Income as a function of w



w from 0 to 10 (given $\beta=1$, $\omega=26$)

Rewriting w as $w = \frac{1-s}{\tau_S}$

$s := 30\%$ $\quad \tau_S := \frac{1}{4}$

Account balances using this substitution

$$F_{Le2}(\beta,\omega,s,\tau_S) := \Lambda$$

$$F_{De2}(\beta,\omega,s,\tau_S) := -\frac{\Lambda\cdot\tau_S\cdot(\beta-r_L)\cdot(\omega-r_D)}{(\beta-r_D)\cdot(s-\omega\cdot\tau_S+r_D\cdot\tau_S-1)}$$

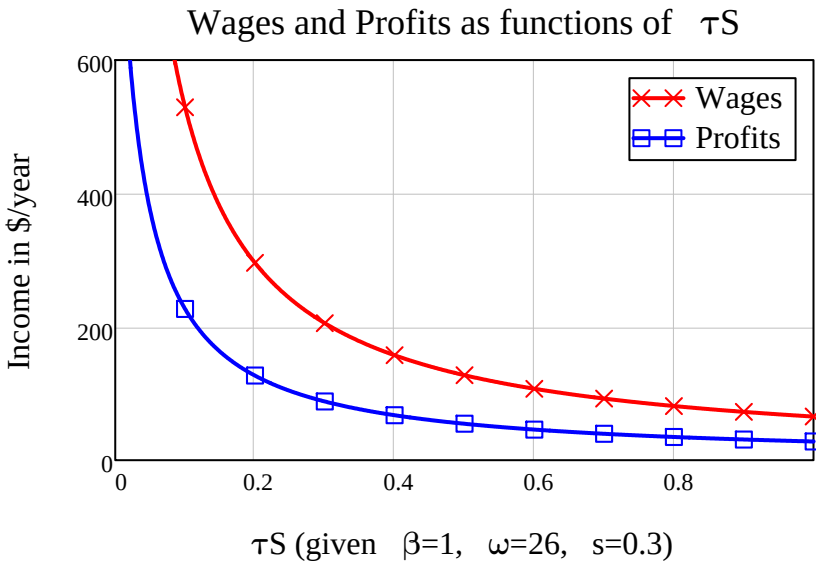
$$W_{De2}(\beta,\omega,s,\tau_S) := \frac{\Lambda\cdot(\beta-r_L)\cdot(s-1)}{(\beta-r_D)\cdot(s-\omega\cdot\tau_S+r_D\cdot\tau_S-1)}$$

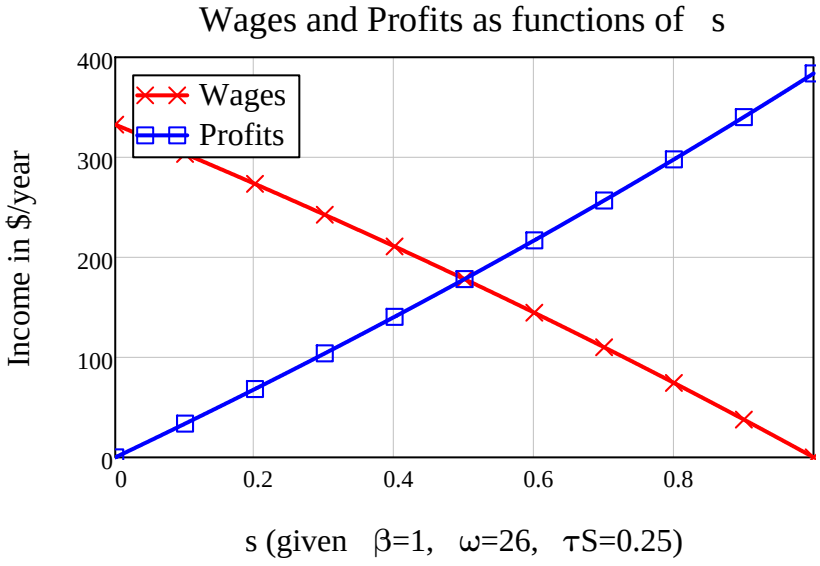
$$B_{Ie2}(\beta,\omega,s,\tau_S) := -\frac{\Lambda\cdot(r_D-r_L)}{\beta-r_D}$$

Equilibrium wages and profits using this substitution

$$Wages_2(\beta,w,s,\tau_S) := \frac{1-s}{\tau_S}\cdot F_{De2}(\beta,\omega,s,\tau_S)$$

$$Profits_2(\beta,w,s,\tau_S) := \frac{s}{\tau_S}\cdot F_{De2}(\beta,\omega,s,\tau_S)$$





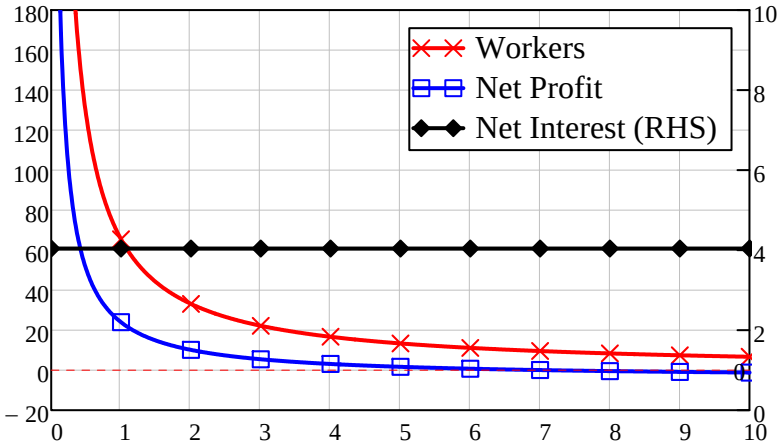
Equilibrium class incomes includung interest transfers

$$\text{Workers}(\beta,\omega,s,\tau_S) := \text{Wages}_2(\beta,w,s,\tau_S) + r_D \cdot W_{De2}(\beta,\omega,s,\tau_S)$$

$$\text{Capitalists}(\beta,\omega,s,\tau_S) := \text{Profits}_2(\beta,w,s,\tau_S) - r_L \cdot \Lambda + r_D \cdot F_{De2}(\beta,\omega,s,\tau_S)$$

$$\text{Bankers}(\beta,\omega,s,\tau_S) := r_L \cdot \Lambda - r_D \cdot \left(F_{De2}(\beta,\omega,s,\tau_S) + W_{De2}(\beta,\omega,s,\tau_S)\right)$$

Class incomes as functions of τS



τS from 0 to 10 (given β=1, ω=26, s=0.3)

	Deposit Accounts	Gross Incomes	Net Incomes
Capitalists	$F_{De2}(\beta,\omega,s,\tau_S) = 86.627$	$Profits_2(\beta,w,s,\tau_S) = 103.952$	$Capitalists(\beta,\omega,s,\tau_S) = 99.819$
Workers	$W_{De2}(\beta,\omega,s,\tau_S) = 9.333$	$Wages_2(\beta,w,s,\tau_S) = 242.555$	$Workers(\beta,\omega,s,\tau_S) = 242.649$
Bankers	$B_{Ie2}(\beta,\omega,s,\tau_S) = 4.04$	$r_L \cdot \Lambda = 5$	$Bankers(\beta,\omega,s,\tau_S) = 4.04$

Dynamic simulation of the model

$s := s \quad \tau_S := \tau_S$ $\qquad C := \frac{1-s}{\tau_S} \cdot F_D(t)$

$$S_1 := \begin{pmatrix} \text{"Type"} & 1 & -1 & -1 & 0 \\ \text{"Name"} & \text{"Firm Loan"} & \text{"Firm Deposit"} & \text{"Worker Deposit"} & \text{"Bank Income"} \\ \text{"Symbol"} & F_L(t) & F_D(t) & W_D(t) & B_I(t) \\ \text{"Loan Transactions"} & A - A & -A & 0 & A \\ \text{"Deposit Interest"} & 0 & B & 0 & -B \\ \text{"Wages"} & 0 & -C & C & 0 \\ \text{"Deposit Interest"} & 0 & 0 & D & -D \\ \text{"Consumption"} & 0 & E + F & -E & -F \end{pmatrix}$$

$$\text{System}(S_1) \rightarrow \begin{bmatrix} \frac{d}{dt}F_L(t) = 0 \\ \frac{d}{dt}F_D(t) = \beta \cdot B_I(t) + r_D \cdot F_D(t) - r_L \cdot F_L(t) + \omega \cdot W_D(t) + \frac{F_D(t) \cdot (s-1)}{\tau_S} \\ \frac{d}{dt}W_D(t) = r_D \cdot W_D(t) - \omega \cdot W_D(t) - \frac{F_D(t) \cdot (s-1)}{\tau_S} \\ \frac{d}{dt}B_I(t) = r_L \cdot F_L(t) - r_D \cdot F_D(t) - \beta \cdot B_I(t) - r_D \cdot W_D(t) \end{bmatrix}$$

$\beta := 1$ $\omega := 26$ $r_L := 5\%$ $r_D := 1\%$ $\Lambda := 100$ $\text{Years} := 10$

Given

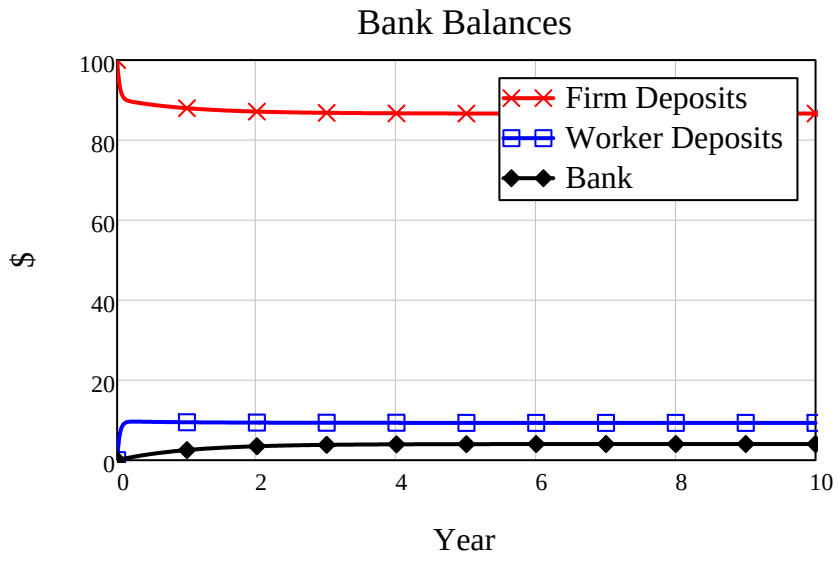
$$\frac{d}{dt}F_L(t) = 0 \qquad F_L(0) = \Lambda$$

$$\frac{d}{dt}F_D(t) = \beta \cdot B_I(t) + r_D \cdot F_D(t) - r_L \cdot F_L(t) + \omega \cdot W_D(t) + \frac{F_D(t) \cdot (s-1)}{\tau_S} \qquad F_D(0) = \Lambda$$

$$\frac{d}{dt}W_D(t) = r_D \cdot W_D(t) - \omega \cdot W_D(t) - \frac{F_D(t) \cdot (s-1)}{\tau_S} \qquad W_D(0) = 0$$

$$\frac{d}{dt}B_I(t) = r_L \cdot F_L(t) - r_D \cdot F_D(t) - \beta \cdot B_I(t) - r_D \cdot W_D(t) \qquad B_I(0) = 0$$

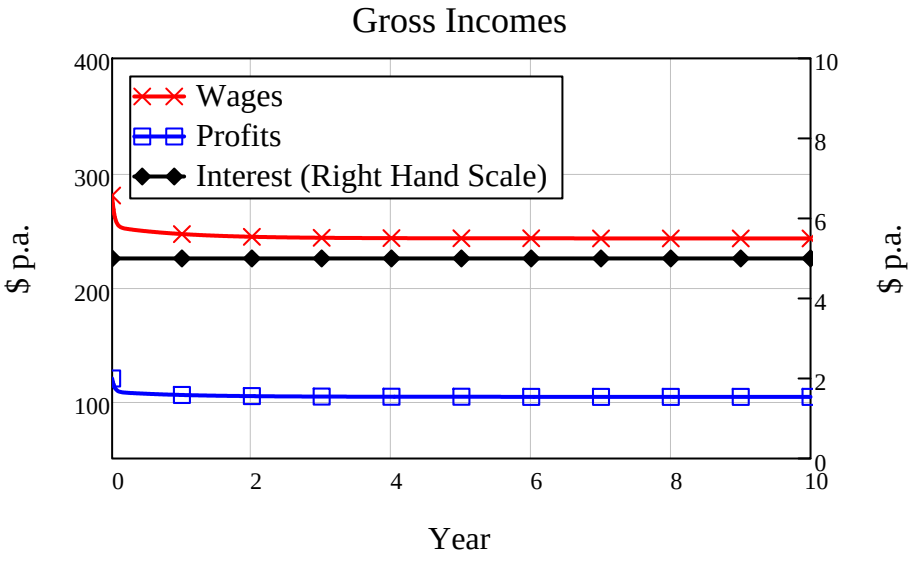
$$\begin{pmatrix} F_L \\ F_D \\ W_D \\ B_I \end{pmatrix} := \text{Odesolve}\left[\begin{pmatrix} F_L \\ F_D \\ W_D \\ B_I \end{pmatrix}, t, \text{Years}\right]$$

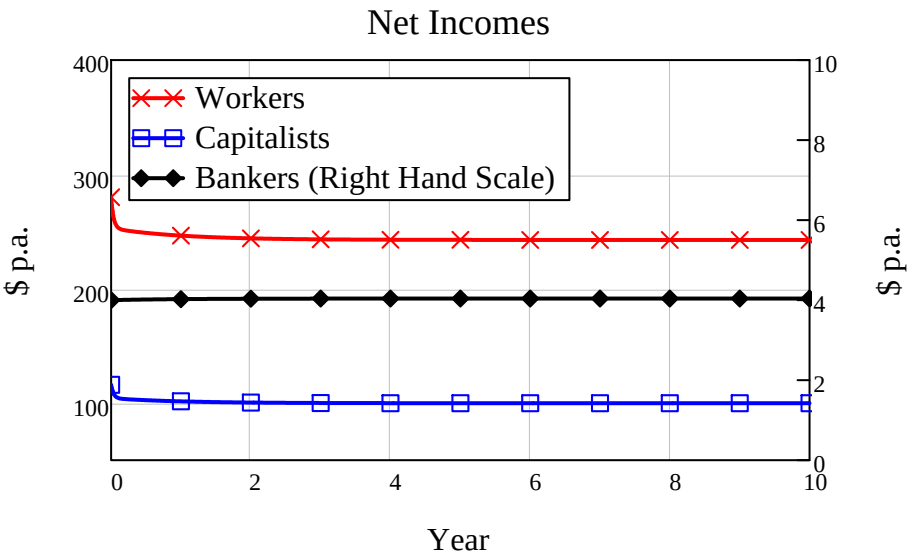


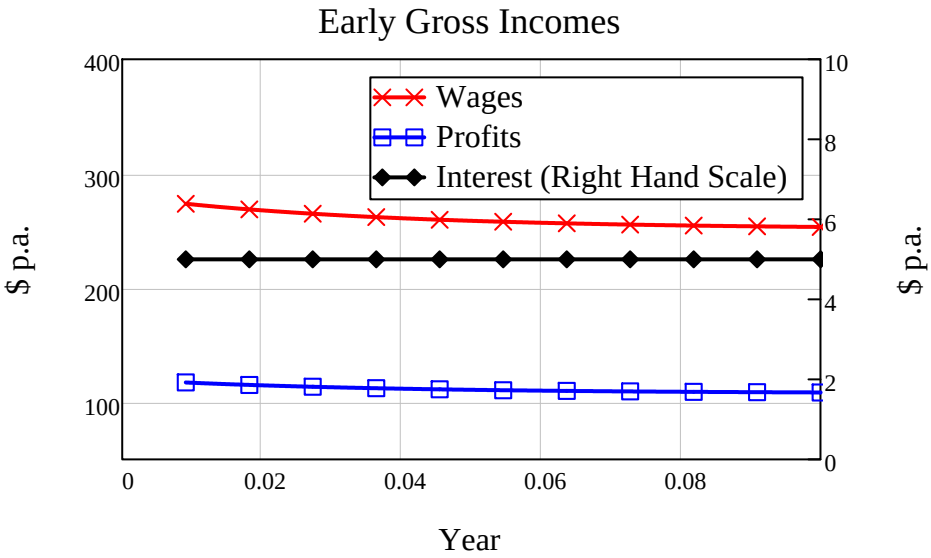
$$\text{Wages}(t) := \frac{1 - s}{\tau_S} \cdot F_D(t) \qquad \text{Workers}(t) := \text{Wages}(t) + r_D \cdot W_D(t)$$

$$\text{Profits}(t) := \frac{s}{\tau_S} \cdot F_D(t) \qquad \text{Capitalists}(t) := \text{Profits}(t) + r_D \cdot F_D(t) - r_L \cdot F_L(t)$$

$$\text{Interest}(t) := r_L \cdot F_L(t) \qquad \text{Bankers}(t) := \text{Interest}(t) - r_D \cdot \left(F_D(t) + W_D(t)\right)$$




Net Incomes
Workers
Capitalists
Bankers (Right Hand Scale)
$ p.a.
$ p.a.
Year
400
300
200
100
10
8
6
4
2
0
0
2
4
6
8
10


Early Gross Incomes
Wages
Profits
Interest (Right Hand Scale)
$ p.a.
$ p.a.
Year
400
300
200
100
10
8
6
4
2
0
0
0.02
0.04
0.06
0.08