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Markov-chain approximations of vector autoregressions: Application of general multivariate-normal integration techniques☆

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ABSTRACT

Discrete Markov chains are helpful for approximating vector autoregressive processes in computational work. We relax G. Tauchen (1986) [Finite state Markov-chain approximations to univariate and vector autoregressions. Economics Letters 20, 177–181] in practice using multivariate-normal integration techniques to allow for arbitrary positive-semidefinite covariance structures. Examples are provided for non-diagonal and singular non-diagonal error covariances.

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1. Introduction

Vector autoregressions (VARs) are useful as a concise specification of the dynamics of an economic system. In some applications, however, this usefulness is limited without a simple method for calculating expectations involving the system. For example, iterative methods for solving a Bellman equation with a VAR specification for the state transitions require computing the expected future values of the states. The implied integration can quickly hinder the solution process.

In this general context, Tauchen (1986) proposes a tractable method for approximating a VAR using a Markov chain over a finite grid. The resulting probabilities allow for simple computation of expectations without integration. The paper makes use of transformed VARs under the assumption of a diagonal covariance structure in the reduced-form error term. This permits the integration of well known univariate distributions.

In this paper, we show that one can treat the problem more conveniently. While other generalizations of Markov approximation techniques have been introduced, these are sometimes set within a much extended environment, such as the treatment of non-linear models with quadrature methods in Tauchen and Hussey (1991). Instead, we focus on simple techniques for the calculation of multivariate probabilities with arbitrary positive-definite covariance structures that allow economists to deal directly with VAR processes without the need to modify their forms to satisfy diagonality assumptions. In addition, researchers can use recent techniques for the calculation of probabilities involving positive semi-definite systems to directly treat processes with singular error covariance. While these types of simplifications are not necessary in theory, in practice and in the context of the solution of a broader economic model they can be quite useful. We present two example VAR-process approximations with non-diagonal and singular non-diagonal error covariance matrices and show that our procedure admirably recovers the traits of the original structural processes.

2. Approximation

In the discussion that follows, we consider a VAR of the form

$$A_0 Z_t = A_1 + A_2 Z_{t-1} + \varepsilon_t,\ \varepsilon_t \sim N(0, \Sigma),$$

with Z_t an $m \times 1$ vector and Σ an arbitrary positive-semidefinite error covariance matrix. The reduced-form specification is

$$Z_t = \overline{A}_1 + \overline{A}_2 Z_{t-1} + \overline{\varepsilon}_t,\ \overline{\varepsilon}_t \sim N\left(0, \overline{\Sigma}\right),$$

where $\overline{A}_i = A_0^{-1} A_i$, $\overline{\varepsilon}_t = A_0^{-1}\varepsilon_t$, and $\overline{\Sigma} = A_0^{-1}\Sigma A_0^{-1'}$. A stationarity assumption for Z_t yields the process covariance Σ^* after iteration on

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$\Sigma \leftarrow \overline{A}_2 \Sigma^* \overline{A}_2' + \overline{\Sigma}$. This VAR structure for Z_t substantially differs from that considered by Tauchen (1986) only in the practical treatment of the error covariance matrix $\overline{\Sigma}$; Tauchen (1986) had required that $\overline{\Sigma}$ be diagonal and positive-definite. The general positive-semidefiniteness of Σ also allows for arbitrary positive-semidefinite structure in $\overline{\Sigma}$. Non-diagonal and singular covariance matrices are present in this class.

To approximate the process Z_t, one sets up a finite grid S consisting of Q possible states, denoted $S_1,...,S_Q \in R^m$. The Markov-chain approximation, M_t, varies over this grid S. By definition, the dynamic properties of M_t are completely determined by the associated $Q \times Q$ transition matrix P such that $P_{i,j} = P(M_{t+1} = S_j | M_t = S_i)$. These transition probabilities are defined in this approximation as $P_{i,j} = P(Z_{t+1} \in V_j | Z_t = S_i)$, where the V_i's are non-overlapping m-dimensional intervals such that $S_i \in V_i$ and $\cup_{i=1}^{Q} V_i = R^m$.

Given the autoregressive structure of Z_t, this definition can be rewritten as

$$P_{i,j} = P\left(\overline{A}_1 + \overline{A}_2 S_i + \overline{\varepsilon}_{t+1} \in V_j\right) = P\left(\overline{\varepsilon}_{t+1} \in V_j'\right),$$

where $V_j' = V_j - \left(\overline{A}_1 + \overline{A}_2 S_i\right)$. Recall that $\overline{\varepsilon}_{t+1} \sim N\left(0, \overline{\Sigma}\right)$. This implies that the probability above can be expressed as the m-dimensional integral

$$P_{i,j} = \int_{V_j'} f_{\overline{\Sigma}}(k) dk$$

with $f_{\overline{\Sigma}}(k)$ equal to the multivariate density for $N\left(0, \overline{\Sigma}\right)$ (when such a density exists).

Integrals of the above form are well understood. Genz (1992) presents a Monte Carlo-type algorithm that can be used in the case of arbitrary positive-*definite* $\overline{\Sigma}$. Genz and Kwong (2000) provide a technique for the case of singular $\overline{\Sigma}$. With these integration methods, quite general covariance structures are tractable within the framework of Tauchen (1986). As a comparison, note that this general multivariate integration is avoided in practice in that work under the assumption of a diagonal error covariance structure by the decomposition of the integral above into

$$\prod_{i=1}^{m} \int_{V_j^{i\prime}} f_i(k_i) dk_i$$

where f_i is the univariate normal density of the i-th component of $\overline{\varepsilon}_{t+1}$ and $V_j^{i\prime}$ is the i-th univariate component of V_j'.

Although the notation and integration techniques apply to an arbitrary selection of the location and number of the points S_i and the integration bounds V_i, these choices are important because they constrain the possible behavior of the approximation M_t. While larger grids with finer grid points allow in principle for more complex and accurate dynamics, they require greater computational time. Tauchen's (1986) uniformly spaced scheme is used in the examples below, but this is not the only available technique; e.g., Adda and Cooper (2003) use equal probability weights to construct a univariate grid.

3. Examples and simulation

This section approximates a trivariate VAR Z_t for two cases: (1) a non-diagonal error covariance matrix, and (2) a singular non-diagonal error covariance matrix. As a check of the reliability of the approximations, we run 1000 simulations for both the original process Z_t and the approximation M_t for each covariance structure. We then compare mean estimated OLS coefficients for each 100-period simulated data set. Calculations were performed in MATLAB, with the required Monte Carlo-type multivariate integration carried out using the function qscmvnv.m (available on the website of Alan Genz) with 1000 random draws.[1]

The specification of the grid is the same across both examples and follows Tauchen (1986). Let q_i be the number of unique i-th dimensional states in S which can be visited by the i-th component M_{it} of the Markov-chain approximation, with the total number of states $Q = \prod_{i=1}^{3} q_i$. We set $q_i = 5$ for each i, yielding 125 total states in the approximation grid S. After iteration, the unconditional standard deviation of each component Z_{it} can be found from the diagonal of the process covariance Σ^*. Given the unconditional mean of Z_t, $\mu = \left(I - \overline{A}_2\right)^{-1} \overline{A}_1$, we space the components of the grid points $S_1,...,S_{125}$ equally around the components μ_i so that two unconditional standard deviations of Z_{it} are spanned in either direction around μ_i.

Each simulation of the process Z_t itself requires draws from $N\left(0, \overline{\Sigma}\right)$ and initialization, chosen here as the mean μ. Simulation of the approximate Markov process M_t follows Adda and Cooper (2003) via draws u from the uniform [0,1] distribution. After calculating the transition probabilities $P_{i,j}$, consider the matrix $\overline{P}$ with $\overline{P}_{i,j} = \sum_{k=1}^{j} P_{i,k}$. Given $M_t = S_i$, the uniform draw $u_{t+1} \in \left(\overline{P}_{i,j-1}, \overline{P}_{i,j}\right]$ implies $M_{t+1} = S_j$ (and $M_{t+1} = S_1$ if $u_{t+1} \leq \overline{P}_{i,1}$). The simulation of M_t is also initialized at μ, noting that $\mu \in S$ in our grid specification.

3.1. Non-diagonal and non-singular

The first example considers a non-diagonal error covariance matrix. Using the reduced-form notation for Z_t, the parameters of the VAR are

$$\overline{A}_1 = \begin{bmatrix} -0.5 \\ 0.9 \\ 0.6 \end{bmatrix}, \overline{A}_2 = \begin{bmatrix} 0.25 & 0.1 & 0.5 \\ -0.5 & 0.09 & -0.75 \\ 0.6 & 0 & 0.15 \end{bmatrix}, \overline{\Sigma} = \begin{bmatrix} 0.4 & 0.18 & 0.3 \\ 0.18 & 0.2 & 0.1 \\ 0.3 & 0.1 & 0.7 \end{bmatrix}.$$

The mean estimates using OLS on the directly simulated data Z_t are

$$\overline{A}_1^D = \begin{bmatrix} -0.51 \\ 0.90 \\ 0.60 \end{bmatrix}, \overline{A}_2^D = \begin{bmatrix} 0.23 & 0.11 & 0.50 \\ -0.51 & 0.09 & -0.75 \\ 0.59 & 0.01 & 0.13 \end{bmatrix}, \overline{\Sigma}^D = \begin{bmatrix} 0.38 & 0.17 & 0.28 \\ 0.17 & 0.19 & 0.09 \\ 0.28 & 0.09 & 0.67 \end{bmatrix}.$$

The mean OLS estimates from the simulation of the Markov approximation M_t are

$$\overline{A}_1^M = \begin{bmatrix} -0.50 \\ 0.91 \\ 0.60 \end{bmatrix}, \overline{A}_2^M = \begin{bmatrix} 0.23 & 0.10 & 0.48 \\ -0.49 & 0.09 & -0.76 \\ 0.57 & 0.01 & 0.14 \end{bmatrix}, \overline{\Sigma}^M = \begin{bmatrix} 0.43 & 0.15 & 0.27 \\ 0.15 & 0.32 & 0.08 \\ 0.27 & 0.08 & 0.73 \end{bmatrix}.$$

3.2. Non-diagonal and singular

The second example uses a non-diagonal and singular error covariance matrix.[2] We assume the same grid S and coefficients $\overline{A}_1$ and $\overline{A}_2$. We change $\overline{\Sigma}$ to the singular matrix,

$$\overline{\Sigma} = \begin{bmatrix} 0.01 & 0.01 & 0 \\ 0.01 & 0.1 & -0.09 \\ 0 & -0.09 & 0.09 \end{bmatrix}.$$

[1] The MATLAB code used to produce all the results contained in this paper is available upon request from the authors.

[2] In results not presented here, we also consider the case of a singular diagonal error covariance matrix. The Markov approximation also performs well in this case.

Such singular non-diagonal covariance structures can arise when converting a VAR to its reduced-form representation, since $\overline{\Sigma} = A_0^{-1}\Sigma A_0^{-1\prime}$.[3] In our case, we consider the singular

$$\Sigma = \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.09 \end{bmatrix}$$

and the non-diagonal

$$A_0^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The mean OLS estimates from direct simulation are

$$\overline{A}_1^D = \begin{bmatrix} -0.49 \\ 0.93 \\ 0.57 \end{bmatrix}, \overline{A}_2^D = \begin{bmatrix} 0.23 & 0.09 & 0.49 \\ -0.49 & 0.06 & -0.77 \\ 0.57 & 0.03 & 0.16 \end{bmatrix}, \overline{\Sigma}^D = \begin{bmatrix} 0.01 & 0.01 & 0.00 \\ 0.01 & 0.10 & -0.09 \\ 0.00 & -0.09 & 0.09 \end{bmatrix}.$$

Simulation from the Markov approximation produces

$$\overline{A}_1^M = \begin{bmatrix} -0.49 \\ 0.91 \\ 0.59 \end{bmatrix}, \overline{A}_2^M = \begin{bmatrix} 0.23 & 0.10 & 0.48 \\ -0.48 & 0.06 & -0.73 \\ 0.55 & 0.02 & 0.14 \end{bmatrix}, \overline{\Sigma}^M = \begin{bmatrix} 0.01 & 0.01 & 0.00 \\ 0.01 & 0.11 & -0.08 \\ 0.00 & -0.08 & 0.09 \end{bmatrix}.$$

3.3. Recovery of system properties

The examples above demonstrate that Markov-chain approximations to VARs with relatively complicated error covariance matrices can be constructed with reasonable confidence in their ability to recover the dynamics of the underlying process. A simple way to measure this ability is the duplication of system parameters after simulation and OLS estimation. The processes considered here have distinctive error covariance structures, but the sign and magnitude of system parameters recovered from simulation of the Markov-chain approximations are very comparable to the results obtained from direct simulation of the VAR.

4. Conclusion

Tauchen (1986) presents a useful and simple method of approximating VAR processes via Markov chains, under the assumption that the reduced-form error term's covariance matrix is diagonal. While this method is general in theory, we show that arbitrary error covariance structures can be considered in practice within the original framework proposed by Tauchen (1986) without initial modification of the underlying process. Doing so simply requires the use of general, readily available multivariate integration methods. We provide two examples of VAR approximation and simulation using this technique that recover dynamic properties of the underlying process virtually as well as direct simulation of the process.

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[3] Of course, singular covariance implies that some components of the error term are redundant. Such a specification can be useful in applied work because it allows for exact equations to be incorporated into the larger dynamic structure of the process. In results not reported, we compared the performance of a non-singular bivariate VAR with a singular trivariate VAR differing only by the inclusion of an identity. The transition probabilities obtained were equivalent, as expected.