The supplemental material for "The Econometrics of the Hodrick-Prescott Filter" by Robert M. de Jong and Neslihan Sakarya

Lemma 4: For any function $h : [0,1] \to \mathbb{R}$,

$$
\begin{aligned}
&T^{-1}\sum_{j=2}^{T}\cos(\pi(j-1)m/T)h((j-1)/T) \qquad (1)\\
&= -(2T)^{-1}h(1/T)\\
&-(2T)^{-1}(-1)^m h((T-1)/T)\\
&-T^{-1}(2\sin(\pi m/(2T))^{-2}(h(2/T)-h(1/T))\cos(\pi m/T)\\
&+T^{-1}(2\sin(\pi m/(2T))^{-2}(h(1-1/T)-h(1-2/T))\cos(\pi m(1-1/T))\\
&+T^{-1}(2\sin(\pi m/(2T)))^{-3}\Delta^2 h(3/T)\sin(3\pi m/(2T))\\
&-T^{-1}(2\sin(\pi m/(2T)))^{-3}\Delta^2 h(1-1/T)\sin(\pi m(T-3/2)/T)\\
&+T^{-1}(2\sin(\pi m/(2T)))^{-3}\sum_{j=2}^{T-3}\Delta^3 h((j+2)/T)\sin(\pi(j+1/2)m/T),
\end{aligned}
$$

and

$$
\begin{aligned}
&T^{-1}\sum_{j=1}^{T}\cos(\pi(j-1)(m-(1/2))/T)\cos(\pi(j-1)/(2T))h((j-1)/T) \qquad (2)\\
&= (T)^{-1}h(0)-(2T)^{-1}h(1/T)\\
&-(2T)^{-1}(h(2/T)-h(1/T))\left\{\frac{\cos(\pi m/T)}{(2\sin(\pi m/(2T)))^2}+\frac{\cos(\pi(m-1)/T)}{(2\sin(\pi(m-1)/(2T)))^2}\right\}\\
&+(2T)^{-1}(h(1-1/T)-h(1-2/T))\left\{\frac{\cos(\pi m(1-1/T))}{(2\sin(\pi m/(2T)))^2}+\frac{\cos(\pi(m-1)(1-1/T))}{(2\sin(\pi(m-1)/(2T)))^2}\right\}\\
&+(2T)^{-1}\Delta^2 h(3/T)\left\{\frac{\sin(3\pi m/(2T))}{(2\sin(\pi m/(2T)))^3}+\frac{\sin(3\pi(m-1)/(2T))}{(2\sin(\pi(m-1)/(2T)))^3}\right\}\\
&-(2T)^{-1}\Delta^2 h(1-1/T)\left\{\frac{\sin(\pi m(T-3/2)/T)}{(2\sin(\pi m/(2T)))^3}+\frac{\sin(\pi(m-1)(T-3/2)/T)}{(2\sin(\pi(m-1)/(2T)))^3}\right\}\\
&+(2T)^{-1}\sum_{j=2}^{T-3}\Delta^3 h((j+2)/T)\left\{\frac{\sin(\pi(j+(1/2))m/T)}{(2\sin(\pi m/(2T)))^3}+\frac{\sin(\pi(j+(1/2))(m-1)/T)}{(2\sin(\pi(m-1)/(2T)))^3}\right\}.
\end{aligned}
$$

Proof of Lemma 4 First, consider the formula (1). Using the below trigonometric identities,

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\sin(\beta)\cos(\alpha)$$

$$\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\sin(\beta)\cos(\alpha);$$

and by setting $\alpha=\pi(j-1)m/T$ and $\beta=\pi m/(2T)$ we can write $\cos(\pi(j-1)m/T)$ as

$$\cos(\pi(j-1)m/T)=(2\sin(\pi m/(2T))^{-1}(\sin(\pi(j-1/2)m/T)-\sin(\pi(j-3/2)m/T)).$$

Therefore,

$$T^{-1}\sum_{j=2}^{T}\cos(\pi(j-1)m/T)h((j-1)/T)$$

$$=T^{-1}(2\sin(\pi m/(2T))^{-1}\sum_{j=2}^{T}h((j-1)/T)(\sin(\pi(j-1/2)m/T)-\sin(\pi(j-3/2)m/T)),$$

and because

$$\sum_{j=2}^{T}a_{j-1}(b_j-b_{j-1})=\sum_{j=2}^{T}a_{j-1}b_j-\sum_{j=2}^{T}a_{j-1}b_{j-1}$$

$$=-a_1b_1+a_{T-1}b_T+\sum_{j=2}^{T-1}a_{j-1}b_j-\sum_{j=2}^{T-1}a_jb_j=-a_1b_1+a_{T-1}b_T-\sum_{j=2}^{T-1}(a_j-a_{j-1})b_j,$$

we now conclude that, setting $a_j=h(j/T)$ and $b_j=\sin(\pi(j-1/2)m/T)$,

$$T^{-1}\sum_{j=2}^{T}\cos(\pi(j-1)m/T)h((j-1)/T)$$

$$=T^{-1}(2\sin(\pi m/(2T))^{-1}\sum_{j=2}^{T}(\sin(\pi(j-1/2)m/T)-\sin(\pi(j-3/2)m/T))h((j-1)/T)$$

$$=-T^{-1}(2\sin(\pi m/(2T))^{-1}h(1/T)\sin(\pi m/(2T))$$

$$+T^{-1}(2\sin(\pi m/(2T))^{-1}h((T-1)/T)\sin(\pi(T-1/2)m/T)$$

$$-T^{-1}(2\sin(\pi m/(2T))^{-1}\sum_{j=2}^{T-1}(h(j/T)-h((j-1)/T))\sin(\pi(j-1/2)m/T).$$

Next, using

$$\cos(\alpha-\beta)-\cos(\alpha+\beta)=2\sin(\beta)\sin(\alpha)$$

and setting $\alpha=\pi(j-1/2)m/T$ and $\beta=\pi m/(2T)$, it follows that

$$\sin(\pi(j-1/2)m/T)=(2\sin(\pi m/(2T))^{-1}(\cos(\pi(j-1)m/T)-\cos(\pi jm/T)),$$

so

$$\sum_{j=2}^{T-1}(h(j/T)-h((j-1)/T))\sin(\pi(j-1/2)m/T) \tag{3}$$

$$=(2\sin(\pi m/(2T))^{-1}\sum_{j=2}^{T-1}(h(j/T)-h((j-1)/T))(\cos(\pi(j-1)m/T)-\cos(\pi jm/T)).$$

Alternatively, using the below identity

$$\sum_{j=2}^{T-1}a_{j-1}(b_{j-1}-b_j)=\sum_{j=2}^{T-1}a_{j-1}b_{j-1}-\sum_{j=2}^{T-1}a_{j-1}b_j$$

$$=a_1b_1-a_{T-2}b_{T-1}+\sum_{j=2}^{T-2}a_jb_j-\sum_{j=2}^{T-2}a_{j-1}b_j=a_1b_1-a_{T-2}b_{T-1}+\sum_{j=2}^{T-2}(a_j-a_{j-1})b_j,$$

and setting $a_{j-1}=h(j/T)-h((j-1)/T)$, and $b_j=\cos(\pi jm/T)$, the equation (3) can be represented as

$$(2\sin(\pi m/(2T)))^{-1}(h(2/T)-h(1/T))\cos(\pi m/T)$$

$$-(2\sin(\pi m/(2T)))^{-1}(h(1-1/T)-h(1-2/T))\cos(\pi m(1-1/T))$$

$$+(2\sin(\pi m/(2T)))^{-1}\sum_{j=2}^{T-2}(h((j+1)/T)-2h(j/T)+h((j-1)/T))\cos(\pi jm/T)$$

and therefore,

$$T^{-1}\sum_{j=2}^{T}\cos(\pi(j-1)m/T)h((j-1)/T)$$

$$= -T^{-1}(2\sin(\pi m/(2T))^{-1}h(1/T)\sin(\pi m/(2T))$$
$$+T^{-1}(2\sin(\pi m/(2T))^{-1}h((T-1)/T)\sin(\pi(T-1/2)m/T)$$
$$-T^{-1}(2\sin(\pi m/(2T))^{-1}\sum_{j=2}^{T-1}(h(j/T)-h((j-1)/T))\sin(\pi(j-1/2)m/T)$$
$$= -T^{-1}(2\sin(\pi m/(2T))^{-1}h(1/T)\sin(\pi m/(2T))$$
$$+T^{-1}(2\sin(\pi m/(2T))^{-1}h((T-1)/T)\sin(\pi(T-1/2)m/T)$$
$$-T^{-1}(2\sin(\pi m/(2T))^{-2}(h(2/T)-h(1/T))\cos(\pi m/T)$$
$$+T^{-1}(2\sin(\pi m/(2T))^{-2}(h(1-1/T)-h(1-2/T))\cos(\pi m(1-1/T))$$
$$-T^{-1}(2\sin(\pi m/(2T)))^{-2}\sum_{j=2}^{T-2}(h((j+1)/T)-2h(j/T)+h((j-1)/T))\cos(\pi jm/T).$$

Using the identity $\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\sin(\beta)\cos(\alpha)$ and setting $\alpha=\pi jm/T$ and $\beta=\pi m/(2T)$

$$\cos(\pi jm/T)=(2\sin(\pi m/(2T))^{-1}(\sin(\pi(j+1/2)m/T)-\sin(\pi(j-1/2)m/T))$$

Now, we can manipulate the following expression

$$\sum_{j=2}^{T-2}(h((j+1)/T)-2h(j/T)+h((j-1)/T))\cos(\pi jm/T)$$

$$=(2\sin(\pi m/(2T))^{-1}\sum_{j=2}^{T-2}(h((j+1)/T)-2h(j/T)+h((j-1)/T))(\sin(\pi(j+1/2)m/T)-\sin(\pi(j-1/2)m/T)).$$

using the identity

$$\sum_{j=2}^{T-2}a_{j-1}(b_j-b_{j-1})=\sum_{j=2}^{T-2}a_{j-1}b_j-\sum_{j=2}^{T-2}a_{j-1}b_{j-1}$$

$$=-a_1b_1+a_{T-3}b_{T-2}-\sum_{j=2}^{T-3}(a_j-a_{j-1})b_j,$$

by setting $a_{j-1} = h((j+1)/T) - 2h(j/T) + h((j-1)/T)$ and $b_j = \sin(\pi(j+1/2)m/T)$ we obtain

$$(2\sin(\pi m/(2T)))^{-1}\sum_{j=2}^{T-2}(h((j+1)/T)-2h(j/T)+h((j-1)/T))(\sin(\pi(j+1/2)m/T)-\sin(\pi(j-1/2)m/T))$$

$$\begin{aligned}
&= -(2\sin(\pi m/(2T)))^{-1}\Delta^2 h(3/T)\sin(3\pi m/(2T)) \\
&+(2\sin(\pi m/(2T)))^{-1}\Delta^2 h(1-1/T)\sin(\pi m(T-3/2)/T) \\
&-(2\sin(\pi m/(2T)))^{-1}\sum_{j=2}^{T-3}\Delta^3 h((j+2)/T)\sin(\pi(j+1/2)m/T),
\end{aligned}$$

where $\Delta^2 h((j+2)/T)$ is $h((j+2)/T) - 2h((j+1)/T) + h(j/T)$, and $\Delta^3 h((j+2)/T)$ is $h((j+2)/T) - 3h((j+1)/T) + 3h(j/T) - h((j-1)/T)$.

Therefore, noting that $\sin(\pi(T-1/2)m/T) = \sin(\pi m - \pi m/(2T)) = -\sin(\pi m/(2T))\cos(\pi m)$ $= (-1)^{m+1}\sin(\pi m/(2T))$,

$$\begin{aligned}
&T^{-1}\sum_{j=2}^{T}\cos(\pi(j-1)m/T)h((j-1)/T) \\
&= -T^{-1}(2\sin(\pi m/(2T))^{-1}h(1/T)\sin(\pi m/(2T)) \\
&+T^{-1}(2\sin(\pi m/(2T))^{-1}h((T-1)/T)\sin(\pi(T-1/2)m/T) \\
&-T^{-1}(2\sin(\pi m/(2T))^{-2}(h(2/T)-h(1/T))\cos(\pi m/T) \\
&+T^{-1}(2\sin(\pi m/(2T))^{-2}(h(1-1/T)-h(1-2/T))\cos(\pi m(1-1/T)) \\
&-T^{-1}(2\sin(\pi m/(2T)))^{-2}\sum_{j=2}^{T-2}\Delta^2 h((j+1)/T)\cos(\pi jm/T) \\
&= -(2T)^{-1}h(1/T) - (2T)^{-1}(-1)^m h((T-1)/T) \\
&-T^{-1}(2\sin(\pi m/(2T))^{-2}(h(2/T)-h(1/T))\cos(\pi m/T) \\
&+T^{-1}(2\sin(\pi m/(2T))^{-2}(h(1-1/T)-h(1-2/T))\cos(\pi m(1-1/T)) \\
&+T^{-1}(2\sin(\pi m/(2T)))^{-3}\Delta^2 h(3/T)\sin(3\pi m/(2T)) \\
&-T^{-1}(2\sin(\pi m/(2T)))^{-3}\Delta^2 h(1-1/T)\sin(\pi m(T-3/2)/T)
\end{aligned}$$

$$+T^{-1}(2\sin(\pi m/(2T)))^{-3}\sum_{j=2}^{T-3}\Delta^3 h((j+2)/T)\sin(\pi(j+1/2)m/T).$$

Note that for $h(r) = 1/(1+16\lambda\sin(\pi r/2)^4)$,

$$T^{-1}\sum_{j=2}^{T}\cos(\pi(j-1)m/T)h((j-1)/T)$$

$$= f_{T\lambda}(m) - (2T)^{-1}h(0) - (2T)^{-1}(-1)^m h(1)$$

$$= f_{T\lambda}(m) - (2T)^{-1} - (2T)^{-1}(-1)^m(1+16\lambda)^{-1}.$$

We proceed with deriving the Equation (2)

$$T^{-1}\sum_{j=1}^{T}\cos(\pi(j-1)(m-(1/2))/T)\cos(\pi(j-1)/(2T))h((j-1)/T), \tag{4}$$

by using the trigonometric identity; $\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta) + \sin(\alpha)\sin(\beta)$ and setting $\alpha = \pi(j-1)(m-(1/2))/T$ and $\beta = \pi(j-1)/(2T)$. We obtain

$$\cos(\pi(j-1)(m-(1/2))/T)\cos(\pi(j-1)/(2T))$$

$$= \cos(\pi(j-1)m/T) + \sin(\pi(j-1)(m-(1/2))/T)\sin(\pi(j-1)/(2T)).$$

Now the Equation (4) is

$$\frac{1}{T}\sum_{j=1}^{T}\cos(\pi(j-1)m/T)h((j-1)/T)$$

$$+\frac{1}{T}\sum_{j=1}^{T}\sin(\pi(j-1)(m-(1/2))/T)\sin(\pi(j-1)/(2T))h((j-1)/T). \tag{5}$$

Additonally, the expression (5) can be simplified by using the identity $\sin(\alpha)\sin(\beta) = (\cos(\alpha-\beta) - \cos(\alpha+\beta))/2$ and setting $\alpha = \pi(j-1)(m-(1/2))/T$ and $\beta = \pi(j-1)/(2T)$. Therefore,

$$\sin(\pi(j-1)(m-(1/2))/T)\sin(\pi(j-1)/(2T))$$

$$= \frac{1}{2}(\cos(\pi(j-1)(m-1)/T) - \cos(\pi(j-1)m/T)).$$

Plugging this result back into the expression (5), we have

$$T^{-1}\sum_{j=1}^{T}\cos(\pi(j-1)(m-(1/2))/T)\cos(\pi(j-1)/(2T))h((j-1)/T)$$

$$=(2T)^{-1}\sum_{j=1}^{T}\cos(\pi(j-1)m/T)h((j-1)/T) \tag{6}$$

$$+(2T)^{-1}\sum_{j=1}^{T}\cos(\pi(j-1)(m-1)/T)h((j-1)/T). \tag{7}$$

The third order difference formulation for the expressions (6) and (7) are already derived in the Equation (1). Therefore, it follows that

$$(2T)^{-1}\sum_{j=1}^{T}\cos(\pi(j-1)m/T)h((j-1)/T)$$

$$=(2T)^{-1}h(0)+(2T)^{-1}\sum_{j=2}^{T}\cos(\pi(j-1)m/T)h((j-1)/T)$$
$$=(2T)^{-1}h(0)-(4T)^{-1}h(1/T)-(4T)^{-1}(-1)^{m}h((T-1)/T)$$
$$-(2T)^{-1}(2\sin(\pi m/(2T)))^{-2}(h(2/T)-h(1/T))\cos(\pi m/T)$$
$$+(2T)^{-1}(2\sin(\pi m/(2T))^{-2}(h(1-1/T)-h(1-2/T))\cos(\pi m(1-1/T))$$
$$+(2T)^{-1}(2\sin(\pi m/(2T)))^{-3}\Delta^{2}h(3/T)\sin(3\pi m/(2T))$$
$$-(2T)^{-1}(2\sin(\pi m/(2T)))^{-3}\Delta^{2}h(1-1/T)\sin(\pi m(T-3/2)/T)$$
$$+(2T)^{-1}(2\sin(\pi m/(2T)))^{-3}\sum_{j=2}^{T-3}\Delta^{3}h((j+2)/T)\sin(\pi(j+1/2)m/T).$$

Similarly, the third order difference formulation for the expression (7) is obtained from the Equation (1)

$$(2T)^{-1}\sum_{j=1}^{T}\cos(\pi(j-1)(m-1)/T)h((j-1)/T)$$

$$
\begin{aligned}
&= (2T)^{-1}h(0) - (4T)^{-1}h(1/T) - (4T)^{-1}(-1)^{(m-1)}h((T-1)/T) \\
&-(2T)^{-1}(2\sin(\pi(m-1)/(2T)))^{-2}(h(2/T) - h(1/T))\cos(\pi(m-1)/T) \\
&+(2T)^{-1}(2\sin(\pi(m-1)/(2T)))^{-2}(h(1-1/T) - h(1-2/T))\cos(\pi(m-1)(1-1/T)) \\
&+(2T)^{-1}(2\sin(\pi(m-1)/(2T)))^{-3}\Delta^2 h(3/T)\sin(3\pi(m-1)/(2T)) \\
&-(2T)^{-1}(2\sin(\pi(m-1)/(2T)))^{-3}\Delta^2 h(1-1/T)\sin(\pi(m-1)(T-3/2)/T) \\
&+(2T)^{-1}(2\sin(\pi(m-1)/(2T)))^{-3}\sum_{j=2}^{T-3}\Delta^3 h((j+2)/T)\sin(\pi(j+1/2)(m-1)/T).
\end{aligned}
$$

Thus, the third order difference formulation for the expression (2) is

$$
T^{-1}\sum_{j=1}^{T}\cos(\pi(j-1)(m-(1/2))/T)\cos(\pi(j-1)/(2T))h((j-1)/T)
$$

$$
\begin{aligned}
&= (2T)^{-1}\sum_{j=1}^{T}\cos(\pi(j-1)m/T)h((j-1)/T) + (2T)^{-1}\sum_{j=1}^{T}\cos(\pi(j-1)(m-1)/T)h((j-1)/T) \\
&= (T)^{-1}h(0) - (2T)^{-1}h(1/T) \\
&-(2T)^{-1}(h(2/T) - h(1/T))\left\{\frac{\cos(\pi m/T)}{(2\sin(\pi m/(2T)))^2} + \frac{\cos(\pi(m-1)/T)}{(2\sin(\pi(m-1)/(2T)))^2}\right\} \\
&+(2T)^{-1}(h(1-1/T) - h(1-2/T))\left\{\frac{\cos(\pi m(1-1/T))}{(2\sin(\pi m/(2T)))^2} + \frac{\cos(\pi(m-1)(1-1/T))}{(2\sin(\pi(m-1)/(2T)))^2}\right\} \\
&+(2T)^{-1}\Delta^2 h(3/T)\left\{\frac{\sin(3\pi m/(2T))}{(2\sin(\pi m/(2T)))^3} + \frac{\sin(3\pi(m-1)/(2T))}{(2\sin(\pi(m-1)/(2T)))^3}\right\} \\
&-(2T)^{-1}\Delta^2 h(1-1/T)\left\{\frac{\sin(\pi m(T-3/2)/T)}{(2\sin(\pi m/(2T)))^3} + \frac{\sin(\pi(m-1)(T-3/2)/T)}{(2\sin(\pi(m-1)/(2T)))^3}\right\} \\
&+(2T)^{-1}\sum_{j=2}^{T-3}\Delta^3 h((j+2)/T)\left\{\frac{\sin(\pi(j+(1/2))m/T)}{(2\sin(\pi m/(2T)))^3} + \frac{\sin(\pi(j+(1/2))(m-1)/T)}{(2\sin(\pi(m-1)/(2T)))^3}\right\}.
\end{aligned}
$$

Note that for $h((j-1)/T) = \sin(\pi(j-1)/(2T))^2/(1+16\lambda\sin(\pi(j-1)/(2T))^4)$ and for $m \geq 2$ we have

$$
g_{T,\lambda}(m) = \sqrt{2}T^{-1}\sum_{j=1}^{T}\cos(\pi(j-1)(m-(1/2))/T)\cos(\pi(j-1)/(2T))h((j-1)/T)
$$

$$
\begin{aligned}
&= -\frac{\sqrt{2}}{2T}(h(1/T) - h(0)) \\
&-\frac{\sqrt{2}}{2T}(h(2/T) - h(1/T)) \left\{ \frac{\cos(\pi m/T)}{(2\sin(\pi m/(2T)))^2} + \frac{\cos(\pi(m-1)/T)}{(2\sin(\pi(m-1)/(2T)))^2} \right\} \\
&+\frac{\sqrt{2}}{2T}(h(1-1/T) - h(1-2/T)) \left\{ \frac{\cos(\pi m(1-1/T))}{(2\sin(\pi m/(2T)))^2} + \frac{\cos(\pi(m-1)(1-1/T))}{(2\sin(\pi(m-1)/(2T)))^2} \right\} \\
&+\frac{\sqrt{2}}{2T}\Delta^2 h(3/T) \left\{ \frac{\sin(3\pi m/(2T))}{2^3\sin(\pi m/(2T))^3} + \frac{\sin(3\pi(m-1)/(2T))}{2^3\sin(\pi(m-1)/(2T))^3} \right\} \\
&-\frac{\sqrt{2}}{2T}\Delta^2 h(1-1/T) \left\{ \frac{\sin(\pi m(T-3/2)/T)}{(2\sin(\pi m/(2T)))^3} + \frac{\sin(\pi(m-1)(T-3/2)/T)}{(2\sin(\pi(m-1)/(2T)))^3} \right\} \\
&+\frac{\sqrt{2}}{2T}\sum_{j=2}^{T-3} \Delta^3 h((j+2)/T) \left\{ \frac{\sin(\pi(j+(1/2))m/T)}{(2\sin(\pi m/(2T)))^3} + \frac{\sin(\pi(j+(1/2))(m-1)/T)}{(2\sin(\pi(m-1)/(2T)))^3} \right\},
\end{aligned}
$$

where $h(0) = 0$.

Note: My conjecture is that we could find a bound for $|f_{T,\lambda}(m)| \leq Cm^{-5}$ for $m \in \{1, 2, \ldots, T\}$ as long as $sup_{r\in[0,1]}|h^{iv}(r)| < \infty$ where $h(r) = (1 + 16\lambda\sin(\pi r/2))$. Note that $h'(0) = h''(0) = h'''(0) = 0$, but $h^{iv}(0) \neq 0$. Additionally, $h'(1) = h'''(1) = 0$, but $h''(1) = 16\pi^2\lambda(1+16\lambda)^{-2}$. ($h^i(1) = 0$ when i is odd)