Memo For Resolving a Linear System of Equations

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I. The aim of this short memo is to resolve the linear system of equations using Gauss elimination method.

$$C_{L,lin}(\alpha_*) = a_1 \cdot \alpha_*{}^3 + b_1 \cdot \alpha_*{}^2 + c_1\alpha_* + d_1$$
$$C_{L,max} = a_1 \cdot \alpha_{C_{L,max}}{}^3 + b_1 \cdot \alpha_{C_{L,max}}{}^2 + c_1\alpha_{C_{L,max}} + d_1$$
$$C_{L_\alpha} = 3a_1\alpha_*{}^2 + 2b_1\alpha_* + c_1$$

$$0 = 3a_1\alpha_{C_{L,max}}{}^2 + 2b_1\alpha_{C_{L,max}} + c_1$$

Note: The vector of unknowns is: $[a_1, b_1, c_1, d_1]$. The constants are: $C_{L,lin}(\alpha_*)$, $C_{L,max}$, α_* and $\alpha_{C_{L,max}}$

To facilitate the writing of the system of equations, we adopt the following change of variables:
$[a_1, b_1, c_1, d_1] = [x, y, z, t]$.
The constants α_*, $\alpha_{C_{L,max}}$, $C_{L,lin}(\alpha_*)$, $C_{L,max}$ and C_{L_α} are replaced by: *a*, *b*, *c*, *d*, and *e*, respectively.
This leads to the following equations:

$$c = x \cdot a^3 + y \cdot a^2 + za + t$$
$$d = x \cdot b^3 + y \cdot b^2 + zb + t$$
$$e = 3xa^2 + 2ya + z$$

$$0 = 3xb^2 + 2yb + z$$

We can represent this system in augmented matrix form:

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{matrix} \rightarrow \left(\begin{array}{cccc|c} 1 & a & a^2 & a^3 & t \\ 1 & b & b^2 & b^3 & z \\ 0 & 1 & 2a & 3a^2 & y \\ 0 & 1 & 2b & 3b^2 & x \end{array}\right) = \begin{pmatrix} c \\ d \\ e \\ 0 \end{pmatrix}$$

$$\begin{matrix} R_1 \\ R_2 - R_1 \\ R_3 \\ R_4 \end{matrix} \rightarrow \left(\begin{array}{cccc|c} 1 & a & a^2 & a^3 & t \\ 0 & b-a & b^2-a^2 & b^3-a^3 & z \\ 0 & 1 & 2a & 3a^2 & y \\ 0 & 1 & 2b & 3b^2 & x \end{array}\right) = \begin{pmatrix} c \\ d-c \\ e \\ 0 \end{pmatrix}$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3(a-b) + R_2 \\ R_4(a-b) + R_2 \end{matrix} \rightarrow \left(\begin{array}{cccc|c} 1 & a & a^2 & a^3 & t \\ 0 & b-a & b^2-a^2 & b^3-a^3 & z \\ 0 & 0 & (a-b)^2 & 3a^2(a-b)+b^3-a^3 & y \\ 0 & 0 & -(a-b)^2 & 3b^2(a-b)+b^3-a^3 & x \end{array}\right) = \begin{pmatrix} c \\ d-c \\ e(a-b)+d-c \\ d-c \end{pmatrix}$$

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 + R_3 \end{matrix} \rightarrow \left(\begin{array}{cccc|c} 1 & a & a^2 & a^3 & t \\ 0 & b-a & b^2-a^2 & b^3-a^3 & z \\ 0 & 0 & (a-b)^2 & 3a^2(a-b)+b^3-a^3 & y \\ 0 & 0 & 0 & 3(b^2+a^2)(a-b)+2b^3-2a^3 & x \end{array}\right) = \begin{pmatrix} c \\ d-c \\ e(a-b)+d-c \\ e(a-b)+2(d-c) \end{pmatrix}$$

We ultimately derive the value of *x* from the last equation of the system:

$$x = \frac{e(a-b) + 2(d-c)}{3(b^2+a^2)(a-b) + 2b^3 - 2a^3} = \frac{e(a-b) + 2(d-c)}{(a-b)^3}$$

From the value of *x*, we can backtrack to obtain the values of the other variables *y*, *z*, and *t*.

Alternatively, we can subtract the two equations from one another to find the value of y:

$$e = 3xa^2 + 2ya + z$$

$$0 = 3xb^2 + 2yb + z$$

From these equations, we can derive:

$$e = 3x(a^2 - b^2) + 2y(a-b)$$

Thus, the value of *y* is given by:

$$y = \frac{e - 3x(a^2-b^2)}{2(a-b)} = \frac{e - 3\dfrac{e(a-b) + 2(d-c)}{(a-b)^3}(a^2-b^2)}{2(a-b)}$$

Next, we can derive the formula for *z* from the equation: $0 = 3xb^2 + 2yb + z$

$$z = -2yb - 3xb^2$$

Substituting for *y*:

$$z = -2b\frac{e - 3\dfrac{e(a-b) + 2(d-c)}{(a-b)^3}(a^2-b^2)}{2(a-b)} - 3b^2\frac{e(a-b) + 2(d-c)}{(a-b)^3}$$

Finally, for the variable *t*, we can use the equation:

$$c = x \cdot a^3 + y \cdot a^2 + za + t$$

Rearranging gives us:

$$t = c - za - y \cdot a^2 - x \cdot a^3$$

Thus, we have:

$$t = c + 2ab\frac{e - 3\dfrac{e(a-b) + 2(d-c)}{(a-b)^3}(a^2-b^2)}{2(a-b)} + 3ab^2\frac{e(a-b) + 2(d-c)}{(a-b)^3}$$
$$- a^2\frac{e - 3\dfrac{e(a-b) + 2(d-c)}{(a-b)^3}(a^2-b^2)}{2(a-b)} - a^3\frac{e(a-b) + 2(d-c)}{(a-b)^3}$$

However, due to the complexity of the expressions, it is impractical to write them out in their entirety. For the sake of simplification, the final solution can be expressed as: $\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} \frac{2(c-d)-e(a-b)}{(b-a)^3} \\ \frac{e-3x(a^2-b^2)}{2(a-b)} \\ -2by - 3b^2x \\ c - az - a^3x - a^2y \end{bmatrix}$

II. now, let's try to resolve the linear system of equations using substitution method:

$c = x \cdot a^3 + y \cdot a^2 + za + t \ldots$ (1)
$d = x \cdot b^3 + y \cdot b^2 + zb + t \ldots$ (2)
$e = 3xa^2 + 2ya + z \ldots$ (3)

$0 = 3xb^2 + 2yb + z \ldots$ (4)

(1)-(2): $c - d = x \cdot (a^3 - b^3) + y \cdot (a^2 - b^2) + z(a-b) \ldots$ (5)

(3)- (4): $e = 3x(a^2 - b^2) + 2y(a-b) \ldots$ (6)

$$y = {e - 3x(a^2 - b^2)}/{2(a - b)}$$

$$y = \frac{e}{2(a-b)} - \frac{3}{2}(a+b)x$$

(3): $z = -3xb^2 - 2yb = -3xb^2 - 2b\left[\frac{e}{2(a-b)} - \frac{3}{2}(a+b)x\right] = \frac{-be}{a-b} + 3b(a+b)x - 3xb^2$

$$z = \frac{-be}{a-b} + 3abx$$

We substitute the values of y and z into Equation (5):

$$c - d = x \cdot (a^3 - b^3) + \left[\frac{e}{2(a-b)} - \frac{3}{2}(a+b)x\right] \cdot (a^2 - b^2) + \left[\frac{-be}{a-b} + 3abx\right](a-b)$$

$$-be + 3abx(a-b) + \frac{e}{2}(a+b) - \frac{3}{2}x(a+b)^2(a-b) + x(a^3 - b^3) = c - d$$

$$2x\left[3a^2b - 3ab^2 - {3}/{2}\,(a-b)(a+b)^2 + a^3 - b^3\right] = 2c - 2d - ea + eb$$

$$x\left[\frac{3}{2}a^2b - \frac{3}{2}ab^2 - \frac{1}{2}a^3 + \frac{1}{2}b^3\right] = 2c - 2d - ea + eb$$

we obtain the value of x:

$$x = \frac{e(a-b) + 2(d-c)}{(a-b)^3}$$

We follow the same procedure described previously to find the values of y, z, and t.