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Conservation Principles and Stability in the Evolution of Drainage Systems

by

Terence Robert Smith

A dissertation submitted to The Johns Hopkins University in conformity with the requirements for the degree of Doctor of Philosophy.

Baltimore, Maryland

1971

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ABSTRACT

Models describing the evolution of drainage basins must be based on sound physical principles. Two models are proposed using the ideas of

1. representing a drainage surface mathematically as $z = z(x,y,t)$
2. approximating sediment transport per unit width (q_s) as some function of surface gradient ($|\nabla z|$) and the discharge of water per unit width (q) :

$$q_s = F(|\nabla z|, q)$$

3. using the principles of conservation of water and sediment.

The first (basic) model is constructed using the additional assumptions that

1. water and sediment move down the surface gradient
2. the rainfall (α) is steady and uniform; there is no evaporation or infiltration; and the substrate is homogeneous.

Hence one obtains the (determinate) system:

$$\nabla \cdot \frac{\nabla z}{|\nabla z|} q = -\alpha \qquad (2)$$

$$\nabla \cdot \frac{\nabla z}{|\nabla z|} F(|\nabla z|, q) = \frac{\partial z}{\partial t} \qquad (3)$$

The (basic) problem is to determine the evolution of a surface which varies in the x-direction, but has no initial y-variation.

Equations (2), (3) are considered in one dimension for the "bedload" transport law $q_s = k q^n S^m$, $n, m > 0$.

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Explicit similarity solutions are obtained for almost all m,n. Explicit steady-state solutions are obtained for all m,n. When $n < 1$, the profile is convex; when $n = 1$, straight; and when $n > 1$, concave. The behaviour of $q_s = F(S,q)$ on the associated steady-state surface is analogous to $q_s = kq^nS^m$, $n < 1$, on convexities and to $q_s = kq^nS^m$, $n > 1$, on concavities.

The one-dimensional steady-state solutions are extended without variation in the y-direction. The basic problem is to determine the stability of these surfaces with respect to small perturbations. The results of this analysis are:

1. the surface is stable with respect to perturbations with no y-variation.
2. adding a y-component to the perturbations changes the direction but not the magnitude of the gradient.
3. for general perturbations, convex portions of a steady-state surface are stable, since, by $q_s = F(S,q)$ behaving like $q_s = kq^nS^m$, $n < 1$, converging fluxes of water deposit sediment. Hence channel forms do not grow.
4. for general perturbations, concave portions are unstable since, by $q_s = F(S,q)$ behaving like $q_s = kq^nS^m$, $n > 1$, converging fluxes of water transport more sediment. Hence channel-like forms grow. Moreover, the smallest wavelengths of perturbation in the y-direction grow fastest.

For this model a realistic transport law $q_s = F(S,q)$ must therefore have a steady-state surface which is convex in the upper portion and concave in the lower.

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A modified (basic) model is constructed in which a non-zero lower bound on the wavelengths of unstable perturbations is assumed. The problem is to determine the evolution of N adjacent congruent valleys, with streams approximated by lines. Using stability theory, it is found that:

1. the model is indeterminate unless a law of lateral stream migration is assumed.
2. for two adjacent valleys with constant-angle side-slopes, there is a finite length, proportional to some power (>1) of the valley width, over which the valleys are stable with respect to each other, but beyond which one captures the other.

A model for lateral migration is proposed by which a stream moves away from that side-slope down which comes the greater flux of sediment. Perturbing a stream with constant-angle side-slopes from a symmetrical condition, it is shown that:

either 1. the stream is insensitive to small variations of sediment influx from either bank, in which case it is neutral with respect to lateral movement and the N-valley problem is solved for the constant slope case;

or 2. the stream is sensitive to such variations and is unstable with respect to such movement along its entire length.

Either alternative may exist in reality.

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A model of a channel based on conservation principles is proposed as a vehicle for solving the lateral movement problem. One version of this model predicts a hydraulic geometry of

$$v \sim Q^{1/17}$$

$$d \sim Q^{5/17}$$

$$w \sim Q^{11/17}$$

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TABLE OF SYMBOLS

a	positive constant
c	positive constant
d	particle diameter
$f(\ )$	arbitrary function
$g(\ )$	arbitrary function
h	depth of channel flow
j	positive constant
k	positive constant
$k^1, \ldots, k^4$	positive constant
m, n	positive constants
q	discharge of water per unit width
q_0	steady-state discharge of water per unit width
q'	perturbed discharge of water per unit width
q_L , q_R	fluxes of water entering channel from left, right banks respectively
q_s	discharge of sediment per unit width
q_s'	perturbed discharge of sediment per unit width
q_{s_L} , q_{s_R}	fluxes of sediment entering channel from left, right banks, respectively
s'	perturbed slope
t	time
v	velocity of flow
x, y, z	coordinates
z_0	basic surface

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A	valley area
B	constant
C	Chezy coefficient
D	constant
D_d	drainage density
F	stream frequency
$F(\ ), G(\ ), H(\ )$	arbitrary functions
L	steady-state rate of surface lowering
M	valley width
N	positive integer
Q	total channel water discharge
Q_s	total channel sediment discharge
R	hydraulic radius
S	slope
S_0	steady-state slope
X_c	critical stable length
W	channel width
$\mathbf{A}$	matrix
α, β	constants
γ	specific weight of water
δ_i	constant $i = 1,\ldots,N$
η	similarity variable
θ	valley slope angle
λ_{ii}	constants $i = 1,\ldots,N$
μ, ν	constants
ξ	variable

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ρ ,	constant
τ_0	shear stress
$\mathcal{Q}$ ( )	function
ψ_i	constant $i = 1,\ldots,N$
$\underline{\Psi}$	matrix

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I. INTRODUCTION

Consider the intuitive notion of landscape. It may be idealised as a surface determined by those laws through which the products of its own degradation are moved. Wherever this transport is largely effected through river channels, it is quite startling by how much the form of the surface displays invariance from one region to the next. Yet these "laws" of similarity were not adequately stated until 1945 (Horton, 1945). The six known laws of surface morphometry are shown in Table 1.

It is perhaps an even more startling fact that analogous laws of form are followed in other natural systems. For example, by the branches of a tree, the arteries and airways of the human lung, systems of cities and even by a gold filament growing from solution under an electrical potential. Each of these systems shares the function of transporting given quantities over a surface or through a volume, according to certain laws.

There have been few attempts to explain the growth and form of drainage basins or even the growth of single channels (see Schumm, 1956, Melton 1958). The static geometry of drainage basins has been frequently described (see Strahler, 1965). But few writers have adequately observed the evolution of this geometry. Several authors have used various techniques to obtain approximations to the laws of morphometry. For example, Leopold and Langbein (1962), Shreve (1966), Scheidegger (1968), Woldenberg (1968), and Smart

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TABLE 1

THE LAWS OF MORPHOMETRY

In a given drainage basin

1. The number of streams of a given order tend closely to approximate an inverse geometric series in which the first term is unity and the ratio is the bifurcation ratio.
2. The average lengths of stream of each order tend closely to approximate a direct geometric series in which the first term is the average length of streams of first order.
3. There is a fairly definite relationship between the slopes of stream and order, which may be expressed as an inverse geometric series law.
4. The mean drainage area of streams of each order tend closely to approximate a direct geometric series in which the first term is the mean area of first order subbasins.
5. The frequency F of streams is related to the drainage density D by $F = \frac{2}{3} D_d^2$

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(1968). Each, however, assumes the existence of a stable channel of finite length. Moreover, each assumes rules by which these lengths interact and assumes the stability of the final form. Nowhere in this work is account taken of the fact that drainage basins are self-forming through the basic mechanisms of sediment transport and erosion.

The operation of "random" effects and variational principles is not disputed, but it is quite apparent that to use the former and seek the latter without considering the basic physical mechanisms is premature. Hence the present study aims to elucidate some of the mechanisms involved in drainage basin evolution as a first step towards a more rational theory of landscape.

The approach involves choosing certain fundamental aspects of drainage surface evolution. These are used to build a model. Such model building inevitably involves the use of idealisations and assumptions. Nevertheless, the implications of a model hopefully suggest new insights into the real world. At the same time, they should indicate at what points a model is insufficient to account for real phenomena. Hence the process may be repeated, and one's understanding increase.

Drainage basins are complex phenomena. Under these conditions, mathematical models become indispensible. First, mathematical logic often provides the only means of following the implications of the assumptions of the model. Second, a general mathematical formulation often avoids the pitfalls of pleading to special mechanisms. In the

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case of drainage basins, this isinterpreted to mean that morphological similarity of complex phenomena under diverse external conditions warrants the assumption of only the most general mathematical postulates (see, for example, D'Arcy Thompson, 1917).

However, it must be stressed that the use of mathematical models at this stage is largely to provide qualitative insight into the problem. It is not to compute predictive numbers. Hence, a strong attempt is made in the text to separate qualitative reasoning and the application of mathematical logic. Consequently, the mathematical sections are formally separated from the qualitative text using the device of stating them as propositions and theorems. These mathematical sections are quite essential to the argument, but it is hoped that the text may prove of use if they are skipped by the reader.

The models constructed in this work are based on three facts. First that a drainage basin may be represented as a mathematical surface. Second, that the principles of mass conservation apply to such a surface, and third that one may state adequate laws connecting the movement of sediment over the surface and certain variables. It happens that models based on these facts provide some insights into the evolution of whole land-surfaces, of valleys and systems of valleys and even of single channels. Finally, let it be said that the ultimate value of any model may be judged by the advance it imparts to one's understanding of the real world.

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II. THE BASIC MODEL

The construction of a model idealising certain aspects of drainage basin evolution is made in two stages. In the first, the evolution of a whole landscape is idealised as an initial value problem in the so-called basic problem. In the second stage, the basic model is constructed. This model essentially formulates the principle of conservation for water and sediment.

A. The basic problem

It is assumed that the process of landscape evolution commences with a certain surface, the basic surface, denoted by $z_0 = z_0(x,y,t)$. It is required that z_0 be non-increasing in the direction of the x-axis, from a maximum elevation at $x = 0$ to a minimum at $x = x_0$, while the surface has no variation of elevation in the y-direction. Moreover, it is postulated that this surface is repeated symmetrically about the axis $x = 0$ (see Fig. 1). At time $t = 0$, rainfall is applied to the surface, which being unchannelled, initially gives rise to a flow that has no component in the y-direction. One may now state the

Basic problem: How does the basic surface evolve with time, and under what conditions does it come to assume a form similar to that of known drainage surfaces?

It is clear that this formulation of the problem is a slight variant of a question posed by W. M. Davis (1909) when building his model of cycles of erosion.

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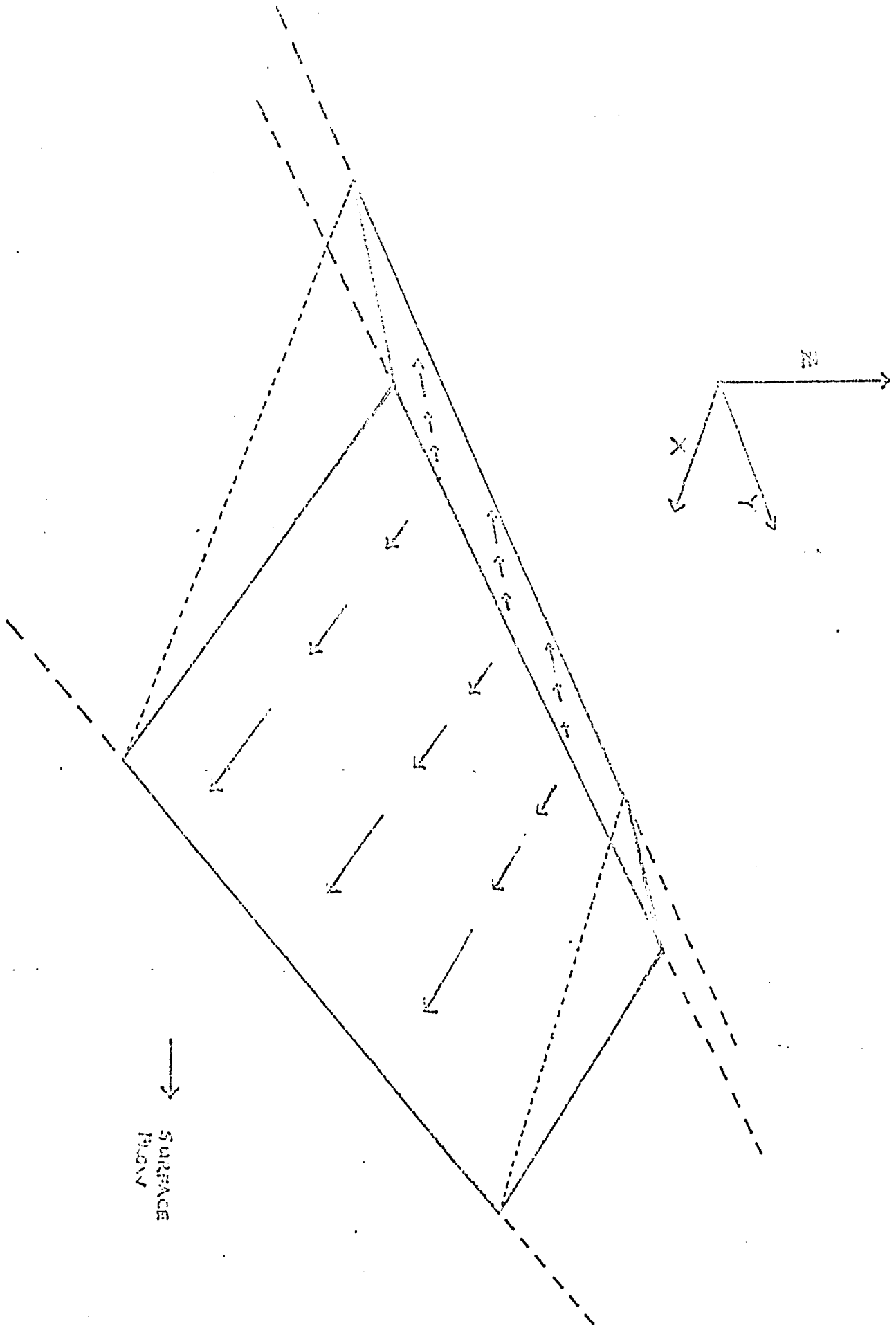

Figure 1. The basic surface

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B. Basic model: statement

The basic problem is investigated using a model based on conservation principles. Several idealisations and assumptions must be made in order that these principles prove of value. The assumptions and idealisations are first listed and then discussed.

A.1 The substrate is homogeneous; the rainfall is uniform and steady; there are no losses of water through evaporation or infiltration.

A.2 The form of a drainage surface may be idealised as a surface $z = z(x, y, t)$ which is twice continuously differentiable in the space domain, and once continuously differentiable in the time domain.

A.3 Water flowing over the surface z moves in a direction down the gradient of the surface If $-\frac{\nabla z}{|\nabla z|}$ is the unit vector in this direction, and $q = q(x, y, t)$ is the magnitude of the discharge of water per unit width in this direction, then one obtains the vector field describing the discharge per unit width over the whole surface:

$$\vec{q} = -\frac{\nabla z}{|\nabla z|} q$$

A.4 Sediment moving over the surface moves in a direction down the gradient of the surface. If $q_s = q_s(x, y, t)$ is the magnitude of the discharge of sediment per unit width, then one obtains the vector field

$$\vec{q}_s = -\frac{\nabla z}{|\nabla z|} q_s$$

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A.5 The magnitude of the discharge of sediment per unit width at any point may be written as some function of the magnitude of the local gradient $(|\nabla z| = S)$ and the magnitude of the local discharge of water per unit width (q). Hence

$$q_s = F(S, q) \tag{1}$$

Moreover, $F(S,q)$ has the following properties:

$$\left.\begin{aligned} F(S,q) &\geq 0 \\ \frac{\partial F}{\partial S} &> 0 \\ \frac{\partial F}{\partial q} &> 0 \end{aligned}\right\} \text{ for all } q, S$$

A.6 The conservation of water flowing over the surface is described by

$$\nabla \cdot \frac{\nabla z}{|\nabla z|} q = -\alpha \tag{2}$$

where α is a constant denoting a uniform and steady input of rainfall.

A.7 The conservation of moving sediment may be written as:

$$\nabla \cdot \frac{\nabla z}{|\nabla z|} F(|\nabla z|, q) = \frac{\partial z}{\partial t} \tag{3}$$

C. The basic model: discussion

Consider each of the assumptions in turn.

A.1 Given a substrate and climate permitting erosion to occur, it is reasonable to assume that the physical mechanisms causing the growth of channels operate under conditions of uniform

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substrate, uniform and steady rainfall and no losses of water through infiltration or evaporation. Evidence to support this contention comes, on the one hand, from the fact that basins of "similar" topologic form occur over considerable ranges of values of these parameters, and on the other, from the implications derivable from these assumptions. It is not denied that variations of these parameters have important consequences for the final form of drainage surfaces, but such effects are considered of secondary importance at this stage of investigation.

A.2 The assumption that $z = z(x, y, t)$ is twice continuously differentiable (i.e. $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial x \partial y}$ are continuous) is strong. Implicit in the assumption is the idea that such a description ignores the granular nature of the surface. It is assumed, however, that the particles are sufficiently small to justify the continuum hypothesis. In the case of fully developed drainage systems, the assumption of continuous first and second space derivatives is not well-justified. However, for a study of the earliest stages of evolution of a basic surface it seems reasonable since the initial surface z_0 and the initial flow q_0 are both twice continuously differentiable.

A.3 The idealisation that water flows downhill would appear to be essentially correct whenever both the depth of flow and the convergence or divergence of flow are small. It ceases to be

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a good approximation, for example, in well-developed channel flow. However, in the early stages of development of the basic surface, convergence and divergence of flow are small, hence the assumption is reasonable under these conditions.

A.4 The idealisation that sediment moves down the gradient of the surface is subject to the same remarks as the analogous assumption concerning the flow of water.

A.5 The assumption that $q_s = F(S, q)$ is the most important in the basic model. Hence it requires considerable justification. First, it is to be noted that sediment size does not enter the function as a variable. It is reasonable to assume however that a particular sediment in transport may be characterised by a single measure (for example, the median diameter).

Consider now the assumption $q_s = F(S, q)$ for the case of bedload transport (and note that since larger particles tend to armour an erosion surface, the overall rate of sediment transport, and hence erosion, may be limited by the transport of the largest particles). In the summary of a report on sediment transportation mechanics, the A.S.C.E. Task Committee for the preparation of the Sedimentation Manual (1971) states (p. 564):

> It appears that sediment transport curves can be approximated by power relations of the form $q_s = \beta q^{n_s}$, β a constant, ... based on experience it also appears that the exponent n_s in this equation lies between 2 and 3.

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The constant B incorporates information on channel slope, which is assumed constant.

Henderson (1966) shows that by using the equation for shear stress

$$\tau_o = \gamma R S$$

γ specific weight of fluid, R hydraulic mean radii, S longitudinal slope; and the Chezy equation

$$V = C R^{\frac{1}{2}} S^{\frac{1}{2}}$$

V mean velocity, C a constant; the Brown form of the Einstein relationship may be written as

$$q_s = k q^2 S^2$$

whilst Raudkivi (1967), using similar procedures demonstrates how a large number of bedload equations may be transformed to:

$$q_s = k q^n S^m \quad , k \text{ a constant.} \qquad (4)$$

It must be noted that such formulae are derived under certain restrictions, such as the assumption that channel-bed form does not change. Nevertheless, it is apparent, both from empirical evidence and semi-rational arguments relating shear stress and bedload transport, that equation (4) is a very reasonable approximation to bedload transport given the present state of knowledge on this subject. Hence allowing the arbitrary form

$$q_s = F(S, q) \qquad (1)$$

to represent such transport may be considered an acceptable assumption. It is likely that equation (1) may prove of value

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in approximating the total sediment transport, since via the Chezy equation and the water continuity equation, S and q are related to the mean values of the variables velocity and depth.

Given the assumption that bed-forms do not change, it is apparent that if $q_s = F(S, q)$, and if slope increases for a given q, then the total transport will increase, since by the Chezy relation, velocity must increase. Hence $\frac{\partial F}{\partial S} > 0$ for all S and q is a reasonable assumption. Similarly, if q increases for a given S, it is reasonable to assume that sediment discharge increases, or $\frac{\partial F}{\partial q} > 0$. A law of the form $q_s = F(S, q)$, having properties $\frac{\partial F}{\partial S} > 0$, $\frac{\partial F}{\partial q} > 0$, is said to be a transport law.

A.6 Equation (2) expresses the continuity of water, given assumptions A.1, A.3, and the further assumption that at the time scale at which surface erosion occurs, the flow of water over the surface is in quasi-equilibrium. This is in the sense that adjustments in the depth of flow occur instantaneously at the time scale of erosion. Hence the rate of change of depth of flow with respect to time may be ignored with respect to the rate of change of surface elevation with respect to time.

A.7 Equation (3) expresses the continuity of sediment given assumptions A.1, A.4, and A.5.

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D. The determinacy of the basic model

Equations (2), (3) may be put into non-dimensional form. Hence they become:

$$\nabla \cdot \frac{\nabla z}{|\nabla z|} q = -1 \qquad (5)$$

$$\nabla \cdot \frac{\nabla z}{|\nabla z|} F(|\nabla z|, q) = \frac{\partial z}{\partial t} \qquad (6)$$

They are considered in this form for the remainder of the work.

It is apparent that if one specifies the requisite initial and boundary conditions, these equations constitute a determinate system, and provide a satisfactory mathematical model for investigation. The essence of this investigation entails a determination of the conditions to be placed on $q_s = F(S, q)$, and then an examination of the correspondence of the model to reality, given these conditions.

It is of some interest to note that equation (6) is in a sense analogous to the parabolic equation:

$$c \nabla \cdot \nabla \theta = \frac{\partial \theta}{\partial t} \qquad (7)$$

whose application to landscape development has been considered by Culling (1963). Consequently, one may expect landscape to behave in a manner analogous to other known diffusion phenomena.

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III. THE ONE-DIMENSIONAL FORM OF THE BASIC MODEL

A. The interpretation of one-dimensional solutions

Some insight into the nature of the basic equations (5), (6) may be gained by studying these equations in a one-dimensional form. This study is made now.

Equations (5), (6) become:

$$\frac{\partial}{\partial x}\left[(-1)\, q\right] = -1 \tag{8}$$

$$\frac{\partial}{\partial x}\left[(-1)\, F\left(-\frac{\partial z}{\partial x}, q\right)\right] = \frac{\partial z}{\partial t} \tag{9}$$

Equation (8) is immediately integrable, and one obtains

$$q = x + c(t) \tag{10}$$

where $c(t)$ is the flux of water at $x = 0$.

It proves important to interpret any solution $z = z(x, t)$ to equation (9) in the following way. Consider the solution $z = z(x, t)$ as a section taken in the x-direction through a surface that has no variation in the y-direction. For example, consider such a section taken through the basic surface (see Fig. 1). Moreover, the solution $z = z(x, t)$ is considered to be repeated symmetrically about the line $x = 0$. Again, the basic surface provides an example.

It may be noted that another interpretation is possible. In this case, one considers a valley of constant fixed width M, with

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sediment transport occurring in a discrete channel in the valley bottom. If one interprets $S = \left|\frac{\partial z}{\partial x}\right|$ as valley slope, and q as the total discharge instead of discharge per unit width, then under certain assumptions, this system is governed by equations analogous to (8), (9). Such a model is developed more fully in section V.D. Until then, the first interpretation of $z = z(x,t)$ will be adopted.

B. The basic boundary conditions

A first aspect of equations (8), (9) concerns the nature of the boundary conditions to be impressed on their solutions. Since $z = z(x,t)$ is interpreted as a surface repeated symmetrically about $x = 0$, it is immediate that the flux to the right of water and sediment at $x = 0$ must be equal and opposite to the flux to the left at $x = 0$. Since these fluxes are positive quantities, they must both be zero. Hence:

$$q(x,t)\Big|_{x=0} = 0 \tag{11}$$

$$F\left(-\frac{\partial z}{\partial x}, x\right)\Big|_{x=0} = 0 \tag{12}$$

These relations may be termed the basic boundary conditions.

From equations (8), (9), applying these conditions, one obtains:

$$q = x \tag{13}$$

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$$F\left(-\frac{\partial z}{\partial x}, x\right) = -\int^{x} \frac{\partial z}{\partial t}\, dx \qquad (14)$$

C. Similarity Solutions for the case $q_s = k q^n S^m$

As a second aspect, consider the nature of solutions to equation (9). Although the arbitrary form of $q_s = F(S, q)$ precludes a general explicit solution to equation (9), it proves instructive to consider an important special case. The special case is the general power-law bedload equation

$$\begin{aligned} q_s &= k\, q^n\, S^m \qquad n, m > 0 \\ &= k\, x^n \left(-\frac{\partial z}{\partial x}\right)^m \end{aligned} \qquad (4)$$

Hence equation (9) becomes:

$$-\frac{\partial}{\partial x}\left[k\, x^n \left(-\frac{\partial z}{\partial x}\right)^m\right] = \frac{\partial z}{\partial t} \qquad (15)$$

The analogy between this equation and the one-dimensional diffusion equation suggests a similarity solution. It is quite remarkable that such a solution exists for almost all values of n,m 0, and this solution may be written explicitly as in the following

Theorem 1: Consider $-\frac{\partial}{\partial x}\left[k\, x^n \left(-\frac{\partial z}{\partial x}\right)^m\right] = \frac{\partial z}{\partial t}$ and the conditions

1. $\lim_{x \to +0} k\, x^n \left(-\frac{\partial z}{\partial x}\right)^m = 0$
2. $\int_0^{\infty} z\, dx = D$
3. $m \neq 0$, $2m \neq n$, $m \neq -1$
   $m > 0$, $n > 0$

} for all time $t > 0$

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Then there is a solution

$$z = \begin{cases} \left(\frac{k}{t}\right)^{\frac{1}{j}} \left[\lambda_0 - m\left(\frac{1}{j}\right)^{\frac{1}{m}} \left(x \left(\frac{k}{t}\right)^{\frac{1}{j}} \right)^{\frac{1-n+m}{m}} \right]^{\frac{m}{m-1}} \left[\frac{m-1}{m}\right]^{\frac{m}{m-1}} \\ \qquad \text{if } 0 \le x \le \left[\frac{\left(\frac{t}{k}\right)^{\frac{1}{j}\left(\frac{1-n+m}{m}\right)} (1-n+m)\lambda_0}{m\left(\frac{1}{j}\right)^{\frac{1}{m}}} \right]^{\frac{m}{1-n+m}} \\ 0, \text{ if } x > \left[\frac{\left(\frac{t}{k}\right)^{\frac{1}{j}\left(\frac{1-n+m}{m}\right)} (1-n+m)\lambda_0}{m\left(\frac{1}{j}\right)^{\frac{1}{m}}} \right]^{\frac{m}{1-n+m}} \end{cases} \qquad (16)$$

where $j = 2m - n$, $\lambda_0 > 0$ depends on D.

Proof: Let $\tau = kt$

$$\lambda = \frac{x}{t^{\frac{1}{j}}}$$

$$z = \frac{1}{t^{\frac{1}{j}}} \varphi(\lambda)$$

then $$\frac{\partial z}{\partial \tau} = -\frac{1}{j} \frac{x}{t^{\frac{1}{j}+1}} \varphi - \frac{1}{t^{\frac{1}{j}}} \varphi' \times \frac{1}{t^{\frac{1}{j}+1}}$$

$$= -\frac{1}{j} \frac{1}{t^{\frac{1}{j}+1}} \left[\varphi + \lambda \varphi' \right]$$

$$\frac{\partial z}{\partial x} = \frac{1}{t^{\frac{1}{j}}} \varphi' \frac{1}{t^{\frac{1}{j}}} = \frac{1}{t^{\frac{2}{j}}} \varphi' \quad -$$

$$\therefore \quad -\frac{1}{j} \frac{1}{t^{\frac{1}{j}+1}} \left[\varphi + \lambda \varphi' \right] = -\frac{1}{t^{\frac{1}{j}}} \frac{\partial}{\partial \lambda} \left[\lambda^n t^{\frac{n}{j}} (-\varphi')^m \frac{1}{t^{\frac{2m}{j}}} \right]$$

$$\frac{1}{j} \frac{\partial [\varphi \lambda]}{\partial \lambda} = \frac{\partial}{\partial \lambda} \left[\lambda^n (-\varphi')^m \right]$$

$$\therefore \quad \frac{1}{j} \varphi \lambda = \lambda^n (-\varphi')^m \qquad \text{by} \quad 1.$$

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$$\left(\frac{1}{j}\right)^{\frac{1}{m}} \lambda^{\frac{1-n}{m}} = -\frac{q'}{q^{\frac{1}{m}}}$$

$$\therefore \quad \left(\frac{1}{j}\right)^{\frac{1}{m}} \frac{\lambda^{\frac{1-n+m}{m}}}{\frac{1-n+m}{m}} = \lambda_0 - \frac{q^{1-\frac{1}{m}}}{1-\frac{1}{m}}$$

$$\therefore \quad \frac{m\, q^{\frac{m-1}{m}}}{m-1} = \lambda_0 - \frac{m\left(\frac{1}{j}\right)^{\frac{1}{m}} \lambda^{\frac{1-n+m}{m}}}{1-n+m}$$

and $$q = \left[\lambda_0 - \frac{m\left(\frac{1}{j}\right)^{\frac{1}{m}} \lambda^{\frac{1-n+m}{m}}}{1-n+m}\right]^{\frac{m}{m-1}} \left[\frac{m-1}{m}\right]^{\frac{m}{m-1}}$$

$$\therefore \quad z = \begin{cases} \frac{1}{t^j}\left[\lambda_0 - \frac{m\left(\frac{1}{j}\right)^{\frac{1}{m}} \lambda^{\frac{1-n+m}{m}}}{1-n+m}\right]^{\frac{m}{m-1}} \left[\frac{m-1}{m}\right]^{\frac{m}{m-1}} & 0 \le \lambda \le \left[\frac{\lambda_0(1-n+m)}{m\left(\frac{1}{j}\right)^{\frac{1}{m}}}\right]^{\frac{m}{1-n+m}} \\ = \\ 0 & \lambda > \left[\frac{\lambda_0(1-n+m)}{m\left(\frac{1}{j}\right)^{\frac{1}{m}}}\right]^{\frac{m}{1-n+m}} \end{cases}$$

let $\lambda = \frac{x}{t^j}$

choose λ_0 such that $\int_0^\infty z\, dx = D$

In the statement of the theorem, $\lim_{x \to +0} k x^n \left(-\frac{\partial z}{\partial x}\right)^m = 0$

states the basic boundary condition of no flux of sediment at $x = 0$, whilst $\int_0^\infty z\, dx = D$ is an integral form of the conservation of

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sediment given a zero flux at $x = 0$ and no sources or sinks at any other point. Figure 2 illustrates two profiles computed for the case $n = m = 2$ (the "Einstein-Brown" equation, Henderson 1966) at differing times.

One may note several facts concerning these solutions. First, a large body of experience relating to equations of parabolic type indicates that when the diffusion coefficient (in this case $k x^n \left(-\frac{\partial z}{\partial x}\right)^{m-1}$) is positive, and when similarity solutions obtain, an arbitrary initial surface assumes the form described by the similarity solutions with some rapidity. Hence, it is reasonable to assume that if these equations constitute a satisfactory model of some aspect of real landscape, then an arbitrary initial land surface would be quickly replaced by forms similar to those described in the solutions.

Second, it is apparent that the solutions depend on the values of n, m. It is possible to characterise some aspects of this dependence. First note certain facts concerning the point $x = x_0$ at which z becomes zero (see Fig. 2). For any time $t > 0$, it is easily seen from equation (14) that $z = 0$ if and only if

$$x_0 = x_0(t) = \left[\frac{1-n+m}{m\left(\frac{1}{j}\right)^{\frac{1}{m}}} \lambda_0 \left(\frac{t}{k}\right)^{\frac{1}{j}} \frac{1-n+m}{m}\right]^{\frac{m}{1-n+m}} \quad (17)$$

Moreover, $\frac{\partial z}{\partial x} = 0$ for this value of x.

For all $x > x_0$, the solution z is defined to be identically

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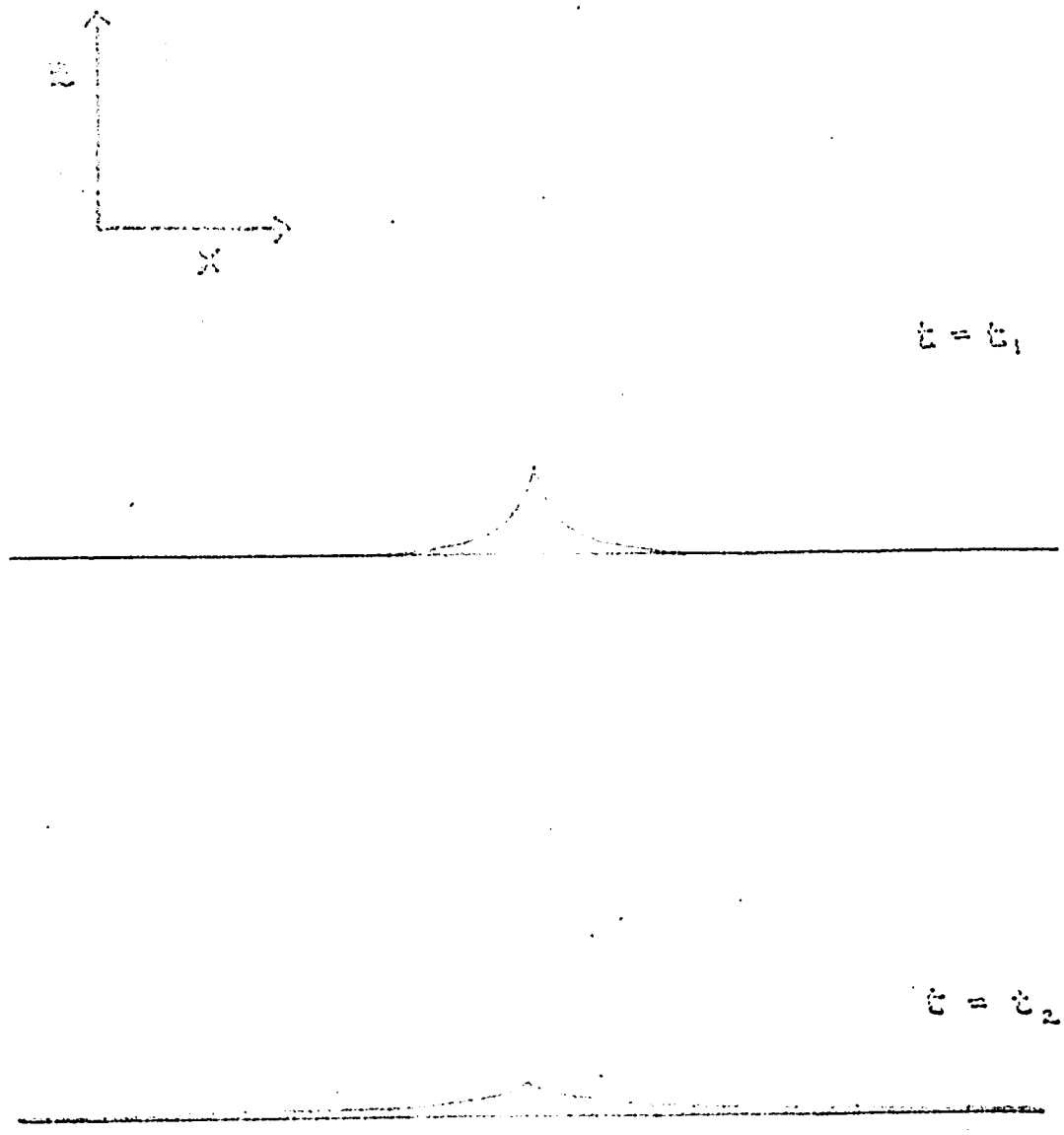

Figure 2. The similarity solution of $\frac{\partial z}{\partial t} = -\frac{\partial}{\partial x}\left[k\, q^2 S^2\right]$

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zero, or else there is a contradiction of the assumption $\int_0^\infty z\,dx = D$. The point $x = x_0(t)$ moves away from the origin at a finite velocity, $\frac{d(x_0(t))}{dt} > 0$. This rate of propagation is determined by the need to keep the amount of sediment to the right of the origin constant (D).

It is now desired to characterise the solutions $z = z(x, t)$ for differing n and m in terms of convexity and concavity. This is easily done, under certain restrictions, in the following:

Lemma 1: If in the solution z to equation (13), given by Theorem 1, $m > 1$ and $\frac{m\left(\frac{1}{J}\right)^{\frac{1}{m}}\left(\frac{k_1 x}{t^{\frac{1}{J}}}\right)^{\frac{1-n+m}{m}}}{1-n+m} > 0$, then

1. there is an inflection point $x_1, 0 < x_1 < x_0$ if and only if $n < 1$
2. the solution z over $0 \leq x \leq x_0$ is concave if $n > 1$.

Proof: Write z for $0 \leq x \leq x_0$ as

$$z = a\left[b - cx^{\frac{1-n+m}{m}}\right]^{\frac{m}{m-1}} \qquad a, b, c > 0$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = a\frac{m}{m-1}\frac{1}{m-1}\left[b - cx^{\frac{1-n+m}{m}}\right]^{\frac{1-m+1}{m-1}}(-c)^2\left(\frac{1-n+m}{m}\right)^2 x^{\frac{2-2n}{m}}$$

$$+ a\frac{m}{m-1}\left[b - cx^{\frac{1-n+m}{m}}\right]^{\frac{1}{m-1}}(-c)\left(\frac{1-n+m}{m}\right)\left(\frac{1-n}{m}\right)x^{\frac{1-n-m}{m}}$$

$\therefore$ to find inflection points, let $\frac{\partial^2 z}{\partial x^2} = 0$

$$\therefore x_1^{\frac{1-n+m}{m}} = \frac{b}{c}\left[\frac{1}{\frac{1-n+m}{(m-1)(1-n)} + 1}\right]$$

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$$\text{But} \qquad x_0^{\frac{1-n+m}{m}} = \frac{b}{c}$$

$$\therefore \quad x_1 = x_0 \left[\frac{1}{\frac{1-n+m}{(m-1)(1-n)} + 1} \right]^{\frac{m}{1-n+m}}$$

$\therefore$ (i) if $0 < n < 1$, then $0 < x_1 < x_0$,

x_1 an internal inflection point

(ii) if $1 < n < m+1$

then $\dfrac{1}{\frac{1-n+m}{(m-1)(1-n)}} < 0$

and $\dfrac{m}{1-n+m} > 0$

$\therefore \quad x_1 > x_0 \;, \quad \dfrac{\partial^2 z}{\partial x^2} > 0$

$\therefore$ profile is concave

(iii) if $n > m+1$

$\dfrac{1}{\frac{1-n+m}{(m-1)(1-n)}} > 0$ but $\dfrac{m}{1-n+m} < 0$

The exponent n is of greater interest than the exponent m in determining the effects of transport law $q_s = k q^n S^m$ (see Lemma 2 below). Moreover, the assumption $m > 1$ is probably not restrictive in the real case. Hence the first restriction in the statement of Lemma 1 is quite satisfactory. The second restriction

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claims interest in those solutions which are non-increasing. This is a completely acceptable restriction.

Interpreting the Lemma 1, one may infer that if $0 < n < 1$, then from the condition that $\frac{\partial z}{\partial x} = 0$ at $x = x_0$, there is an upper convex portion to the solution on $0 < x < x_1$, and a lower concave portion, $x_1 \leq x \leq x_0$, since $x = x_1$ is an inflection point. If $n > 1$, the profile is everywhere concave (as in Fig. 2, $n = 2$).

It is of interest to note here that on the one hand the ASCE Task Committee (1971), Henderson (1966) and Raudkivi (1967) all agree that $n > 1$, whilst on the other, most stream valleys have concave profiles.

A third point of interest concerning the similarity solutions is the observation (see Fig. 2) that the solutions "decline" with time in the sense of Davis (1909).

D. The steady-state solutions

The last aspect of the one-dimensional equations concerns the nature of the steady-state solutions to equation (9). Hence, one requires the following:

Definition 1: A steady state solution to equations (2), (3), is defined to be a solution in which the surface erodes without change of shape.

Hence, the surface is everywhere eroding at the same rate, or equivalently, $\frac{\partial z}{\partial t} = L < 0$, L a constant. In this case, it is clear that at the boundaries of any region on which such a solution is defined, and whenever the solution z is zero at the

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boundaries, one must postulate a certain removal of sediment. If this condition is not fulfilled, by continuity there will be an accumulation of sediment at some point, in which case L cannot be a negative constant.

As before, the arbitrary form of $q_s = F(S, q)$ prevents a full realisation of such solutions, but if one reverts once more to the special case $q_s = k q^n S^m$, explicit, steady-state solutions to equation (15) may be found as in the following:

Lemma 2 Consider

$$-\frac{\partial}{\partial x}\left[x^n\left(-\frac{\partial z}{\partial x}\right)^m\right] = \frac{\partial z}{\partial t}$$

where $\lim_{x \to +0} k\, x^n \left(-\frac{\partial z}{\partial x}\right)^m = 0$

Then when $\frac{\partial z}{\partial t} = L$, $L < 0$

there are solutions

$$z = \frac{L}{k} t - \frac{m\left(-\frac{L}{k}\right)^{\frac{1}{m}} x^{\frac{1-n+m}{m}}}{1-n+m} + C_0$$

C_0 an arbitrary constant.

Proof:

$$\frac{L}{k} = -\frac{\partial}{\partial x}\left[x^n\left(-\frac{\partial z}{\partial x}\right)^m\right]$$

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$$\therefore \quad \left(-\frac{L}{k}\right) x = x^n \left(-\frac{\partial z}{\partial x}\right)^m$$

$$\left(-\frac{L}{k}\right) x^{1-n} = \left(-\frac{\partial z}{\partial x}\right)^m$$

$$\therefore \quad \left(-\frac{L}{k}\right)^{\frac{1}{m}} x^{\frac{1-n}{m}} = -\frac{\partial z}{\partial x}$$

$$\therefore \quad \frac{m\left(-\frac{L}{k}\right)^{\frac{1}{m}} x^{\frac{1-n+m}{m}}}{1-n+m} = -z + C_0 \quad 1$$

$$\therefore \quad z = \frac{L}{k} t - \frac{m\left(-\frac{L}{k}\right)^{\frac{1}{m}} x^{\frac{1-n+m}{m}}}{1-n+m} + C_0$$

is the required solution.

The importance of these solutions (18) results from two facts. First, they form an essential basis for stability analysis (see section IV.A below). Second, they demonstrate a most important physical principle. Consider n in $q_s = k q^n S^m$. It

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is clear from Lemma 2 that the steady state surface is convex if and only if $n < 1$, straight if and only if $n = 1$, and concave if and only if $n > 1$. (Compare this result with that for the general time-dependent solutions of Theorem 1.)

Now writing equation (15) with $\frac{\partial z}{\partial t} = L$, and applying the basic boundary condition, one obtains

$$L x = k x^n \left(-\frac{\partial z}{\partial x}\right)^m = q_s \tag{19}$$

Hence in the steady-state case, the flux of sediment must increase linearly in x. But by equation (13) the flux of water also increases linearly in x. Thus for the case $n = 1$, the necessary increase in sediment flux is entirely accounted for by the increase in water flux. Hence the profile is straight. But if $n < 1$, the sediment flux increases less than linearly with discharge, which itself increases linearly. Hence the slope must increase downstream in order to maintain a sediment discharge linear in x. Therefore the profile is convex. In the case $n > 1$, the converse holds and the profile is concave.

These facts offer an important insight into the action of transport laws in general. For consider now the case $q_s = F(S, q) = F\left(-\frac{\partial z}{\partial x}, x\right)$, and in particular consider the steady-state surface associated with such a law. This surface must be composed of some sequence of convex, straight and concave segments. Hence, using an analogy with the law $q_s = k q^n S^m$, it seems most reasonable to interpret the various segments of the

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associated steady surface in the same way as for the law $q_s = k q^n S^m$. This motivates the following:

Definition 2: Consider the steady state solution to equation (15).

In neighbourhoods where the profile is convex, it is said that the transport law $q_s = F(S,q)$ behaves like $q_s = k q^n S^m$ with $n < 1$. In neighbourhoods where the profile is concave, it is said that $q_s = F(S,q)$ behaves like $q_s = k q^n S^m$, with $n > 1$.

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IV. STABILITY, THE BASIC MODEL AND THE EVOLUTION OF THE BASIC SURFACE

A. Drainage basin evolution and the notion of stability

The treatment of the basic equations reduced to one space variable leads directly to a consideration of the full equations. It is here that the non-linearity of the equations is truly apparent, for while the direction down the gradient in the one-dimensional case is represented by (-1), it now becomes the two-dimensional vector

$$\left(-\frac{\frac{\partial z}{\partial x}}{\left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right)^{\frac{1}{2}}} \mathbf{i} - \frac{\frac{\partial z}{\partial y}}{\left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right)^{\frac{1}{2}}} \right)$$

Computer solutions would appear as the only recourse, yet some knowledge of the solutions of the basic equations is prerequisite to such an attack, for this requires a system of equations to possess "reasonable" stability characteristics. The situation is analogous to that of the real world, since the well-defined geometries of many drainage basins demonstrate that they possess "reasonable" stability characteristics.

In order to formulate the notion of stability more precisely, it is useful to consider first a physical interpretation and then a mathematical statement of the idea. Physically, the interpretation of stability is very simple; as an example, consider a drainage basin which has achieved a steady-state of operation, that is to say, which is everywhere downcutting at a uniform rate. Then consider an experiment in which at time $t = t_0$, the surface is instantaneously

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modified by an amount that is very small, but arbitrary. One then says that the form of the basin is stable, if, when it recommences operation, any small perturbation that was impressed, is removed during the further operation of the system, which therefore reverts to its former invariance. On the other hand, one says that the surface is unstable if any one of the infinitesimal amplitude perturbations grow with time, so changing the previous form of the basin.

Several points should be noted. First, it must be emphasised that the condition for stability must apply to every possible perturbation, while only one particular perturbation need grow for one to say that the basin is unstable. Second, the formulation of the problem is limited to perturbations whose amplitude is very small. It should be noted that the basic problem may now be formulated in terms of this notion of stability. For if one specifies that the basic surface be in a steady state, according to a chosen transport law, it is sufficient to find suitable mathematical techniques for applying perturbations to the surface. Then one may follow their development mathematically using the basic equations.

Consider briefly the general mathematical formulation of the stability problem, in a case where one is faced with n equations in n unknowns. In the first stage, one constructs n steady-state solutions X_{oi}, $i = 1, \dots, n$. These solutions are perturbed by hypothesising solutions of the form $X_i = X_{oi} + x_i'$ where x_i' is very small compared to X_{oi}. Substituting this solution into the full, time-dependent equations, and then dropping

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terms of quadratic order or higher, one obtains two equations for each previous equation, the first describing the known steady behaviour, and the second (separable from the first by the relative smallness of its terms) being a linear equation in the disturbances. One then seeks a complete set of solutions, whereby any arbitrary disturbance may be represented. Since time enters the equations only through the derivatives, one may posit solutions containing the term $e^{\lambda_i t}$. Hence given the boundary conditions to the problem, the solution becomes a matter of determining the normal modes of the system, and hence of the permissible values of λ_i.

If $\lambda_i < 0$ for all normal modes, the system is said to be stable. But if $\lambda_i > 0$ for any mode, the system is said to be unstable.

It is extremely important to realise that if the system is unstable the assumption of small amplitude disturbances prohibits the use of the linearised equations in following the evolution of the system for any length of time.

It may also be noted that one has no difficulty in suggesting the origin of small amplitude disturbances.

B. The stability of the one-dimensional basic equation

As before, it proves instructive to apply the method to the one-dimensional problem before proceeding to the two-dimensional case. The problem, therefore, concerns the stability of a steady basic surface which has been perturbed only along the x-axis (that is, there is no y-variation in these perturbations). It is a remarkable and

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important fact that for any transport law, the solutions are stable, as proved in

Proposition 1: Let $q_s = F(S, q)$ be any Transport Law. Then the system:

$$\frac{\partial q}{\partial x} = 1$$

$$\frac{\partial}{\partial x} F\left(-\frac{\partial z}{\partial x}, x\right) = -\frac{\partial z}{\partial t}$$

with boundary conditions

$$q|_{x=0} = 0$$

$$q_s|_{x=0} = 0$$

$$z|_{x=x_0} = 0$$

is stable with respect to infinitesimal amplitude perturbations.

Proof: Let S_0, q_0 be the steady slope and discharge. Then

$$\frac{\partial}{\partial x}(q_0 + q') = 1 \quad \text{implies} \quad q' = 0$$

Moreover:

$$\frac{\partial z'}{\partial t} = \frac{\partial}{\partial x}\left[\frac{\partial F}{\partial S}\frac{\partial z'}{\partial x}\right]$$

$$\text{implies} \quad \int_0^{x_0} z' \frac{\partial z'}{\partial t} dx = \int_0^{x_0} z' \frac{\partial z'}{\partial x}\frac{\partial F_s}{\partial x} dx + \int_0^{x_0} z' \frac{\partial^2 z'}{\partial x^2} F_s \, dx$$

integrating by parts and using the basic conditions that

$$\frac{\partial F}{\partial S}\frac{\partial z'}{\partial x}\Big|_{x=0} = \frac{\partial z}{\partial x}\Big|_{x=x_0} = 0$$

$$\frac{\partial}{\partial t}\int_0^{x_0} z'^2 dx = -\int_0^{x_0} \frac{\partial F}{\partial S}\left(\frac{\partial z'}{\partial x}\right)^2 dx$$

Now $\frac{\partial F}{\partial S} = F_s > 0$ by definition, and $\left(\frac{\partial z'}{\partial x}\right)^2 > 0$

$$\text{implies} \quad \int_0^{x_0} F_s \left(\frac{\partial z'}{\partial x}\right)^2 dx > 0$$

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One may interpret this result in two ways. First $\frac{z'^2}{2}$ may be considered as the "energy" of the perturbation. Hence, in showing that $\frac{\partial}{\partial t}\left[\int_0^{x_0} \frac{z'^2}{2} dx\right] < 0$, the proposition demonstrates that the total energy of the perturbations is decreasing with time. Second, the equation

$$\frac{\partial}{\partial x}(q_0 + q') = 1 \qquad (20)$$

shows that the perturbation in the water flux, q', is zero. Consequently any perturbation in the sediment flux can occur only as a result of a perturbation of slope. But an increase in slope causes an increase in sediment transport and hence an increase in erosion, while a decrease in slope results in a decrease in erosion. By this process, perturbations are removed. Their behaviour during removal is analogous to the behaviour of knick-points in river channels as described by Brush and Wolman (1960).

C. The basic proposition concerning surface stability

When arbitrary infinitesimal amplitude perturbations are applied to a two-dimensional steady surface, the variation in the y-direction adds a great deal of complexity. Using the transport law $q_s = F(S, q)$ one may write the linearised basic equations as:

$$0 = -\frac{\partial q'}{\partial x} + \frac{x}{S_0}\frac{\partial^2 z'}{\partial y^2} \qquad (21)$$

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$$\frac{\partial z'}{\partial t} = \frac{1}{S_0}\left[F - x F_{q_0}\right]\frac{\partial^2 z'}{\partial y^2} + \frac{\partial}{\partial x}\left[F_{S_0}\frac{\partial z'}{\partial x}\right] - \frac{\partial F_{q_0}}{\partial x} q' \quad (22)$$

where primes denote perturbations and S_0, q_0 are the steady slope and discharge respectively. From these, one obtains the equation for z':

$$\frac{1}{B_1}\frac{\partial z'}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{B_1}\frac{\partial z'}{\partial t}\right) = \frac{\partial}{\partial x}\left(\frac{A}{B_1}\frac{\partial^2 z'}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{F_{S_0}}{B_1}\frac{\partial z'}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{F_{S_0}}{B_1}\frac{\partial^2 z'}{\partial x^2}\right) + \frac{x}{S_0}\frac{\partial^2 z'}{\partial y^2} \quad (23)$$

where $A = \frac{x}{S_0} F_{S_0} \frac{\partial S_0}{\partial x}$, $B_1 = -\frac{\partial F_{q_0}}{\partial x}$

If one postulates a solution of the form $z' = \hat{z}(x)\, e^{i\omega y + ct}$

equation (23) is equivalent to a third-order ordinary differential equation with variable and unspecified coefficients. As such it is intractible. However, a fairly complete understanding of the stability characteristics of the basic model may be constructed by considering four special cases:

a) What happens as the wavelength of perturbation in the y-direction $\left(\frac{1}{\omega}\right)$ is made small relative to the wavelength of perturbation in the x-direction?

b) What happens, under constant rainfall conditions, in a neighbourhood of constant slope?

c) What is the stability result for the particular transport law

$$q_s = k\, q^n S^m \quad ?$$

d) What happens when one assumes a minimal length below which perturbations in the y-direction are stable?

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The first two cases constitute the key result of the whole work, while the second two cases indicate a useful interpretation of this result (case (d) is considered in the next chapter).

Consider now the case as the wavelength of perturbation in the y-direction grows small with respect to the wavelength of perturbation in the x-direction. This assumption allows an approximate solution to the stability problem, as stated in the following:

Basic Proposition: (Proposition 2) Let $q_s = F(S, q)$ be a transport law. Then for ω sufficiently large in

$$z' = \hat{z}(x)\, e^{i\omega y + ct}$$

$\frac{dS_0}{dx} > 0$ implies $c < 0$ implies stability

$\frac{dS_0}{dx} < 0$ implies $c > 0$ implies instability

Hence: (a) Convex sections of the steady basic surface are stable with respect to disturbances of small wavelength in the y-direction,

while (b) concave sections are unstable.

Proof: Consider

$$\frac{1}{B_1} z'_{tx} + \left(\frac{1}{B_1}\right)_x z'_t = \left(\frac{A}{B_1}\right)_x z'_{yy} + \left(\frac{A}{B_1}\right) z'_{yyx} + \left(\frac{F_{S_0}}{B_1} z'_x\right)_x + \left(\frac{F_{S_0}}{B_1} z'_{xy}\right)_x + \frac{x}{S_0} z'_{yy}$$

Assume solution $z' = \hat{z}(x)\, e^{i\omega y + ct}$, $c = c'\omega^2$, then

$$0 = \hat{z}\left[-\left(\frac{c'\omega^2}{B_1}\right)_x - \left(\frac{A}{B_1}\right)_x \omega^2 - \left(\frac{x}{S_0}\right)\omega^2\right] + \hat{z}'\left[-\frac{c'\omega^2}{B_1} - \frac{A}{B_1}\omega^2\right] + \hat{z}'\left(\frac{F_{S_0}}{B_1}\right)_x + \hat{z}''\left[\frac{F_{S_0}}{B_1} + \left(\frac{F_{S_0}}{B_1}\right)_x\right] + \hat{z}'''\frac{F_{S_0}}{B_1}$$

$$= \hat{z} q + \hat{z}' p + \hat{z}'\left(\frac{F_{S_0}}{B_1}\right)_x + \hat{z}''\left[\frac{F_{S_0}}{B_1} + \left(\frac{F_{S_0}}{B_1}\right)_x\right] + \hat{z}'''\frac{F_{S_0}}{B_1} \qquad (ii)$$

For ω^2 large, ignore terms $O(<\omega^2)$ except $\hat{z}'''\frac{F_{S_0}}{B_1}$

$$\therefore 0 = \hat{z}'''\frac{F_{S_0}}{B_1} + p\hat{z}' + q\hat{z} \qquad (iii)$$

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Assume for moment constant coefficients in (iii') (iv)

$\therefore$ obtain $r^3 + p'r + q' = 0$

if $r^2 \sim O(p')$ then $q' \sim O(0)$ .. A good estimate if $\left|\frac{q}{(\epsilon p)^{\frac{3}{2}}}\right|$ small

if $r \sim O(1)$ then $r \sim O\left(-\frac{q'}{p'}\right)$. A good estimate if $\left|\frac{q'^2}{p'^3}\right|$ small

Hence obtain local solution to (iii')

$$\hat{z}_0 = c_1 e^{\sqrt{-p'}\,x} + c_2 e^{-\sqrt{-p'}\,x} + c_3 e^{-\frac{q'}{p'}x} \quad (v)$$

Now seek to extend solution by varying the coefficients. Since $\sqrt{-p'} \sim O(\omega)$ and $-\frac{q'}{p'} \sim O(1)$, the first two terms on the right of (v) vary more quickly than the third. Hence approximate solutions for these terms with

$$\hat{z}_1 = \delta(x)\, e^{+\int^x \sqrt{(\epsilon p')}\,dx} = \delta(x)\, e^{\int^x r\,dx}$$

$$\hat{z}_2 = \delta(x)\, e^{-\int^x \sqrt{(\epsilon p')}\,dx} = \delta(x)\, e^{-\int^x r\,dx}$$

where $r = \sqrt{-p'} = \sqrt{\frac{C+A}{F_{s0}}}$ (vi)

Substituting (vi) into (ii) and equating powers of ω :

α^3 : $r^2 = -\frac{B}{F_{s0}} p' = \frac{C+A}{F_{s0}}$

α^2 : $0 = \delta q' + \delta' p' + \delta r^2 \left[\frac{F_{s0}}{B_1} + \left(\frac{F_{s0}}{B_1}\right)_x\right] + \delta' r^2 \frac{F_{s0}}{B_1} + 3\delta r r' \frac{F_{s0}}{B_1}$

$$\therefore \quad \delta(x) = e^{-\int^x \left(\frac{q' + r^2\left[\frac{F_{s0}}{B_1} + \left(\frac{F_{s0}}{B_1}\right)_x\right] + 3rr'\frac{F_{s0}}{B_1}}{p' + \frac{F_{s0}}{B_1} 3r^2}\right) dx} = e^{-\int^x q(x)\,dx}$$

$$\therefore \quad \hat{z}_1 = e^{-\int q(x)\,dx + \omega \int r\,dx}$$

$$\hat{z}_2 = e^{-\int q(x)\,dx}\, 1 - \omega \int r\,dx$$

For third solution, drop all terms $O(<\omega^2)$ in (ii')

$$\therefore \quad \hat{z}_3 = e^{-\int^x \frac{q'}{p'}\,dx}$$

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$\therefore$ obtain approximate solution :

$$\hat{z} = c_1 e^{-\int q\,dx + \omega \int r\,dx} + c_2 e^{-\int q\,dx - \omega \int r\,dx} + c_3 e^{-\int \frac{q'}{p'} d}$$

Boundary condions are :

(a) $\hat{z}$ bounded, by need of complete set of funct

(b) $\hat{z}(0) = 0$

(c) $\hat{z}(x_0) = 0$

No (a) implies $c_1 = 0$, since $C + A > 0$ implies r real, and need $\omega \to \infty$.

Applying (b), (c) obtain :

$$\int_0^{x_0} \left[q - q'/p' + \omega r\right] dx = 0$$

$$\therefore \int_0^{x_0} r\,dx = \frac{1}{\omega} \int_0^{x_0} \left[\frac{q'}{p'} - q\right] dx$$

If $C + A < 0$, L.H.S. is complex, but R.H.S. real by definition of q'/p', q.

$\therefore \int_0^{x_0} i r'\,dx = 0$, but $r' > 0$. $\therefore r' = 0$, $\therefore r = 0$

As $\int_0^{x_0} r\,dx \to 0$, $r \to 0$ and as $r \to 0$, $q'/p' - q \to 0$

since $q \to q'/p'$

Since $\frac{\partial E}{\partial s_0} > 0$, $C \to -A$ as $\omega \to \infty$.

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It is also possible to prove a stability result for any wavelength of perturbation in the y-direction for neighbourhoods of the steady basic surface which have a constant slope. For this problem it is useful to construct the

Definition 3: A steady basic surface is said to be y-neutral if the stability equation reduces to $\frac{\partial z'}{\partial t} = \frac{\partial}{\partial x}\left(F_{s_0} \frac{\partial z'}{\partial x}\right)$

This says that the surface behaves as if it were unaware of perturbations in the y-direction. Hence one may prove the

Proposition 3: Let $q_s = F(S, q)$ be a transport law for a surface over which rainfall is uniform and steady. Then $\frac{dS_0}{dx} = 0$ in a neighbourhood of $X = X_0$ if and only if for any perturbation, the surface is y-neutral in that neighbourhood.

Proof:

(i) $\frac{dS_0}{dx} = 0$

$\therefore$ by $\quad L = F_{q_0} + F_{s_0} \frac{dS_0}{dx}$

$$L = F_{q_0}$$

$\therefore$ by

$$\frac{\partial z'}{\partial t} = \frac{x}{S_0}\left[L - F_{q_0}\right] \frac{\partial^2 z}{\partial y^2} + \frac{\partial}{\partial x}\left[F_{s_0} \frac{\partial z'}{\partial x}\right] - q' \frac{\partial}{\partial x}\left[F_{q_0}\right]$$

$$\frac{\partial z'}{\partial t} = \frac{\partial}{\partial x}\left[F_{s_0} \frac{\partial z'}{\partial x}\right]$$

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(ii) Case (a) $\frac{x}{S_0}\left[L - F_{q_0}\right]\frac{\partial^2 z'}{\partial y^2} = q'\frac{\partial}{\partial x}\left[F_{q_0}\right] = 0$

iff $F_{q_0} = L$

$\therefore$ by $F_{S_0} > 0$, $\frac{dS_0}{dx} = 0$

Case (b) $\frac{x}{S_0}\left[L - F_{q_0}\right]\frac{\partial^2 z'}{\partial y^2} = q'\frac{\partial}{\partial x}\left[F_{q_0}\right] \neq 0$

Then by $\frac{\partial q'}{\partial x} = \frac{x}{S_0}\frac{\partial^2 z'}{\partial y^2}$

$$\frac{1}{q'}\frac{\partial q'}{\partial x} = -\frac{\frac{\partial}{\partial x}\left[L - F_{q_0}\right]}{L - F_{q_0}}$$

$$\therefore \ln q' = \ln\left(L - F_{q_0}\right)^{-1} + g(y,t)$$

$$\therefore q' = \frac{1}{F_{S_0}\frac{dS_0}{dx}} f(y,t)$$

But this is True iff the perturbations at $x = x_0$ have the given form.

$\therefore$ case (b) cannot hold for all perturbations.

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With regard to the third special case (c) mentioned above, it is easy to prove

Proposition 4: For any transport law of the form $q_s = q^n \varphi(s)$, where $n > 0$, the associated steady basic surface is:

(1) stable if $n < 1$

(2) y-neutral if $n = 1$

(3) unstable if $n > 1$.

Moreover, in the unstable case, the rate of growth increases as the wavelength of disturbance in the y-direction decreases.

Proof:

From $h_x = x^n \varphi(s)$

and $F(s,q) = q^n \varphi(s)$

it follows that

$$F_{q_0} = n x^{n-1} \varphi(s)$$

$$\therefore \frac{\partial F_{q_0}}{\partial x} = 0$$

Moreover $F - x F_{q_0} = 1 - n$

$\therefore$ using equation (22), one obtains

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$$\frac{\partial z'}{\partial t} = \frac{1}{S_o}(1-n)\frac{\partial^2 z'}{\partial y^2} + \frac{\partial}{\partial x}\left(F_{S_o}\frac{\partial z'}{\partial x}\right)$$

It is easy to verify that this equation is stable if $n<1$, unstable if $n>1$.

It should be noted that this latter proposition solves the stability problem for an arbitrary disturbance, in contrast to the basic proposition.

D. The interpretation of the basic proposition

Sufficient information has now been collected to allow of a reasonable interpretation of the basic proposition. But first consider the following remarks. It is instructive to interpret the effect of perturbations on the steady basic surface. Consider the magnitude of the gradient, $|\nabla z| = \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}}$. On perturbing this function of the steady basic surface one obtains

$$|\nabla z| = S_o + \frac{\partial z'}{\partial x} \qquad (24)$$

But, on perturbing the gradient,

$$\nabla z = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$$

one obtains:

$$\nabla z = \left(-S_o + \frac{\partial z'}{\partial x}, \frac{\partial z'}{\partial y}\right) \qquad (25)$$

Hence equation (25) shows that a component of slope in the y-direction occurs for the first time. However, equation (24) shows that the

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overall slope of the surface is unaffected by this y-component. In other words, one obtains the same change in the magnitude of flux of sediment due to the change in slope as one obtains in perturbing the one-dimensional case. On the other hand, the directions of the fluxes of water and sediment are modifed by the new y-component of slope. Hence neighbourhoods of convergence and divergence in the flux of water and sediment are produced. See Fig. 3.

It may also be remarked that as far as the question of channel growth is concerned, it is sufficient to consider neighbourhoods of convergence.

It proves convenient to interpret the basic proposition in terms of the three classes of segment (convex, straight, concave) that occur on a steady basic surface.

First, Proposition 3 showed that the straight segments $\left(\frac{dS_0}{dx} = 0\right)$ are y-neutral for the general transport law $q_s = F(S, q)$. This fact is proved again for the special transport law $q_s = q^n \, q(S)$ with $n = 1$ in Proposition 5. Note that in both cases the proofs hold for arbitrary disturbances. Consider the case $q_s = k q^n S^m$ $n = 1$, and a neighbourhood of convergence on the perturbed surface. Since $n = 1$, two converging fluxes of water carry the same total volume of sediment when joined as when separate (that is, by the

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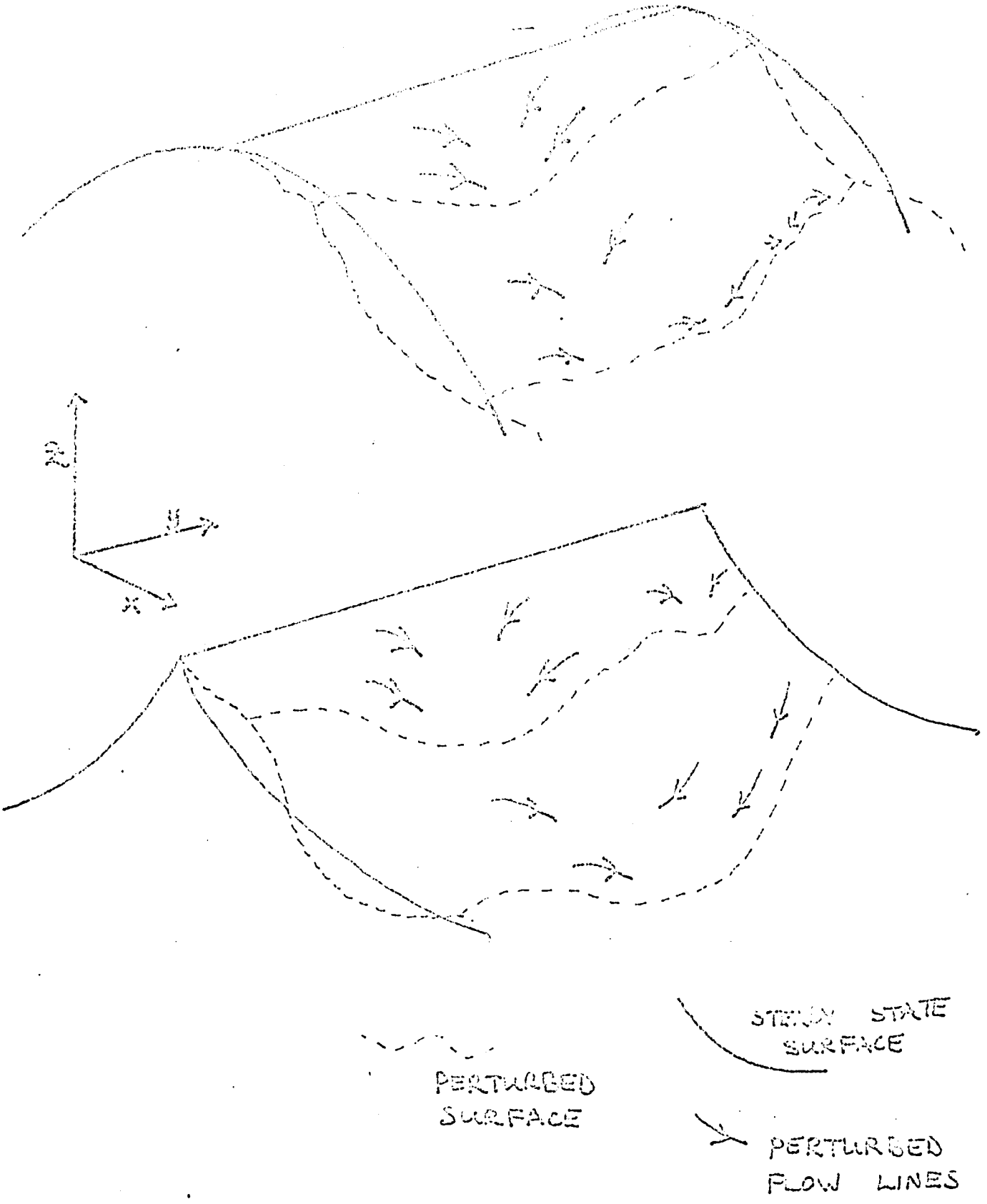
Figure 3. The perturbed steady state surface



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linear dependence of q_s on q in $q_s = kq^n S^m$). Hence convergence of water fluxes in this case causes neither additional erosion nor deposition. Hence although perturbations in the y-direction exist, the surface behaves (in terms of erosion of deposition) as though unaware of such perturbations. Proposition 3 indicates similar behaviour of the surface for the general law $q_s = F(S, q)$). Definition 2, which stated that the law $q_s = F(S, q)$ behaves like $q_s = kq^n S^m$ $n = 1$ in straight segments of the steady basic surface was motivated by the concept of the two laws giving rise to analogous behaviour in such regions. Hence it seems most reasonable to extend the mechanism explaining the stability result for

$q_s = kq^n S^m$, $n = 1$ to the general law

$q_s = F(S, q)$ in straight segments.

It is easy now to invoke mechanisms explaining the stability results in convex and concave regions. When the steady basic surface is convex $\frac{dS_o}{dx} > 0$, small wavelength perturbations in the y-direction are removed for the case $q_s = F(S, q)$, while all wavelength perturbations in the y-direction are removed for the case $q_s = kq^n S^m$, $n < 1$. In the latter case, it is evident that converging fluxes of water carry less total sediment when joined than when separate (since $n < 1$). Hence, deposition occurs in neighbourhoods of convergence, and perturbations in the y-direction

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are removed. Once more, and using the motivation of Definition 2, it is most reasonable to extend this mechanism to the general case $q_s = F(S, q)$ to explain the stability result for convex segments.

Lastly, on concave portions of the surface, small wavelength perturbations in the y-direction grow for the case $q_s = F(S, q)$, while for the case $q_s = k q^n S^m$, $n > 1$, they grow for all wavelengths of disturbance. For the latter case, since $n > 1$, the total sediment transporting capacity of two volumes of water is increased when they join together. Thus additional erosion occurs in regions of convergence. Hence channel-like forms grow. Again, it is reasonable to extend this mechanism to the general case $q_s = F(S, q)$. Both the basic proposition and proposition 4 predict that the smallest wavelengths of disturbance in the y-direction are the fastest growing. This fact may be interpreted in the following sense. Consider equation (21)

$$\frac{\partial q'}{\partial x} = \frac{x}{S_0} \frac{\partial^2 z'}{\partial y^2}$$

This equation states that in neighbourhoods of convergence, the rate of increase of the perturbed water flux in the "downstream" direction is maximal for those perturbations whose curvature in the y-direction is greatest. But given a fixed amplitude of disturbance, this curvature must be greatest for the smallest wavelength of disturbance.

A final matter in need of interpretation concerns the significance of allowing the wavelength of disturbance in the y-direction to

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become very small. Since the surface is composed of particulate matter, the "continuum" hypothesis must break down at some scale. Hence it is meaningful to say that a channel grows by sediment transport only if it is wider than the diameter d of the largest particle it carries. Hence $\frac{1}{\omega} > d$. However this fact is nowhere implied in the model.

E. The extension of the basic proposition to arbitrary disturbances

The basic proposition was proved only for the case of small wavelength disturbances. However, there is good reason to believe that one may extend it to cover any wavelength of disturbance. Note that in this connection it is sufficient to show that no wavelength of perturbation grows on the convex portion of a steady basic surface. (This is true because instability is proved for some wavelengths on concavities and y-stability is proved for all wavelengths on straight sections.)

First, Lemma 2 demonstrates that for $q_s = k q^n S^m$, the steady basic surface is convex for $n < 1$. It would seem that a convex neighbourhood of the steady surface associated with $q_s = F(S, q)$ may be locally well approximated by a neighbourhood of the surface associated with $q_s = k q^n S^m$ for some $n < 1, m$. But the stability result for the latter law was proved for all wavelengths of disturbance. This suggests that convexities for the general law $q_s = F(S, q)$ are stable for all wavelengths.

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Second, Proposition 3 showed that in straight sections of a surface associated with the law $q_s = F(S, q)$, y-neutrality held for all disturbances, as for the case $q_s = k q^n S^m$, $n = 1$. But in the latter case, such neighbourhoods (or even points where $\frac{dS_o}{dx} = 0$) provide a dividing segment (or line) between stable and unstable neighbourhoods. It is reasonable to assume that the same holds for $q_s = F(S, q)$, which would imply that convexities are stable.

Finally, consider the mechanism invoked to explain the stability result, and the mechanism which suggested why the smallest perturbations should be the fastest growing. Since small perturbations (which cause a greater rate of convergence of water than large perturbations)do not grow on a convexity, it is hard to explain why longer wavelength perturbations should grow.

Hence the following

Basic hypothesis: The basic proposition extends to all wavelengths of disturbance in the y-direction.

F. Landscape forming transport laws

Given a sediment transport law $q_s = F(S, q)$ whose associated steady surface is everywhere straight or convex, it is apparent that channels never develop from infinitesimal amplitude perturbations. Conversely, for laws which have an associated steady surface that is everywhere concave, there can never be stability, since the smallest wavelength perturbations are always growing at the

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greatest rate. Hence if the basic model is to apply to the real case, it is essential to have a transport law which has both convexities and concavities in its associated steady surface. Using the simplest configuration, one may make the

Definition 4: A transport law $q_s = F(S, q)$ is said to be a landscape-forming transport law if the associated steady basic surface is convex in the upper portion and concave in the lower portion.

As an example of such a law, consider

$$q_s = k q^n S^m + \dot{a} S \quad , \quad \begin{matrix} k, a > 0 \\ n > 1 \end{matrix} \qquad (26)$$

G. Two flaws in the basic model

It is now assumed that $q_s = F(S, q)$ is a landscape-forming transport law. Using this assumption, the basic model suggests the following tentative theory of the development of drainage surfaces. Two fundamental weaknesses in the model become apparent.

Consider a basic surface. The basic hypothesis claims the existence of a finite length (call this the X_c distance after Horton, 1945) below which the surface remains unchannelled. Beyond this length (that is, in the "concave" section) y-direction perturbations begin to grow into rill-like features. As these rills deepen, they presumably leave intervening ridges, which, if less wide than twice the distance X_c, remain unrilled. However, the basic model contains no implication of a lower non-zero bound on the

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wavelength of unstable perturbations. It is therefore difficult to explain why such ridges should be left according to this model. Hence it is probable that the basic model is in need of modification. For example, it may prove possible to model dynamical effects. It is known that the transport of sediment per unit width decreases significantly (due to wall effects) when the width of a channel becomes less than five times its depth. Such a fact may well prevent the smallest perturbations from growing fastest, implying the existence of a finite amplitude of disturbance whose growth-rate is maximal.

A theory of the later stages of channel-system development reveals the second main objection to the model. Final stabilisation of form must occur for the model to prove reasonable. In the case of channels this must result from the development of a balance between the rate of channel down-cutting and the rate of sediment inflow from the side-slopes. But assumption A.3 of the model (water flows down the gradient of the surface) indicates that channels tend to approach forms where all the water is concentrated into a line. Hence discharge per unit width approaches infinity. It is difficult, therefore, to envisage the required balance in terms of the basic model. Hence at some point the model must be modified to take account of the constraints imposed by the existence of a free-water surface.

The basic proposition also indicates the possibility of explaining channel growth in terms of a theory which involves a variational principle. It seems that the initial evolution of the system is directed towards maximising sediment transport in some

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sense. For rills grow on the concavity because they increase the total transporting capacity of the flow, while perturbations are damped out on the convexities because converging and diverging flows carry less sediment than flows without convergence and divergence. Since this theory is only supported by the linearised equations, it is unclear as to whether it can be extended to later development. It is also unclear what constraints must be placed on any such variational principle.

H. The basic model and reality

Despite its two shortcomings, the basic model is in nice agreement with some aspects of reality. First, much real landscape in profile is composed of simple sequences of convexities and concavities. Gilbert (1909) first pointed out that different transport principles must apply to the two classes of segment, a fact now suggested by the model. Second, Horton (1945) suggested that a critical belt of no erosion (his X_c distance) was essential for the growth of channel systems. There is much empirical evidence to support a somewhat modified form of this hypothesis. Again, the basic model suggests that such a stable length is indeed essential for a satisfactory theory of channel growth. Moreover, it is clear that this length has a profound effect on drainage density. On this point it is of interest to note that empirical evidence shows higher rainfalls corresponding to larger drainage densities. On the other hand the model predicts a relationship between rainfall and the distance as in the following:

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Proposition 5: For a landscape-forming transport law, with a point of inflection, an increase in rainfall, α, leads to a decrease in the stable length, X_c, for a given steady-state rate of downcutting, L.

Proof: Consider $L_x = F(x, S_{10}), \frac{dS_{10}}{dx} = 0$ at $x = x_0$

$L_x = F(x, S_{20}) \frac{dS_{20}}{dx} = 0$ at $x = x_1 \neq x_0$

at $x = x_0$, and using $L = \frac{\partial F}{\partial q} + \frac{\partial F}{\partial S}\frac{dS}{dx}$,

$$L = F_q$$

$$= \rho F_q + F_{S_{20}} \frac{dS_{20}}{dx}$$

$$\therefore \frac{F_q}{F_{S_{20}}}(\rho - 1) = -\frac{dS_{20}}{dx}$$

if $\rho > 1$ (higher rainfall), $\frac{dS_{20}}{dx} < 0$

$\therefore$ x_0 lies on the concavity

$\therefore$ distance to point of inflection is reduced.

It is intuitively reasonable to suppose that a decrease in the X_c distance leads to an increase in drainage density. Hence the model agrees nicely with reality on this point.

The existence of the stable length, X_c, and hence of convexities on the steady surface results from the fact that

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$q_s = F(S, q)$ behaves in such regions like $q_s = k q^n S^m$, $n < 1$. It is not difficult, however, to suggest mechanisms which would cause the transport of sediment to depend on q as if it were less than linear in q in some region. For example, one may postulate a finite substrate permeability, which would result in lowered (or zero) sediment transport by water in regions close to divides. Or one may postulate a certain critical discharge needed to move particles of a given size.

In summary, the basic model proves enlightening on certain aspects of the initial growth of channel systems. In particular, it is enlightening as to what form a transport law $q_s = F(S, q)$ must take in order to provide a satisfactory law explaining channel growth. At the same time, the model and its shortcomings suggest certain directions for other research.

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V. A MODIFIED BASIC MODEL, THE INTEGRATION OF CHANNEL SYSTEMS AND THE PROBLEM OF LATERAL MIGRATION OF CHANNELS

A. The need of a modified basic model

The basic proposition demonstrates that given a landscape-forming transport law and the unstable region of the associated surface, the smallest perturbations are the fastest growing. Hence computer methods applied to the basic equations would prove of small value. It is probable, however, that the very complexity of the drainage basin problem will eventually force the use of computor techniques. The following analysis is based on this assumption.

It is reasonable to assume that in a uniform and steady climate, any drainage surface tends to approach a well-defined and reasonably stable form. This suggests developing a model which on the one hand contains many of the essential features of the basic model, and on the other, possesses the property that under computation, unstable perturbations grow into well-defined forms. Such a model is now developed.

B. The stability assumption

Following the discussion of the basic proposition, it is reasonable to make the following

Stability assumption: For the unstable regions of a steady basic surface, there exists a critical wavelength of disturbance in the y-direction below which the surface is stable.

First, note that this assumption avoids the first major weakness of the basic model (section IV.G). Second, it claims the existence of

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a "valley" whose dimensions are in some sense minimal. The assumption proves to have deep implications.

The stability assumption is introduced by modifying the basic model with a certain degree of discreteness. It is necessary, therefore, to modify both the basic problem and the basic model.

C. The modified basic problem and the N-valley problem

Consider a continuous surface $z = z(x,y,t)$, where, for a given y, z decreases from a maximum elevation at $x = 0$ to a minimum at $x = x_0$. Let this surface be serrated with valleys (see Fig. 4), such that all valleys are congruent and each has a channel and divides which run in the x-direction. Assume that this surface is repeated symmetrically about the line $x = 0$, and call the surface the modified basic surface.

If rainfall commences at time $t = 0$, one may state the

Modified basic problem: How does the modified basic surface evolve with time and how do the initial valleys interact to produce forms similar to those of real drainage basins?

As before, the method is to perturb a steady-state form of the modified basic surface. It proves convenient to perturb the surface in the following way. Take a set of N adjacent valleys. Perturb each one independently. Then repeat this unit of N perturbed valleys to the right and to the left of the first set (see Fig. 5). By the symmetry of the surface, it is sufficient to follow the development of one set of N valleys. Call this the N-valley problem.

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Figure 4. The modified basic surface

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D. The modified basic model: statement

It is now essential to construct a model describing the evolution of a single valley (see Fig. 6). Such a model must embody the principles of mass conservation, and must also contain a statement of the stability assumption. Once more, the model is presented as a list of assumptions followed by a critical interpretation.

B.1 The substrate is uniform; there is a steady and uniform rainfall; and there are no losses of water through evaporation or infiltration.

B.2 A valley may be represented as a surface which is continuous everywhere and twice continuously differentiable almost everywhere.

B.3 The channel lies at the intersection of the two side-slope elements and is represented as a line.

B.4 Water flowing over the side-slopes moves in the y-direction.

B.5 Sediment transported over the side-slopes moves in the y-direction.

B.6 The transport of sediment over the side-slopes is given by a law of the form

$$q_s = Q(S, q) \tag{27}$$

$$Q \geq 0$$

$$\frac{\partial Q}{\partial S} > 0$$

$$\frac{\partial Q}{\partial q} > 0$$

$Q(S,q)$ behaves like $kq^n S^m$, $n < 1$

B.7 The conservation of water on the side slopes may be written as

$$\frac{\partial q}{\partial y} = I \tag{28}$$

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B.8 The conservation of sediment on the side-slopes may be written as

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial y}\left[q\,(S, q_s)\right] \tag{29}$$

B.9 The law of sediment transport in the channel is given by:

$$Q_s = k' Q^n S^m \tag{30}$$

Q_s the total sediment transport, Q the total discharge, S the channel slope.

B.10 The law of conservation of water in the channel may be written as

$$\frac{\partial A}{\partial x} = y_1 - y_0 \tag{31}$$

$A = A(x,t)$ the area of valley above a given x . y_0, y_1 are the y-coordinates of the left and right divides respectively.

B.11 The law of conservation of sediment in the channel is given by

$$\int_{y_0}^{y_1} \frac{\partial z}{\partial t}\, dy = \frac{\partial}{\partial x} Q_s \tag{32}$$

or equivalently

$$\frac{\partial Q_s}{\partial x} = q_{s_L} + q_{s_R} \tag{32}'$$

q_{s_L}, q_{s_R} the fluxes of sediment from the left and right side-slopes respectively.

B.12 The basic boundary condition holds on all divides.

E. The modified basic model: discussion

B.1 This set of assumptions has been discussed in connection with assumption A.1 of the basic model.

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B.2 This assumption is in fact a considerable weakening of assumption A.2 of the basic model, and is consequently more acceptable.

B.3 The assumption of line channels is based on two facts. First, real channels have a width very much smaller than the width of their valleys, hence they may be approximated as lines. Second, in the discussion of assumption A.5 of the basic model, it was pointed out that the law $q_s = kq^n S^m$, $n > 1$, is a reasonable law describing bed-load transport in channels. But such a law leads to instabilities which grow fastest for the smallest wavelength. Hence, using the stability assumption, one may either discard that law or represent streams as lines. The last alternative seems preferable.

B.4 The assumption that water flowing over the side-slopes moves only in the y-direction is reasonable given that the channel deviates by fairly small amounts from a straight line in the x-direction, and that the channel slope is small with respect to the side-slope. Although these assumptions are not tenable under all conditions, it is suggested that they are sufficiently accurate for an initial stability analysis of the implications of the stability assumption. Under these conditions the channel departs from a straight line by a small amount.

B.5 The same discussion applies to the case of sediment moving over the side-slopes.

B.6 The side-slope transport law is a general transport law as defined in assumption A.5 of the basic model, but with the

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restriction that it does not allow small amplitude disturbances to grow on the side-slopes whenever their projected length is less than a certain number (the X_c distance discussed above in section IV.H). By the basic proposition, such a condition holds if $q_s = G(S, q)$ behaves like $q_s = k q^n S^m$ $n < 1$.

B.7 The equation for water conservation on the side-slopes is a direct consequence of assumptions B.1 and B.4.

B.8 The equation for sediment conservation over the side-slopes is easily derived given assumptions B.1, B.5 and B.6.

B.9 By assumption B.3, channels are represented as lines. By the discussion of assumption A.5 in the basic model, it is reasonable to approximate sediment transport in a channel with $q_s = k q^n S^m$ $n > 1$. It is therefore necessary to derive a relationship between total sediment discharge and total discharge, because the discharge of sediment per unit width for a channel of width is not defined.

By hydraulic geometry (Leopold & Maddock, 1953) the width W of a channel is related to total discharge Q by a relationship:

$$W = a Q^b \qquad a \text{ constant}, \quad b \sim \tfrac{1}{2}$$

Using

$$q_s = k q^n S^m$$
$$W q = Q$$
$$W q_s = Q_s$$

obtain $Q_s = k Q^p S^q$, $p > 1$ if $n > 1$, $b = \tfrac{1}{2}$

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B.10 The law of conservation of water in the channel is a direct consequence of assumptions B.1, B.4, and B.7.

B.11 Similarly, the law of conservation of sediment in the channel is easily derived given B.1, B.5, B.8, and B.9.

B.12 The assumption of the basic boundary condition at the divides is obviously reasonable.

E. The indeterminacy of the modified basic model and lateral channel migration.

The modified basic model is a discrete approximation to the basic model. Unlike the latter, however, it is mathematically indeterminate. This fact is seen in the following

Demonstration: Consider a single valley whose divides have been fixed, and where y_s, the y-coordinate of the line channel has also been fixed. Consider a slope transport law of the form $q_s = a S$, $a > 0$. The problem of the evolution of the system is analogous to the problem of heat conduction in a long bar with insulated sides and ends and a heat source at $y = y_s$. This problem is well-posed. Now, for a side-slope transport law $q_s = q(S)$, there is no reason to expect different behaviour. Hence, if $y = y_s$ moves with time, it must do so by a relationship not stated in the model.

The indeterminacy results from a loss of information, incurred by passing from the continuous model to the discrete model. In this process a line channel is obtained by integrating over a finite length.

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Hence, one has arrived at the classic problem of the lateral movement of channels. Evidently, a general theory of drainage basin evolution is heavily dependent on its solution. In particular the N-valley problem is not soluble. Nevertheless, it is a surprising fact that some important inferences concerning this problem are obtainable without reference to lateral movement.

B. The N-valley, θ-slope problem and valley capture

The continuity equations of the modified basic model are not very tractable. Hence the model is further modified by replacing assumptions B.6 and B.8 with

(B.6,B.8)' The side-slopes are everywhere maintained at a constant angle θ .

First, it should be noted that such a condition holds in many stream-systems (see for example Strahler, 1950, Schumm, 1956, Howard, 1970), where the angle θ approximates the angle of repose of side-slope materials. The assumption essentially states that above a certain angle, sediment transport increases markedly, and this agrees well with the angle of repose examples. Second, the purpose of the new model is to obtain a qualitative idea of the implications of the stability assumption, and it is presumed that a suitable special case of the modified basic model is sufficient for such a purpose.

Call this the θ -slope model. Consider it in the context of the N-stream problem. It is easy to see that once the total width of the N-stream unit is fixed (NM) and once the (y,z) coordinates of each line stream are given, the geometry of the

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whole N-stream system is determined by the θ -slope assumption. Hence the easy

Proposition 6: The geometry of the N-stream θ -slope model is described, up to the determination of the y-coordinate of each stream, by:

$$\frac{\partial Q_{s2i}}{\partial x} = \frac{\partial z_{2i}}{\partial t}\left[y_{2i+1} - y_{2i-1}\right] + \frac{\partial y_{2i}}{\partial t}\left[z_{2i-1} - z_{2i+1}\right] \tag{33}$$

$$i = 1, \ldots, N$$

$$\frac{\partial A_{2i}}{\partial x} = y_{2i+1} - y_{2i-1} \tag{34}$$

$$\sum_{i=1}^{N} A_{2i} = NM\,x \tag{35}$$

$$(z_2 - z_1) = \tan\theta\,(y_2 - y_1)$$
$$(z_3 - z_2) = \tan\theta\,(y_2 - y_3)$$
$$\vdots \qquad\qquad \vdots$$
$$(z_{2N} - z_1) = \tan\theta\,(y_1 + NM - y_{2N})$$

where $(i \geq 1)$ z_{2i}, y_{2i} are coordinates of channels,

$(i \geq 1)$ $z_{2i\pm1}, y_{2i\pm1}$ are coordinates of divides,

$(z_1, y_1), (z_1, y_1 + NM)$ are coordinates of end points,

A_{2i} is drainage area to stream (z_{2i}, y_{2i}).

Proof: Equation (1) obtained by trapezoidal rule for areas and conservation of sediment. (See Fig. 5)

These equations are studied for their stability characteristics. Hence it is necessary to construct a steady surface. This consists

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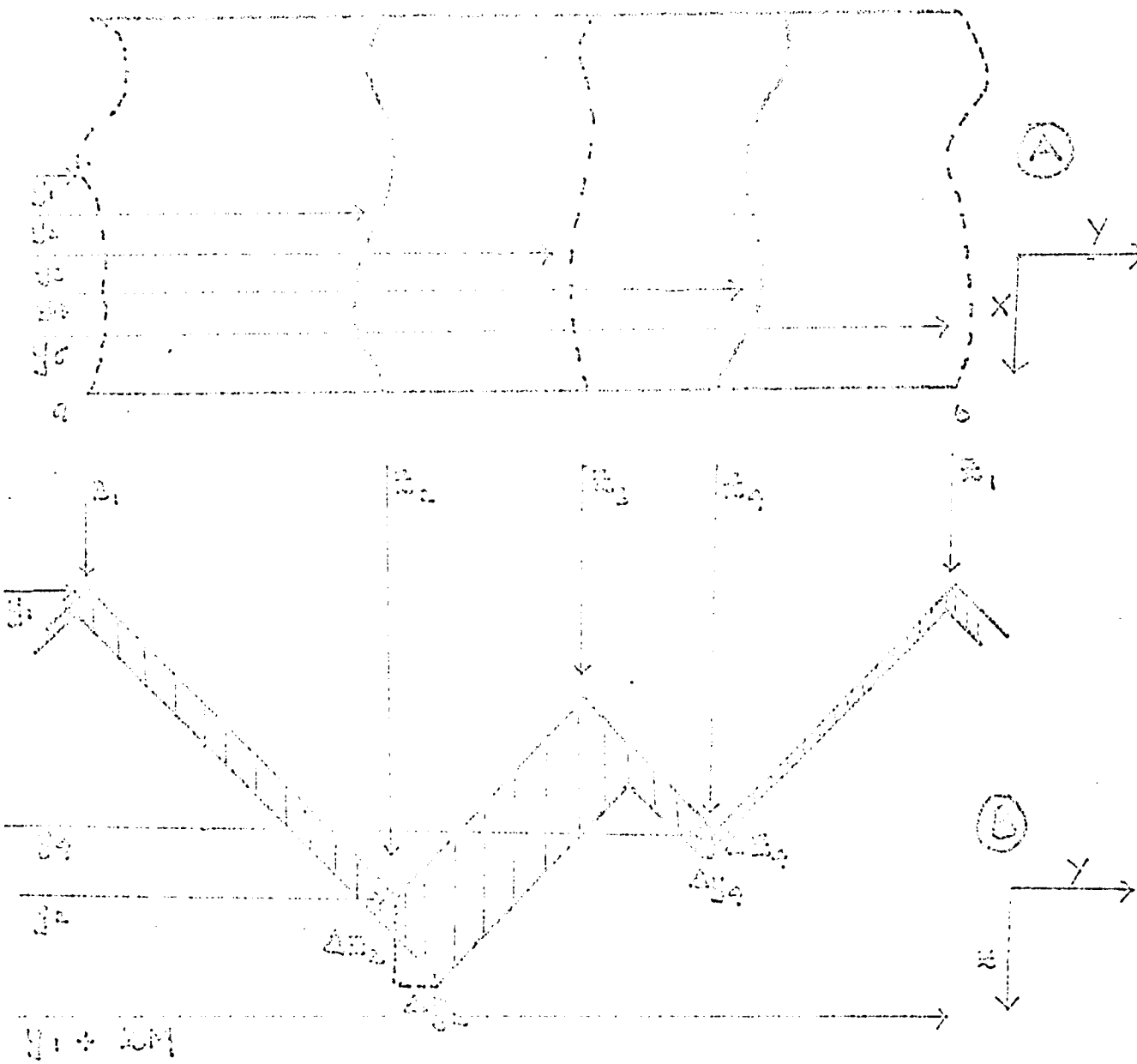
Figure 5. The 2-stream, β-slope model



(A) PLAN OF 2-VALLEY MODEL

(B) SECTION THROUGH a,b OF 2 VALLEY MODEL.

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of N-streams flowing in straight lines down the x-axis, each having a valley of constant width M. The profile of each channel (and by the θ -slope assumption, each divide has a similar profile) is given by Lemma 2, since the steady-state discharge in each stream is Mx. Hence, perturbing the equations of the previous proposition, one obtains the linearised equations of the N-stream θ -slope model:

$$m \frac{\partial z'_{2i}}{\partial t} = \frac{\partial}{\partial x}\left[k(M_x)^p S_0^{q-1} q \frac{\partial z_{2i}}{\partial x}\right] + L(p-1)\frac{\partial A'_{2i}}{\partial x}$$

$$\sum_{i=1}^{N} A'_{2i} = 0$$

where primes denote perturbed variables. Consider now the stability problem for various cases of N. When N = 1, $\frac{\partial A'}{\partial x} = 0$, and one obtains:

$$M \frac{\partial z'}{\partial t} = \frac{\partial}{\partial x}\left[k(Mx)^p S_0^{q-1} \frac{\partial z'}{\partial x}\right] \qquad (38)$$

which is just the one-dimensional diffusion equation with a positive diffusion coefficient. Hence, by Proposition 1, it is stable. As in the one-dimensional case of the basic model, the imposed symmetry in the y-direction allows no perturbation to stream discharge, hence there is no mechanism to counter the stabilising tendency due to slope variations. It must be noted that this statement of stability contains no implications concerning the stability of the y-coordinate of the stream.

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For the case $N = 2$, it is a surprising fact that the linearised equations allow a decoupling of the variables z, y. Hence one obtains an equation in $z_2' + z_4'$, and another in $z_2' - z_4'$. Consequently one may solve explicitly for z_2', z_4'. Thus one obtains the significant result of the following:

Proposition 7: For the two-stream θ-slope model, there exists a finite length, given by

$$x_0 = B\left[\frac{(-p+q+1)^2}{q}\frac{1}{(p-1)}\left(\frac{k}{L}\right)^{\frac{1}{q}}\right]^{\frac{q}{-p+q+1}}\left[\tan\theta\right]^{\frac{-q}{-p+q+1}}M^{\frac{p+q-}{-p+q}}$$

$$= B'\frac{M\ \tan\theta}{S_0(x_0)} \qquad (39)$$

where $B, B' > 0$ constant, $L = k(Mx)^p S_0^q$.

Over the distance x_0, the two streams are stable with respect to infinite amplitude perturbations, but beyond x_0, they are unstable (or, equivalently, one valley grows at the expense of the other).

Proof: The linearised equations for the 2-stream θ-slope model are:

$$\frac{\partial}{\partial x}\left[k(Mx)^p S_0^q\left(p\frac{A_1'}{M} + q\frac{\partial z_2'}{S_0\,\partial x}\right)\right] = L\cot\theta\,(z_2' - z_4') + M\frac{\partial z_2'}{\partial t} \quad (i)$$

$$\frac{\partial}{\partial x}\left[k(Mx)^p S_0^q\left(p\frac{A_4'}{M} + \frac{q}{S_0}\frac{\partial z_4'}{\partial x}\right)\right] = L\cot\theta\,(z_4' - z_2') + M\frac{\partial z_4'}{\partial t} \quad (ii)$$

$$A_2' = -A_4'$$

(i) + (ii) $\Rightarrow$ $\varphi_1 = z_2' + z_4'$ is stable by Proposition 1

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Consider (i) − (ii)

$$\frac{\partial}{\partial x}\left[k(M_x)^p S_0^q\left(\frac{p}{M_x}(A_2'-A_4') + \frac{q}{S_0}\left(\frac{\partial z_2'}{\partial x} - \frac{\partial z_4'}{\partial x}\right)\right)\right] = 2L\cot\theta\,(z_2'-z_4') + M\left(\frac{\partial z_2}{\partial t} - \frac{\partial z_4}{\partial t}\right)$$

Let $z_2' - z_4' = \varphi_2$:

$$\therefore \frac{\partial}{\partial x}\left[k(M_x)^p S_0^p\left(\frac{1}{M_x}(A_2'-A_4) + \frac{q}{S_0}\frac{\partial \varphi_2}{\partial x}\right)\right] = 2L\cot\theta\,\varphi_2 + M\frac{\partial \varphi_2}{\partial t}$$

Now $$\frac{\partial}{\partial x}\left[k(M_x)^p S_0^q \frac{p}{M_x}(A_2'-A_4')\right] = \frac{\partial}{\partial x}\left[2A_1' L p\right]$$

$$= 2L_p \varphi_2 \cot\theta$$

Hence $$2k\cot\theta(p-1)\varphi_2 + \frac{\partial}{\partial x}\left[k(M_x)^p S_0^q \frac{q}{S_0}\frac{\partial \varphi_2}{\partial x}\right] = M\frac{\partial \varphi_2}{\partial t}$$

$$\therefore \quad 2L\cot\theta(p-1)\varphi_2 + \frac{\partial}{\partial x}\left[\frac{Lq}{(LM_x)^{\frac{1}{q}}}(k(M_x)^p)^{\frac{1}{q}}\frac{\partial \varphi_2}{\partial x}\right] = \frac{\partial \varphi_2}{\partial t}$$

$$\therefore \quad 2L\cot\theta(p-1)\varphi_2 + \frac{\partial}{\partial x}\left[Lq\left(\frac{k}{L}\right)^{\frac{1}{q}} M^{\frac{p-1}{q}} x^{\frac{q+p-1}{q}}\frac{\partial \varphi_2}{\partial x}\right] = \frac{\partial \varphi_2}{\partial t}$$

Let $c_1 = \frac{2L}{M}\cot\theta(p-1)$, $c_2 = L^{\frac{q-1}{q}} k^{\frac{1}{q}} q\, M^{\frac{p-1}{q}}$

$$\therefore \quad c_1\varphi_2 + c_2\frac{\partial}{\partial x}\left[x^{\frac{p+q-1}{q}}\frac{\partial \varphi_2}{\partial x}\right] = \frac{\partial \varphi_2}{\partial t}$$

Let $\varphi_2 = \hat{\varphi}(x)\, e^{\lambda t}$

$$\therefore \quad (c_1-\lambda)\hat{\varphi} + c_2\frac{\partial}{\partial x}\left[x^{p+\frac{q-1}{q}}\frac{\partial \hat{\varphi}}{\partial x}\right] = 0$$

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Let $w = x^{\frac{1+q-p}{q}} = x^{\beta}$

$\therefore \quad 0 = (c_1 - \lambda) w \hat{\varphi} + c_2 \beta w \hat{\varphi}' + c_2 \beta^2 w^2 \hat{\varphi}''$

Let $k_1 \equiv a$ $\qquad k_1 = \frac{1}{\beta}$

$b \equiv 0$

p arbitrary $\qquad k_2 = \frac{c_1 - \lambda}{c_2 \beta^2}$

$c \equiv 0$

$d \equiv k_2$

$q = \frac{1}{2}$

$\alpha = \frac{1-k_1}{2}$, $\beta = 0$, $\mu = 2\sqrt{k_2}$, $\nu = 1 - k_1$

Then, applying Theorem 1, P. 363, Wylie (1966); obtain

$$\hat{\varphi} = w^{\frac{1-k_1}{2}} \left[b_1 J_{1-k_1}(2\sqrt{k_2}\, w^{\frac{1}{2}}) + b_2 Y_{1-k_1}(2\sqrt{k_2}\, w^{\frac{1}{2}}) \right] e^{\lambda t}$$

Using boundary conditions $q_s = 0$ at $w = 0$

and taking δ_i as smallest positive value such that $J_{k_1 - 1}(\delta_i) = 0$, obtain $\lambda < 0$ if the wavelength of perturbat^n

$$w_0 < \frac{\delta_i^2 c_2 \beta^2}{4}$$

Let $X_0^{\beta} = w_0 = \frac{\delta_i^2 \beta^2}{4} \frac{c_2}{c_1} = \frac{\delta_i^2 \beta^2}{4} \left(\frac{K}{L}\right) \bar{q}^{\frac{1}{2(p-1)}} \frac{q}{\cot\theta} \frac{M^{\frac{p+q-1}{q}}}{}$

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$$\therefore \quad x_o = \left[\frac{S_i^2 \beta^2 q}{8} \left(\frac{k}{L}\right)^{\frac{1}{q}} \left(\frac{1}{p-1}\right)\right]^{\frac{q}{p+q+1}} \left[\tan\theta\right]^{\frac{q}{-p+q+1}} M^{\frac{p+q-1}{-p+q+1}}$$

Also by $L_x = k (M_x)^p S_o^q$

$$x_o = \frac{S_i^2 \beta^2 q}{8(p-1)} \frac{M \tan\theta}{S_o(x_o)}$$

[N.B for $p = 2 = q$, $x_o = B M^3$]

First, consider an interpretation of this result. When the wavelength of disturbance is sufficiently small, the proposition states that the stabilising tendency inherent in the one-dimensional diffusion equation is dominant over the effects of water concentration. This is because given a certain amplitude of disturbance, as the wavelength becomes shorter, the slope (of the channel) becomes greater, so making the diffusive stabilising action stronger. But as the wavelength becomes longer, the slope decreases. However, since the discharge of the channels has been perturbed, their erosive power is modified. Hence at a certain wavelength, when the perturbation to slope is sufficiently small, this positive feedback to erosion overcomes the tendency to stability induced by the slope effects. Therefore one valley grows positively, and the other negatively (by the constraint on total width 2M).

Second, it is apparent that this result is an implication of the stability assumption. It should be noted that the stable length is proportional to some power (greater than unity for "reasonable"

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values of p,q in $Q_s = K Q^p S^q$) of the width M. By the stability assumption, M is less than the wavelength below which a basic surface is stable.

Third, there is a strong relationship between this result and the stability result for the basic model. The N-stream θ -slope model may be interpreted as a linear approximation of the unstable region of a perturbed basic surface. Thus one may relate M to $\frac{1}{\omega}$. Consider the case N = 2 in the θ -slope model. As M approaches zero, the behaviour predicted in the basic model is obtained, since the stable length approaches zero (or, equivalently, the unstable region is unstable everywhere). But the 2-stream model further predicts that if M is non-zero, a finite stable length exists. Hence, if the analogy between the continuous basic model and the 2-stream θ -slope model is adequate, one has further evidence justifying the basic proposition.

The formal analogy between the basic model and the N-stream θ -slope problem is seen when the linearised equations for each system are compared:

(1) Basic model:

$$\frac{\partial z'}{\partial t} - L\frac{\partial q'}{\partial x} = \frac{1}{S_0}\frac{\partial}{\partial x}\left[\bar{F}_q q'\right] + \frac{1}{S_0}\frac{\partial}{\partial x}\left[F_s s'\right] \qquad (40)$$

(2) N-stream θ -slope model:

$$\frac{\partial z'_{2i}}{\partial t} + \frac{L}{M}\frac{\partial A'_{2i}}{\partial x} = \frac{1}{M}\frac{\partial}{\partial x}\left[\bar{F}_A A'_{2i}\right] + \frac{1}{M}\frac{\partial}{\partial x}\left[F_s s'\right] \qquad (41)$$

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H. A hypothesis of lateral channel migration

For the case $N \geq 3$ in the N-stream θ-slope model, the z-coordinates of the channels no longer separate into equations independent of the y-coordinates. Hence one is forced to consider the problem of lateral movement.

Since the modified basic model is principally a statement of mass-conservation laws, it is reasonable to assume that the lateral movement of a channel depends only on the relative influxes of sediment and water from either side-slope. In this case, it is likely that a stream moves away from that slope which contributes the greatest flux of sediment to the channel. Melton (1960) documents good evidence to support this contention.

Hence one may write

$$\frac{\partial y_{2i}}{\partial t} = H\left(\frac{q_{s_{L2i}}}{q_{s_{R2i}}}\right), \quad \frac{\partial y_{2i}}{\partial t} > 0 \text{ if } \frac{q_{s_{L2i}}}{q_{s_{R2i}}} > 1 \tag{42}$$

if one assumes that such an adjustment is instantaneous, one obtains:

$$q_{s_{L2i}} = q_{s_{R2i}} \tag{43}$$

for each stream. Equivalently, one may say that a stream tends to equalise the sediment fluxes from its side-slopes. The 2-stream θ-slope model is sufficient to test for the stability characteristics of this model, as one may verify by the following

Proposition 8: For the 2-stream θ-slope model, the hypothesis $q_{s_{L2i}} = q_{s_{R2i}}$ for each stream implies that there is no stable length with respect to lateral motion of the channel.

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$q_{sL} = q_{sR}$ implies

$$\left(\frac{\partial z_2}{\partial t} - \tan\theta \frac{\partial y_2}{\partial t}\right)(z_2 - z_1) = \left(\frac{\partial z_2}{\partial t} + \frac{\partial y_2}{\partial t}\tan\alpha\right)(z_2 - z_4) \quad \text{(i)}$$

$$\left(\frac{\partial z_4}{\partial t} - \tan\theta \frac{\partial y_4}{\partial t}\right)(z_4 - z_3) = \left(\frac{\partial z_4}{\partial t} + \tan\theta \frac{\partial y_4}{\partial t}\right)(z_4 - z_1) \quad \text{(ii)}$$

Perturbing from steady-state with $y_2 = \frac{M}{2}$, $y_4 = \frac{3M}{2}$

$$\frac{\partial z_2'}{\partial t}(z_3' - z_1') = \frac{\partial y_2'}{\partial t}\tan\theta\,(z_2' - z_1' + z_2' - z_3') \quad \text{(iii)}$$

$$\frac{\partial z_4'}{\partial t}(z_1' - z_3') = \frac{\partial y_4'}{\partial t}\tan\theta(z_4' - z_1' + z_4' - z_3') \quad \text{(iv)}$$

(iii) − (iv) implies

$$2\cot^2\theta \frac{L}{M}(z_3' - z_1') = \frac{\partial y_2'}{\partial t} - \frac{\partial y_4'}{\partial t}$$

$$\therefore \quad 2\cot^2\theta \frac{L}{M} = \left(\frac{\partial(y_2' - y_4')}{\partial t}\right)\frac{1}{(y_2' - y_4')}$$

$$\therefore \quad (y_2' - y_4') = f_d(x)\, e^{\frac{2L}{M}\tan\theta\, t}$$

Since the instability is independent of x, there is no stable length. One may interpret the result as follows (see Fig. 5). A shorter slope must cut down further than a longer slope to provide a given volume of sediment. By the geometry, the longer slope becomes still longer.

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It is easy to apply a similar analysis to the law (42)

$\left(\frac{\partial y_{2i}}{\partial t} = H\left(\frac{q_{s_{1,i}}}{q_{s_{2,i}}}\right) \right)$. Under this law, two stability results are possible. For a law $H(\xi)$ such that $\left.\frac{\partial H}{\partial \xi}\right|_{\xi=1}$ is not very small, the whole stream length is unstable with respect to sideways movement. If $\left.\frac{\partial H}{\partial \xi}\right|_{\xi=1}$ is very small, the whole stream is neutral (that is, $\frac{\partial y'_{2i}}{\partial t} = 0$), and perturbations grow neither positively nor negatively. The condition $\left.\frac{\partial H}{\partial \xi}\right|_{\xi=1}$ implies that when a stream is initially receiving equal sediment fluxes from either bank, any small perturbation to these fluxes does not immediately cause the stream to move sideways.

Consider the implications of these two cases. For the unstable case, it is probable that the result extends from the θ -slope model to the general N-stream model, since there is no inherent mechanism in the modified basic model to counter instability in the long run. Hence the existence of stable river systems is evidence against the hypothesis. It is possible, however, that the process is extremely slow with respect to the time-scale of observation.

An immediate implication of the second (stable) case is that one may solve the N-stream θ -slope problem, as in the:

Proposition 9: For the N-stream θ -slope model, with neutral behaviour of the y-coordinates with respect to infinitesimal amplitude perturbations, the minimum stable length for the

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streams is given by

$$x_0 = B' \frac{1}{(2-\Psi_i)} \frac{M \tan\theta}{S_0(x_0)}$$

where Ψ_i is the smallest value satisfying the following (Nth order) equation:

$$\det \underline{\Psi} = \det \begin{pmatrix} -\Psi & 1 & 0 & \dots & 0 & 1 \\ 1 & -\Psi & 1 & \dots & 0 & 0 \\ \vdots & & & & & \\ 1 & 0 & 0 & \dots & 1 & -\Psi \end{pmatrix} = 0$$

Proof:

For the $2i$ th stream, have the linearised equation:

$$\frac{\partial z_{2i}}{\partial t} - \frac{1}{M} \frac{\partial}{\partial x}\left[f(x) \frac{\partial z_{2i}}{\partial x}\right] - (p-1)\frac{h}{M}\cot\theta\, z_{2i} = -(p-1)\frac{h}{M}\cot\theta \frac{z_{2i+2}+z_{2i}}{2}$$

or

$$D z_{2i} = z_{2i-2} + z_{2i-2}, \quad D \text{ the operator}$$

$$\therefore \quad \underline{A} \begin{pmatrix} z_{2N} + z_4 \\ \vdots \\ z_{2N-2} + z_2 \end{pmatrix} = \underline{A} \begin{pmatrix} D z_2 \\ \vdots \\ D z_{2N} \end{pmatrix}$$

Chose $\underline{A}$ such that:

$$\begin{aligned} a_{i2} + a_{iN} &= \Psi_i a_{i1} \\ &\vdots \\ a_{iN-1} + a_{i1} &= \Psi_i a_{iN} \end{aligned} \qquad i = 1, \dots, N$$

If Ψ_i satisfies $\det \underline{\Psi} = 0$, Then obtain

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N linearly independent variables

$$u_{2i} = \sum_{j=1}^{N} a_{ij} z_{2j} \quad \text{such that}$$

$$\Psi_i u_i = D z_{2i}$$

$$\therefore \frac{\partial u_{2i}}{\partial t} = \frac{1}{M} \frac{\partial}{\partial x}\left[f(x) \frac{\partial u_{2i}}{\partial x}\right] + (p-1)\frac{\omega\tau\theta}{M}\left(\frac{2-\Psi_i}{2}\right) u_{2i}$$

and this is stable for:

$$\chi < \frac{2}{(2-\Psi_i)} \frac{S_i^2 \beta^2 q}{4(p-1)} \frac{M \tan\theta}{S_0(x_0)}$$

The solutions z_{2i} are uniquely determined linear combinations of the u_{2i}.

The roots of $\det \underline{\Psi} = 0$ have been computed for $N = 1,2,3,4,6$ (see Table 2). Several points may be noted. First, the factor of the 2-stream problem, $\frac{M \tan\theta}{S_0(x_0)}$, reappears. Second, the number of distinct roots to $\det \underline{\Psi} = 0$ would appear to be generally less than N, and may be interpreted as the number of topologically distinct ways in which N valleys can interact according to the model. Third, one may interpret the stability result with the same argument as for the 2-stream problem, with the result once more depending on the stability assumption.

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TABLE 2

Roots of $\det \underline{\underline{\Psi}} = 0$ for N = 1,2,3,4,6.

N	Roots of $\det \underline{\underline{\Psi}} = 0$	Number of distinct roots
1	$\Psi_1 = 2$	1
2	$\Psi_1 = 2, \Psi_2 = -2$	2
3	$\Psi_1 = -1, \Psi_2 = -1, \Psi_2 = 2$	3
4	$\Psi_1 = 0, \Psi_2 = 0 \quad \Psi_3 = 2, \Psi_4 = -2$	3
5	$\Psi_1 = 2$	?
6	$\Psi_1 = 1, \Psi_2 = 1, \Psi_3 = -1, \Psi_4 = -1$ $\Psi_5 = 2, \Psi_6 = -2$	4

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The interest of the model is in the fact that as valley capture occurs at points beyond the computed length, perturbations to the sediment influx ratio become finite, under which condition sideways movement and hence stream coalescence must occur by the law

$$\frac{\partial y_{2i}}{\partial t} = H\left(\frac{q_{s_{L2i}}}{q_{s_{R2i}}}\right) \quad , \quad \frac{\partial y_{2i}}{\partial t} > 0 \text{ if } \frac{q_{s_{L2i}}}{q_{s_{R2i}}} > 1$$

In summary, however, it must be noted that the hypothesised law and its two stability cases are conjecture. Hence there is no satisfactory theory of lateral movement and its stability.

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6. A CHANNEL MODEL

A. A channel model and lateral migration

The purpose of this section is to propose a model whereby the problem of sideways movement may be solved. Although it has not been possible to find a satisfactory solution, the model does give rise to a most interesting result.

The model will be considered in the sequence of its stages of construction.

B. The channel model: statement

First, one must model a channel of finite width. Again, this may be idealised as a surface subject to the laws of conservation of sediment and water. However, the basic model is no longer applicable, since the direction of water movement in a channel is far better approximated by the slope of the free surface than by the slope of the bed. The model will be presented as a list of assumptions followed by a critical interpretation (see Fig. 6).

C.1 Water moves in the direction of the free surface.

C.2 Sediment moving in a channel has two components of direction:

a) the direction of the water surface gradient,

b) the direction of the channel bed gradient.

C.3 The magnitude of discharge of water per unit width (q) is given by

$$q = k''' h^{3/2} S^{\frac{1}{2}}$$

k''' constant, h the depth, S the slope of the water surface.

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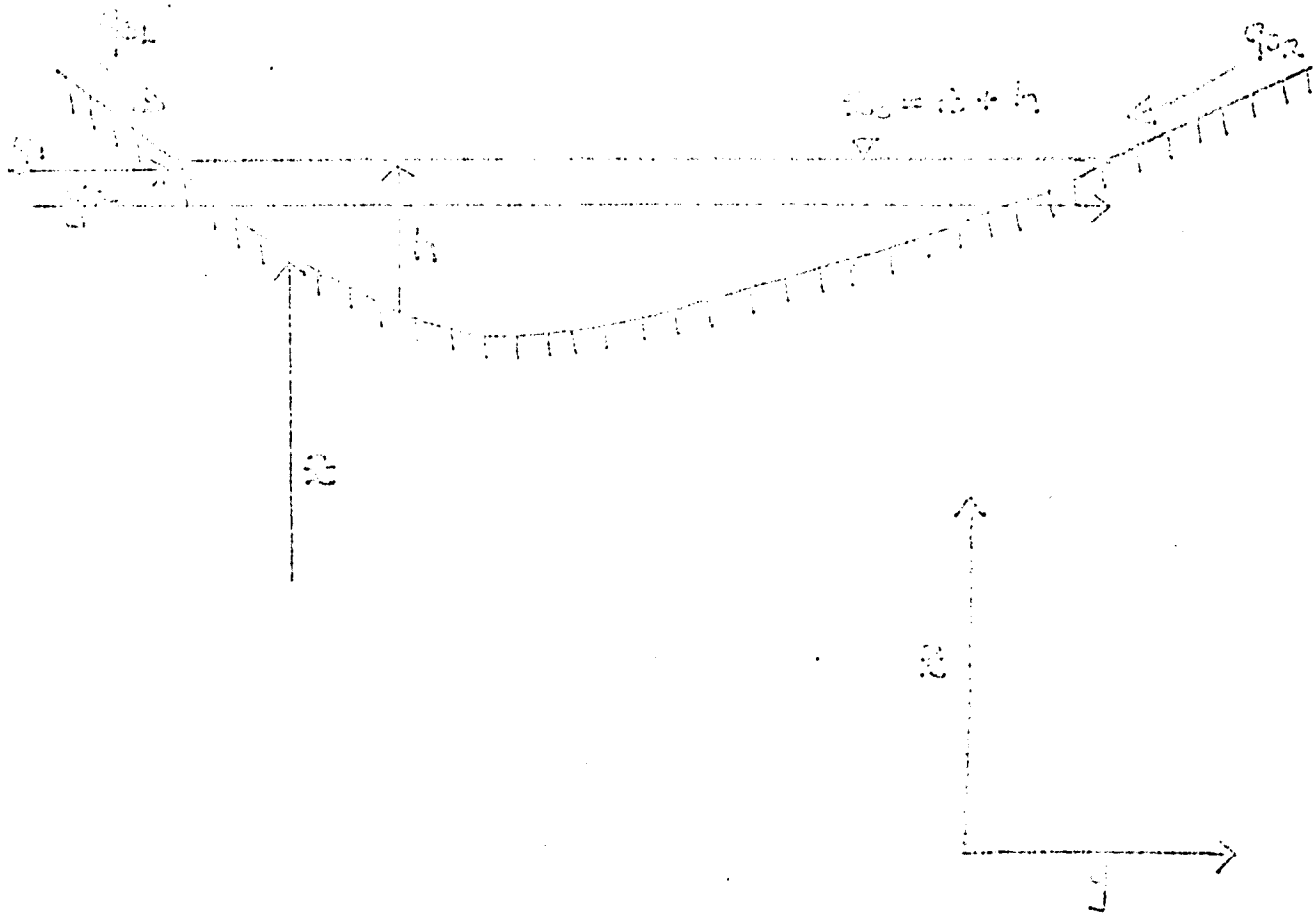
Figure 6. The channel



C.4 The magnitude of sediment discharge per unit width is given by

$$q_{s_z} = k' q^n S^m$$

in the direction of the water surface gradient.

$$q_{s_a} = k' D' q^n S^m |\nabla z| \qquad k', D' \text{ constants.}$$

in the direction of the channel bed gradient.

C.5 The conservation of water in the channel is given by:

$$\nabla \cdot \frac{\nabla(z+h)}{|\nabla(z+h)|} q = 0$$

C.6 The conservation of sediment in the channel is given by:

$$\text{Average}\left[\frac{\partial z}{\partial t} = \nabla \cdot \left(\frac{\nabla(z+h)}{|\nabla(z+h)|} R_1 k''' q^n S^m - \frac{\nabla z}{|\nabla z|} R_2 D' k''' q^n S^m \nabla z \right) \right]$$

where R_1, R_2 are functions of a random variable.

C.7 There are boundary conditions:

a) $h(y_1) = h(y_2) = 0$

b) $\int_{y_1}^{y_2} q \, dy = Q$ (total water discharge)

c) $\int_{L}^{y_1} \int_{y_1}^{y_2} q_s \, dy \, dx = Q_s$ (total sediment discharge).

C. The channel model: discussion

C.1 is a fairly reasonable assumption based on average conditions.

C.2 is similarly reasonable.

C.3 is obtained using the Chezy relationship, and is hence reasonably satisfactory.

C.4 The sediment transport in the direction of the water surface gradient is approximated by the general law suggested by

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Raudkivi (1967). The sediment transport in the direction of the channel bed gradient is less easy to describe. However, if one views sediment transport on the bed as a continual exchange of particles and as related to turbulence, then it is clear that given a cross-channel gradient, there will be a net diffusion process down the gradient of the surface. Hence it seems permissible to write this component of sediment transport as some variable coefficient k multiplying the downstream sediment transport, and as a first approximation, it seems reasonable to write .

$$k = -D' \nabla z \quad , \quad D' \text{ constant}$$

C.5 is straightforward, assuming no infiltration of water, and no significant flux of water through the free surface, which merely implies that most water enters the channel along its sides.

C.6 is again straightforward, except for the random variables R_1, R_2. An exact balance of sediment transport in a small area of channel is probably not described by exactly deterministic equations, because of turbulence, natural perturbations, etc. But on average such a balance may be described.

C.7 A channel is obviously characterised by boundaries where the depth of flow approaches zero (a). (b) follows directly from C.5, and (c) follows by the averaging process suggested in C.6.

The second stage is to assume that the channel form is in quasi-equilibrium on a time-scale at which the side-slopes, long-profile, etc., are changing. This is a most reasonable assumption (Wolman

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1955; Schumm & Lichty 1964). Hence in the third stage, one may obtain the rate of sideways movement in terms of the boundary conditions at the edges, i.e. $q_{s_1}, q_{s_2}, q_1, q_2$. The fourth stage is to embed this result in the chosen N-valley model, which then becomes determinate.

D. The channel model and hydraulic geometry

Approximations to the model have not proved of much use in solving the problem. However, one reasonable approximation does cast light on another problem, that of hydraulic geometry (Leopold and Maddock, 1953), showing that the problem is more simply deterministic than was previously imagined (see Langbein, 1964). Consider (without discussion) the following approximation to the channel model:

C.1' z_0, the free surface of the stream, is a function of x only.

C.2' the component of direction of sediment transport, q_{s_d}, down the gradient of the channel bed, moves in the y-direction only.

C.5' given by $$\int_{y_1}^{y_2} q\, dy = Q = M_x$$

C.6' $$\frac{\partial z}{\partial t} = -\frac{\partial}{\partial y}\left[k_2 S_0^p h \frac{\partial h}{\partial y}\right] - \frac{\partial}{\partial x}\left[k_1 S_0^p h^q\right]$$

C.7' Boundary conditions: a) $$h(y_1) = h(y_2) = 0$$

b) $$\int_{y_1}^{y_2} q_{s_d}\, dy = \frac{\partial Q_s}{\partial x} = L_x$$

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The other assumptions remain the same. Then assuming a steady-state $\frac{\partial z}{\partial t} = 0$ (or, equivalently, the sediment carried away in the channel is exactly balanced by that introduced at the boundaries, q_{s_1}, q_{s_2}, it is possible to seek for a solution of C.6' in terms of similarity variables. Using:

$$S_0 = C x^M$$
$$h = x^N \phi\left(\frac{y}{x^\alpha}\right)$$

and solving M, N, α using equations C.6', C.5', C.7'(c), it is possible to reduce equation C.6' to an ordinary differential equation in $\eta = \frac{y}{x^\alpha}$. But the main interest lies in the values of M, N, α, for these represent the exponents of a hydraulic geometry. Using the "reasonable" law $q_{s_d} = k q^2 S^2$ (the equivalent of the Einstein bed-load equation) one obtains the relations

v = velocity $\sim Q^{1/17}$

d = depth $\sim Q^{5/17}$

w = width $\sim Q^{11/17}$

which agree very roughly with reality. Hence one may readily conclude that channel-form is largely a function of the sediment transport law operative. The result also suggests the reasonableness of the proposed approach to lateral migration of channels.

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VII. SUMMARY AND CONCLUSIONS

1. All previous models of drainage basin evolution are unsatisfactory. For the most part, they ignore the underlying physical mechanisms of erosion and sediment transport.
2. The nature and complexity of the problem suggest the ultimate need of computer techniques. To this end, it is essential to construct models that both adequately describe the real world and are suited to the application of computer techniques.
3. Several important qualitative insights into the process of basin evolution are provided by models based on three ideas. First that a basin may be represented as a mathematical surface $z = z(x,y,t)$. Second that the transport of sediment per unit width over this surface may be approximated by some function of the surface gradient and the discharge of water per unit width. Third, that mass is conserved during the evolution of the surface.
4. The basic model is based on these assumptions. Using infinitesimal amplitude perturbation theory, the model implies:
   a. Given a steady-state drainage surface associated with a particular sediment transport law, $q_s = F(S, q)$, the surface is stable wherever convex, and unstable wherever concave. Moreover, unstable perturbations grow faster as the wavelength of disturbance in the y-direction decreases.
   b. On convexities of the steady-state surface, $q_s = F(S, q)$ behaves like $q_s = k q^n S^m$, $n < 1$. Therefore any

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perturbations which cause the convergence of water give rise to a lowered sediment transporting capacity per unit volume of water. Hence perturbations are removed by deposition. On concavities, however, $q_s = F(S, q)$ behaves like $q_s = k q^n S^m$, $n > 1$. Hence perturbations causing the convergence of water grow by erosion, since the sediment transporting capacity per unit volume of water is increased.

c. On convexities, the behaviour is analogous to that of gradient dynamical systems going forward in time. On concavities, the behaviour is analogous to gradient dynamical systems going backward in time.

d. In the initial stage of development, a drainage surface is transformed in such a way as to maximise sediment transport.

e. For the proposed model to prove realistic, $q_s = F(S, q)$ must be of such a form as to give rise to a steady surface, convex in its upper portion and concave in its lower portion.

f. Under this assumption, the model provides a mechanistic insight into the initial growth of channel systems. At the same time, it is in correspondence with reality in that many landscape features are convexo-concave in profile, with no channelling of the convexities but with channelling of the concavities.

g. The model is unrealistic in that the smallest perturbations grow fastest, and that a free water surface is nowhere implied.

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5. The basic model must be modified in order to eliminate the instability of disturbances of arbitrarily small wavelength. One may either model some process which effectively rules out this problem, or construct a model postulating the existence of stable valleys of finite width. Adopting the latter approach, and approximating streams as lines, one may conclude from a modified basic model that:
   a. In order to model the evolution of N congruent valleys, it is necessary to specify the law by which streams migrate laterally.
   b. For a constant angle side-slope model with two initially parallel and congruent valleys, there is a stable length over which the valleys may exist side-by-side, but beyond which one valley grows at the expense of the other. The stable length is proportional to some power (greater than unity) of the valley width.
   c. The behaviour of the perturbed basic surface and the N-valley surface are analogous.
6. A model describing lateral migration may be based on the assumption that a channel moves away from that side-slope down which comes the greater sediment flux. Using this hypothesis in connection with the modified basic model (with constant angle side-slopes), one may conclude that
   a. if a stream is sensitive to small changes in sediment flux from either side-slope, then it is unstable with respect to sideways movement along its entire length.

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b. if the stream is not sensitive, then it is neutral with respect to lateral movement. Hence a satisfactory solution to the N-valley problem is possible.

Either alternative is possible in reality.

6. A model of a channel may be based on conservation principles, and may prove useful in determining a law of lateral channel migration. Such a model provides a reasonable estimate of hydraulic geometry.

7. The results justify the use of models based on conservation principles. In particular, they justify the use of a transport law of the form $q_s = F(S, q)$. However, such a law is in need of further modification.

8. Several directions for future research are indicated:

   a. The introduction of some dynamical assumptions; for example, with respect to investigating a lower limit for unstable wavelengths.

   b. An investigation of **lateral** channel migration in terms of sediment transport laws.

   c. An investigation of the effects of climatic and geologic variation.

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