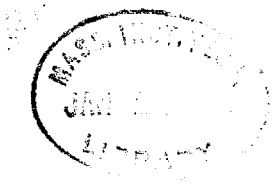
DESIGN OF DYNAMIC ANALYZER

by



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1942

Submitted in Partial Fulfillment of the Requirements for

the Master of Science Degree in Mechanical Engineering

from the

Massachusetts Institute of Technology

1946

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ACKNOWLEDGEMENT

The author wishes to express his gratitude to Professor John A. Hrones for valuable advice and assistance in conducting this thesis.

CONTENTS

TITLE PAGE

ACKNOWLEDGEMENT

TABLE OF CONTENTS

INTRODUCTION 1

HISTORY 3

EQUIPMENT 5

ANALYTICAL CONSIDERATIONS 9

MECHANICAL SOLUTION 14

DESIGN 24

CONCLUSION 31

BIBLIOGRAPHY

DRAWINGS

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DESIGN OF DYNAMIC ANALYZER

INTRODUCTION

The general equation for the forced vibration of an elastic system with one degree of freedom can by written as

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = \frac{F_{af}}{m}\sin\omega_f t$$

where x = displacement of the vibrating body from a neutral position

$\dot{x} = \frac{dx}{dt}$ = velocity of the body

$\ddot{x} = \frac{d^2t}{dt^2}$ = acceleration of the body

ω_n = natural undamped frequency of the vibrating system

$\xi = \frac{c}{c_c}$ = damping constant of system/ critical value of damping

F_{af} = amplitude of forcing force

m = mass of body

ω_f = forcing frequency

Fig. 1 pictures diagrammatically the apparatus necessary to produce such a vibration, with a forcing force acting on the mass.

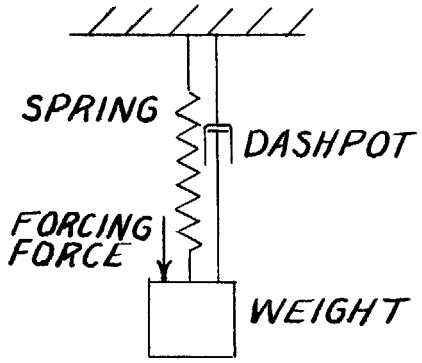

FIG. 1

This equation is fundamental in the study of vibrations. While it is readily solved by analytical methods it also yields to solution by mechanical means. Mechanical computers such as the differential analyzers at Massachusetts Institute of Technology could produce a graphical solution of the equation. Such a solution would have the general form shown in Fig. 2. The exact configuration would depend, of course, on

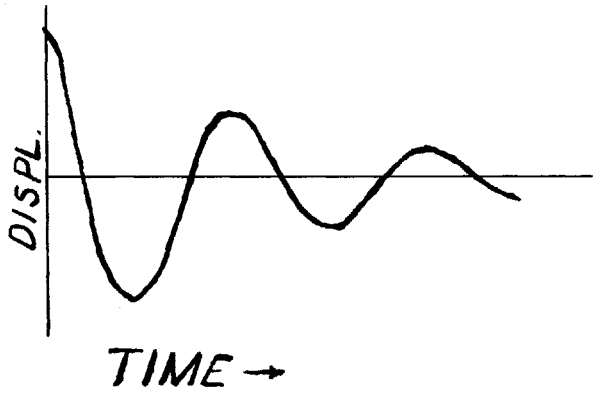

FIG. 2

the values assigned to the constants in the equation.

It has been considered helpful to the study of the vibrations problem to be able to produce readily the graphical solution to this equation. Therefore this thesis was undertaken with the object of designing a mechanical computer to produce the graphical solution of this equation. The recent receipt by the controls laboratory of the machine design department of considerable quantities of computing devices

and small gears, shafts, bearings, and brackets makes this a propitious time to consider design and construction of such a device.

HISTORY

The idea of the use of mechanical analysis in the more advanced mathematical processes is not a new one. Leibnitz envisaged it comprehensively over 200 years ago. The far-reaching project of utilizing complex mechanical interrelationships as substitutes for intricate processes of reasoning owes its inception to an inventor of the calculus itself.

The most important element in any device to solve differential equations (i.e., differential analyzer) is the integrator. Apparently the first integrator was invented by Hermann and Lammle of Munich in 1814. This was a two-disk type integrator and has become the classical integrator; it is shown schematically in Fig. 3.

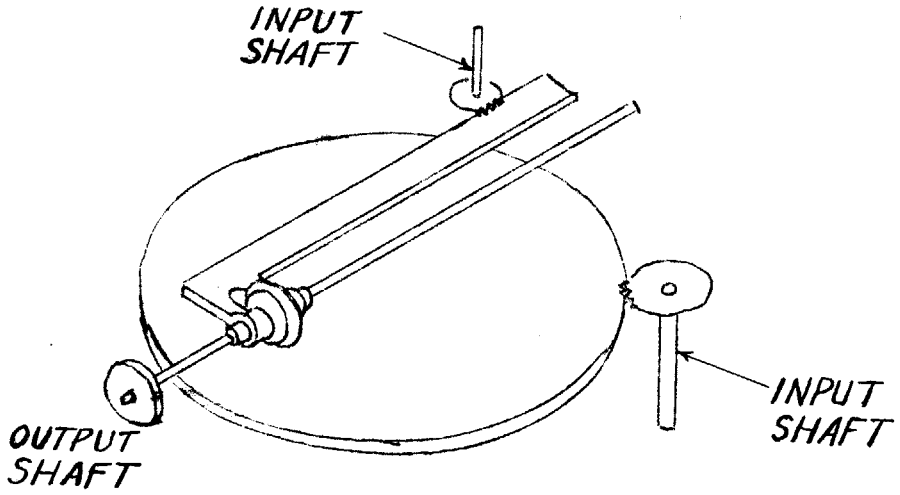

FIG. 3

Analysis of this integrator is similar to that of the ball type integrator, which is described later on. This device has a serious practical disadvantage in that output torque is limited by the adhesion between the two disks, which in turn is determined by the pressure between them. However, since disk T must continually change radial position relative to disk D, friction must be kept to a minimum; thus the two requirements for good operation are antagonistic.

Professor James Thomsom of England devised the forerunner of the present ball-type integrator in 1876, and his brother William Thomson (Lord Kelvin) used this integrator to devise a mechanism to solve many types of differential equations. Lord Kelvin employed several interconnected integrators, and claimed he could solve differential equations of all orders.

Continuing the development of the integrator, in 1927 Dr. Vannevar Bush and associates announced the development of a device which they called a mechanical integraph. This machine plotted continuously the integral of two functions. It employed the principle of the electrical integrating watthour-meter of Thomson.

(While they employed Thomson's principle of the watt-

hour-meter, Dr. Bush and associates were unaware of the early work of Thomson in which he described his method for using a similar machine to solve differential equations.)

By tracing the curves of two functions and introducing this mechanical movement into the integraph, the machine could produce a solution to the equation

$$F(x)=\int_a^x f_1(x)f_2(x)dx$$

in the form of a plot of F(x) vs. x. Also it was found that with certain connections the machine could solve for the function ϕ when f is known in

$$\phi(x)=\int_a^x f(x)\phi(x)dx$$

Since the development of this original machine at M.I.T. many improvements have been made. Great advancement in continuous computing mechanisms has occurred in the field of gun fire-control apparatus. Now differential analyzers and other continuous computing devices have reached a very high degree of precision.

EQUIPMENT

As stated in the introduction, equipment already available for this project includes computing mechanisms, gears, shafts, and bearings. These computing mechanisms include in-

tegrating units and differential gears. Naturally this computer was designed around the equipment available.

Integrators. The integrators, which were formerly employed in fire control devices, are of the ball type and are pictured in Fig. 4. Gear A, Fig. 4, is driven at constant speed and drives a hardened steel disk about 2½" in diameter. Rolling on the disk and driven by it is a hardened steel ball about ¼" diameter, and this ball in turn drives a second ball (H, Fig. 4d) through rolling contact. This second ball drives the output shaft B. Pressure at all of these contact points is maintained by spring-loading the cover plate, which is hinged at F.

The speed of shaft B in relation to shaft A is governed by the distance of the balls from the center of the disk. This distance is controlled by shaft C. A pinion of shaft C drives a rack cut in the carriage (Fig. 4d) carrying the balls and thus moves the balls along the output shaft.

As stated in Dr. Bush's recent paper, the integrator is a mechanism having two input shafts and one output shaft, arranged so that the ratio of an incremental rotation of the output shaft (B) to an incremental rotation of one of the input shafts (C) is a continuous linear function of the angular position of the second input shaft (A). Thus, if

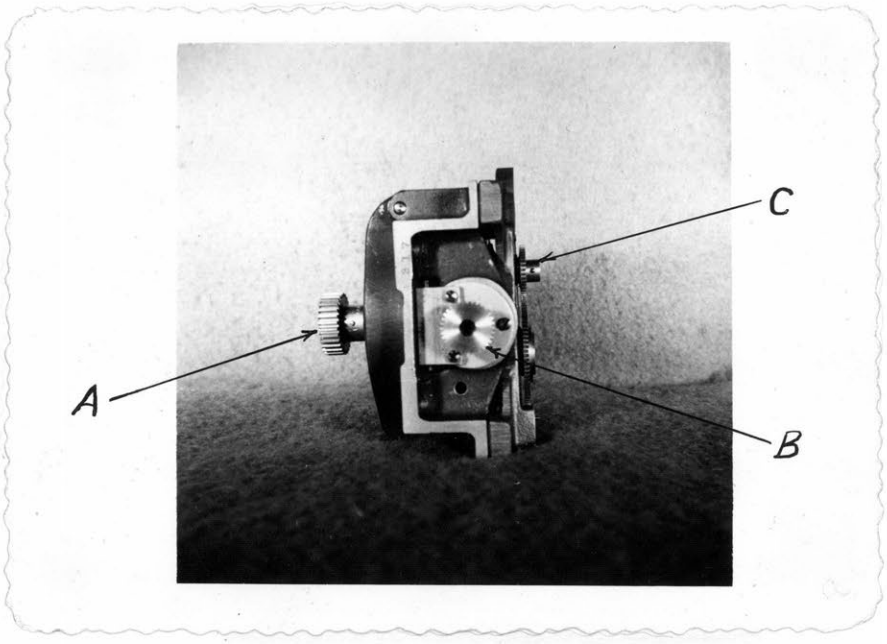

FIG. 4a

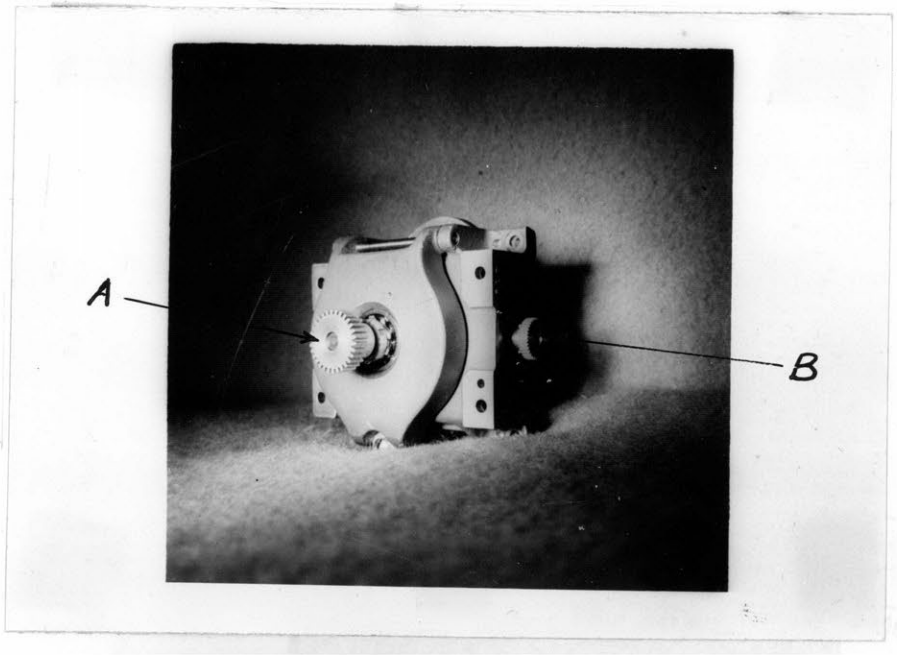

FIG. 4b

FIG. 4 ~ INTEGRATING UNIT

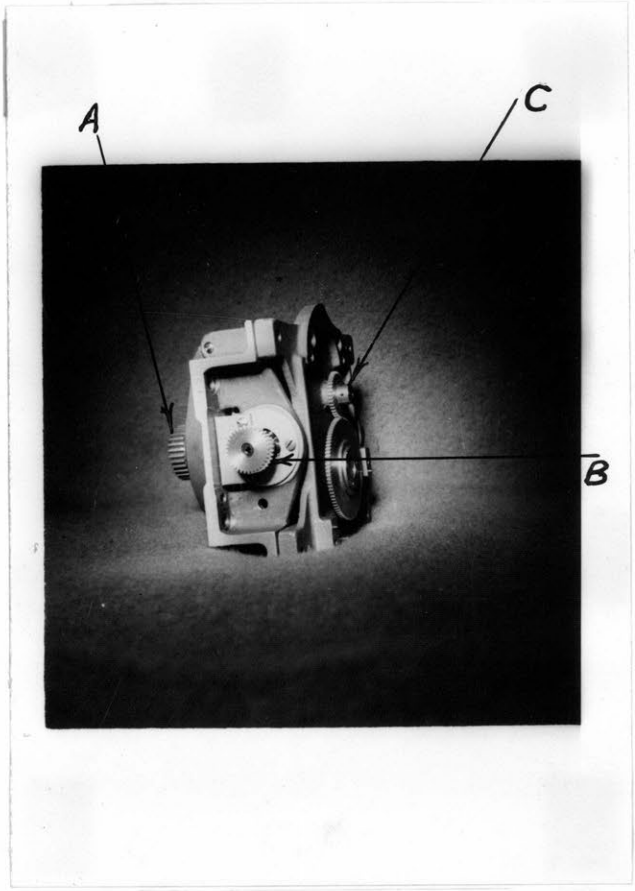

FIG. 4c

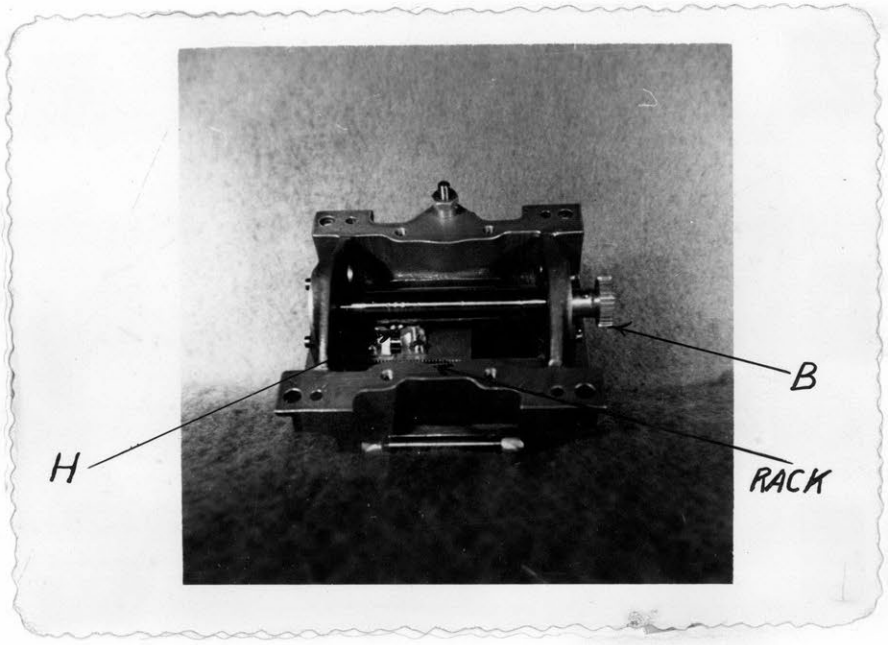

FIG. 4d

z = angular position of the output shaft

v = angular position of one input shaft

y = angular position of second input shaft

k = constant of the integrator,

the integrator mechanism relates these quantities in accordance with the expression

$$\frac{\Delta z}{\Delta y} = kv$$

At any instant this expression takes the derivative form

$$\frac{dz}{dy} = kv$$

from which is obtained the equation of the integrator

$$z = k\int_{y_1}^{y} v\,dy$$

And if angular displacement of the two input shafts continuously represent variables, then Z represents (to some scale) the integral of one variable with respect to the other.

If the variable y represents time (as in this project), then the integration process may be seen another way. Starting from the realization that the peripheral velocity of shaft B at the ball is equal to the peripheral velocity of the disk at the ball (assuming no slippage),

$$\dot{z} r_1 = v\dot{y} \ ,$$

where $\dot{z}$ = angular velocity of shaft B

r_1 = radius of shaft B

v = displacement of ball from center of disk

$\dot{y}$ = angular velocity of shaft A .

Then

$$r_1 \frac{dz}{dt} = v\dot{y}$$

$$\dot{z} = \frac{\dot{y}}{r_1}\int v\,dt$$

Here $\dot{y}$ and r_1 are constant, so z is proportional to $\int v dt$.

Differential Gears. Differentials are employed in this problem to add or subtract two variables. While spur gear differentials are employed here, Fig. 5 shows a bevel gear differential for clarity.

Referring to Fig. 5, if two variables x and y are input to the end gears as shown (as angular displacements, of course), the gears Aand B roll on the end gears carrying the spider around so that angular displacement of the spider z represents the algebraicaddition of variables x and y. If the end gears rotate in the same direction, the differential adds; if they rotate in opposite directions it subtracts. There is always a change in scale (revolutions per unit of variable represented) in going througha differential, the spider having twice the value per revolution of either of the inputs.

Available also in the controls laboratory are a great

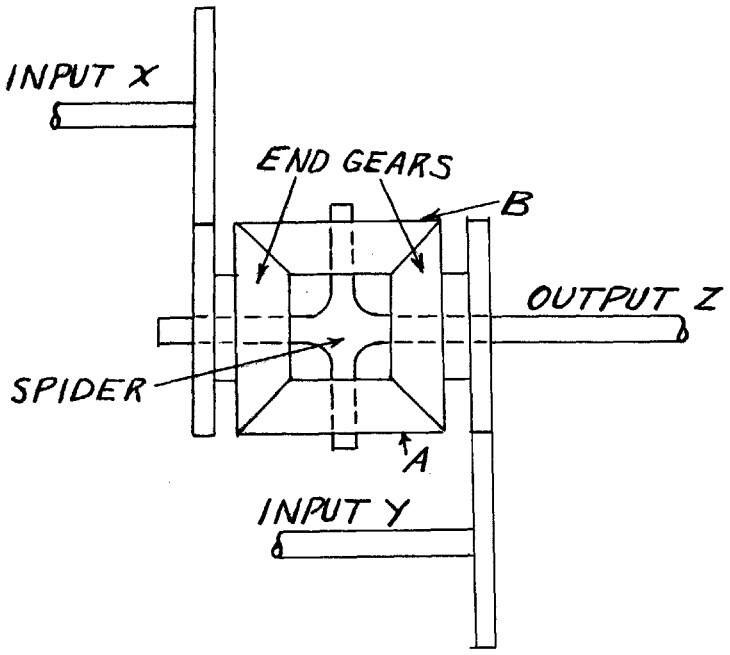

FIG. 5~ DIFFERENTIAL GEAR

variety of gears of light metal. These include 48 and 32 pitch spur gears of many small sizes and journal diameters, bevel, helical, and worm gears. Shafting of light metal is also available in many sizes and lengths, and a variety of ball bearings and mounting brackets is on hand. The use of the light metal members greatly reduces the power required, which is an important consideration here since the torque output of the integrators is fairly limited.

ANALYTICAL CONSIDERATIONS

On page 1 the equation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{F_{af}}{m}\sin\omega_f t$$

was given as the equation whose solution we desire to produce mechanically. Considering this equation analytically, the general solution is

$$x = e^{-\frac{c}{2m}t}(C_1\sin qt + C_2\cos qt)$$

$$+ \frac{F_{af}/K}{\sqrt{\left(1-\frac{\omega_f^2}{\omega_n^2}\right)^2 + \left(2\frac{c}{c_c}\frac{\omega_f}{\omega_n}\right)^2}}\sin(\omega_f t - \phi)$$

where C_1 and C_2 are constants depending on initial conditions,

$q = \sqrt{\omega_n^2 - \frac{c^2}{4m^2}}$ = natural damped frequency

ϕ = phase angle = $\tan^{-1} \frac{c\omega_f}{K - m\omega_f^2}$

$= \tan^{-1} \frac{2\frac{c}{c_c}\frac{\omega_f}{\omega_n}}{1 - \omega_f^2/\omega_n^2}$

e = base of natural logs.

The other symbols have the notation given on page 1.

In order to determine C_1 and C_2, assume initial conditions of $x = 0$, $\dot{x} = 0$, at $t = 0$. This is arrived at by considering the system to be a spring-suspended mass acted upon by an external harmonic force. The mass is held tight in a clamp while the external force is acting. At $t = 0$ the clamp is suddenly released, at the moment when the forced vibration would just have its maximum amplitude. From these initial conditions it follows that at the instant of release the mass has no deflection and no velocity. Thus at $t = 0$ the forced vibration has $x = x_0$ and $\dot{x} = 0$. These conditions can be satisfied only by starting a <u>free</u> vibration with $x = -x_0$ and $\dot{x} = 0$. Then the combined or total motion will start at zero with sero velocity. Fig. 6 shows the combination of free and forced vibrations.

Then

$$0 = C_2 - \frac{F_{af}/K \sin\phi}{\sqrt{\left(1 - \frac{\omega_f^2}{\omega_n^2}\right)^2 + \left(2\frac{c}{c_c}\frac{\omega_f}{\omega_n}\right)^2}}$$

$$C_2 = \frac{F_{af}/K \sin\phi}{\sqrt{\quad}}$$

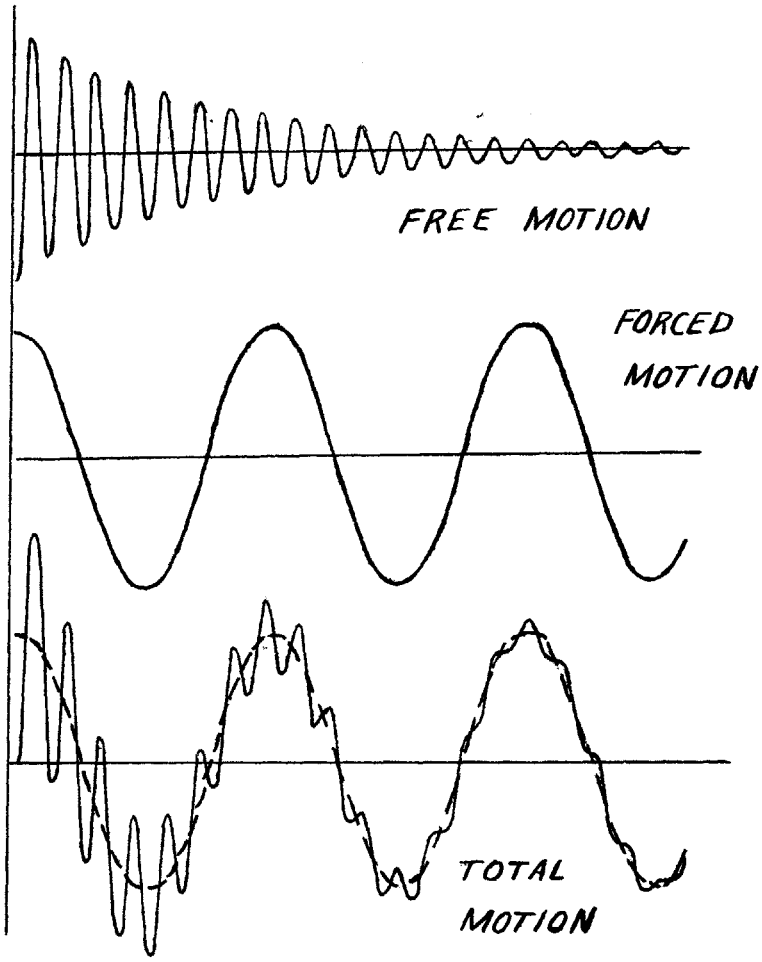

FIG. 6 ~ COMBINATION OF FREE AND FORCED VIBRATION

Differentiating,

$$\dot{x} = qC_1 e^{-\frac{c}{2m}t}\cos qt - \frac{c}{2m}C_1 e^{-\frac{c}{2m}t}\sin qt - qC_2 e^{-\frac{c}{2m}t}\sin qt - \frac{c}{2m}C_2 e^{-\frac{c}{2m}t}\cos qt + \frac{F_{af}/K\ \omega_f \cos(\omega_f t - \phi)}{\sqrt{\left(1-\frac{\omega_f^2}{\omega_n^2}\right)^2+\left(2\frac{c}{c_c}\frac{\omega_f}{\omega_n}\right)^2}}$$

at $t = 0$,

$$0 = qC_1 - \frac{c}{2m}C_2 + \frac{F_{af}/K\ \omega_f \cos\phi}{\sqrt{\left(1-\frac{\omega_f^2}{\omega_n^2}\right)^2+\left(2\frac{c}{c_c}\frac{\omega_f}{\omega_n}\right)^2}}$$

$$C_1 = \frac{c}{2mq}\frac{F_{af}/K\ \sin\phi}{\sqrt{\left(1-\omega_f^2/\omega_n^2\right)^2+\left(2\,c/c_c\,\omega_f/\omega_n\right)^2}}$$

When $\omega_f = \omega_n$ the natural and the forced vibrations are in resonance, and as might be expected it can be shown that the maximum displacement occurs with this conditions. Considering first the natural vibration, $e^{-\frac{c}{2m}t}(C_1\sin qt + C_2\cos qt)$ is a maximum when C_1 and C_2 are a maximum. Since $\tan\phi = \infty$ (and therefore $\phi = \pi/2$) when $\omega_f = \omega_n$, a look at the above expressions for C_1 and C_2 shows that both C_1 and C_2 are maximum with this condition.

Considering the forced vibration, the term

$$\frac{F_{af}/K\ \sin(\omega_f t - \phi)}{\sqrt{\left(1-\frac{\omega_f^2}{\omega_n^2}\right)^2+\left(2\frac{c}{c_c}\frac{\omega_f}{\omega_n}\right)^2}}$$

is a maximum when the radical is a minimum. By trial and error this was found to occur when $\omega_f = \omega_n$ and c/c_c is a minimum. Therefore maximum displacement for both the free vibration and the

forced vibration occur when $\omega_f = \omega_n$.

But although the forcing frequency and the damping ratio for x_{max} are thus readily established, it is difficult to determine x_{max} accurately. Assuming initial conditions and values for the constants, the time at which x_{max} occurs could be found only by trial and error. This was found to be a very cumbersome process. Table 1 lists the values of x determined for given constants and various values of t. Obviously the results are inconclusive, and to chart accurately the behavior of this equation over a cycle would be an enormous task. Then if $\dot{x}_{max}$ and $\ddot{x}_{max}$ were to be determined similarly by first taking derivatives, even greater difficulties would be encountered.

But as we desire these maximum values only as a starting point in the design of the analyzer, great accuracy isn't necessary. Therefore we try to find an approximation.

Referring to the discussion on initial conditions, it was stated that the free vibration began with $x = -x_o$ and the forced vibration with $x = x_o$. Since the maximum amplitude of a damped free vibration occurs on the first cycle, the combined value of the free and forced vibrations could never exceed $2x_o$, or twice the amplitude of the steady state vibration. Thus if we write

$$x_{ss} = x_o \sin \omega_f t$$

where x_{ss} = steady state displacement, then

$$x_{max} = 2x_o$$

VALUES ASSUMED:

$x_{STATIC} = 1.041$ IN.

$\zeta = 0.1$

$\omega_f = \omega_n = 20$ RAD/SEC.

TIME (SEC)	$x_{TRANS.}$ (IN.)	x_{STEADY} (IN.)	x_{TOTAL} (IN.)
0			1.041
1	6.25	−2.07	4.18
2	6.14	3.47	9.61
3	5.82	4.96	10.78
4	5.41	0.62	6.03
.05	4.18	−2.81	1.37
.20	−5.38	3.41	−1.97
.10	−1.37	2.16	0.79
.15	−5.78	5.15	−0.63
.25	−0.66	−1.44	−2.09
.06	3.21	−1.88	1.33
.07	2.14	−0.88	1.26
.08	0.66	0	0.66
.09	−0.22	1.19	0.97
.11	−2.54	3.07	0.53
.12	−3.58	3.84	0.30
.13	−4.49	4.46	−0.03
.14	−5.23	4.90	−0.33
.16	−6.13	5.15	−0.98
.17	−6.61	5.20	−1.41
.18	−6.19	5.03	−1.16
.19	−5.89	4.66	−1.23

TABLE I ~ VALUES OF DISPLACEMENT VS. TIME FOR ASSUMED CONDITIONS

is definitely on the conservative side.

Similarly we can write

$$x_{total} = x_{transient} + x_{ss}$$

$$= x_o \sin \omega_n t + x_o \sin \omega_f t$$

Here again $x_{max} = 2x_o$ is a limiting value. Taking derivatives,

$$\dot{x}_{total} = x_o \omega_n \cos \omega_n t + \omega_f x_o \cos \omega_f t$$

$$\dot{x}_{max} = x_o \omega_n + \omega_f x_o$$

$$= \frac{x_{max}}{2} (\omega_n + \omega_f)$$

$$\ddot{x}_{total} = -x_o \omega_n^2 \sin \omega_n t - x_o \omega_f^2 \sin \omega_f t$$

$$\ddot{x}_{max} = x_o \omega_n^2 + x_o \omega_f^2$$

$$= \frac{x_{max}}{2} (\omega_n^2 + \omega_f^2)$$

Therefore if we select a value of x_{max}-- either arbitrarily or by choice of physical constants of the system-- then $\dot{x}_{max}$ and $\ddot{x}_{max}$ are readily determined. The above expressions can be simplified by recalling that x_{max} occurs when $\omega_f = \omega_n$. Therefore we can write

$$\dot{x}_{max} = \frac{x_{max}}{2} (2\omega_n) = \omega_n x_{max}$$

and

$$\ddot{x}_{max} = \frac{x_{max}}{2} (2\omega_n^2) = \omega_n^2 x_{max}$$

This presumes that $\dot{x}_{max}$ and $\ddot{x}_{max}$ occur when $\omega_f = \omega_n$, and this is at least approximately correct. The value of x_{max} drops off rapidly as ω_f increases above ω_n, which more than compensates

for increased value of the quantity $(\omega_n + \omega_f)$.

MECHANICAL SOLUTION

In mechanisms of the continuous computing type, all mathematical quantities are represented by movements of machine elements. The scale factor of any shaft is defined as the number of revolutions turned by that shaft per unit of the variable represented. If a shaft represents a variable y at a scale factor of 500 (called a 500y shaft), and y varies by three units, the shaft will make 1500 revolutions.

A gear ratio operates directly upon the scale factor. Thus if a 100y shaft is geared down at a 1:2 speed ratio, the output shaft is a 50y shaft.

A constant coefficient can be introduced either by means of a gear ratio or by change of scale factor. Thus, if a 500y shaft is available, and the problem requires $\frac{1}{4}y$, a 1:4 gear ratio may be employed to produce

$$\frac{500}{4} y = 500\left(\frac{y}{4}\right)$$

Often, however, it is unnecessary to use an actual gear, for a shaft labelled 500y can also be used as

$$500 \cdot 4 \frac{y}{4} = 2000\left(\frac{y}{4}\right)$$

Variables which are to be added or subtracted must enter the differential gearing at the same scale factor.

With the integrators on hand, 4.42 radians rotation of

the rack pinion drives the carriage from the middle ot the extreme position. Therefore the scale factor of a variable entering on the rack pinion must be such that extreme excursion of the variable from zero is represented by a shaft rotation equal to or less than 4.42 radians.

Link-up. We can readily construct a diagram showing the link-up of the principle units necessary to solve the equation of this problem. First a time shaft is supplied driven at constant speed, which is used to drive the disks of the integrators used. Then if we imagine a shaft representing $\ddot{x}$ to drive the carriage of one of the integrators (without considering what drives this shaft), the output shaft of this integrator represents $\dot{x}$. Using this output to drive the carriage of a second integrator, x is produced on the output shaft. The angular displacement of this shaft can then be converted to straight line motion and utilized to chart x against time.

But it is necessary to complete the circuit and supply a source of power for the $\ddot{x}$ shaft. Refferring to Fig. 7, which shows the completed diagram, x and $\dot{x}$ are multiplied by ω_f^2 and $2\zeta\omega_f$ respectively (as in the equation) through use of gearing, and then they are added at differential B. The sine function is produced through use of two integrators with "feedback" connections, with $\omega_f t$ instead of t being represented on the con-

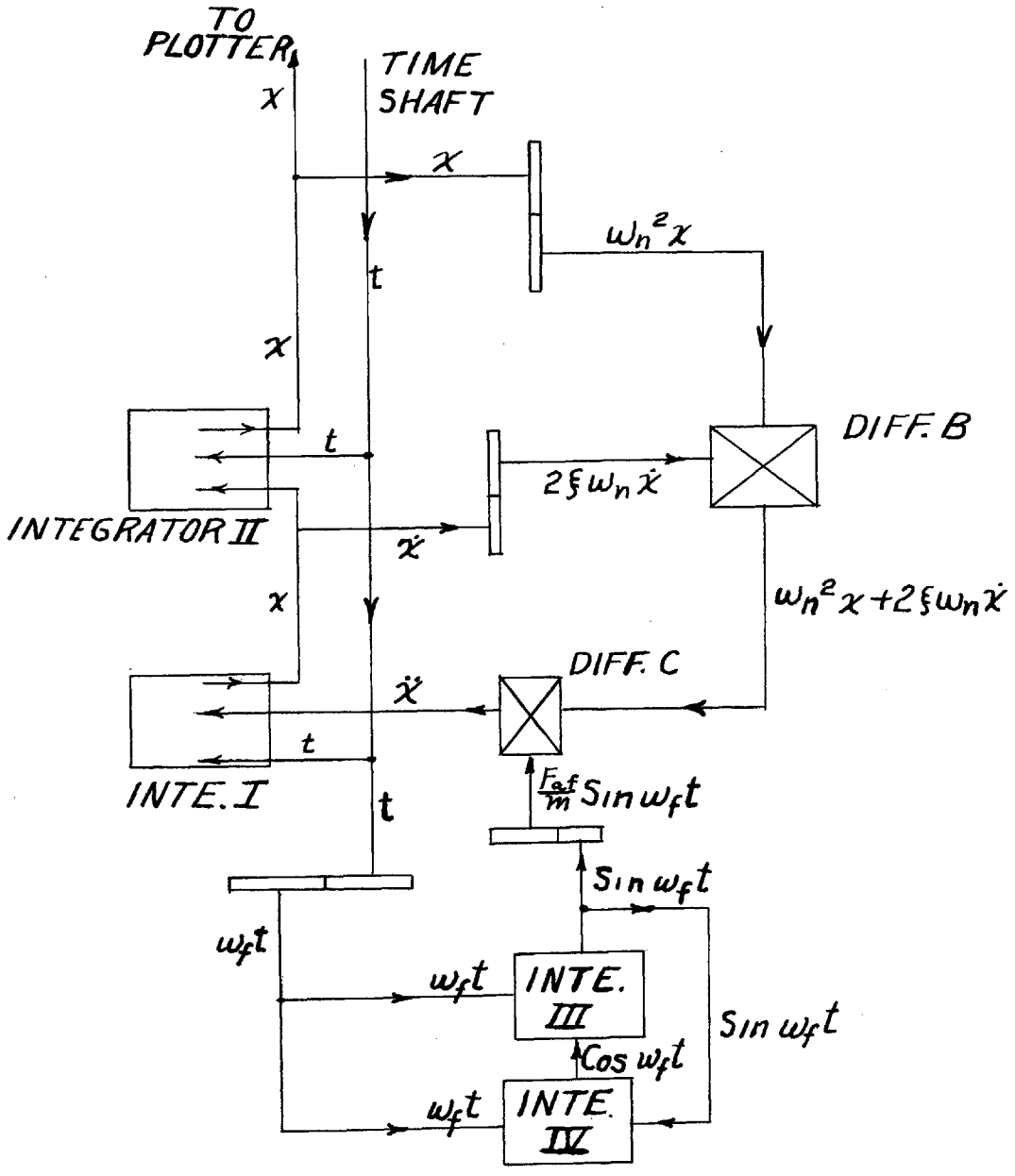

FIG. 7 ~ LINK-UP OF PRINCIPLE UNITS

stant speed shaft. Sin $\omega_f t$ is multiplied by the constant F_{af}/m and then it is added to $\omega_n^2 x + 2\zeta\omega_n$ at differential C. This sum is then equal to $\ddot{x}$ as in the equation, so the output of differential C becomes the $\ddot{x}$ shaft. This last connection is equivalent to the equal sign in the equation, and makes the machine perform the desired computation.

Of course this is a simplified explanation of the design process. It ignores the possible scale factor and gear ratio difficulties involved.

To solve this system, it was found best to begin on the output side, i.e., on the x-shaft going to the plotter. Hence we start by assuming values for x_{max} and $(z_x)_{max}$ (the maximum revolution of the x shaft), and work back to determine $\dot{x}_{max}$ and $\ddot{x}_{max}$ using the approximation previously described on page 13. From these limiting values and by assigning values to the constant in the equation we can proceed to a solution.

Repeating the equation of the integrator,

$$z = \frac{\dot{y}}{r_1}\int v\,dt$$

Here

$$v = \theta_v r_2$$

where θ_v = angular displacement of pinion

r_2 = radius of pinion.

Then

$$z = \frac{\dot{y} r_2^2}{r_1}\int \theta_v\,dt$$

On the integrator producing x,

$$\theta_v = K\dot{x}$$

Then $$Z_x = \frac{K\dot{y} r_2}{r_1}\int \dot{x}\,dt$$

$$= \frac{K\dot{y} r_2}{r_1}\, x$$

To find k,

$$(\theta_v)_{max} = 4.42 \text{ rad.} = K\dot{x}_{max}$$

$$K = \frac{4.42}{\dot{x}_{max}}$$

$$Z_x = \frac{\dot{y} r_2 \cdot 4.42}{r_1 \cdot \dot{x}_{max}}\, x$$

Now we assume reasonable values for $\dot{y}$, $(z_x)_{max}$, and x_{max}:

$\dot{y}$ = 200 rpm

$(z_x)_{max}$ = 5 revolutions

x_{max} = 5 inches

With the integrators on hand,

$$r_2 = \frac{1}{2\pi} \text{ in. } \& \; r_1 = 1/4 \text{ in.}$$

Then $$Z_x = \left(\frac{200}{60} 2\pi\right) \frac{1/2\pi}{1/4} \frac{4.42}{\dot{x}_{max}}\, x$$

$$= \frac{59x}{\dot{x}_{max}}$$

$$1 \text{ rad. of } Z_x = \frac{x_{max}}{59} \text{ units of } x$$

$$1 \text{ rev.} = \frac{2\pi \dot{x}_{max}}{59}$$

$$(Z_x)_{max} = 5 \text{ rev.} = \frac{10\pi}{59}\dot{x}_{max} = .5325\, \dot{x}_{max} \text{ units}$$

or $$x_{max} = .5325\, \dot{x}_{max}$$

Then using the aproximation

$$\ddot{x}_{max} = \omega_n \dot{x}_{max},$$

$$\omega_n = \frac{1}{.5325} = 1.88 \text{ rad./sec.}$$

$$\dot{x}_{max} = \omega_n x_{max} = 1.88 \cdot 5 = 9.4 \text{ in/sec}$$

and $$\ddot{x}_{max} = \omega_n^2 x_{max} = 1.88^2 \cdot 5 = 17.65 \text{ in/sec.}$$

In order to solve the system completely with scale factors and gear ratios we write the equation of the integrator as

$$S z = K \int A v \, d(Bt)$$

Here S, A, and B are scale factors for the respective variable shafts, and K is a constant of the integrator. This can be rewritten as

$$S z = KAB \int v \, dt$$

Hence, the scale factor of z is given by

$$S = KAB.$$

We can find S, A, and B from the previously determined maximum values of the variables, and thus find K. Using the above integrator equation in this form we can use directly the desired scale factors.

Using the case of Integrator 1,

$$v = \ddot{x} \text{ and } z = \dot{x}.$$

$$\ddot{x}_{max} = 17.65 \text{ in/sec}^2 = 4.42 \text{ rad.}$$

$$1 \text{ unit of } \ddot{x} = .0399 \text{ rev.} = A$$

$$\dot{y} = 200 \text{ rpm} = 20.95 \text{ rad./sec.}$$

$$1 \text{ unit of } t = 20.95 \text{ rad.} = 3.33 \text{ rev.} = B$$

$$Z_{\dot{x}} = \dot{y} \frac{r_2}{r_1} \frac{4.42}{\ddot{x}_{max}} \dot{x}$$

$$= \left(\frac{200}{60} 2\pi\right) \frac{1/2\pi}{1/4} \frac{4.42}{17.65} \dot{x} = 3.34 \dot{x}$$

$$1 \text{ unit of } \dot{x} = 3.34 \text{ rad.} = .532 \text{ rev.} = S$$

$$\text{Then} \quad .532 Z = K \int .0399 \ddot{x} \, d(3.33t)$$

$$.532 Z = .133 K \dot{x}$$

$$K = \frac{.532}{.133} = 4.0,$$

$$\text{since} \quad KAB = S .$$

To continue with the solution we follow the procedure outlined in Dr. Bush's paper of October 1945, "A New Type of Differential Analyzer". Then the system of Fig. 7 appears as in Fig. 8. The horizontal lines represent shafts with the symbols on the right indicating the variable on each shaft and its scale factor. The boxes represent integrators and the verti-

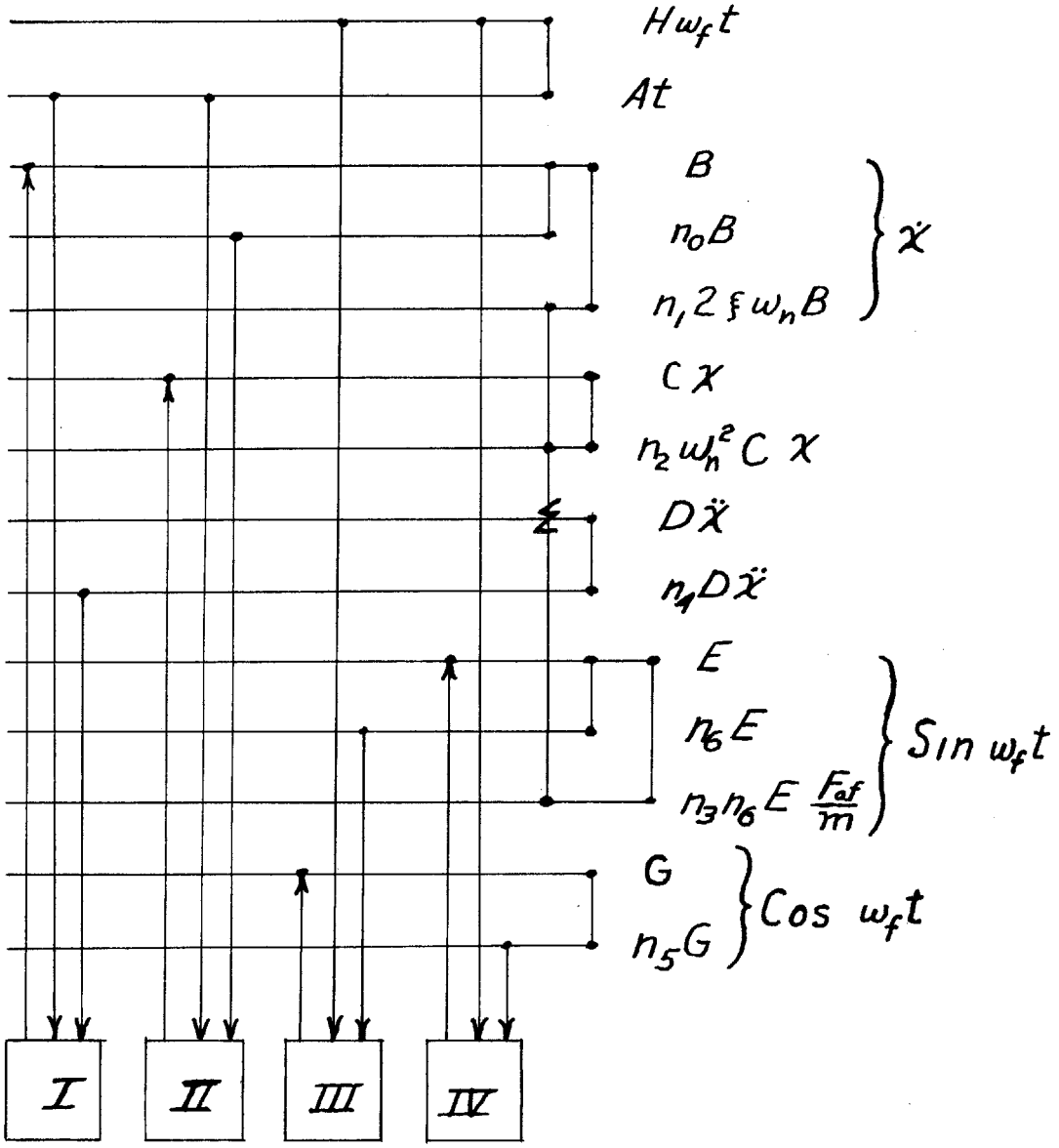

FIG. 8 ~ CONNECTIONS FOR MECHANICAL SOLUTION OF

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = \frac{F_{af}}{m} Sin\ \omega_f t$$

cal lines are the input and output shafts to the integrators, with interconnections to the other shafts indicated by heavy dots. The vertical lines on the right side indicate gearing between two shafts.

Then starting with integrator 1, we input time (the At shaft) and $\ddot{x}$ at some scale factor. The output is $B\dot{x}$, i.e., $\dot{x}$ at a scale factor of B revolutions per unit of $\dot{x}$. Gearing $B\dot{x}$ at ratio n_o so that

$$\dot{x}_{max} = 4.42 \text{ radians},$$

we input $\dot{x}$ into Integrator II at scale factor n_oB along with At. The output of II is Cx.

The $H\omega_f t$ shaft is produced by gearing from the At shaft, since ω_f is assigned a constant value. $\text{Sin}\,\omega_f t$ and $H\omega_f t$ are input into Integrator III to produce $G\text{Cos}\,\omega_f t$, which is geared at ratio n_5 and input into Integrator IV to produce $E\text{Sin}\,\omega_f t$. The scale factor of the $\text{Sin}\,\omega_f t$ input into Integrator III is then n_6E.

The $\dot{x}$, x, and $\text{Sin}\,\omega_f t$ are all multiplied by constants, so that we then have shafts $2n_1\zeta\omega_n Bx$, $n_2\omega_n{}^2Cx$, and $n_3n_6E\frac{Faf}{m}\text{Sin}\ t$. These are added to produce $D\ddot{x}$, and then the input into Integrator I is $Dn_4\ddot{x}$. This completes the system.

Then substituting scale factors in the form of the integrator equation on page 18, we start to solve the system:

Output of Integrator I:

$K\int Dn_4\ddot{x}\ d(At) = 4n_4DA\dot{x} = B\dot{x}$ (1)

Output of Integrator II:

$K\int (n_oB\dot{x})d(At) = 4n_oBAx = Cx$ (2)

Output of Integrator III:

$K\int (n_6E\ \text{Sin}\ \omega_f t)d(H\omega_f t) = 4n_6EH\ \text{Cos}\ \omega_f t = G\ \text{Cos}\ \omega_f t$ (3)

Output of Integrator IV:

$K\int (n_5G\ \text{Cos}\ \omega_f t)d(H\omega_f t) = 4n_5GH\ \text{Sin}\ \omega_f t = E\ \text{Sin}\ \omega_f t$ (4)

Driving variables: $\ddot{x}$, $\dot{x}$, $\text{Sin}\ \omega_f t$, $\text{Cos}\ \omega_f t$

$\ddot{x} \leqslant 17.65\ \text{in/sec}^2$

$\dot{x} \leqslant 9.4\ \text{in/sec}^2$

$\text{Sin}\ \omega_f t \leqslant 1$

$\text{Cos}\ \omega_f t \leqslant 1$

$Dn_4 \cdot 17.65 = 4.42$ radians $= .704$ rev.; $Dn_4 = .0398$ (5)

$n_oB \cdot 9.4 = .704$; $n_oB = .0748$ (6)

$n_6E \cdot 1 = .704$; $n_6E = .704$ (7)

$n_5G \cdot 1 = .704$; $n_5G = .704$ (8)

from (1) and (5) -- $4 \cdot .0398A = B = .1592A$

from (2) and (6) -- $4 \cdot .0748A = C = .2992A$

then $B = .533C$

from (3) and (7) -- $4 \cdot .704H = G = .2816H$

from (4) and (8) -- $4 \cdot .704H = E = .2816H$

then $G = E$

and $n_6 = n_5$.

At $\dot{y} = 200$ rpm, $A = 3.33$ rev. per unit of variable

then $B = .53$

$C = 1.00$

$n_6 = .0748/.53 = .141$

At differential B, Fig. 7, the entering variables must have the same scale factors:

$$n_2 C = n_1 B$$

Similarly at differential C,

$$n_2 C/2 = n_1 B/2 = D = 2 n_3 n_6 E$$

Arbitrarily letting $n_4 = 1/7$,

$D = .0398 \cdot 7 = .2786$

$n_1 = 2 \cdot .2786/.53 = 1.052$

$n_2 = 2 \cdot .2786/1.0 = .5592$

Using (7)--

$$.2786 = 2 \cdot n_3 \cdot .704$$

$$n_3 = .198$$

Letting $n_5 = 1$, then $G = .704 = E$

$n_6 = 1$

$H = E/.2816 = 2.5$

Thus all the scale factors are determined for the conditions we have assumed. Now the gear ratios thus established are examined to see if they are within reason.

Gear ratios. On the $\dot{x}$ shaft the gear ratio is $2 n_1 \zeta \omega_n$. With $1 \geq \zeta \geq .1$, limiting values are .3956 and 3.956.

On the x shaft, gear ratio is $n_2\omega_n^2$

$$n_2\omega_n^2 = .5592 \cdot 1.88^2 = 1.976$$

On the Sin $\omega_f t$ shaft the gear ratio is

$$n_3 n_6 F_{af}/m = .198 F_{af}/m$$

If we let F_{af}/m equal 5, the gear ratio is 1; therefore we let F_{af} be 5.

On the time shaft the gear ratio is $H\omega_f/A$. With $2\omega_n \geq \omega_f \geq .25\omega_n$, limiting values are .352 and 2.82.

To produce the variable ratios required for ζ and ω_n, we use integrating umits since they are available. By simply hand feeding the carriage drive, we can establish any desired speed ratio of output shaft to input (constant speed) shaft from 2.8 to 0 with an integrating unit. This setting on each unit will remain fixed while a solution is performed with any combination of ζ and ω_n. Where the ratio above is greater than 2.8, we'll have to gear the unit externally.

This solution seems to be very satisfactory; several sets of values were tried, but these results were the most reasonable. The gear ratios are small, the largest being about 7 to 1.

With $\omega_n = 1.88$ rad./sec. and $\omega_f = \omega_n$, a cycle of the vibrating system occurs in about $3\frac{1}{3}$ seconds, and ten cycles would

be produced in about 35 seconds. With extreme values of ω_f ($.25\omega_n$ and $2\omega_n$) the time required to produce ten cycles would range from 17 seconds to two minutes 20 seconds. Therefore a graphical solution to the general equation with a particular set of constants could be produced in a reasonably short time.

DESIGN

Many factors entered into the actual design of the mechanical system after the analytical work was done. Since the output torque of the integrating units is fairly limited, the gear trains on the output side of the integratorswere designed to be as small as possible. This necessitated more extensive gearing on the input side of the integrators, but the source of power could be altered to take care of this. Of course the use of light gears and shafts helps defeat the torque problem.

Designing for minimum gear trains also will necessitate more elaborate mountings of shafts and integrators. Integrators were stationed in various positions and on various elevations, and therefore some right angle brackets will be required. Spur gears were used wherever possible so that axial thrust would be no problem.

In solving the gear trains, directions of rotation into

integrating units and differentials wasn't much of a problem. The direction of rotation could be changed at many places after the gear trains were fixed, including the time shaft side of integrators and some of the carriage drive inputs.

The general procedure on gearing was to select gears to give the proper speed ratio from the stock on hand. Any necessary idlers were located approximately, and then an exact size of gear was selected. The exact center location of the idler was then determined graphically.

Design of Sub-Assembly A. Sub-Assembly A, Plate 1, was designed from the link-up of Fig. 7. Primary considerations were

1) Connecting the output of Integrator I with the rack drive of Integrator II and with the disk drive of Integrator III by spur gears,

2) Connecting the outputs of Integrators II and III to Differential I by spur gears,

3) Connecting handily to Differential II with the output of Differential I and the sine shaft, and

4) Allowing access to run the $\ddot{x}$ shaft back to drive the rack of Integrator I.

Of course all connections were made as simple as possible. The methods of mounting units and the connections from the time shaft, while given secondary importance, were considered simultaneously. The assembly is designed to be mounted on one flat plate.

Most integrating units in the controls lab now have a 34-tooth, 48-pitch gear on the output shaft; a 40-tooth, 48-pitch gear on the rack drive and a 90-tooth, 48-pitch gear as its idler; and a 30-tooth, 32-pitch gear on the disk drive. Wherever possible these gears were utilized in this design.

Starting with the output of Integrator I, the gear ratio into Inte. II is $n_o = .141$, or approximately a 7:1 reduction. We could not drive directly from gear 2,Plate 1, to gear 12 because a mounting post is necessary at the corner of Inte. II. Therefore the curvaceous gear train shown was arrived at. Gear 10 rotates .141 times as fact as Gear 2 and is the same size as Gear 12.

The drive to Inte. III has a 2:1 speed reduction. Then with the reduction possible with the integrator the gear ratio necessary to introduce ξ --- .3956 to 3.956--- can be obtained. A handwheel assembly, available in the controls lab, can be bolted to Inte. III so that the carriage may be

positioned easily by hand.

The outputs of Inte. II and III are input into Differential I through single idlers. The differentials available have 48-tooth, 48-pitch gears, so the gears 14 and 17 are determined to give the proper ratios to feed $\omega_n^2 x$ and $2\zeta\omega_n x$ into the differential. The idlers were selected from stock on hand to be as small as possible.

The inputs into Differential II are at 1:1. The output $\ddot{x}$ is then geared at 1:7 to the rack, or carriage, drive of Inte. I. This reduction is accomplished with gears 19, 18, 22, and 23. Gear 22 has 40 teeth and Gear 23 has 90, which is the reverse of the gears in those respective positions now on the integrators; Gears 18 and 19 were then selected to give the 7:1 reduction.

The relative elevations of the integrators is such that gears 2, 12, and 4 are in the same horizontal plane. The relative positions horizontally were adjusted for compactness and simplicity of the assembly.

No member is directly above another member, so that vertical supports for each unit can be installed. For this reason the differentials were placed out to the left rather than closer to Inte. I. Obviously Inte. II and III were positioned longitudinally to feed directly into Different-

ial I.

Design of Sub-Assembly B. Sub-Assembly B, Plate 2, produces the sine function, which is fed into Differential II in Sub-Assembly A. This sub-assembly is designed around the direct connection of the output of Integrator V with the carriage drive input of Inte. VI. Then the output shaft of VI and the carriage drive shaft of V are parallel, and the feedback connection can be made by spur gears.

Inte. IV multiplies the time shaft by ω_f . Here again the carriage drive is hand fed and the handwheel assembly is advantageous. Since the output of IV drives the disks of both V and VI, a bevel gear connection is necessary. The drive to V is through a single idler, and to VI through the bevels, an idler shaft, and two idler gears.

The relative elevations of V and VI are determined by making the output shaft of V and the input shaft of VI at the same elevation. Then the output shaft of IV is placed level with the disk drive of V.

Shafts such as that shown as the output of IV can be directly coupled to the integrator shaft, which projects only a fraction of an inch.

Assembly. Plate 3 shows the complete assembly. It includes the sub-assemblies previously described, the time shaft and its connections, and the plotting device and its

drives.

The time shaft is motor-driven and runs the length of the assembly. From a spur gear and 2 bevel gears mounted on it the time shaft drives the disk of three integrators. Inte. II is driven directly; Inte. IV is driven through a vertical shaft and two idlers; Inte. I is driven through an intermediate gear which also serves to drive the plotting drum through a gear train.

It was decided to drive the drum from the time shaft and execute reciprocating motion with a rack driven from the x shaft, rather than drive the drum from the x shaft. This was done so that the lesser load would be on the x shaft.

Then the speed of the drum was determined so that a plot of at least several cycles is made in one revolution of the drum. In this way a plot is made by placing a sheet of paper arounnd the drum and letting the machine run for one revolution of the drum. Otherwise the plot would overlap as the drum revolved, or the paper would have to be continuously fed over the drum.

With ω_f a minimum the vibration cycles are slowest. Then $\omega_f = .25\omega_n$, and one cycle occurs in 13.3 seconds. Therefore, at least a minute is required for a good plot, and for the drum to turn at one rpm requires a 200:1 re-

duction from the time shaft. The gear train actually shown, using three steps, has a 213:1 reduction.

The rack pinion, which oscillates to produce the ordinate of the displacement-time curve, is driven from the output of Inte. II through a set of bevel gears. For 5 in. max. travel (i.e., $x_{max} = 5$ in.),

$$\text{pinion diam. } d = \frac{5}{\pi v} = \frac{1.591}{v}$$

$$\text{or } v = 1.591/d$$

where v = max. rev. of the rack pinion.

With $(z_x)_{max} = 5$ rev.,

$$\text{gear ratio to rack pinion} = \frac{1.591}{5d} = \frac{.3182}{d}$$

With a 96-tooth gear on Inte. II, we install a 224-tooth gear to mate with it on the x shaft. For this to be the correct gear ratio (assuming the bevels are 1:1) then

$$\frac{.3182}{d} = \frac{96}{224} = .428$$

$$d = .743$$

This is 35.6 teeth for 48-pitch gear; a 36-tooth gear is used.

The rack is cut 12 inches long to allow for five inches travel each way. The rack carries a bracket holding a Foxboro Recording Pen which is in contact with the plotting paper. The height of the pen is even with the axis of the drum.

CONCLUSION

The drawings of Plates 1,2 , and 3, resulting from the mathematical analysis and design of this thesis, are more or less working drawings to be used in further design and construction of this machine. While functional improvements can probably be made by further study, on paper the functional design seems very satisfactory. The machine produces a graphical solution of reasonable size of the general equation, and in a short period of time; and it solves the equation for a range of values of the damping and the forcing frequency so that the effect of varying these quantities may be observed.

The interconnections between units seem fairly simple, and the units could all be mounted from one flat plate. While the accuracy obtainable with this machine probably wouldn't exceed nine parts in ten, it is sufficient to show the contour of the graphical solution. The members specified in the drawing are practically all available in the controls laboratory, so that cost of this machine would be small.

It was originally intended that this thesi s would include design of a complete assembly showing mountings, bearings, shafting, and couplings. Lack of time prevented

this from being accomplished, although some work was done on it. Therefore a large mechanical design problem must be worked out before actual construction of the machine can be undertaken.

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