REPRESENTATIONS OF FINITELY GENERATED NILPOTENT GROUPS
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Representations

of

Finitely Generated Nilpotent Groups

by Ian D. Brown

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Summary

We are studying those unitary representations of a finitely generated nilpotent group G which are induced from finite dimensional representations of subgroups. An arbitrary representation ρ turns out to be of the above kind if and only if G contains a subgroup H such that $\rho|H$ contains a character of finite multiplicity.

We show that any representation of the above kind is actually induced from a one dimensional representation of a subgroup. We derive a criterion for when such induced representations are irreducible and when the irreducible ones are equivalent.

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Introduction

This thesis is a study of certain irreducible representations of finitely generated nilpotent groups. By "representation" we will always mean a representation of the group in the group of unitary operators on some separable Hilbert space. The irreducible representations we study are those induced (definition below) from finite dimensional representations of subgroups. We will describe the equivalences among these representations and derive one intrinsic characterization of them.

An induced representation (on these groups) is defined as follows:

Definition 0.1: Let H be a subgroup of the given group G, ρ a representation of H in the unitary operators on the Hilbert space $\mathcal{H}(\rho)$. Let $\mathcal{N}(\rho \uparrow G)$ be the set of all functions $f: G \longrightarrow \mathcal{H}(\rho)$ with the properties:

1) $f(hg) = \rho(h)f(g)$ for all $h \in H$

2) $\sum_{g \in G/H} ||f(g)||^2 < \infty$, where this sum runs over a set of representatives of G/H.

Let G act on $\mathcal{H}(\rho \uparrow G)$ by

$$(gf)(g') = f(g'g)$$ - i.e. right translation.

This action is called the representation of G induced

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from ρ, and is denoted $\rho \uparrow G$. One can check that $\mathcal{H}(\rho \uparrow G)$ is separable if $\mathcal{H}(\rho)$ is, and $\rho \uparrow G$ is unitary if ρ is.

We now must describe certain results of G. W. Mackey which we will state only for discrete groups, but which have a somewhat wider application (cf. Mackey, (2)).

The lemmas are vital steps in the derivation of an intertwining number theorem (Theorem 0.8) for discrete groups. We start with

Definition 0.2: If L, M are representations of a group G, an intertwining operator for L and M is a bounded linear operator $T: \mathcal{H}(L) \longrightarrow \mathcal{H}(M)$ such that $TL(g) = M(g)T$ for all $g \in G$.

Then we have

Lemma 0.3: Let U, V be representations of subgroups H, K respectively of the group G, and T be an intertwining operator for $U\uparrow G$ and $V\uparrow G$. Then there exists a function A from $G \times G$ to the bounded linear operators from $\mathcal{H}(U)$ to $\mathcal{H}(V)$ with the properties:

a) $A(kxg, hyg) = V(k)A(x,y)U(h)^{-1}$ for all $k \in K$, $h \in H$, x, y, g in G.

b) there exists a constant $N > 0$ such that

$$\sum_{x \in K\backslash G} ||A(x,y)v||^2/||v||^2 \leq N \quad \text{for all } y \in G,\ v \in \mathcal{H}(U)$$

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c) there exists a constant $N' > 0$ such that

$$\sum_{y \in G/H} ||A(x,y)^* v||^2 / ||v||^2 \leq N' \quad \text{for all } x \in G \text{ and } v \in \mathcal{H}(V).$$

d) $$(Tf)(x) = \sum_{y \in H\backslash G} A(x,y) f(y) \quad \text{for all } f \in \mathcal{H}(U\uparrow G)$$

e) $$(T^* f)(y) = \sum_{x \in K\backslash G} A(x,y)^* f(x) \quad \text{for all } f \in \mathcal{H}(V\uparrow G).$$

Proof: In fact, A is defined as follows. Let f_y^v be the element of $\mathcal{H}(U\uparrow G)$ which is non-zero only on the coset Hy, and $f_y^v(y) = v \in \mathcal{H}(U)$. Then $(Tf_y^v)(y)$ is a vector in $\mathcal{H}(V)$ for all $x \in G$, and we call it $A(x,y)v$. In (2) p. 585, Mackey shows that this A has the above five properties.

To state the next lemma, we need

Definition 0.4: The intertwining number, $i(L, M)$ of two representations L, M of G is the dimension of the space of intertwining operators for L and M.

Lemma 0.5: With the notation of Lemma 0.3, $i(U\uparrow G, V\uparrow G)$ is equal to the dimension of the space of all functions A satisfying conditions a), b), and c). Further, this is the sum over all $K \times H : \tilde{G}$ double cosets D in $G \times G$ (where $\tilde{G}$ is the diagonal subgroup) of the dimension d_D of the space of A satisfying a), b) and c) and vanishing off D.

Proof: Mackey, (2), lemmas B and C.

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Now we need some notation. For $g \in G$, let $H^g = g^{-1}Hg$ and $U^g(x) = U(g\,x\,g^{-1})$ for all $x \in G$. Then we can state

Lemma 0.6: $i(U^d | H^d \cap K, V | H^d \cap K)$ is equal to the dimension of the space of all functions A satisfying condition a) only and vanishing off the double coset D, where $d = xy^{-1}$, $(x,y) \in D$.

Proof: Mackey, (2), lemma D.

Hence one could calculate the intertwining number for $U\uparrow G$ and $V\uparrow G$ in terms of information about U and V if one knew which of the A in lemma 0.6 satisfy b) and c) of 0.3. The following lemma is a partial answer.

Lemma 0.7: If $[H^d : H^d \cap K] < \infty$ and $[K : H^d \cap K] < \infty$, and A satisfies 0.3 a) and vanishes off $K \times H \cdot (d,e) \cdot \tilde{G}$, then A satisfies b) and c) of 0.3. If U, V are one dimensional and A satisfies a), b) and c) of 0.3, then if $A(d,e) \neq 0$, $[H^d : H^d \cap K] < \infty$ and $[K : H^d \cap K] < \infty$.

Proof: Mackey, (2), lemma E.

Putting all these facts together, and noting that (d,e), (d', e) are in the same $K \times H : \tilde{G}$ double coset if and only if d, d' are in the same $K : H$ double coset, we have Mackey's theorem:

Theorem 0.8: Let U, V be representations respectively of the subgroups H, K of a countable group G. Then

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$$\sum_{d \in D_f} i(U^d | H^d \cap K,\ V | H^d \cap K) \leq i(U\uparrow G, V\uparrow G) \leq \sum_{d \in D} i(U^d/H^d \cap K, V/H^d \cap K)$$

where D is a complete set of double coset representatives of $K:H$ in G and $d \in D_f$ if and only if both $[H^d:H^d \cap K]$ and $[K:H^d \cap K]$ are finite. If U, V are one dimensional the first inequality is an equality.

A slight modification of this theorem will be our chief tool for studying the irreducible representations of finitely generated nilpotent groups that are induced from finite dimensional representations. The first task of the next chapter will be to create this modification.

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Chapter I

Throughout this G will be a finitely generated nilpotent group. The purpose of this chapter is to decide which finite dimensional representations of subgroups of G induce to irreducible representations, and to describe all the equivalences between these irreducible representations.

We begin with a simple lemma on finite dimensional unitary operators

Lemma 1.1: If $\mathcal{H}_1$, $\mathcal{H}_2$ are finite dimensional inner product spaces, $B: \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is linear, and U is a unitary operator on $\mathcal{H}_1$ such that

$$\sum_{n\in Z} ||BU^n v||^2 < \infty \quad \text{for all} \quad v \in \mathcal{H}_1 ,$$

then B = 0.

Proof: Let $\mathcal{H}_1 = \text{Ker}B \oplus W$ with W orthogonal to Ker B. If any limit point of $\{U^n w: n \in Z\}$ (where $w \in W$) is not in Ker B then there exists a sequence $w_i \in W$ (namely, $P_W U^{n_i} w$, where P_W is the orthogonal projection onto W) with all $||w_i|| > a$ for some $a > 0$ and $\Sigma||Bw_i||^2 < \infty$. (Note that $B = BP_W$). But B is one to one continuous on W - hence such a sequence is impossible, since it would mean $Bw_i \longrightarrow 0$. Thus all limit points of $\{U^n w: n \in Z\}$ are in Ker B. Let V be the subspace of $\mathcal{H}_1$ generated by these limit points as w

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runs over W. The set of all such points is invariant under U - hence $UV \subset V$. Then $UV^{\perp} \subset V^{\perp}$. But $V \subset \text{Ker } B$. Hence $W \subset V^{\perp}$. Then $UW \subset V^{\perp}$. Then the limit points of $U^n w$ are in $V^{\perp} \cap V = 0$. Since U is unitary, we must conclude that $W = 0$. That is, $B = 0$.

We also need

Lemma 1.2: If G is a torsion free finitely generated nilpotent group with subgroup H such that for each $g \in G$ there is a positive integer k such that $g^k \in H$, then $[G:H] < \infty$.

Proof: By Malcev's results, (3), G imbeds as a lattice in some real nilpotent Lie group N. Then G contains a set of elements g_i such that $\log g_i$ are a basis of the Lie algebra of N. Then $g_i^{k_i} \in H$ and $\log g_i^{k_i}$ are also a basis for the Lie algebra. Thus H is a lattice of N. But if G/H is infinite, this yields an infinite set of points in N/H with no limit point, violating the compactness of N/H. Thus $[G:H] < \infty$.

With the aid of these two lemmas we can prove the following variation of Theorem 0.8.

Theorem 1.3: Using the notation of the introduction, if H and K are subgroups of G and U, V finite dimensional representations of them respectively, then

$$i(U\uparrow G, V\uparrow G) = \sum_{d \in D_f} i(U^d | H^d \cap K, V | H^d \cap K)$$

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i.e. we have the equality of the last line of theorem 0.8 for any finite dimensional U and V.

Proof: By lemma 0.5 we need to show that the d_D mentioned there is 0 if the double coset is not in D_f, and the appropriate intertwining number if it is in D_f. By combining lemma 0.6 and the first statement of lemma 0.7, the last half of this sentence is true. Hence it suffices to show that if $d \notin D_f$, the dimension of the space of A non-zero on $K \times H \cdot (d,e) \cdot \tilde{G}$ satisfying a), b) and c) is zero.

But this space is contained in the space of A of lemma 0.6. In fact the space we want is contained in the space of operators A generated from B which intertwine $U^d|H^d \cap K$ and $V|H^d \cap K$. We can see this as follows: let $A(kx_0g, hy_0g) = V(k)\, B\, U(h)^{-1}$ where $x_0y_0^{-1} = d$. (This from Mackey's proof of lemma 0.6). Here $BU^d(k) = V(k)B$ for all $k \in H^d \cap K$. Thus it remains to show that no A thus generated, for $d \notin D_f$, possesses both properties b) and c).

Note that $A(x,y) \neq 0 \iff x = kx_0g$, $y = hy_0g$

$$\iff xy^{-1} = kx_0y_0^{-1}h$$
$$= kdh$$
$$\iff xy^{-1} \in KdH$$

Then $x_1x_2^{-1} \in K \iff k_1dh_1h_2^{-1}d^{-1}k_2^{-1} \in K$

assuming $x_1y^{-1} = k_1dh_1$, $x_2y^{-1} = k_2dh_2$

$$\iff dh_1h_2^{-1}d^{-1} \in K$$

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Condition b) requires that

$$\sum_{x \in G/K} ||A(x,y)v||^2 \leq N||v||^2$$

Here $||A(x,y)v|| = ||A(x_o, hy_o)v||$ since v is unitary, and the sum runs over $h \in H/H \cap K^{d-1}$ by the above argument. Thus we require

$$\sum_{h \in H/H \cap K^{d-1}} ||BU(h)^{-1}v||^2 < N||v||^2$$

If $H^d/H^d \cap K$ is infinite, so is $H/H \cap K^{d-1}$ and by lemma 1.2 there is $h_o \in H$ such that no power of h_o is in $H \cap K^{d-1}$. Thus condition b) means, at least, that

$$\sum_{m \in Z} ||BU(h_o{}^m)^{-1}v||^2 < \infty .$$

By lemma 1.1, $B = 0$. Hence if A possesses property b) and is non-zero, then $[H^d : H^d \cap K] < \infty$. Similarly, property c) implies $[K : H^d \cap K] < \infty$. The proof of the theorem is complete.

To present our analysis of the irreducible representations induced from finite dimensional representations we need another concept.

Definition 1.4: A subgroup $H \subset G$ is complete if for any $x \in G$ such that $x^n \in H$ for some integer $n \neq 0$, we have $x \in H$. We call the smallest complete subgroup containing a subgroup H the completion of H, and we denote it by H^*.

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The main theorem, then, of this section, is the following:

Theorem 1.5: Let G be a finitely generated torsion free nilpotent group. Then:

1) If H is a complete subgroup of G, V a finite dimensional representation of H and there is no $g \in N_G(H) - H$, where $N_G(H)$ is the normalizer of H in G, such that V^g is equivalent to V, then $V\uparrow G$ is irreducible.

2) If H is a subgroup of G and V is a finite dimensional representation of H such that $V\uparrow G$ is irreducible, then $V\uparrow G$ is equivalent to a representation obtained as in 1).

3) If V_1, V_2 are finite dimensional irreducible representations of complete subgroups H_1, H_2 respectively satisfying the condition in 1), then $V_1\uparrow G$ is equivalent to $V_2\uparrow G$ if and only if there is $g \in G$ such that $H_1^g = H_2$ and V_1^g is equivalent to V_2.

Proof: 1) First we show that if $G \supset K \supset H$, K complete, and $V\uparrow K$ is irreducible, then $V\uparrow N_G(K)$ is irreducible. Suppose not. Then

$$1 < i(V\uparrow N_G(K),\ V\uparrow N_G(K)) \leq \sum_{d \in D} i(V\uparrow K^d,\ V\uparrow K) \quad \text{where}$$

$d \in D$ run over representatives of K in $N_G(K)$. Thus there must be $d \in N_G(K) - K$ such that $V\uparrow K^d$ is equivalent to $V\uparrow G$. But one can show that $(V\uparrow K)^d$ is equivalent to $V^d\uparrow K$ where V^d is now defined on H^d. But, from theorem 1.3 we know that

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$$i(V\uparrow K,\ V^d\uparrow K) = \sum_{x\in D_f} i(V^x \mid H^x \cap H^d,\ V^d \mid H^x \cap H^d)$$

where D_f is the set of $H:H^d$ double coset representatives in K such that $[H^x:H^x\cap H^d] < \infty$ and $[H^d:H^x\cap H^d] < \infty$. But H^x and H^d are complete, which means that $x \in D_f$ if and only if $H^x = H^d$. Then $V\uparrow K$ is equivalent to $V^d\uparrow K$ if and only if $H^x = H^d$ and $V^x = V^d$ - i.e. if and only if $H^{xd^{-1}} = H$ and $V^{xd^{-1}} = V$. But this means that $xd^{-1} \in N_G(H)$ and, by the hypothesis, that $xd^{-1} \in H$. Since $x \in K$ this means that $d \in K$ which is a contradiction. Hence $V\uparrow N_G(K)$ is irreducible.

By applying this to $N_G(H)$, then its normalizer, ... etc., since G is nilpotent, in a finite number of steps we conclude that $V\uparrow G$ is irreducible.

2) Since $V\uparrow G$ is irreducible, $V\uparrow H^*$ is. Let $V\uparrow H^* = U$. Then $V\uparrow G = V\uparrow H^*\uparrow G$ is one of the representations in 1) if there is no $g \in N_G(H^*) - H^*$ such that U^g is equivalent to U. Suppose there is one. Then

$$i(U\uparrow N_G(H^*),\ U\uparrow N_G(H^*)) = \sum_{d\in N_G(H^*)/H^*} i(U^d,\ U) \geq 2$$

since U^e and U^g are equivalent to U. Then $U\uparrow N_G(H^*)$ is not irreducible - contradicting the irreducibility of $V\uparrow G$. Hence if $V\uparrow G$ is irreducible it is equivalent to one of the representations in 1).

3) $$i(V_1\uparrow G,\ V_2\uparrow G) = \sum_{d\in D_f} i(V_1{}^d \mid H_1{}^d \cap H_2,\ V_2 \mid H_1{}^d \cap H_2)$$

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and $d \in D_f$ if and only if $H_1^d = H_2$ (since they are complete) and the intertwining number is non-zero if and only if V_1^d is equivalent to V_2.

Now we know which finite dimensional representations induce irreducible representations. We should note that a well known theorem for irreducible representations of finite nilpotent groups carries over to finite dimensional irreducible representations of any nilpotent group.

Theorem 1.6: Let V be a finite dimensional irreducible representation of the nilpotent group G. Then V is induced from a one dimensional representation of some subgroup of G.

Proof: Consider V as a representation of $G_o = G/\text{Ker } V$. If V is not one dimensional, G_o is not abelian. Since G_o is nilpotent there is an abelian normal non-central subgroup $A \subset G_o$. Then V/A is a sum of characters. Call one of these χ. Let $F = \{g \in G: \chi^g = \chi\}$. If $g \in G-F$, g carries $\mathcal{H}(\chi)$, the subspace of $\mathcal{H}(V)$ on which A acts according to χ, onto $\mathcal{H}(\chi^g)$. Hence $[G_o:F] < \infty$. Since F preserves $\mathcal{H}(\chi)$ we can let the representation of F on $\mathcal{H}(\chi)$ be U. Now note that $F \neq G_o$ since V is faithful and A is not central - hence not a multiple of a single character. Hence if we can prove that $V \simeq U\uparrow G_o$, we are done by induction on the dimension of $\mathcal{H}(V)$.

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Let $f_i^v \in \mathcal{H}(U\uparrow G)$ be defined by $f_i^v(g_k) = \delta_{ik}v$ where $v \in \mathcal{H}(\gamma)$, the g_k are representatives for G_o/F and δ_{ik} is the Kroneker δ. Then the f_i^v span $\mathcal{H}(U\uparrow G)$. Let $Wf_i^v = V(g_i^{-1})v$. Then $W: \mathcal{H}(U\uparrow G) \longrightarrow \mathcal{H}(V)$, one can check that W commutes with the action of G_o and is unitary, using the fact that $V(g_i)\mathcal{H}(\gamma) \perp \mathcal{H}(\gamma)$ for $g_i \neq e$. Thus we conclude that V is induced as a representation of G_o, from a faithful representation of some subgroup of G_o, hence, by induction, from a character γ_o on some subgroup H of G_o. Then, as a representation of G, V is induced from γ_o' on $\pi^{-1}(H)$ where $\pi: G \longrightarrow G_o$ and $\gamma_o'(g) = \gamma_o(\pi(g))$.

We can then put 1.5 and 1.6 together to get the complete result.

Theorem 1.7: Let G be a torsion free finitely generated nilpotent group. Then

1) If H is a subgroup of G, γ a character (one dimensional representation) on H, and there is no $g \in N_G(H^*) - H$ such that $\gamma^g = \gamma$ on $H^g \cap H$, then $\gamma\uparrow G$ is irreducible.

2) If H is a subgroup of G, V a finite dimensional representation of H such that $V\uparrow G$ is irreducible, then $V\uparrow G$ is equivalent to a representation obtained as in 1).

3) If γ_1, γ_2 are characters on subgroups H_1, H_2 respectively satisfying the condition in 1), then $\gamma_1\uparrow G$ is equivalent to $\gamma_2\uparrow G$ if and only if there is a $g \in G$ such that

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$H_1^{*g} = H_2^*$ and $\chi_1^g = \chi_2$ on $H_1^g \cap H_2$.

Proof: 1) Let $\chi \uparrow H^* = V$. Then, by 1.5, it suffices to show that V^g is not equivalent to V for all g in $N_G(H^*) - H^*$. By 1.3

$$i(V^g, V) = \sum_{d \in D_f} i(\chi^{gd} | H^{gd} \cap H, \chi | H^{gd} \cap H) \quad \text{and} \quad d \in D_f$$

when $[H^{gd} : H^{gd} \cap H] < \infty$ and $[H : H^{gd} \cap H] < \infty$. But $H^{gd}/H^{gd} \cap H$ is finite $\iff$ $H^{*gd}/H^{gd} \cap H$ is finite

$\iff$ $H^{*gd}/H^{*gd} \cap H^*$ is finite

$\iff$ $gd \in N_G(H^*)$.

Similarly for the second index. This proves 1).

2) is directly from 2) of 1.5 and theorem 1.6.

3) $$i(\chi_1 \uparrow G, \chi_2 \uparrow G) = \sum_{d \in D_f} i(\chi_1^d | H_1^d \cap H_2, \chi_2 | H_1^d \cap H_2)$$

$\neq 0$ if and only if there is a $d \in G$ such that $[H_1^d : H_1^d \cap H_2]$ and $[H_2 : H_1^d \cap H_2]$ are both finite and $\chi_1^d = \chi_2$ on $H_1^d \cap H_2$. But $H_1^d/H_1^d \cap H_2$ is finite $\iff$ $H_1^{d*}/H_1^d \cap H_2$ is finite

$\iff$ $H_1^{*d}/(H_1^d \cap H_2)^*$ is finite

$\iff$ $H_1^{*d}/H_1^{*d} \cap H_2^*$ is finite

$\iff$ $H_1^{*d} = H_2$

and similarly for $H_2/H_1^d \cap H_2$ finite.

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Chapter II

In this chapter we will establish a criterion for determining whether an arbitrary irreducible representation of a finitely generated nilpotent group is induced from a one dimensional representation of some subgroup. By 1.6, of course, this criterion identifies irreducible representations induced from any finite dimensional representation.

We must begin with a couple of lemmas about representations and their subgroups. The first says that if the restriction of a representation preserves a one dimensional subspace, then the character defined thereby induces a representation not disjoint from the original representation. More precisely, we have

Lemma 2.1: Let ρ be a representation of G, $v \in \mathcal{H}(\rho)$ such that $\rho(H)v \subset \mathbb{C}v$ where H is some subgroup, and $\rho(g)v \perp v$ for all $g \in G - H$. Then if $\chi(h)v = \rho(h)v$, $i(\chi\uparrow G, \rho) > 0$.

Proof: We merely have to construct one non-zero intertwining operator.

Let $U = \chi\uparrow G$, g_i be right coset representatives of H and f_i^v be in $\mathcal{H}(U)$ defined by $f_i^v(g_j) = v \quad i = j$ and $f_i^v(g_j) = 0 \quad i \neq j$

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Then $U(g)f_i^v = f_k^w$ where $w = \chi(g_k g g_i^{-1})v$ and g_k is the representative of Hg_ig^{-1}. To see this,

$$U(g)f_i^v(hg_j) = f_i^v(hg_jg)$$
$$= \chi(h)f_i^v(g_jg)$$
$$= \chi(h)f_i^v(g_jgg_l^{-1}g_l) \quad \text{where} \quad g_l$$

represents Hg_jg

$$= \begin{cases} 0 & l \neq i \\ \chi(h)\ \chi(g_k g g_i^{-1})v & l = i \end{cases} \quad \text{since}$$

$l = i$ if and only if $g_jg \in Hg_i \iff g_j \in Hg_ig^{-1}$

$$= \chi(h)f_k^w(g_j) \quad \text{as claimed.} \quad \text{Now define}$$

$W: \mathcal{H}(U) \longrightarrow \mathcal{H}(\rho)$ by

$$W(f_i^v) = \rho(g_i^{-1})v$$

Then $WU(g)f_i^v = Wf_k^w$

$$= \rho(g_k^{-1})\ \rho\ (g_k g g_i^{-1})v$$
$$= \rho(g)\ \rho\ (g_i^{-1})v$$
$$= \rho(g)Wf_i^v$$

Define W on all of $\mathcal{H}(U)$ by linearity. Then if W is bounded it is an intertwining operator. In fact, W is an isometry. First note that for $f \in \mathcal{H}(U)$,

$$f = \Sigma\ f_i^{f(g_i)}$$

Thus $(Wf, Wf_o) = (\Sigma\ \rho(g_i^{-1})f(g_i),\ \Sigma\ \rho(g_j^{-1})f_o(g_j))$

$$= \sum_i\ (\rho(g_i^{-1})f(g_i),\ \rho(g_i^{-1})f_o(g_i))$$

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by the perpendicularity hypothesis of the lemma,

$$= \sum_i (f(g_i), f_o(g_i))$$

$$= (f, f_o) \cdot$$

The other lemma is concerned with the multiplicity of characters in the restriction of a representation.

Lemma 2.2: If ρ is a representation of a nilpotent group H and χ is a one dimensional subrepresentation of ρ, and H_o is a normal subgroup of H such that $[H:H_o]$ is finite with the property that $\chi|H_o$ occurs an infinite number of times in $\rho|H_o$ (i.e. $i(\chi|H_o, \rho|H_o)$ is infinite), then χ occurs an infinite number of times in ρ.

Proof: Choose an integer k such that $H^k \subset H_o$, $H^{k-1} \not\subset H_o$ ($H^i = [H, H^{i-1}]$). Choose $h \in H^{k-1} - H_o$. Then the group H_1 generated by h and H_o is normal in H. Also, for some positive integer n, $h^n \in H_o$. Let $v_i \in \mathcal{H}(\rho)$ be an infinite independent set on which H_o acts as $\chi|H_o$. Then the subspace spanned by the $\{h^i v_r : i = 1 \ldots n\}$ contains a vector on which H_1 acts as $\chi|H_1$. Namely, if $a = \chi(h)$ and

$$w = \sum_{i=0}^{n-1} a^{n-i} \rho(h^i) v_r \quad , \text{ then}$$

$h^p w = \chi(h)^p w$ for all integers p. Since there is an infinite number of independent v_r, there are an infinite number of independent such w - hence $\chi|H_1$ is of infinite multiplicity in $\rho|H_1$. Continuing this construction a finite number of times, we reach the conclusion of the lemma.

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These two lemmas will be very useful in what follows. We now come to our criterion.

Definition 2.3: We will say a representation of a group G has the finite multiplicity property if there is some $g \in G$, $v \in \mathcal{H}(\rho)$ such that $\rho(g)v \in \mathbb{C}v$, and if $H = [g \in G: \rho(g)v \in \mathbb{C}v]$ and $\rho(h)v = \chi(h)v$ for all $h \in H$, then χ only occurs a finite number of times in $\rho|H$ - i.e. $i(\chi, \rho|H)$ is finite.

With this we can state the main theorem of this chapter.

Theorem 2.4: If G is a finitely generated nilpotent group, an irreducible representation ρ of G has the finite multiplicity property if and only if ρ is induced from a one dimensional representation of some subgroup of G.

Proof: Claim: There is a subgroup $H \subset G$ and a vector v in $\mathcal{H}(\rho)$ such that $\rho(h)v = \chi(h)v$ for all $h \in H$ ($\chi(h) \in \mathbb{C}$) with the additional property that $\chi \uparrow G$ is irreducible.

If this is so, $|N_G(H^*)$ is irreducible. But this means, by theorem 0.8, that $\chi^g \neq \chi$ on $H^g \cap H$ for any $g \in N_G(H^*) - H$. Choose $h \in H^g \cap H$ such that $\chi^g(h) \neq \chi(h)$. Then

$$\begin{aligned}(\rho(g)v, v) &= (\rho(hg)v, \rho(h)v) \\ &= (\rho(g)\chi^g(h)v, \chi(h)v) \\ &= \chi^g(h)\,\overline{\chi(h)}\,(\rho(g)v, v)\end{aligned}$$

Since $\chi^g(h) \neq \chi(h)$, then $(\rho(g)v, v) = 0$.

Let $K = N_G(H^*)$. Then $\chi \uparrow G$ is an irreducible subrepresentation

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of the representation of K given by the closed linear span of $\rho(K)v$, by lemma 2.1.

Let $N_G(K) = K_1$. Let $\chi\uparrow K = U$ and choose $g \in K_1 - K$. Then by theorem 0.8, U^g is not equivalent to U since $U\uparrow G$ is irreducible.

Let $\mathcal{H}_o$ be a subspace of $\mathcal{H}(\rho)$ on which K acts according to U. Then $g\mathcal{H}_o \perp \mathcal{H}_o$ for $g \in K_1 - K$, since, if not, the projection of $g\mathcal{H}_o$ to $\mathcal{H}_o$ is an intertwining operator for U^g and U.

By the form of U there is a vector $v_o \in \mathcal{H}_o$ such that $\rho(k)v_o \perp v_o$ for all $k \in K - H$ and $\rho(h)v_o = \chi(h)v_o$ for all $h \in H$. Thus $\rho(g)v_o \perp v_o$ for all $g \in K_1 - H$. Hence, by lemma 2.1, $\chi\uparrow K_1$ is a subrepresentation of the representation of K_1 on the closed linear span of $\rho(K_1)v_o$.

Since G is nilpotent, a finite number of applications of this argument shows that $\chi\uparrow G$ is a subrepresentation of ρ. But ρ is irreducible. Thus $\chi\uparrow G$ is equivalent to ρ.

Thus the "only if" part of the theorem is proven if we can establish the claim.

By theorem 0.8, we would be done if $\chi^g \neq \chi$ on $H^g \cap H$ for all $g \in N_G(H^*)$, where H, χ are as in definition 2.3. We will produce a new triple H', χ', v' with this property.

The first step is to satisfy this requirement for $g \in N_G(H^*) - H^*$. Let $H_o = \bigcap_{g \in N_G(H^*)} g^{-1}Hg$. Thus $[H:H_o]$ is finite. Let $\chi_o = \chi|H_o$ and $L = \{g \in N_G(H^*): \chi_o{}^g = \chi_o\}$.

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Then, using lemma 2.2, the dimension of the space spanned by Lv is finite, since $\rho|H_o$ acts as χ_o thereon.

Then $\rho(L)$ on this subspace is a nilpotent unitary subgroup of $GL(n,\mathbb{C})$ for some n. Hence the connected component of its algebraic closure is diagonalizable. Thus L has a subgroup L_o such that $[L:L_o]$ is finite and there is a v_o in the span of $\rho(L)v$ such that $tv_o \in \mathbb{C}v_o$ for all $t \in L_o$. Now if there is a $g \in N_G(H^*) - H^*$ such that $\chi^g = \chi$ on $H^g \cap H$, then $\chi^g = \chi$ on H_o and $g \in L$. Then rank $L >$ rank H (rank = minimal number of generators) and rank $L_o >$ rank H.

Since G is finitely generated, we can begin with an H with the required properties and maximum rank, and the above argument shows that there is no $g \in N_G(H^*) - H^*$ such that $\chi^g = \chi$ on $H^g \cap H$. In fact, our argument is stronger - there is no $g \in N_G(H^*) - H^*$ such that $\chi^g = \chi$ on H_o.

Now let $L = \{g \in H^*: \chi^g|H_o = \chi|H_o\}$. Let $L^i = [L,L^{i-1}]$, $L^m \subset H_o$ and $L^{m-1} \not\subset H_o$. Choose $g \in L^{m-1} - H_o$ and let H_1 be the group generated by g and H_o. Then H_1 is normal in L since H_o is. Since $L \subset H^*$, $g^k \in H_o$ for some positive integer k. Let a be a k'th root of $\chi(g^k)$ and define w as in the proof of lemma 2.2. Then $\rho(g)w = aw$ and $\rho(h)w = \chi(h)w$ for all $h \in H_o$. Thus χ extends to a character χ_1 of H_1. Let $L_1 = \{t \in L: \chi_1{}^t = \chi_1\}$ and continue this process. Since $[H^*:H_o]$ is finite, this terminates with $L_k = H_k$, say. Then for $g \in H^* - L_k$, $\chi^g \neq \chi$ on H_o if $g \in H^* - L$, $\chi_1{}^g \neq \chi_1$ on $H_1 \supset H_o$ if $g \in L - L_1$, ..., $\chi_k{}^g \neq \chi_k$ on $H_k \supset H_{k-1}$ if

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$g \in L_{k-1} - L_k$. Thus $\chi_k{}^g \neq \chi_k$ on H_k for any $g \in H^* - L_k = H^* - H_k$. From the above paragraph, $\chi_k{}^g \neq \chi_k$ on $H_o \subset H_k$ for any $g \in N_G(H^*) - H^*$. Thus $\chi_k{}^g \neq \chi_k$ on $H_k{}^g \cap H_k$ for any $g \in N_G(H^*) - H_k$. Since $H^* = H_k^*$ this is precisely the subgroup - representation pair we were seeking. Hence we have the claim, and thus the "only if" part of the theorem.

Suppose $f \in \mathcal{H}(\chi\uparrow G)$ and $hf = \chi(h)f$. As in the proof of 2.1, let g_i be the right coset representatives of H. Let $f_i(g_j) = 1 \quad i = j$

$= 0 \quad i \neq j$ and define f_i elsewhere so that it is in $\mathcal{H}(\chi\uparrow G)$. Then $f = \Sigma f(g_i)f_i$ and $hf_i = \chi(g_k h g_i^{-1})f_k$ where g_k is the representative of Hg_ih^{-1}. Then

$$(1) \qquad hf = \Sigma f(g_i)\,\chi(g_k h g_i^{-1})f_k$$

$$= \Sigma\ \chi(h)f(g_s)f_s$$

The f_i are an orthonormal basis of $\mathcal{H}(\chi\uparrow G)$. Thus $|f(g_i)| = |f(g_k)|$ for all $h \in H$. Thus for f to be in $\mathcal{H}(\chi\uparrow G)$, Hg_i must have a finite orbit in the cosets where $f(g_i) \neq 0$ for f to have a finite norm. Thus there is a finite integer n such that $g_ih^n \in Hg_i$. i.e. $g_ih^ng_i^{-1} \in H$. Thus $g_ihg_i^{-1} \in H^*$ for each $h \in H$. Thus $g_iHg_i^{-1} \subset H^*$. Taking * on both sides, $g_iH^*g_i^{-1} = H^*$, hence $g_i \in N_G(H^*)$.

But $\chi\uparrow G$ is irreducible if and only if $i(\ \ |G,\ |G) = 1$ and by theorem 0.8, this is if and only if

$$\sum_{\substack{d\in D_f \\ d\neq e}} i(\chi^d|H^d\cap H,\ \chi|H^d\cap H) = 0$$

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Now $d \in D_f$ if and only if $d \in N_G(H^*)$. Hence for $d \in N_G(H^*)$ there is an $h \in H^d \cap H$ such that $\chi^d(h) \neq \chi(h)$. Now

$$\begin{aligned} hf(d) &= f(dh) = f(dhd^{-1}d) \\ &= \chi^d(h)f(d) \quad \text{for } h \in H^d \cap H \\ &\neq \chi(h)f(d) \quad \text{unless } f(d) = 0 \end{aligned}$$

Hence f can only be non-zero on one right H coset - H itself. There it is determined once its value at e is. Hence χ has multiplicity 1 in $\chi \uparrow G|H$.

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Chapter III

The natural question to ask at this point is whether every irreducible representation of a finitely generated nilpotent group is equivalent to one induced from a character on some subgroup. What we will show in this chapter is that this is not so. We will produce an example of a representation which is not equivalent to any induced from a character.

Our example will be a representation of the group of 3×3 matrices of the form

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{where } x, y, z \text{ are integers}$$

We will denote this matrix by (x,y,z) so that

$$(x,y,z)(x',y',z') = (x+x',\ y+y',\ z+z'+xy')$$

First we describe a general form for a representation of G. A particular example of this will turn out to be the required counter-example. For f in $L^2(S_1, \lambda)$ where S_1 is the circle and λ is Lebesgue measure, let

$$U_s(x,y,z)f(t) = e^{i(ty+bz)}s(x,t)f(t+xb) \quad \text{where } b \text{ is}$$

not in $\mathbb{Q}\pi$ ($\mathbb{Q}$ is the rational numbers). This is a representation for any measurable $s: Z \times S_1 \longrightarrow S_1$, where the image circle is the unit circle of the complex numbers, if s satisfies the property

((1)) $\qquad s(x+x',\ t) = s(x,t)s(x',\ t+xb)$

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Note that, in fact, $s(1, t)$ can be an arbitrary measurable function, and then $s(x,t)$ is completely defined by ((1)).

Let us now find out which U_s are equivalent, and at the same time show that they are irreducible.

Lemma 3.1: The U_s are irreducible and U_{s_1} is equivalent to U_{s_2} if and only if there is measurable $f:S_1 \longrightarrow S_1$ such that

$$s_1(1,t) = f(t)\overline{f(t + b)}s_2(1,t)$$

Proof: Suppose $V:L^2(S_1,\lambda) \longrightarrow L^2(S_1,\lambda)$ is unitary and $U_{s_1}V = VU_{s_2}$ - i.e. $U_{s_1}(x,y,z)V = VU_{s_2}(x,y,z)$ for all integers x,y,z. Then

$$VU_{s_2}(0,y,0)f = U_{s_1}(0,y,0)Vf$$

Thus V commutes with multiplication by e^{iyt} for any integer y. Thus, by a known theorem of functional analysis (cf. Naimark, (4), p. 499, I) $Vf(t) = v(t)f(t)$ for some measurable $v:S_1 \longrightarrow S_1$. Then $VU_{s_2}(x,0,0) = U_{s_1}(x,0,0)V$ implies $v(t)s_2(x,t)f(t+xb) = s_1(x,t)v(t+xb)f(t+xb)$ for any f in $L^2(S_1,\lambda)$ for all $t \in S_1$ and $x \in Z$. Thus $v(t)s_2(x,t) = s_1(x,t)v(t+xb)$ is necessary for equivalence, and by the decomposition of V above it can easily be shown to be sufficient. It is also easy to see that if $s(x,t)$ is constructed from $s(1,t)$ by ((1)), then this equation is equivalent to

$v(t)s_2(1,t) = s_1(1,t)v(t+b)$, as required. If $s_1 = s_2$, then this means that $v(t) = v(t+b)$ - i.e. v is invariant

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under translation by b. But this translation is an ergodic action on the circle. Hence, by a standard theorem of ergodic theory, v is constant a.e. with respect to λ. Thus V is in fact a scalar multiple of the identity operator, which shows that U_s is irreducible.

Now let us look at a character χ and subgroup H. U_s is infinite dimensional. Hence if $\chi\uparrow G$ is equivalent to U_s, $[G:H]$ is infinite. Further, by theorem 0.8,

$$i(\chi\uparrow G, \chi\uparrow G) = \sum_{d\in D_f} i(\chi^d | H^d\cap H, \chi | H^d\cap H)$$

If $d = (0,0,z)$, d is central so that $H^d = H$ and $\chi^d = \chi$. Thus if $i(\chi\uparrow G, \chi\uparrow G) = 1$, there can be only one $H:H$ double coset representative of the form $d = (0,0,z)$. Hence $(0,0,Z) \subset H$. From this and the fact that $[G:H]$ is infinite, we can see that $H = \{(nk,np,z):n,z \text{ in } Z\}$. Call this subgroup H_{kp}. If χ is a character on H, since $(0,0,Z)$ is central, $\chi((0,0,z)) = e^{iwz}$ for some w for all $z \in Z$. If I is the identity operator, then $\chi\uparrow G((0,0,z)) = e^{iwz}I$. But $U_s((0,0,z)) = e^{ibz}$. Hence $w = b$ if $\chi\uparrow G \simeq U_s$.

Lemma 3.2: If U_s is equivalent to $\chi\uparrow G$ for χ a character on H_{kp}, then $k = 1$.

Proof: Let $K = H_{01}$. Parameterize the elements of H_{kp} by n and z. Let $[n,z] = (nk,np,z)$. Then $[n,z][n',z'] = (n+n', z+z' + nn'kp)$. Then one can check that any character on H_{kp} is of the form χ_{ab} where

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$$\chi_{ab}(nk,np,z) = \exp i(na-bn(n-1)kp/2+bz)$$

First let us examine $\chi_{ab}\uparrow G|K$ where χ_{ab} is a character on H_{kp} and $k \neq 0$. Then the H_{kp}:K double coset representatives may be chosen $\{(x,0,0):0 \leq x < k\}$. One can check that if $d = (x,0,0)$, $\chi_{ab}{}^{d} = \chi_{a+xpb,b}$. Then by Mackey's theorem (Mackey, (2), p. 135, Theorem 3.5)

$$\chi_{ab}\uparrow G|K = \sum_{x=0}^{k-1} [\chi_{ab}{}^{d}|(0,0,Z)]\uparrow K$$

$$= \sum_{x=0}^{k-1} [\chi_{0b}|(0,0,Z)]\uparrow K$$

= k copies of the regular representation of $(0,Z,0)$ direct product with ψ_b where $\psi_b(0,0,z) = e^{ibz}$. If $k \neq 1$, this has a commuting algebra which is not abelian. A projection onto one of the copies does not commute with the isomorphism between the copies. On the other hand,

$U_s(0,y,z)f(t) = e^{i(yt+bz)}f(t)$, which is precisely the regular representation of $(0,Z,0)$ direct product with ψ_b. Using the theorem in Naimark ((4), p. 499, I), it is easy to see that this representation has an abelian commuting algebra. Hence to prove the result it remains to dispose of the case where $k = 0$. Then $\chi_{ab}\uparrow G|K = \sum_{x=-\infty}^{\infty} \chi_{ab}{}^{d}$ by Mackey's theorem, where $d = (x,0,0)$ if $p = 1$. For $k = 0$ and $p \neq 1$ or -1 it is not hard to show, using theorem 0.8, that $\chi_{ab}\uparrow G$ is not irreducible.

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Now if U_s and $\chi_{ab}\uparrow G$ were equivalent for χ_{ab} defined on H_{01}, then $U_s|K$ and $\chi_{ab}\uparrow G|K$ would be equivalent, and by a theorem of Mackey (Mackey, (2), p. 103) Lebesgue measure would be measure isomorphic to the measure which weights each point of the orbit of a as 1. But this is a contradiction, since Lebesgue measure would then be atomic. This concludes the lemma.

Now some U_s are equivalent to $\chi\uparrow G$ for some χ. In fact we have the following lemma.

Lemma 3.3: If χ_{ab} is a character on H_{1p}, $\chi_{ab}\uparrow G$ is equivalent to U_s where $s(1,t) = e^{i(a-tp)}$

Proof: For $f \in \mathcal{H}(\chi_{ab}\uparrow G)$ let $f'(m) = f(0,m,0)$. Then $f \longrightarrow f'$ is an isomorphism $\mathcal{H}(\chi_{ab}\uparrow G) \longrightarrow \mathcal{L}_2$ and $\chi_{ab}\uparrow G(x,y,z)f((0,m,0)$

$$= f(x,m+y,z)$$
$$= f((x,xp,z-xm-xy+x^2p)(0,m+y-xp,0))$$
$$= \chi_{ab}(x,xp,z-xm-xy+x^2p)f'(m+y-xp)$$

Taking Fourier transform,

$$\chi_{ab}\uparrow G(x,y,z)\hat{f'}(t)$$
$$= \sum_m e^{-itm}\ \chi_{ab}\uparrow G(x,y,z)f'(m)$$
$$= \sum_m e^{-itm}\chi_{ab}(x,xp,z-xm-xy+x^2p)f'(m+y-xp)$$
$$= \sum_m \exp i[t(m-y+xp) - bx(m-y+xp)+xa-bx(x-1)p/2 + b(z-xy+x^2p)]\ f'(m)$$

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$$= e^{i(ty+bz)} \exp i[xa-bx(x-1)p/2-txp] \quad f'(t+xb)$$

Thus comparing to the form of U_s ,

$$s(1,t) = e^{i(a-tp)} .$$

Corollary : U_s is equivalent to a representation induced from a character on some subgroup if and only if there is some real a and integer p and measurable $f: S_1 \longrightarrow S_1$ such that

$$s(1,t) = e^{i(a-pt)} f(t)\overline{f(t+b)}$$

What we are going to show is that we can choose $s(1,t)$ so that this equation is impossible.

To do this, we need

Lemma 3.4: If χ_{ab} is a character on H_{10} and χ_{uv} a character on H_{1p} for $p \neq 0$, then $\chi_{ab}\uparrow G$ is not equivalent to $\chi_{uv}\uparrow G$ for any choice of a, b, u, v.

Proof: $i(\chi_{ab}\uparrow G, \chi_{uv}\uparrow G)$

$$= \sum_{d \in D_f} i(\chi_{ab}{}^{d} | H_{10}{}^{d} \cap H_{1p} , \chi_{uv} | H_{10}{}^{d} \cap H_{1p})$$

But H_{10} is normal and $H_{10} \cap H_{1p} = (0,0,Z)$. Hence for no d is $[H_{10}{}^{d} : H_{10}{}^{d} \cap H_{1p}]$ finite. Thus D_f is empty and the representations are not equivalent.

Let us note what this says according to the equivalence formula for the U_s. Substituting $s_1(1,t) = e^{ia}$ and $s_2(i,t) = e^{i(u-pt)}$ we find that

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$e^{ia} = e^{i(u-pt)} f(t)\overline{f(t+b)}$ is false for any measurable $f: S_1 \longrightarrow S_1$, $p \neq 0$, and any a, u.

But recall that if U_s is induced from χ there must be an $f \in \mathcal{H}(U_s)$ such that $U_s(h)f = \chi(h)f$ for all $h \in H$, the subgroup on which χ is defined. Thus there must be an $f \in L^2(S_1, \lambda)$ such that $U_s(n,np,z)f(t)$

$$= \exp i[na-bn(n-1)p/2+bz]\, f(t).$$

i.e.

$$e^{i(ntp+bz)} s(n,t) f(t+nb) = \exp i[na-bn(n-1)p/2+bz]\, f(t).$$

Taking absolute values, this means $|f|$ is constant on orbits under translation by b. Hence $|f|$ is constant e.a. by ergodicity. Hence we can assume $f: S_1 \longrightarrow S_1$. Then, for $n = 1$, this says

$$s(1,t) = e^{ia} f(t)\overline{f(t+b)}\ e^{-itp}.$$

U_s is an irreducible representation if $s(1,t) = e^{it/2}$

Then $e^{it/2} = e^{ia}\, e^{-itp}\, f(t)\, \overline{f(t+b)}$

Then $e^{it(1+2p)} = e^{i2a}\, f(t)^2\, \overline{f(t+b)}^2$

Since $1+2p \neq 0$ for any p, this is a contradiction. Hence this U_s is not induced from a character.

It is interesting to note that any irreducible representation of G can be written as a U_s, as follows. Let μ be any

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measure on S_1 ergodic and non-singular under translation by b. Let $L_2(S_1, \mu, H_k)$ be measurable functions on S_1 with images in real k-dimensional space with the property that $||f||$ is in $L_2(S_1, \mu)$ for some norm on $\mathbb{R}^k$. Then we can write any irreducible representation as a U_s where $s(x,t)$ is now an unitary operator of dimension k. The general equivalence formula for these is exactly as for the one dimensional case, where, here, $v(t)$ are k-dimensional unitary operators.

One can show that if γ_{ab} is induced from H_{kp}, $k \neq 0$, the result is a U_s for dimension k and μ Lebesgue. If γ_{ab} is induced from $(0,Z,Z)$ the result is a U_s for dimension 1 with μ a measure which weights each point of the b orbit of a as 1. If b is in $Q\pi$, γ_{ab} is induced from a subgroup of finite rank, and then the measure occurring weights each point of the (finite) b orbit of a as 1.

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