THE RESPONSE OF CONTINUOUS SYSTEMS TO RANDOM NOISE FIELDS

by Richard H. Lyon

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ABSTRACT

In this work we are concerned with the excitation of continuous systems by random noise fields. The excitation is interpreted as the average or mean square displacement, velocity, etc., and is obtained from a knowledge of certain statistical properties of the source.

The thesis begins with an historical account of the developing use of the equation of motion with random source terms, the so-called Langevin equation. This equation is then used to calculate the correlation functions for the system response when correlations of the same order of the source are specified.

The formalism is then applied to the finite string for both stationary and moving noise fields. It is found that correlation lengths due to the source and viscosity of the surrounding medium strongly affect the excitation for various wavelengths. As an experimental test a thin metal ribbon is placed in a flowing turbulent field, and its excitation for various values of flow is examined. A qualitative agreement with the predicted results is obtained.

The analysis is then applied to infinite strings where the interpretation is uncomplicated by the effect of boundaries. Substantially similar results as for the finite string are obtained. If one observes

the excitation of the string as a function of wavelength, the effect of correlation lengths in the noise field is very striking.

The thesis is concluded by an attempt to create the random noise fields assumed previously by a superposition of elementary sources. This superposition creates an ensemble of source functions which may then be averaged over to obtain the source correlation. An attempt to create a representation of the turbulent field by this method completes the work.

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Title: Assistant Professor of Physics

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I HISTORICAL BACKGROUND

1.1 INTRODUCTION

In this chapter we shall trace the development of some of the ideas concerned with the excitation of physical systems by sources having random or unpredictable time dependence. Such an examination leads to the study of equations of motion having random source functions and of schemes for drawing physical information from such equations.

Historically this kind of analysis came about from the interest in Brownian motion, although if there had been high gain amplifiers in 1900, resistor noise and its excitation of electronic circuits could just as easily have supplied the physical motivation for the study.

1.2 BROWNIAN MOTION

In 1827 Robert Brown noticed that small particles immersed in fluid were subject to a strange erratic motion, which he interpreted as resulting from molecular impacts with the surrounding medium. The irregular motion of a particle was treated at first very naturally by the method of random flights which it visually resembles. In 1880 Rayleigh[1] showed that for a large number of steps the random walk problem becomes equivalent to the solution of a partial differential equation of the parabolic type, i.e., a diffusion equation. In 1905 Einstein[2] used such an equation to solve the distribution of free Brownian particles when they are started from rest at a specified time and position. This distribution is effec-

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[1]Reference numbers refer to the Bibliography.

tively the probability of finding the particle in some small region after a time t. By requiring that the particle have a Maxwellian velocity distribution after an infinite lapse of time, in accordance with the equipartition requirement, he was able to get an expression for the mean square displacement at time t of a Brownian particle. Einstein's result was

$$\langle x^2 \rangle_{ave} = \frac{2RT}{Nf} t ,$$

where t is the elapsed time, R the gas constant, T the temperature, f the friction coefficient for a small sphere, and N Avagadro's number. This equation allowed for the possibility of determining Avagadro's number -- which Perrin did and for which he was awarded the Nobel Prize in 1926.

In 1906 Marjan von Smoluchowski,[3] a Polish physicist, extended the work of Einstein to a more complex diffusion equation to take in the effect of gravity and other external fields. His results on the mean square displacement differed from Einstein's by a factor of $\frac{64}{27}$, and the new value was supported by measurements done by The Svedberg shortly therafter. Thus the random flight attack on the problem came to an impasse.

The first attack upon the dynamics of the particle was made by P. Langevin[4] in 1908. Langevin wrote the equation of motion for the free mass point as

$$m \frac{d^2x}{dt^2} = -f \frac{dx}{dt} + X(t) ,$$

where the term on the left is the acceleration reaction, and the terms

on the right represent the forces due to the surrounding fluid. The term proportional to the first derivative of x is the force due to viscous drag as given by Stokes,[5] and it represents the average force presented to the particles by molecular impacts. The remaining term X(t) is a non-predictable, random function of time* and represents the fluctuating force resulting from the collisions. This equation, and all such equations of motion having random sources, have come to be known as Langevin equations. By assuming the equipartition of energy, Langevin was able to calcualte the mean square displacement of a particle after a time t and obtained Einstein's result above. In the paper referred to he questioned the accuracy of Svedberg's work and the applicability of the Stokes coefficient to small particles. Later experimental investigations have confirmed the results of Einstein and Langevin.

The next major step came in 1917 when L. S. Ornstein[6] used the Langevin equation to calculate the mean values of the displacement and velocity. Essentially what is done is to obtain formally the solution of the equation of motion by variation of parameters. Then one raises each side of the equation to the desired power and averages. This requires a knowledge of the correlation properties of the source function X(t), which are derived from assumptions concerning the statistics of that function. The integrals may then usually be carried out over the correlation functions and the mean values obtained.

This plan for obtaining mean values represents the philosophy which is used in Chapter II for the derivation of response correlation functions

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*The random function X(t) is sometimes called fortuitous. This is not a value judgment on its presence but is another term for what is also called a "chance variable."

when source correlations are known. However, we are getting ahead of the story.

In the paper above Ornstein[6] assumed an external field and obtained the diffusion equation developed by Smoluchowski[3] by considering the Langevin equation for the special case of a harmonically bound particle. This diffusion equation is known as the Fokker-Planck equation. Its coefficients are obtained by integrating the Langevin equation over short time intervals and is more generally applicable than the diffusion equations resulting from a random flight analogy. The way was then open for people to apply the work to generalized harmonic oscillators,[7] coupled electrical networks,[8] chains of coupled particles,[9] finite strings and bars, and finally to continuous systems representable by linear operators.

In 1927 Ornstein[10] calculated the Brownian motion of a finite string by breaking it up into modes and treating each mode as a harmonic oscillator. Using previous results for the single oscillator and assuming equipartition, he was able to obtain the mean square displacement at the midpoint of the string after equilibrium had been reached. The next year Houdjik[11] did the same for the finite bar. It is my impression that these represent the first applications of the Langevin equation to continuous systems.

In 1931 G. A. Van Lear, Jr., and G. E. Uhlenbeck[12] did a more general piece of work on the finite string and bar, extending the work of Ornstein and Houdjik to the non-stationary region of time. They calculated as a function of time the mean square displacement of the midpoint of the string starting from rest. Their work as it applies to the finite

string anticipates the theoretical formulation in this thesis. Nevertheless, their bias toward the problem of Brownian motion leads them to perform their ensemble averages over an ensemble of strings rather than sources. This distinction, if unimportant in its results, is important in terms of the ease of calculating the correlation functions. For example, in Chapter VI source correlations are calculated directly by averaging over an ensemble of source functions.

In this connection should be mentioned the work of G. A. Krutkow,[13] who likewise considered the Brownian motion of finite strings, from a more mathematical point of view. Krutkow has been easily the most outstanding Russian contributor to the theory of Brownian motion.

Brownian motion represents a rather special case of a random noise field, as we shall point out in Chapter III. It is possible to enter other noise fields into the work of Lear and Uhlenbeck, but in 1931 such problems as the excitation of elastic boundaries by turbulent flow were not so pressing as they are now.

1.3 THE STATISTICS OF TIME FUNCTIONS

In developments parallel to those in Brownian motion, people were becoming very much interested in the properties of functions like $X(t)$. The Fourier analysis of functions which do not die down in time and are without periodic components was performed by N. Wiener[14] in 1930. The correlation function which was used by Ornstein[6] turns out to be the cosine transform of the power spectrum in frequency of $X(t)$.

Several excellent papers have appeared in this area, particularly as the results apply to noise theory and stochastic processes in elec-

trical engineering. A standard in the field is that by S. O. Rice,[15] who has worked out the statistical properties of electrical currents for many applications. Another paper particularly useful for electrical engineering work is a report by Y. W. Lee.[16] Most of the work mentioned here assumes statistically stationary time processes (except for some examples in Rice), but some interest has been displayed in the extension of the concepts of power spectra and correlation functions to non-stationary time functions.[17,18]

In a series of interesting papers[19-22] Eckart has developed a theory of the propagation of correlation functions in continuous media. The idea is that correlation functions are found to obey a partial differential equation similar in form to the Langevin equation but of a higher dimensionality. This is then solved in terms of source convolutions. This formally appears very similar to the results of Chapter II. This approach, however, has two limitations which that of Chapter II does not have. The first is the complexity of solutions of the "Langevin equation" for correlation functions, which is usually much harder to solve than the equation of motion itself. The second is that the source function must have a finite convolution, which requires that the source noise field $f(\vec{r},t)$ have a Fourier integral transform. These are both rather severe limitations. The derivation in Chapter II does not possess either.

1.4 CONCLUSION

The work in this thesis is designed to be an extension of the theories which have been worked out so far. It is felt that there is required

a more generally applicable formulation of the propagation of correlation functions and mean values than has existed before. In addition, some practical methods for testing the results are needed which will be more in keeping with present-day interests, particularly in acoustics. Some of these contemporary problems are excitation by flowing turbulence, the noise generated by a region of cavitation, excitations of large auditoria by applause, and other such problems. The work which follows offers a formalism which enables one to attack many problems such as these and at the same time is intended to have a physical motivation for the proofs and examples.

II THE PROPAGATION OF MEAN VALUES

2.1 INTRODUCTION

One of the problems encountered when one starts to work with physical systems excited by sources having random properties is that the questions to be asked are not obvious. In the case of a coherent source, one might ask about the directivity, the propagated energy, absorption, phase and group velocities, diffraction patterns, etc. For incoherent sources, one expects energy to be radiated, absorption to occur, and so on, but certainly diffraction patterns will be blurred and phase velocities for such complex signals may be almost meaningless. In addition, there are properties of the signal which, although existent in the coherent problem, assume major significance for noise problems. In short, one must ask the right questions in order to have the analysis produce meaningful results.

2.2 THE MEANING OF NOISE

The term "noise" has a variety of connotations which extend from the sixty-cycle hum of a defective amplifier to the hissing signal of an untuned f-m receiver. The first falls under the category of unwanted signals -- signals which interfere with the transmission of information between a transmitter and a receiver. The second has this property also, but it has another characteristic which connects it more closely with the purpose of this thesis: namely, the property of randomness.

A. Randomness

By randomness let us mean that the signal varies in a non-predictable, and consequently a non-repeatable, manner as a function of one or more of its arguments. In acoustics problems these variables will generally be time and space. That is, if the requirements for the creation of the signal are arranged and all the conditions at our disposal are reproduced exactly,* then the resulting signal will be unlike the one which preceded it in another experiment. Obviously, what is lacking on our part is information concerning the state of the physical system, a lack of information which produces the randomness of the results of the experiment. Nevertheless, the randomness does occur, and we must find some way of expressing our ignorance (with the paradoxical result that we shall be pleased with ourselves when we have done so!).

B. Concept of an Ensemble

As a result of a long series of repetitions of the experiment under identical conditions (again, those at our disposal), we will have a collection of signals which are unlike each other in detail. I say detail because we shall find that there may be certain properties of the signal which are very nearly the same for all the samples collected. We shall call this collection of signals an ensemble, and each of the individual results, a member of the ensemble.

In general, we expect that our efforts to control (and by control we mean the reproduction of the measurement of an external parameter) the conditions are not entirely for nought, for these rather gross

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*These are sometimes referred to as "similarly prepared systems."

"boundary conditions" govern, and hence fix, certain properties of all the members individually and of the ensemble as a whole. One property which we have control over is the set of joint probability distributions which in general is an infinite set of functions, not entirely unrelated to each other.

C. The Distribution Functions

For the moment let us assume that the random properties of the ensemble are only exhibited in terms of a single variable, namely that of time. If we also assume that each of the experiments above ran for a time T, we may label the beginning of each member, 0; and the end, T. Then we call the distance between a point on each experimental time record and the beginning, t. Hence, the value of each member of the ensemble at the time t is a number which may be tabulated. If we make a plot of the density of values at time t versus the value of the signal, letting the number of members of the ensemble increase without limit while keeping the area under the plot at unity, then we have a plot of the distribution of values of the members of the ensemble at time t. We shall call this plot the first-order probability distribution. If we call the value of the random variable f at the time t, then this is denoted by $W_1(f,t)$. By our normalization, it is clear that this must satisfy

$$\int_{-\infty}^{+\infty} W_1(f,t)\, df = 1. \qquad 2.01$$

This process is indicated for an ensemble of five signals in Figure 2.01.

Similarly, one may select two positions in time, t_1 and t_2, and form a two-dimensional probability density such that the value f_1 at time t_1, and the value f_2 at time t_2, are governed by the joint probability dis-

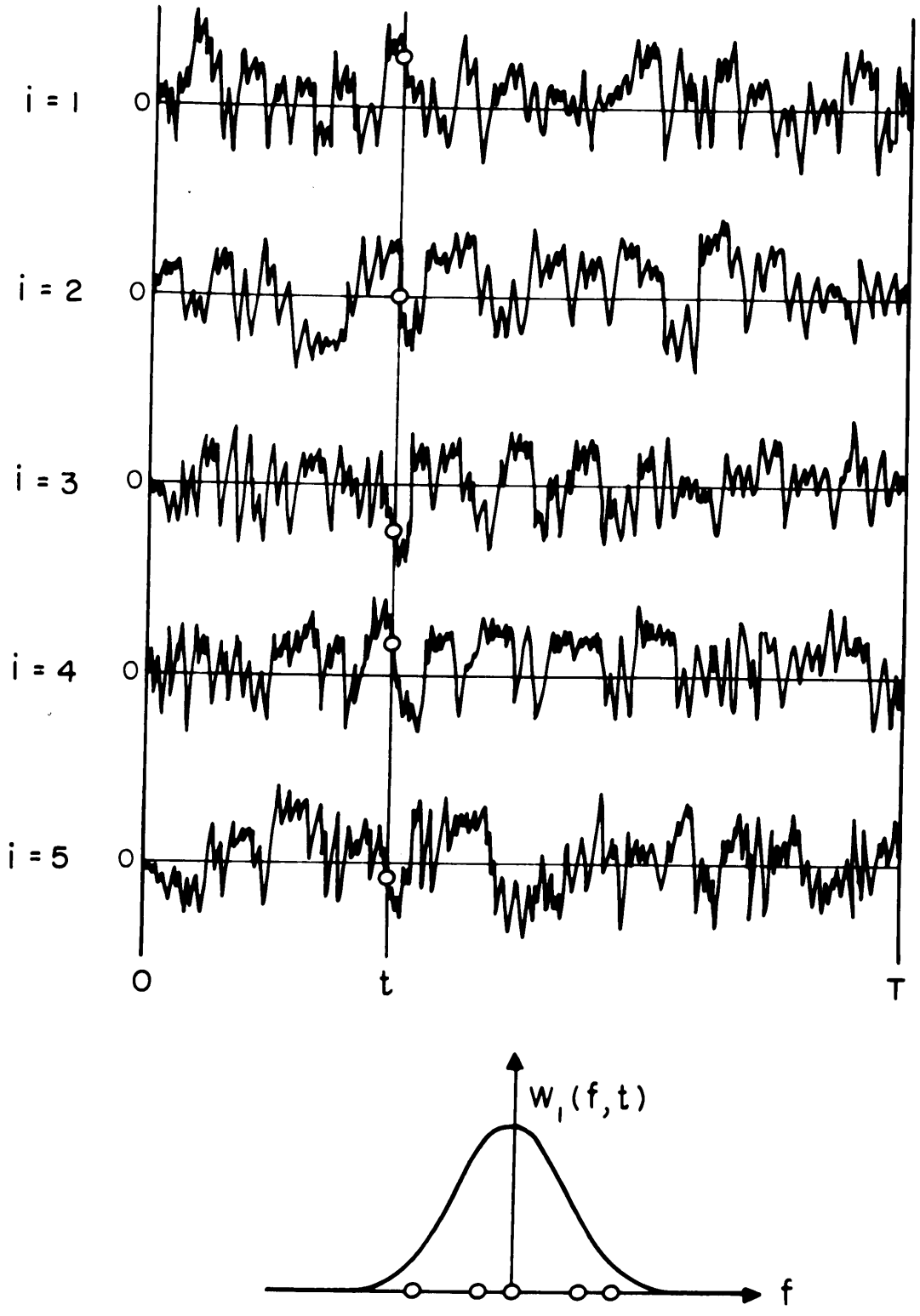

Figure 2.01 Ensemble of Five Random Functions.

tribution

$$W_2(f_1,t_1 \mid f_2,t_2) ,$$

with the property

$$\int df_1 df_2 W_2(f_1,t_1 \mid f_2,t_2) = 1. \qquad 2.02$$

Clearly, one also has the condition,

$$\int df_2 W_2(f_1,t_1 \mid f_2,t_2) = W_1(f_1,t_1). \qquad 2.03$$

In general, then, one has for n times $t_i (i = 1,2,\cdots,n)$ the n^{th} order joint probability distribution $W_n(f_1,t_1 \mid f_2,t_2 \mid \cdots \mid f_n,t_n)$. In the nature of these distributions, one has the following properties:

(a) $W_n \geq 0$

(b) W_n is a symmetric function in the sets of variables (f_i,t_i)

(c) $W_k(f_1,t_1 \mid \cdots \mid f_k,t_k) = \int df_{k+1} \cdots df_n W_n(f_1,t_1 \mid \cdots \mid f_n,t_n)$. 2.04

The property (c) puts a very definite limitation on the W_n's one may have, since by integrating over f_k, t_k must drop out. The complete set of W_n's (as n increases without limit) is said to completely define the random process. On its surface this would not seem to be a particularly helpful situation since an infinite set of probability distributions is a substantial amount of information to be gathered. However, W_1 gives us a fair amount of information about a process, and W_2 is usually sufficient to answer most of the questions which arise concerning physical processes.

As we shall see later, there are special cases in which this sim-

plifies even more. Very often the distribution W_2 depends only on the difference between t_1 and t_2. This is called a stationary process. If the random dependence is on position as well, and the spatial dependence is on the relative separation $|\vec{r}_1 - \vec{r}_2|$, the system is called homogeneous. Another important situation arises when W_2 is the only unique probability distribution. Then W_2 is the highest order distribution necessary to determine all the others, namely, $W_3(1,2,3) = \frac{W_2(1,2)W_2(2,3)}{W_1(2)}$. Such a situation is called a "Markoff process." "Process" is probably an unfortunate choice of terms here since it is the number of dynamical variables for the system under excitation which are included, and not the random properties of the process, which determines the Markoffian character. One should say that the work with the probability distributions done in this chapter is essentially unaffected by whether or not one has a Markoff process.

D. Distributions in Space and Time

In most of the work in this thesis the noise sources will have a space dependence as well as a time dependence. This is true whether or not at a given instant of time the source is a random function of the space coordinates (just as one may have a probability distribution for a completely well-defined function of time, albeit a delta function). In general, however, the distributions will be like

$$W_n(f_1,\vec{r}_1,t_1| \cdots | f_n,\vec{r}_n,t_n) ,$$

and the relationships governing their behavior are basically unchanged from those in the previous section. When we have such a distribution

of random sources in space, we shall say that a noise field is present. Since most of the work in the thesis is concerned with the excitation of continuous systems by noise fields, then joint probabilities with the spatial dependence are intended.

E. Ensemble Averages

With a set of probability distributions like this, one would expect to be able to calculate mean values. In practically any experimental situation one obtains by sufficient repetition of the experiment a set of average results and a distribution of points about this average. This is substantially what we ask for here. From the distribution $W_1(f,\vec{r},t)$ one may obtain mean values of any degree for the variable f, namely

$$\langle f^n \rangle_{ave} = \int_{-\infty}^{\infty} f^n W_1(f,\vec{r},t)df \ , \qquad 2.05$$

where the brackets $\langle \cdots \rangle_{ave}$ from here on refer to the ensemble average. This average of course will depend on the "unintegrated" variables $\vec{r}$ and t. This is not surprising, but in the important situations of stationary and homogeneous processes one or both of these variables will drop out.

One may wish to know the average value of the product of the variable f at two or more different times. As such, one may write the expression

$$\langle f_1^{\alpha} f_2^{\beta} \cdots f_n^{\nu} \rangle_{ave} = \int f_1^{\alpha} f_2^{\beta} \cdots f_n^{\nu} W_n(f_1,t_1,\vec{r}_1| \cdots |f_n,t_n,\vec{r}_n) \qquad 2.06$$

which in general will depend on the set of variables $(\vec{r}_i,t_i)$; i = 1,2,···,n.

It is here assumed that the W_n's are known. As we shall see later, this is not always necessary, but equivalent conditions may be substituted. Particularly when one knows the mean and the variance of a distribution, he may calculate mean squares without any further knowledge of the distribution, since $\langle f^2 \rangle_{ave} = \langle (f - \langle f \rangle_{ave})^2 \rangle_{ave} + \langle f \rangle^2_{ave}$.

2.3 THE RESPONSE OF PHYSICAL SYSTEMS TO RANDOM NOISE FIELDS

When we speak of the response of a system to a source having random properties, we are not concerned with the exact shape of signal which is produced. This in general repeats itself in no detectable way, so that anything we might wish to infer from such an examination would be useless. Such unpredictability leads us to consider what questions may be meaningful to ask.

Since we have, in principle at least, the W_n's at our disposal, we might ask questions concerning the mean values which are important from physical considerations. Let us consider a noise field acting as a source on a physical system. Any member of the ensemble of functions $f(\vec{r},t)$ will produce a corresponding response $\phi(\vec{r},t)$ in the system. Let us represent the system by the partial differential equation describing its motion, that is

$$\mathcal{L}\, \phi(\vec{r},t) = -4\pi f(\vec{r},t) \,, \qquad 2.07$$

where $\mathcal{L}$ is a linear operator which includes boundary conditions. Formally the solution to this equation is

$$\phi(\vec{r},t) = -4\pi \mathcal{L}^{-1} f(\vec{r},t) \,, \qquad 2.075$$

where, now, $\mathcal{L}^{-1}$ means an integration of the impulse Green's function

times the source function over the region of interest plus an integration over the surfaces of the region and the initial conditions. If we spell this out in more detail, when $\mathcal{L}$ is the wave operator, one has

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\,\phi(\vec{r},t) = -4\pi f(\vec{r},t)\,, \qquad 2.08$$

and the solution is [23]

$$\phi(\vec{r},t) = \int_0^{t^+} dt_o \int dv_o G(\vec{r},t\,|\,\vec{r}_o,t_o)\, f(\vec{r}_o,t_o) + \int_0^{t^+} dt_o \oint d\vec{S}_o \cdot (G\nabla_o\phi - \phi\nabla_o G)$$
$$- \frac{1}{c^2}\int dv_o \left(\frac{\partial G}{\partial t_o}\Big|_{t_o=0}\phi_o(\vec{r}_o) - G_{t_o=0}\frac{\partial\phi}{\partial t_o}\Big|_{t_o=0}\right). \qquad 2.085$$

Above we have assumed that the system is started at $t = 0$ with initial values and boundary conditions specified, and t^+ means $t+\epsilon$ to ensure the integration over any singularities which might occur when $t_o = t$.

Let us ignore boundary conditions and initial conditions (which we may do for many important physical situations), and we find the solution to be

$$\phi^{(i)}(\vec{r},t) = \int_0^{t^+} dt_o \int dv_o G(\vec{r},t\,|\,\vec{r}_o,t_o)\, f^{(i)}(\vec{r}_o,t_o)\,, \qquad 2.09$$

where $f^{(i)}$ is a particular member of the ensemble of source functions. If we know the distribution of f, we may obtain the average value of ϕ,

$$\langle\phi(\vec{r},t)\rangle_{ave} = \int_0^{t^+} dt_o \int dv_o\, G(\vec{r},t\,|\,\vec{r}_o,t_o)\,\langle f(r_o,t_o)\rangle_{ave}\,. \qquad 2.10$$

This result arises from the linearity of the solution,

$$\lim_{N\to\infty}\frac{\sum_{i=1}^{N}\phi^{(i)}(\vec{r},t)}{N} = \int_0^{t^+} dt_o \int dv_o\, G(\vec{r},t\,|\,\vec{r}_o,t_o)\,\lim_{N\to\infty}\frac{\sum_{i=1}^{N} f^{(i)}(\vec{r}_o,t_o)}{N}\,. \qquad 2.11$$

It is clear, however, that this may be interpreted as

$$\langle \phi(\vec{r},t) \rangle_{ave} = \int_0^{t^+} dt_o \int dv_o \; G(\vec{r},t \mid \vec{r}_o,t_o) \int_{-\infty}^{\infty} f \; W_1(f,\vec{r}_o,t_o) df \; , \quad 2.115$$

which is remarkable since the function $W_1(f,\vec{r}_o,t_o)$ has passed right under the integrals over the variables $\vec{r}_o,t_o$, and f has lost its dependence on $\vec{r}_o,t_o$ and become a variable of integration itself. If the average value of the solution over many experiments as a function of $\vec{r},t$ is required, it may be obtained in this manner.

Let us now go on to the physically interesting second order mean values.

A. Second Order Mean Values

Usually in acoustics one is not so much interested in the mean value of a solution as he is its mean square value. The reason is, of course, that mean squares of the response or its derivative in time or space is directly related to energy, or energy flow, or similar quantities. For example, one may write

$$\phi^{(i)}(\vec{r},t)\phi^{(i)}(\vec{r}',t') = \int_0^{t^+}\int_0^{t'^+} dt_o dt_o' \iint dv_o dv_o' \; G(\vec{r},t \mid \vec{r}_o,t_o)$$

$$G(\vec{r}',t' \mid \vec{r}_o',t_o') \cdot f^{(i)}(\vec{r}_o,t_o) \; f^{(i)}(\vec{r}_o',t_o') \; ,$$

and by averaging over the ensemble again one has

$$\langle \phi(\vec{r},t) \; \phi(\vec{r}',t') \rangle_{ave} = \int_0^{t^+}\int_0^{t'^+} dt_o dt_o' \iint dv_o dv_o' \; G(\vec{r},t \mid \vec{r}_o,t_o)$$

$$G(\vec{r}',t' \mid \vec{r}_o',t_o) \; \langle f(\vec{r}_o,t_o) f(\vec{r}_o',t'_o) \rangle_{ave} \; . \quad 2.12$$

This equation we shall use many times in the following chapters. An expression like this calculates the correlation between the random response

at two different positions and times. When $\vec{r}=\vec{r}'$ and $t=t'$, this becomes the mean square response to the ensemble of source functions. The methods for obtaining mean values of products of derivatives of ϕ are clear.

B. The Stationary Time Process

Thus far, whenever we have spoken of a mean value, we have meant the ensemble average. This average is satisfactory for situations in which an experiment is repeated many times, for example in nuclear scattering problems. Many times, however, what is done is the taking of a time average of a signal, or of its square, either with instruments designed to perform such averages or by observation of the mean displacement of a fluctuating meter reading. We now wish to explore the relationships between these time averages and the ensemble average with which we have been occupied. If we take the ensemble which we have obtained and examine the joint probability distribution $W_2(f_1,\vec{r}_1,t_1 \mid f_2,\vec{r}_2,t_2)$, we may be struck by certain symmetry properties among the variables $\vec{r}_1,t_1$ and $\vec{r}_2,t_2$. In particular, we may have its complete time dependence through the variable $\tau \equiv t_1-t_2$. If this is so, the process is said to be stationary in time. In terms of the space variables, an x dependence merely on $\xi = x_1-x_2$ is said to be homogeneous in x, etc. A dependence on $\rho \equiv |r_1-r_2|$ alone denotes a field which is said to be completely homogeneous and isotropic. The requirement that W_2 be symmetric in the variables $f_1,\vec{r}_1,t_1$ and $f_2,\vec{r}_2,t_2$ forces the dependence to be even in τ for the stationary time case, even in ξ for the homogeneous in x case, and so on.

Suppose we pick two points $\vec{r}_1$ and $\vec{r}_2$ and consider the members of the ensemble for these points. If the members are stationary in time, the correlation between the functions f_1 and f_2, as described under Section A (Second Order Mean Values), is a function of the two points $\vec{r}_1$ and $\vec{r}_2$ and of the delay τ . Since the parity in τ is not affected by the averaging process, the correlation is likewise an even function of τ . We wish to show that the ensemble averaging process may be replaced by a time averaging process over a single member of the ensemble. To show this, we must invoke the ergodic hypothesis.

The ergodic hypothesis states that in a system bounded by a set of fixed parameters (the constant conditions we have spoken of) the system will reach arbitrarily close to any condition consistent with those parameters and spend an amount of time in the region directly proportional to the probability (i.e., the ensemble probability W_n) of its attaining that condition. More in line with what we have been saying, in a system operating under fixed conditions, the value of the function f will approach arbitrarily close to a selected value f_1 and will spend an amount of time in the interval $f_1, f_1 + df_1$ proportional to the probability $W_1(f_1)df_1$. This makes the time average equivalent to the ensemble average for a stationary time process.

We shall denote the time average by $\langle \cdots \rangle$ and form it thus:

$$\left\langle f(\vec{r},t) \right\rangle = \frac{1}{T} \int_0^T f(\vec{r},t)\, dt. \qquad 2.13$$

In the earlier work then, for stationary processes, we have

$$\left\langle \phi(\vec{r},t)\phi(\vec{r}',t) \right\rangle_{ave} = \int_0^t \int_0^{t'} dt_o dt_o' \iint dv_o dv_o' \; G(\vec{r},t \mid \vec{r}_o,t_o)$$

$$G(\vec{r}',t' \mid \vec{r}_o',t_o') \left\langle f(\vec{r}_o,t_o)\; f(\vec{r}_o',t_o') \right\rangle \quad , \qquad 2.14$$

where the average on the l.h.s. is still an ensemble average. We must make this distinction since the system was started at $t=0$ with no motion. Thus far the transients may not have died out. If we had set the lower variable of integration at $t=-\infty$, the system would have forgotten how it started, and the average on both sides would be a time average. One will notice, however, that the time average on the r.h.s. could not have been applied directly since it must be integrated over the variables t_o, t_o'.

More complicated cases than this may arise, however. Suppose that ϕ is the pressure in an infinite acoustic medium due to the radiation of random noise sources localized in a finite region of space centered at the origin. If the sources are turned on at $t=0$, the sources will achieve a stationary state immediately, certainly after a time T (long compared to any fluctuations in the signal). If the velocity of waves in the medium is c, then for regions of space $|\vec{r}| \ll cT$ the response ϕ will be stationary. However, for regions $|\vec{r}| \gg cT$ the response will be just beginning, and near $|\vec{r}| = cT$ the response will be in transition to the stationary condition. The point is that one may use stationary assumptions concerning the source, but whether or not one may use such an assumption for the response depends very much on the kind of domain and the region of the domain in which he is working.

One might argue for replacing the ensemble average by an average over space as is done for turbulence,[24] but in most physical situations

we stand at fixed positions and record time functions and average them in time.

C. Convergence Properties of the Solutions

We are aware that one may not just take any kind of time dependence for the source and apply it to any system and expect the integrals to converge. The convergence of a solution ϕ to a source f is made subject to certain requirements of the integral of the square of f over the region. These same requirements govern the convergence of the correlation response, $\langle \phi(\vec{r},t)\phi(\vec{r}',t') \rangle_{ave}$, since this is the average of the product of two solutions.

There are in general only four separate convergence situations which are of interest to us as they arise in a physical situation. They are

$$\text{(a)} \quad \lim_{V,T\to\infty} \int_V dv \int_0^T dt\, f^2(\vec{r},t) \quad \text{is finite}$$

$$\text{(b)} \quad \lim_{T,V\to\infty} \frac{1}{V} \int_V dv \int_0^T dt\, f^2(\vec{r},t) \quad \text{is finite} \qquad 2.15$$

$$\text{(c)} \quad \lim_{V,T\to\infty} \frac{1}{T} \int_V dv \int_0^T dt\, f^2(\vec{r},t) \quad \text{is finite}$$

$$\text{(d)} \quad \lim_{V,T\to\infty} \frac{1}{VT} \int_V dv \int_0^T dt\, f^2(\vec{r},t) \quad \text{is finite.}$$

If (b), (c), or (d) is true, (a) is infinite. If (d) is true, (b) and (c) are infinite. If (a), (b), or (c) is true, then (d) vanishes. If (b) is true, (c) vanishes, and vice versa. The time averages and integrals are obviously closely related to finite energy or finite power. The space integrals are associated with finite energy or energy density. They

come out this way.

(a) Finite energy

(b) Finite energy density

(c) Finite power

(d) Finite power, power density.

The terms power and energy should not be taken too literally. One may have a finite mean square response with no power being transferred at all, for example in an oscillating resonant system having no losses. These are very closely related to whether or not f is representable by a Fourier series or a Fourier integral. For each of the finite quantities above we may represent f by a Fourier integral in the unaveraged variable, the transform of which is a kind of power or energy spectrum in frequency and/or wavelength. Here again the properties of f, or of its higher order mean values, are not reflected directly in the response ϕ. We must first integrate over a Green's function which may fall in either class (a) or (c) above -- in (a) for infinite regions and finite non-conservative regions, and in (c) for finite conservative regions. The convergence properties of ϕ depend on the combined properties of f and the Green's function G. In any physical situation, however, the convergence is assured unless one incorrectly specifies the source or Green's function.

D. Fourier Representations

The Fourier representation for stationary random functions, either of finite or infinite extent in space and/or time, has been the subject of many papers and a few books in recent years, and it seems unnecessary

to duplicate the developments here. We shall recall a few of the major results which are of particular importance for the present work.

The auto-correlation function is defined by

$$\psi(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)f(t+\tau)dt \ , \qquad 2.16$$

which is the Fourier cosine transform of the power spectrum of f, namely

$$\Psi(\omega) = 2\int_0^\infty \psi(\tau)\cos\omega\tau\, d\tau \ , \qquad 2.17$$

where

$$\Psi(\omega) = \lim_{T\to\infty} \frac{2|F_T(\omega)|^2}{T} \ ,$$

and

$$F_T(\omega) = \int_0^T f_T(t)\, e^{i\omega t}\, dt.$$

Here f_T is zero outside $(0,T)$, and all the limiting processes are assumed to exist. For case (d) above one must include the space transform

$$\psi(\vec{\xi},\tau) = \lim_{V,T\to\infty} \frac{1}{VT}\int_0^T dt \int_V dv\ f(\vec{r},t)f(\vec{r}+\vec{\xi},t+\tau) \ , \qquad 2.18$$

and

$$\Psi(\vec{\kappa},\omega) = 2\int_0^\infty d\tau \cos\omega\tau \int d\vec{\kappa}\ \cos(\vec{\kappa}\cdot\vec{\xi})\,\psi(\vec{\xi},\tau)$$

$$= \lim_{V,T\to\infty} \frac{16\,|F_{T,V}(\vec{\kappa},\omega)|^2}{VT} \qquad 2.19$$

and

$$F_{VT} = \int_V dv \int_0^T dt\ f_{VT}(\vec{r},t)\, e^{i(\omega t - \vec{\kappa}\cdot\vec{r})} \ ,$$

where f_{VT} is zero outside V;0,T.

Due to the stationary character of these expressions in time (and/or space), by the ergodic hypothesis they are the same for each member of the ensemble. For functions which are class L^2 in a certain variable,* this is not the case; and one must perform an average over the ensemble in those variables. For example, consider the emitting region mentioned above. At each point the source is a stationary time series. Hence the spectrum in frequency may be obtained by a single time record from each source point. However, the spectrum in wavelength ($\vec{\kappa}$) must be an average over the ensemble of source realizations.

The number of results which one may obtain from the analysis of the propagation of mean squares and the representations by Fourier expansion is almost unlimited. Merely by setting up the solution in terms of integrations over space, time, frequency, and wave number and integrating in various orders, many useful (and many more not-so-useful) formulas may be obtained. One useful result, when one knows the power spectrum of the source, is obtained as follows:

$$\left\langle \phi(\vec{r},t)\phi(\vec{r}',t') \right\rangle_{ave} = \iint_V dv_o dv'_o \iint dt_o dt'_o \; G(\vec{r},t \mid \vec{r}_o,t_o)$$

$$G(\vec{r}',t' \mid \vec{r}'_o,t'_o) \left\langle f(\vec{r}_o,t_o) f(\vec{r}'_o,t'_o) \right\rangle_{ave} .$$

The source field is assumed to be stationary in time and homogeneous in space so that we may write,

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*A function h is said to be class L^2 in the variable y when $\int_{\text{all } y} h^2(y)\, dy$ exists and is finite.

$$\left\langle f(\vec{r}_o,t_o)f(\vec{r}'_o,t'_o)\right\rangle_{ave} = \lim_{VT\to\infty} \frac{1}{2VT}\int_{-T}^{T} dt \int_V dv\ f(\vec{r}_o,t_o)f(\vec{r}_o+\vec{\xi},\ t_o+\tau) \qquad 2.20$$

$$= \psi(\vec{\xi},\tau)\ .$$

The power density spectrum is given by

$$\Psi\ (\vec{\kappa},\omega) = \int_{-\infty}^{\infty} d\vec{\xi} \int_{-\infty}^{\infty} d\tau \cos\omega\tau\ \cos\ (\vec{\kappa}\cdot\vec{\xi})\psi(\vec{\xi},\tau)\ ,$$

where

$$\Psi\ (\vec{\kappa},\omega) = \lim_{V,T\to\infty} \frac{16\left|F(\vec{\kappa},\omega)\right|^2}{VT}\ ,$$

and

$$f(\vec{r},t) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} d\vec{\kappa} \int_{-\infty}^{\infty} d\omega\, e^{i(\vec{\kappa}\cdot\vec{r}-\omega t)}\ F(\vec{\kappa},\omega)\ .$$

If we should happen to know the spectrum of the source in wave number and frequency, we could use it to get the response correlation in the following way.

$$\psi\ (\vec{\xi},\tau) = \frac{1}{\pi^4}\int_0^{\infty} d\omega\, d\vec{\kappa}\Psi(\vec{\kappa},\omega)\ \cos\ (\omega\tau - \vec{\kappa}\cdot\vec{\xi})$$

$$= \frac{1}{(16)^2\pi^4}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\vec{\kappa}\Psi(\vec{\kappa},\omega)\ e^{i(\omega\tau - \vec{\kappa}\cdot\vec{\xi})}\ .$$

Hence,

$$\left\langle \phi(\vec{r},t)\phi(\vec{r}',t')\right\rangle = \frac{1}{2^8\pi^4} \int_{-\infty}^{\infty} d\mu \int_{-\infty}^{\infty} d\zeta \int_V d\vec{\xi}\, d\vec{\rho}\ G(\vec{r},t\mid\vec{r}_o,t_o) \qquad 2.21$$

$$G(\vec{r}',t'\mid\vec{r}'_o,t'_o) \iint_{-\infty}^{\infty} d\omega\, d\vec{\kappa}\Psi(\vec{\kappa},\omega)\ e^{i(\omega\tau - \vec{\kappa}\cdot\vec{\xi})}\ ,$$

where

$$\mu = t_o + t_o', \quad \zeta = t_o - t_o', \quad \vec{\rho} = \vec{r}_o + \vec{r}_o', \quad \vec{\sigma} = \vec{r}_o - \vec{r}_o' .$$

Now, $\int d\vec{\rho} \int d\mu \, G(\vec{r},t \mid \vec{r}_o,t_o)\, G(\vec{r}',t' \mid \vec{r}_o',t_o')$ is what might properly be called the convolution of the product $G(\vec{r},t \mid \vec{r}_o,t_o)\, G(\vec{r}',t' \mid \vec{r}_o',t_o')$. The transform, given by

$$\int_{-\infty}^{\infty} d\zeta \int_V d\vec{\xi} \left\{ \int_{-\infty}^{\infty} d\mu \int_V d\vec{\rho}\, G(\vec{r},t \mid \vec{r}_o,t_o)\, G(\vec{r}',t' \mid \vec{r}_o',t_o') \right\} e^{i(\omega\tau - \vec{\kappa}\cdot\vec{\xi})} ,$$

will be denoted by $\mathcal{G}(\vec{r},\vec{r}',t,t' \mid \vec{\kappa},\omega)$, and we have

$$2^8 \, 4 \left\langle \phi(\vec{r},t)\phi(\vec{r}',t') \right\rangle_{ave} = \int_{-\infty}^{\infty} d\omega \int d\vec{\kappa}\, \mathcal{G}\, \Psi . \qquad 2.22$$

Other relationships like this are possible, and one must examine his knowledge of the problem in order to determine which formalism is the most feasible.

We have now considered enough of the theory of the propagation of mean values in order to proceed with the work in Chapters III and IV. We have at times restricted ourselves to problems in which the joint probability distributions are invariant to translational shifts of the time axis in cases where we wished to replace ensemble averages by time averages. In general, however, we have not been restrictive on this matter and have allowed freely for the non-stationary time process as well. Before going on to applications of the methods, it would be well to consider the non-stationary process.

E. Non-Stationary Time Process

In many problems of statistical dynamics situations arise in which the statistical properties of the random functions (source, response,

or both) are not stationary in time. As we have pointed out before, the response to a source, which itself is a stationary time process, need not be stationary if the transients have not had a chance to die out. In order to fix our thoughts, let us take some examples.

Consider a room, with finite absorption at all frequencies, being excited by a speaker producing a random output. Suppose the speaker is turned on at time $t=0$. Then a receiver (microphone) in the room will first pick up the direct signal, a little later the first reflection, and so on. As long as new reflections are received and the energy output of the speaker has not come into equilibrium with the absorption at the walls, the microphone output will be non-stationary since the power spectrum is changing with time. As equilibrium is reached, the response becomes a stationary time process.

As opposed to the situation where a stationary source produces a non-stationary response, let us consider just the inverse of this. If one places a speaker in a room having perfectly hard walls and excites the room with a burst of random noise from $t=0$ to $t=t$, the response will be non-stationary until many reflections from the walls have occurred and the process becomes stationary.

We can also have sources which are non-stationary because the probability distributions are time dependent, as in the case of noise from a whirling propellor blade. Usually the response to such sources will be non-stationary as well. The shot effect from a vacuum tube current undergoing grid voltage variations is a good example of this. We actually use the time dependent probability to calculate amplification factors, and so on.

We wish to tie in this non-stationary time process with the Fourier analysis ideas in the last section. Fourier analysis is good because it transforms the time and space averages which replace ensemble averages for stationary processes. This limitation to stationary processes is not absolute, however, if we introduce, following Page,[17] the concept of the instantaneous power spectrum.

The concept of power is that of a rate of change of energy in time. This may very well change with time, just as a discharging condensor delivers a power which changes in time to a resistor shunted across its terminals. Hence, if one considers a signal, he may obtain the instantaneous power spectrum by taking the time derivative of the energy delivered up to the time t. If the signal is random, one takes the derivative of the ensemble average of the energy spectrum. The following derivation of the instantaneous power is close to that of Lampard,[18] but is more consistent with the notation and approach in this thesis.

Let us first consider the one-dimensional space situation, so that the ensemble member is $f^{(i)}(x,t)$. Let us assume that $f^{(i)}$ started at a large negative time, -T, and continues up to the present,t. We wish to know the power spectrum at the present.

$$f^{(i)}(x,t_o) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F^{(i)}(x,\omega)\, e^{-i\omega t_o}\, d\omega \ ,$$

and 2.23

$$f^{(i)}(x',t_o') = \frac{1}{2\pi} \int_{-\infty}^{\infty} F^{(i)}(x',\omega')\ e^{-i\omega' t_o'}\, d\omega' \ .$$

Then, if x=x',

$$\left|F^{(i)}(x,\omega)\right|^2 = \int_{-T}^{t} \int_{-T}^{t} f^{(i)}(x,t_o)\, f^{(i)}(x,t_o')\, e^{i\omega(t_o-t_o')}\, dt_o dt_o' \ .$$

We take the ensemble average and introduce

$$\psi_x(t_o,t_o') \equiv \left\langle f^{(i)}(x,t_o)\, f^{(i)}(x,t_o') \right\rangle_{ave} \qquad 2.24$$

and let

$$\mu = t_o + t_o', \quad \zeta = t_o - t_o' ;$$

then

$$\left\langle \left| F^{(i)}(x,\omega) \right|^2 \right\rangle_{ave} = \frac{1}{2} \left[\int_{-2T}^{t-T} d\mu \int_{-\mu-2T}^{\mu+2T} d\zeta + \int_{t-T}^{2t} d\mu \int_{\mu-2T}^{2t-\mu} d\zeta \right]$$

$$\cdot \psi_x(t_o,t_o')\, e^{i\omega\zeta} ,$$

and

$$\frac{\partial}{\partial t} \left\langle \left| F(x,\omega) \right|^2 \right\rangle_{ave} = 2 \int_0^\infty \psi_x(t,t-\gamma) \cos\omega\gamma \, d\gamma \equiv W(x,\omega) , \qquad 2.25$$

as we let $T \to \infty$. In the same way, one may get the instantaneous power density, given as

$$W(k,\omega) = 2 \int_0^\infty d\gamma \int_0^\infty d\alpha \Big[\psi(x,t \mid x-\alpha, t-\gamma) \cos(k\alpha - \omega\gamma) \qquad 2.26$$

$$+ \psi(x-\alpha, t \mid x, t-\gamma) \cos(k\alpha + \omega\gamma) \Big] ,$$

where

$$\psi(x,t \mid x',t') \equiv \left\langle f(x,t) f(x',t') \right\rangle_{ave} . \qquad 2.27$$

One may easily carry this on to two or three dimensions, but the number of correlation functions to be obtained increases twofold for each new dimension. Hence, there are four for two dimensions and eight for three

dimensions.

2.4 CONCLUSION

This gives us sufficient tools to study a class of problems in which continuous systems are excited by noise fields having well defined correlation properties. We now pass on to the first of those examples in Chapter III.

III RANDOM EXCITATION OF FINITE STRINGS

3.1 INTRODUCTION

In this chapter we shall apply some of the formalism developed in the last chapter to the special case of the damped, finite string. The plan will be to assume a source correlation $\langle f(\vec{r}_o,t_o)f(\vec{r}_o',t_o')\rangle_{ave}$ and with the knowledge of the Green's function for the system calculate the mean square response. Although from our analysis it is possible to calculate the response correlation functions $\langle \phi(\vec{r},t)\phi(\vec{r}',t')\rangle_{ave}$, we shall in all cases restrict our attention to the mean square response -- partly because that represents the interesting experimental quantity, partly because it substantially eases the computation.

3.2 EXAMPLE 1, THE FORCE RANDOM IN SPACE AND TIME

Let us begin our examples with a very simple situation. We assume that a finite, damped string is being forced by a field which is purely random in both space and time. Purely random in time means that the signal at a given instant is completely uncorrelated with itself at any later time. We then denote such a correlation by a δ -function, namely

$$\langle f(t_o)f(t_o')\rangle_{ave} = D\,\delta(t_o-t_o') \quad ,$$

where 2D is the spectral power density of the source. Hence, for our one-dimensional system the correlation is

$$\langle f(x_o,t_o)f(x_o',t_o')\rangle = D\,\delta(x_o-x_o')\,\delta(t_o-t_o') \quad .$$

A. The Finite String

The wave equation for the finite string is

$$T\frac{\partial^2 \phi}{\partial x^2} - \rho_\ell \frac{\partial^2 \phi}{\partial t^2} - \eta \frac{\partial \phi}{\partial t} = -4\pi f \quad ,$$

where η is the friction coefficient. Dividing through by ρ_ℓ gives

$$c^2 \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial t^2} - \beta \frac{\partial \phi}{\partial t} = -\frac{4\pi}{\rho_\ell} f \quad , \qquad 3.01$$

where

$$c^2 = \frac{T}{\rho_\ell} \quad , \quad \beta = \frac{\eta}{\rho_\ell} \quad .$$

In this equation ϕ is the displacement, η is the viscous coefficient, T is the tension, ρ_ℓ is the linear density, and c the phase velocity. The string is considered fastened at the points $x = 0$, $x = L$.

We expand the solution ϕ into a sum of space eigenfunctions,

$$\phi = \sum_{m=1}^{\infty} \Phi_m (t) \psi_m (x) \quad , \qquad 3.02$$

where

$$\psi_m (x) = \sqrt{\frac{2}{L}} \sin \frac{m\pi x}{L} = \sqrt{\frac{2}{L}} \sin k_m x \quad , \qquad 3.025$$

and expand the linear forcing function f likewise:

$$f = \sum_{m=1}^{\infty} F_m (t) \psi_m (x). \qquad 3.03$$

Φ_m is the displacement of the m^{th} mode, and F_m is the force active on that mode. Hence, we have

$$\omega_m^2 \Phi_m - \frac{\partial^2 \Phi_m}{\partial t^2} - \beta \frac{\partial \Phi_m}{\partial t} = \frac{-4\pi}{\rho_\ell} F_m(t) \quad , \qquad 3.04$$

where

$$\omega_m = \frac{m\pi c}{L} = k_m c$$

and is the frequency of the m^{th} mode. We define the impulse Green's function $\mathcal{G}(x,t \mid x_o,t_o)$ as the solution to the equation

$$c^2 \frac{\partial^2 \mathcal{G}}{\partial x^2} - \frac{\partial^2 \mathcal{G}}{\partial t^2} - \beta \frac{\partial \mathcal{G}}{\partial t} = -4\pi \delta(x-x_o)\delta(t-t_o) \quad . \tag{3.05}$$

If we choose to expand the Green's function in eigenfunctions, we may write

$$\mathcal{G} = \sum_m G_m(x_o,t_o;t)\,\psi_m(x) \quad , \tag{3.06}$$

and recognizing that

$$\delta(x-x_o) = \sum_m \psi_m(x)\psi_m(x_o) \quad ,$$

we have

$$\frac{\partial^2 G_m}{\partial t^2} + \beta \frac{\partial G_m}{\partial t} + \omega_m^2 G_m = 4\pi \psi_m(x_o)\delta(t-t_o) \quad . \tag{3.07}$$

G_m is seen to be an impulse Green's function for the m^{th} mode. By the method of variation of parameters, one gets the solution of this to be

$$G_m = \begin{cases} -\frac{2\pi i}{\omega_{1,m}} \psi_m(x_o)\, Q_m(t-t_o) & t > t_o \\ 0 & t < t_o \quad , \end{cases} \tag{3.08}$$

where

$$Q_m(t-t_o) = 2ie^{-\frac{\beta}{2}(t-t_o)} \sin \omega_{1,m}(t-t_o) \tag{3.085}$$

and

$$\omega_{1,m} = \sqrt{\omega_m^2 - \beta^2/4} \quad ,$$

which is the frequency of the m^{th} mode as it is reduced by the action of the viscosity.

B. The Source

As we have said, the source correlation function is given by

$$\left\langle f(x_o,t_o)f(x_o',t_o')\right\rangle = D\,\delta(x_o-x_o')\delta(t_o-t_o') \quad . \qquad 3.09$$

This has the unfortunate property of having infinite power, as one may ascertain by recognizing that the mean square force at a point is just

$$\left\langle f^2(x_o,t_o)\right\rangle = D\,\delta(0)\cdot\delta(0) \quad .$$

The meaning of this is that energy is spread over all frequencies and all wavelengths with constant density, as may be seen by taking the cosine transform in space and time to get $\Psi(\varkappa,\omega)$ as we did in Chapter II. However, any particular mode of the string has only a finite pass band. As long as the actual source has a sensibly constant spectrum over this band (in frequency and wavelength), the δ-function will represent the actual response and will greatly ease the integration over certain of the space and time variables. Also, we shall assume that the source has been turned on at $t = -\infty$ so that the response has become a stationary process.

C. Calculation of Mean Square Displacements

From Chapter II the correlation function for the response is

$$\langle \phi(x,t)\phi(x',t') \rangle = \frac{-4\pi^2}{\rho_\ell^2} \sum_{n,m=1}^{\infty} \frac{1}{\omega_{1,n}\omega_{1,m}} \psi_n(x)\psi_m(x') \int_{-\infty}^{t} dt_o \int_{-\infty}^{t'} dt_o' \, Q_n(t-t_o)Q_m(t-t_o')R_{nm}(t_o;t_o') \quad , \tag{3.10}$$

where

$$R_{nm}(t_o;t_o') = \int_0^L dx_o \int_0^L dx_o' \psi_n(x_o)\psi_m(x_o') \langle f(x_o,t_o)f(x_o',t_o') \rangle \quad . \tag{3.105}$$

$R_{nm}(t_o;t_o')$ is seen to be just the correlation of the two source functions $F_n(t_o)$, $F_m(t_o')$, and is the n,m element of a correlation matrix. The R_{nm}'s are inconvenient to use in this example but will be used in the next where the correlation does not have such a nice form as in this problem. As in Chapter II we introduce the variables

$$\rho = x_o + x_o' \qquad \mu = t_o + t_o'$$

$$\sigma = x_o - x_o' \qquad \zeta = t_o - t_o' \quad . \tag{3.11}$$

From the Jacobian of the transformation, one has

$$dx_o\, dx_o' = \frac{1}{2} d\rho\, d\sigma \quad ; \text{ and } \quad dt_o\, dt_o' = \frac{1}{2} d\mu\, d\zeta \quad .$$

We shall use this transformation repeatedly in other problems. For the mean square value of ϕ, we set $x = x'$, $t = t'$, and get

$$\langle \phi^2(x,t) \rangle = \frac{4\pi^2 D}{\rho_\ell^2} \sum_{n,m} \frac{1}{\omega_{1,n}\omega_{1,m}} \psi_n(x)\psi_m(x) \iint_0^L \psi_n(x_o)\psi_m(x_o')\, dx_o dx_o'$$

$$\cdot \frac{1}{2}\int_{-\infty}^{2t} d\mu\, e^{-\frac{\beta}{2}(2t-\mu)} \int_{\mu-2t}^{2t-\mu} d\zeta \cdot 4 \sin \omega_{1,n}\left\{t-\tfrac{1}{2}(\mu+\zeta)\right\}$$

$$\sin \omega_{1,m}\left\{t-\tfrac{1}{2}(\mu-\zeta)\right\}\delta(\sigma)\,\delta(\zeta) \quad . \qquad 3.12$$

The derivation of this expression may be obtained by placing R_{nm} of 3.105 into 3.10. Then the expression for the source correlation from 3.09 is substituted, and the change of variables in 3.11 is used. Finally the eigenfunctions as defined by 3.025 and the Q_m's from 3.085 are substituted.

The limits of integration on μ, ζ come from the rotation of the axes produced by the change in variables as shown in Figure 3.01a. A quantity which has experimental as well as physical meaning is the mean square displacement of the m^{th} mode, which gives the contribution to the mean potential energy of the string from that mode. From our notation this is $\langle \Phi_m^2 \rangle$, which is obtained by picking out from $\langle \phi^2 \rangle$ (aside from space dependence) the term for which m = n. This is then

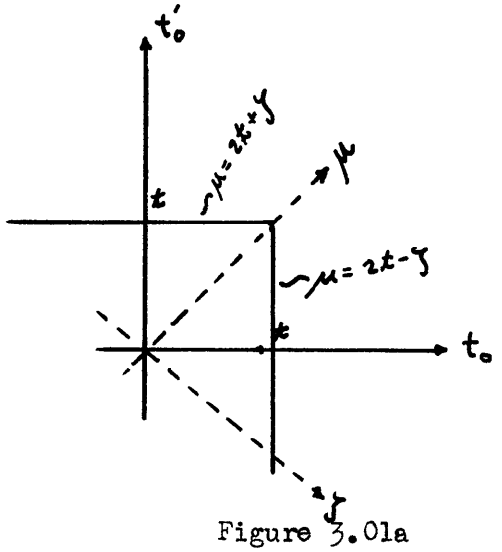

Figure 3.01a

$$\langle \Phi_m^2 \rangle = \frac{8\pi^2 D}{\rho \ell^2} \cdot \frac{1}{\omega_{1,m}^2} \iint_0^L dx_o dx_o' \psi_m(x_o)\psi_m(x_o') \int_{-\infty}^{2t} e^{-\frac{\beta}{2}(2t-\mu)} d\mu$$

$$\cdot \sin^2 \frac{\omega_{1,m}}{2}(2t-\mu) \cdot \delta(\sigma) \quad ,$$

where we have carried out the integration over ζ . Setting $\gamma = 2t - \mu$, one has

$$\langle \Phi_m^2 \rangle = \frac{4\pi^2 D}{\rho_\ell^2 \omega_{1,m}^2} \iint_0^L dx_o dx_o' \psi_m(x_o) \psi_m(x_o') \int_0^\infty e^{-\frac{\beta}{2}(\gamma)} d\gamma (1-\cos \omega_{1,m}\gamma) \delta(\sigma) \qquad 3.13$$

$$= \frac{4\pi^2 D}{\beta \rho_\ell^2 \omega_m^2} \int_0^{2L} d\rho \int d\sigma \frac{1}{L} (\cos k_m \sigma - \cos k_m \rho) \delta(\sigma) \ ,$$

where the integration over ρ , σ extends over the shaded square in Figure 3.01b. The eigenfunction product has been replaced by its trigonometric identity, and the integration over γ has been carried out. The integration over σ leaves

$$\langle \Phi_m^2 \rangle = \frac{4\pi^2 D}{\beta \rho_\ell^2 \omega_m^2 L} \int_0^{2L} d\rho \, (1-\cos k_m \rho).$$

The cosine integrates out and leaves us with

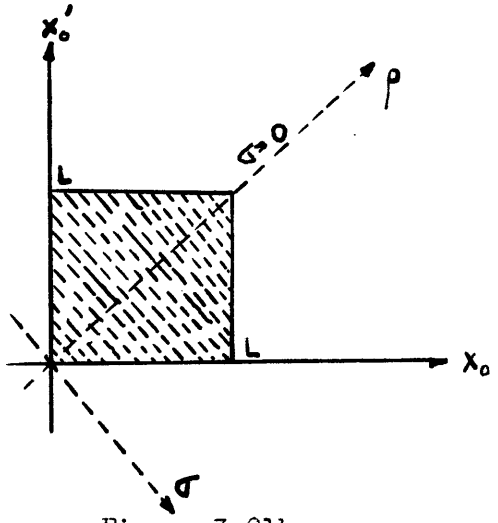

Figure 3.01b

$$\langle \Phi_m^2 \rangle = \frac{8\pi^2 D}{\beta \rho_\ell^2 \omega_m^2} \ . \qquad 3.14$$

For the terms $n \neq m$, the eigenfunction product is

$$\frac{2}{L} \sin k_n x_o \sin k_m x_o' = \frac{1}{2} \left\{ \cos \left(k_{n-m} \frac{\rho}{2} + k_{n+m} \frac{\sigma}{2} \right) - \cos \left(k_{n+m} \frac{\rho}{2} + k_{n-m} \frac{\sigma}{2} \right) \right. .$$

The integration over σ gives $\frac{1}{L} \left\{ \cos\left(k_{n-m} \frac{\rho}{2} \right) - \cos\left(k_{n+m} \frac{\rho}{2} \right) \right\}$, and the integration from $0 \to 2L$ on ρ gives $2\delta_{nm}$. Hence, in the expression 3.14 $\langle \Phi_m^2 \rangle$ represents the total motion of the string. The potential

energy of the m^{th} mode is $\frac{1}{2}\rho\,\omega_m^2\langle\Phi_m^2\rangle$, and if we set this equal to $\frac{1}{2}kT$ we can evaluate D and obtain an expression for the Brownian motion of the finite string, as Lear and Uhlenbeck have done.* The reader is referred to their paper for a more extended analysis of this example.

3.3 EXAMPLE 2, THE PIECEWISE DELAYED EXCITATION

Now that the general procedure in obtaining mean squares has been demonstrated, let us consider the types of problems we have in mind as the goal of our labor in this chapter. We ultimately want to know how a moving, random pressure field, such as turbulent flow, would excite a finite string -- or what is equivalent, a thin metal ribbon. One possible approximation to a moving noise field which one can make is to force part of a string with a random noise field, and then delay the random function and apply it to another part of the string.

A. The Correlation of the Source

The string will be defined by the same wave equation as in Section 3.2 - A. Again we shall assume that the source was "turned on" at $t = -\infty$ so that the response will be stationary. The source consists of uniform forcing of the string from $0 \rightarrow \frac{L}{2}$ by the purely random function $f(t_o)$. It is then delayed by a time τ_o and fed to the second half $\frac{L}{2} \rightarrow L$ as shown in Figure 3.02a. We ask what is the mean energy fed into the m^{th} mode as a function of the delay τ_o. The correlation field will be

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*Lear and Uhlenbeck, "The Brownian Motion of Strings and Elastic Rods." THE PHYSICAL REVIEW, Volume 38, page 1583. In this paper, they have essentially obtained the results of equations 3.10, 3.105, and 3.14.

described in terms of the R_{nm}'s of Section 3.2 - C. If we assume that the correlation of f is given by

$$\left\langle f(t_o)f(t_o')\right\rangle = D\,\delta(\sigma) \quad ,$$

we may express the correlation by the "field" in Figure 3.02b. The coefficients $R_{nm}(\zeta)$ are obtained essentially from the process indicated in equation 3.105. Hence we must evaluate the integral

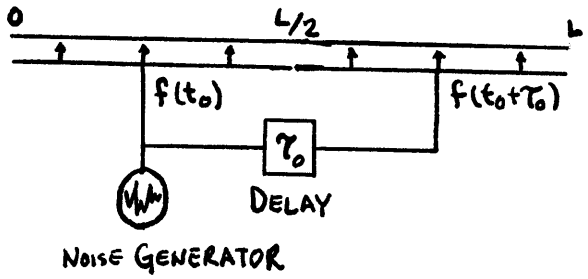


Figure 3.02a

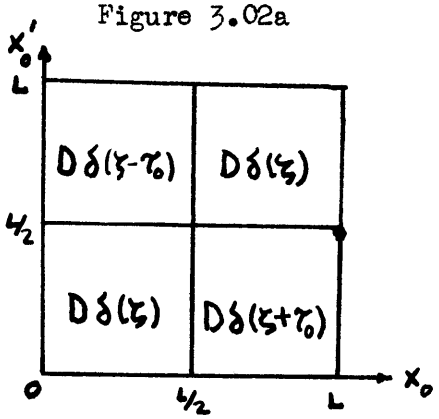


Figure 3.02b

$$R_{nm}(\zeta) = \iint_0^L dx_o dx_o' \psi_n(x_o)\psi_m(x_o')\left\langle f(x_o,t_o)f(x_o',t_o)\right\rangle_{ave} \quad , \qquad 3.105$$

where the correlation above is given by the field of Figure 3.02b. Since we are effectively expanding a product of step functions, what we need are the integrals

$$\int_0^{L/2} \sin k_m x_o\, dx_o = -\frac{1}{k_m}\left(\cos\frac{m\pi}{2} - 1\right)$$

and

$$\int_{L/2}^{L} \sin k_m x_o\, dx_o = -\frac{1}{k_m}\left(\cos m\pi - \cos\frac{m\pi}{2}\right) .$$

From these, then, and from 3.105 we have

$$R_{nm}(\zeta) = \frac{2D}{k_n k_m L}\left[\delta(\zeta)\left\{(\cos\frac{n\pi}{2} - 1)(\cos\frac{m\pi}{2} - 1)\ (\cos m\pi - \cos\frac{m\pi}{2})\right.\right.$$

$$\left.(\cos n\pi - \cos\frac{n\pi}{2})\right\} + \delta(\zeta - \tau_o)(\cos\frac{n\pi}{2} - 1)(\cos m\pi - \cos\frac{m\pi}{2})$$

$$\left. + \delta(\zeta + \tau_o)(\cos\frac{m\pi}{2} - 1)(\cos n\pi - \cos\frac{n\pi}{2})\right] .$$

For either n or m = 4ℓ, where ℓ is any positive integer ($\neq 0$), $R_{nm} = 0$. This is clear when one observes the symmetry of the set up in Figure 3.02a. If we have m and n even (twice an odd integer), then

$$\text{(a)} \qquad R_{nm}(\zeta) = \frac{8DL}{\pi^2 mn}\left\{2\delta(\zeta) - \delta(\zeta - \tau_o) - \delta(\zeta + \tau_o)\right\} .$$

For m odd, n even,

$$\text{(b)} \qquad R_{nm}(\zeta) = \frac{4DL}{\pi^2 mn}\left\{\delta(\zeta - \tau_o) - \delta(\zeta + \tau_o)\right\} .$$

For m even, n odd,

3.15

$$\text{(c)} \qquad R_{nm}(\zeta) = \frac{4DL}{\pi^2 mn}\left\{\delta(\zeta + \tau_o) - \delta(\zeta - \tau_o)\right\} .$$

And for m and n odd,

$$\text{(d)} \qquad R_{nm}(\zeta) = \frac{2DL}{\pi^2 mn}\left\{2\delta(\zeta) + \delta(\zeta + \tau_o) + \delta(\zeta - \tau_o)\right\} .$$

B. The Calculation of the Energy of the Modes

In this example we shall calculate both the mean potential energy for the m^{th} mode and the mean kinetic energy. The mean potential energy is obtained from $\langle \Phi_m^{\,2} \rangle$. From 3.10, by selecting out the terms for which n = m, this is seen to be

$$\langle \Phi_m^2 \rangle = \frac{8\pi^2}{\rho_\ell^2 \omega_{1,m}^2} \int_{-\infty}^{2t} d\mu\, e^{-\frac{\beta}{2}(2t-\mu)} \sin \frac{\omega_{1,m}}{2}(2t-\mu-\zeta)$$

$$\sin \frac{\omega_{1,m}}{2}(2t-\mu+\zeta) \int_{\mu-2t}^{2t-\mu} d\zeta\, R_{nm}(\zeta) \quad . \tag{3.16}$$

Now, for m even, we have from 3.15 (a)

$$\langle \Phi_m^2 \rangle_e = \frac{8\pi^2}{\rho_\ell^2 \omega_{1,m}^2} \int_{-\infty}^{2t} d\mu\, e^{-\frac{\beta}{2}(2t-\mu)} \sin \frac{\omega_{1,m}}{2}(2t-\mu-\zeta)$$

$$\sin \frac{\omega_{1,m}}{2}(2t-\mu+\zeta) \int_{\mu-2t}^{2t-\mu} d\zeta \cdot \frac{8LD}{\pi^2 m^2} \left\{ 2\delta(\zeta) - \delta(\zeta+\tau_o) - \delta(\zeta-\tau_o) \right\}$$

$$= \frac{64LD}{m^2 \rho_\ell^2 \omega_{1,m}^2} \left[\int_{-\infty}^{2t} d\mu\, e^{-\frac{\beta}{2}(2t-\mu)} \left\{ 1-\cos \omega_{1,m}(2t-\mu) \right\} \right.$$

$$\left. -2\int_{-\infty}^{2t-|\tau_o|} d\quad e^{-\frac{\beta}{2}(2t-\mu)} \sin \frac{\omega_{1,m}}{2}(2t-\mu-\tau_o) \sin \frac{\omega_{1,m}}{2}(2t-\mu+\tau_o) \right] .$$

This integration is straightforward and yields

$$\langle \Phi_m^2 \rangle_e = \frac{64LD}{\rho_\ell^2 \omega_{1,m} \omega_m^2 \beta m^2} e^{-\beta|\tau_o|/2}$$

$$\cdot \left\{ 2\omega_{1,m}(e^{\beta|\tau_o|/2} - \cos \omega_{1,m}\tau_o) \right.$$

$$\left. - \beta \sin \omega_{1,m}|\tau_o| \right\} . \tag{3.17}$$

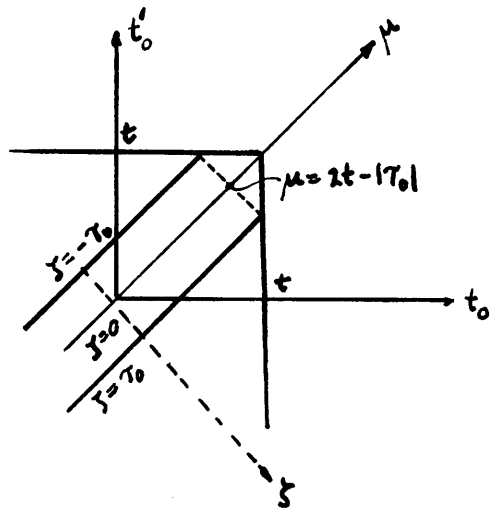


Figure 3.03

For m odd, we had in 3.15 (d)

$$R_{nm}(\zeta) = \frac{2LD}{\pi^2 m^2} \left\{ 2\delta(\zeta) + \delta(\zeta+\tau_o) + \delta(\zeta-\tau_o) \right\} ,$$

which gives

$$\langle \Phi_m^2 \rangle_o = \frac{16LD}{\rho_\ell^2 \omega_{1,m}{}^2 m^2} \left[\int_{-\infty}^{2t} d\mu\, e^{-\frac{\beta}{2}(2t-\mu)} \left\{ 1-\cos \omega_{1,m}(2t-\mu) \right\} \right.$$

$$\left. + 2 \int_{-\infty}^{2t-|\tau_o|} d\mu\, e^{-\frac{\beta}{2}(2t-\mu)} \sin \frac{\omega_{1,m}}{2}(2t-\mu-\tau_o) \sin \frac{\omega_{1,m}}{2}(2t-\mu+\tau_o) \right] ,$$

and the result of this integration is

$$\langle \Phi_m^2 \rangle_o = \frac{16LD}{\rho_\ell^2 \omega_{1,m}{}^2 m^2 \beta \omega_m^2} e^{-\beta|\tau_o|/2} \left\{ 2\omega_{1,m} \left(e^{\beta|\tau_o|/2} + \cos \omega_{1,m} \tau_o \right) \right. \qquad 3.18$$

$$\left. + \beta \sin \omega_{1,m} |\tau_o| \right\} .$$

Let us now calculate the kinetic energy for the m^{th} mode. The mean square transverse velocity is given by

$$\langle U_m^2(t) \rangle = \frac{-4\pi^2}{\rho_\ell^2 \omega_{1,m}^2} \iint_{-\infty}^{t} dt_o dt_o' R_{mm}(\zeta)\, \dot{Q}_m(t-t_o)\, \dot{Q}_m(t-t_o') , \qquad 3.19$$

where $U_m = \frac{\partial \phi_m}{\partial t}$ and $\dot{Q}_m(t-t_o) = \frac{\partial Q_m(t-t_o)}{\partial t}$. $\dot{Q}_m$ is essentially the velocity response of the m^{th} mode to an impulse. Hence,

$$\dot{Q}_m(t-t_o) = 2ie^{-\frac{\beta}{2}(t-t_o)} \left\{ -\frac{\beta}{2} \sin \omega_{1,m}(t-t_o) + \omega_{1,m} \cos \omega_{1,m}(t-t_o) \right\} .$$

For <u>even m</u>, this becomes, after some algebra,

$$\langle U_m^2 \rangle_e = \frac{64LD}{\rho_\ell^2 m^2 \beta \omega_{1,m}} e^{-\beta|\tau_o|/2} \left\{ 2\omega_{1,m} \left(e^{\beta|\tau_o|/2} - \cos \omega_{1,m} \tau_o \right) + \beta \sin \omega_{1,m} |\tau_o| \right\} , \qquad 3.20$$

and similarly, for odd m,

$$\langle U_m^2 \rangle_o = \frac{16LD}{\rho_\ell^2 m^2 \beta \omega_{1,m}} e^{-\beta|\tau_o|/2} \left\{ 2\omega_{1,m}(e^{\beta|\tau_o|/2} + \cos\omega_{1,m}\tau_o) - \beta \sin \omega_{1,m}(\tau_o) \right\} . \qquad 3.21$$

Since the mean potential energy is given by

$$V_m = \frac{1}{2} \rho_\ell \omega_m^2 \langle \bar{\Phi}_m^2 \rangle ,$$

and the mean kinetic energy by

$$K_m = \frac{1}{2} \rho_\ell \langle U_m^2 \rangle ,$$

we notice that for both the even and the odd modes the kinetic and potential energy for a mode are different by the term $2\beta \sin \omega_{1,m}(\tau_o)$. As the viscosity diminishes, this difference becomes smaller in relation to the values of the energies involved. The total energy for the even modes is

$$E_{m,e} = V_{m,e} + K_{m,e} = \frac{128LD}{\rho_\ell m^2 \beta} (1 - e^{-\beta|\tau_o|/2} \cos \omega_{1,m}\tau_o) , \qquad 3.22$$

and for the odd modes it is

$$E_{m,o} = V_{m,o} + K_{m,o} = \frac{32LD}{\rho_\ell m^2 \beta} (1 + e^{-\beta|\tau_o|/2} \cos \omega_{1,m}\tau_o) . \qquad 3.23$$

One can easily verify that the ratio $\frac{\omega_m}{\beta}$ is the Q factor of the m^{th} mode.* In Figure 3.04a we have plotted the function $(1 - e^{-\beta|\tau_o|/2} \cos \omega_{1,m}\tau_o)$

*Examine the impedance for the m^{th} mode as given by Morse, Vibration and Sound, Second Edition.

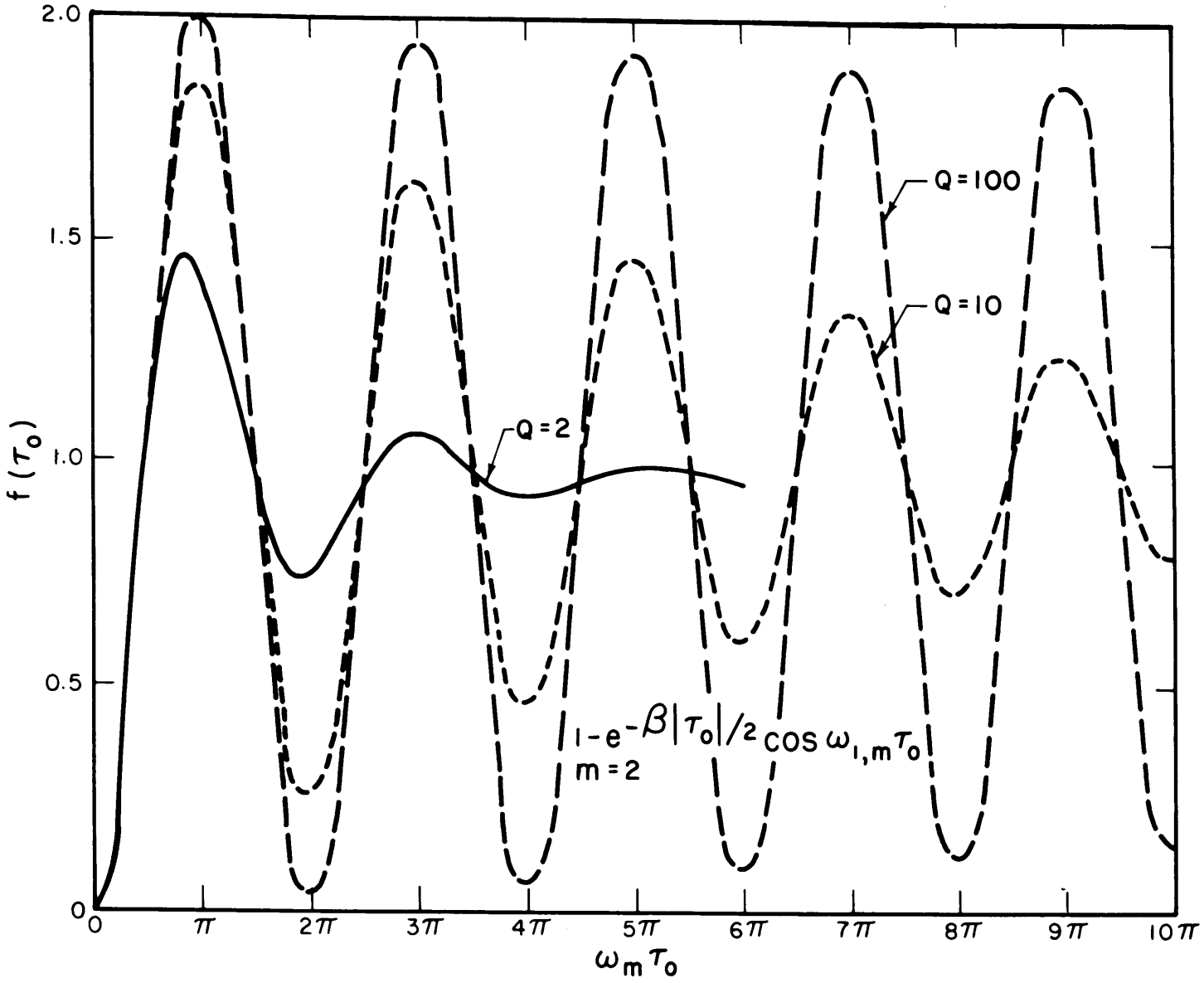


Figure 304a Average Energy in Even Modes.

for Q values of 2, 10, 100. This gives the dependence of the energy in the second mode in units of $\frac{32LD}{\rho\beta}$ as a function of τ_0. In Figure 3.04b we have plotted the function $(1+e^{-\beta|\tau_0|/2}\cos\omega_{1,n}\tau_0)$ for Q values of 1, 5, 50. This gives, in the same units as before, the dependence on τ_0 of the energy in the fundamental mode. As one would expect, for $\tau_0 = 0$ there is no energy fed to the even modes, but there is a maximum fed to the odd ones. As τ_0 advances, say for the second mode, the energy increases until it reaches a maximum at $\tau = \frac{\pi}{2\omega_{1,2}}$, which is a half period for this mode. This maximum represents a sort of coincidence effect between the source and the mode. It suggests that if we move a random noise field along so that its velocity equals the phase velocity of waves on the string, a maximum of excitation will be attained. We shall study this problem further in later examples.

Another interesting aspect of the curves is that for high damping (or low Q) the maxima and minima die out and approach an asymptote for large τ_0 which would be the excitation if the string were driven by two incoherent random sources over its two halves. The damping, then, produces a "forgetting" effect such that if one waits too long before applying the signal to the second half, the string "forgets" that the signal was ever applied before.

C. Experimental Verification

In an attempt to determine whether the coincidence effects predicted by the preceding analysis were experimentally observable, an apparatus like that diagramed in Figure 3.05 was set up. The source is a thermal noise generator, and the signal is fed to an electrostatic plate which

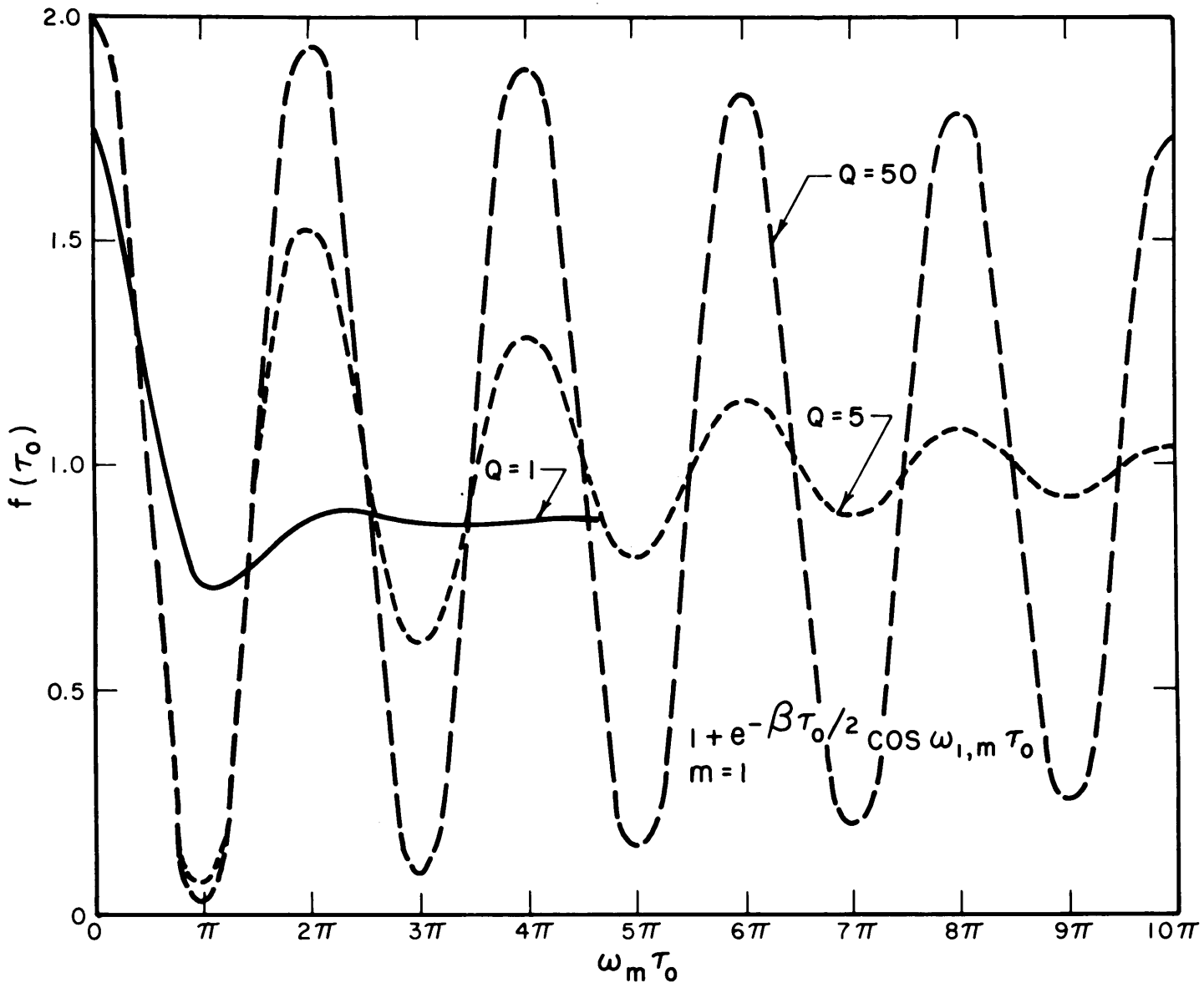


Figure 304b Average Energy in Odd Modes.

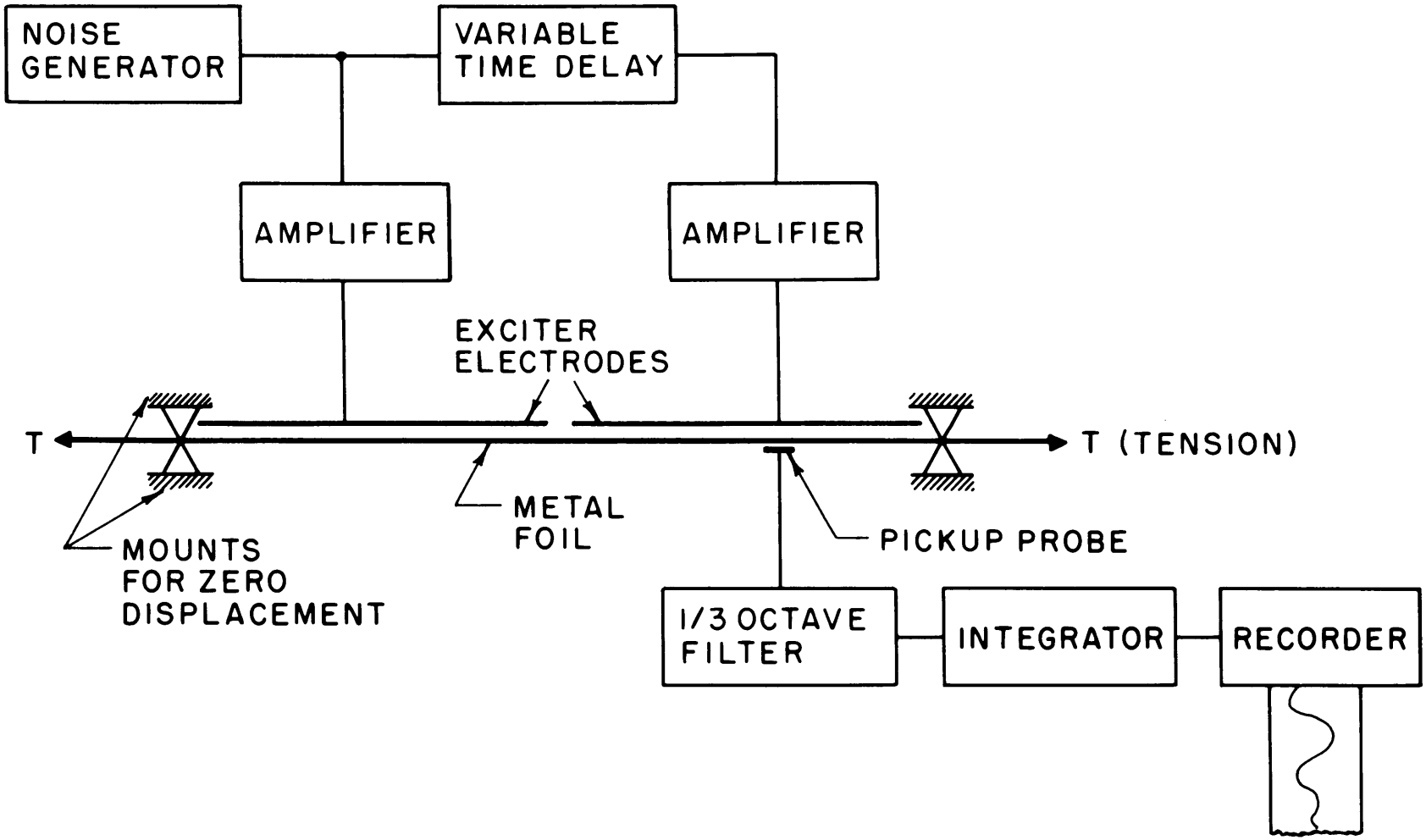
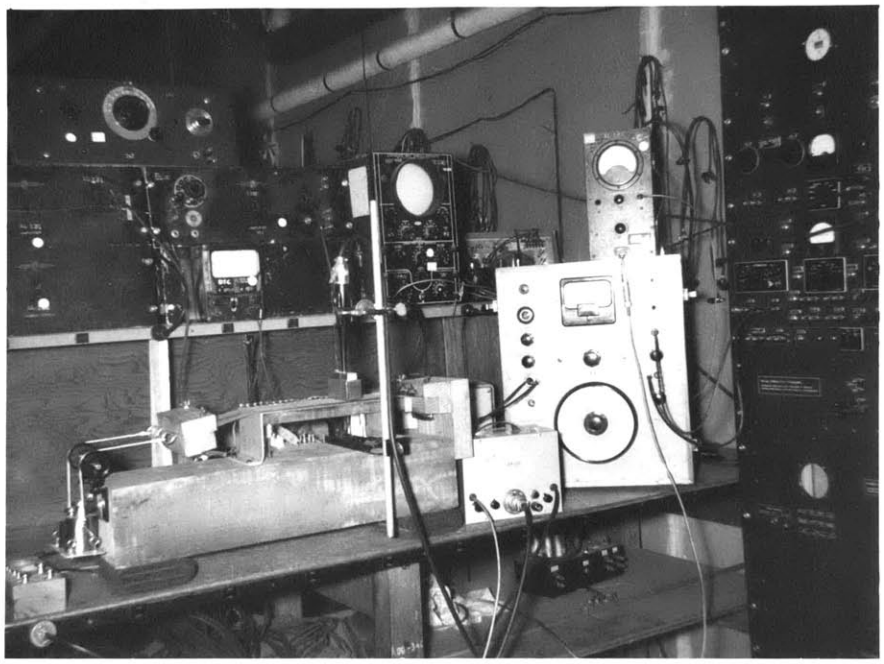


Figure 3.05 Diagram of Experimental Set-Up of Example 2.

drives an aluminum foil ribbon over half of its length. The signal is delayed by a variable time delay system[25] and fed to another electrostatic plate which drives the ribbon over the second half. The motion of the ribbon is detected by means of a modified Altec-Lansing microphone base used as a capacitance sensitive probe. A particular mode of vibration is selected by utilizing the natural selectivity of the mode and placing a third octave band filter about the resonant frequency on the output of the probe. The mean square response is obtained electronically by the squaring and integrating facilities of the Goff correlator.[26] The set up is shown in the photographs of Figure 3.06.

What is measured is essentially the mean square velocity $\langle v_m^2 \rangle$. However, for reasonably high Q the functional dependence is very nearly that of Figure 3.04. In Figure 3.07 we have plotted the results of this experiment for the second and third modes of the string as a function of time delay. The rather high background level is thought to be due to inhomogeneities of tension in the three-inch-wide ribbon. For the second mode the measured Q was 10, and for the third it was 15. The resonant frequencies were 105 cycles and 160 cycles respectively, the discrepancy in harmonicity very likely being due to motion of the supports. The agreement in terms of the shape of the curves appears to be pretty good, with the coincidence effect plainly visible.

Now that we have taken the initial step toward dealing with the problem of the moving field and have acquired some familiarity with the procedures of calculation, let us move on to the next example.



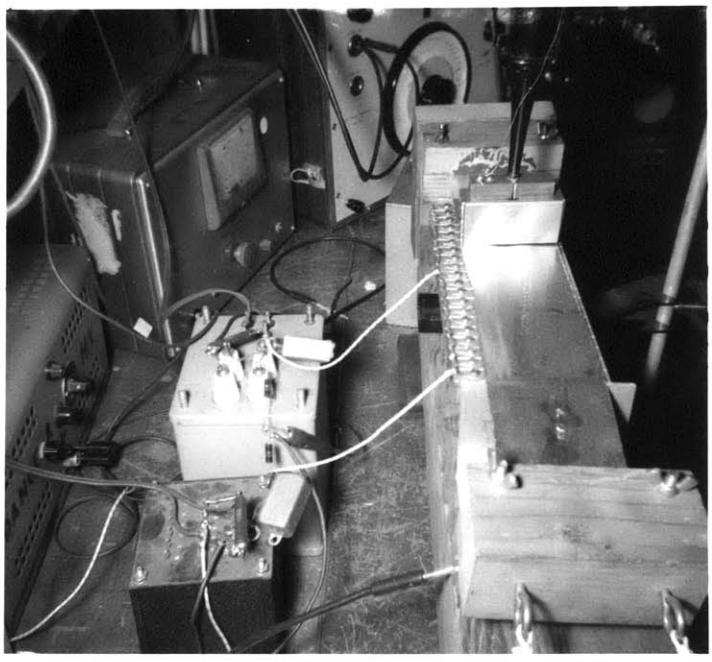


Figure 3.06. Photographs of the Experimental Set-Up of Example 2.

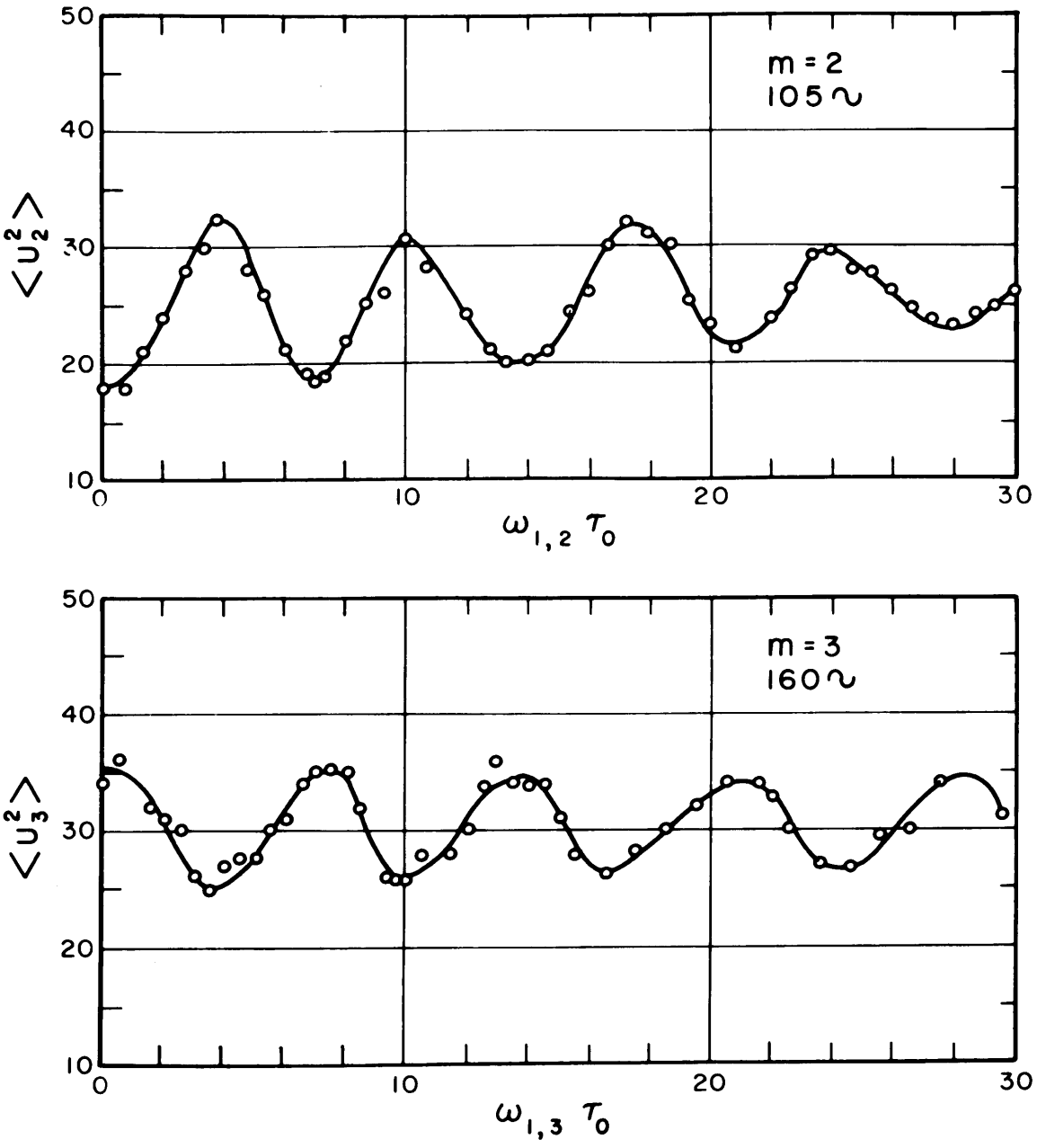


Figure 3.07 Measured Mean Square Response for Second and Third Modes.

3.4 EXAMPLE 3, THE PURELY RANDOM MOVING NOISE FIELD

In this example we shall examine the response of a finite string when excited by a moving noise field which is purely random in space. Although this is impossible to realize physically, it still can lead to useful results and is the simplest kind of correlation function to integrate. Again, we are interested in coincidence effects and will be interested in what happens when the flow velocity of the noise field is near the phase velocity of waves on the string.

A. The Correlation of the Source

We consider a force field in space (one-dimensional) which does not change with time but is a purely random function of x. That is,

$$\left\langle f(x_o) f(x_o') \right\rangle_{ave} = D\,\delta(\sigma) \quad .$$

We then allow this field to be dragged along at a velocity v, to the right, so that the correlation becomes

$$\left\langle f(x_o - vt_o) f(x_o' - vt_o') \right\rangle_{ave} = D\,\delta(\sigma - v\zeta).$$

This is the correlation function for the source which we shall use for our calculations.

B. Calculation of the Mean Square Response

With this kind of source and the Green's function we have been using, the mean square displacement of the m^{th} mode is

$$\langle \Phi_m^2 \rangle = \frac{8\pi^2 D}{\rho_\ell^2 \omega_{1,m}^2} \int_0^L \int_0^L dx_o dx_o' \, \psi_m(x_o)\psi_m(x_o') \int_{-\infty}^{2t} d\mu \, e^{-\frac{\beta}{2}(2t-\mu)}$$

$$\int_{\mu-2t}^{2t-\mu} d\zeta \sin\omega_{1,m}\left\{t-\tfrac{1}{2}(\mu+\zeta)\right\} \cdot \sin\omega_{1,m}\left\{t-\tfrac{1}{2}(\mu-\zeta)\right\}$$

$$\cdot \delta(\sigma - v\zeta) \, ,$$

which is just like equation 3.12 except for the modified source correlation function. We are interested in obtaining $\langle \Phi_m^2 \rangle$ as a function of the flow velocity v. First, we carry out the integration over ζ ($\equiv t_o - t_o'$), which merely substitutes σ/v for ζ everywhere in the integrand and modifies the limits of integration on μ as shown in Figure 3.08a.

$$\langle \Phi_m^2 \rangle = \frac{4\pi^2 D}{\rho_\ell^2 v \omega_{1,m}^2} \int_0^L \int dx_o dx_o' \psi_m(x_o)\psi_m(x_o') \int_{-\infty}^{2t-|\sigma|/v} d\mu \, e^{-\frac{\beta}{2}(2t-\mu)} \qquad 3.25$$

$$\left\{\cos\omega_{1,m}\,\sigma/v - \cos\omega_{1,m}(2t-\mu)\right\} .$$

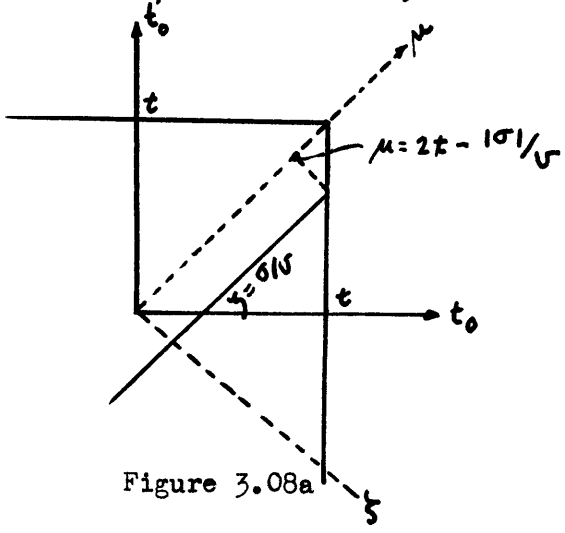


Figure 3.08a

If we substitute $\gamma = 2t-\mu$ and integrate over γ, we have

$$\langle \Phi_m^2 \rangle = \frac{4\pi^2 D}{v\omega_{1,m}\omega_m^2 L} \int_0^L \int dx_o dx_o'$$

$$e^{-\beta|\sigma|/2v}\left(\frac{2\omega_{1,m}}{\beta}\cos\omega_{1,m}\frac{\sigma}{v}\right.$$

$$\left. + \sin\omega_{1,m}\frac{|\sigma|}{v}\right)\cdot(\cos k_m\sigma - \cos k_m\rho) \, . \qquad 3.26$$

The integration over the space variables extends over the region shown

in Figure 3.08b. It will be noticed that the integrand is symmetric about the lines $\rho = L$ and $\sigma = 0$. Hence we may integrate over the shaded triangle and multiply by 4. Our expression becomes

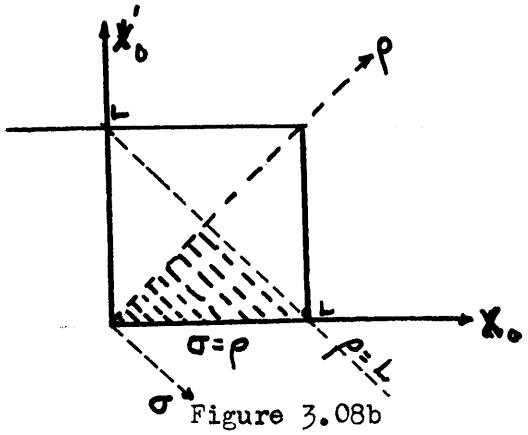


Figure 3.08b

$$\frac{8\pi^2 D}{v\omega_{1,m}\omega_m^2 L\rho_\ell^2}\int_0^L d\rho\int_0^\rho d\sigma\, e^{-\beta\sigma/2v}\left(\frac{2\omega_{1,m}}{\beta}\cos\omega_{1,m}\sigma/v + \sin\omega_{1,m}\sigma/v\right)$$

$$\cdot(\cos k_m\sigma - \cos k_m\rho)\ .$$

After a good deal of algebra, this becomes

$$\langle\Phi_m^2\rangle = \frac{8\pi^2\alpha D}{\rho_\ell^2 k_m^5 L}\left[\frac{2\alpha^2(8\alpha^2\delta_m^2+12\delta_m^2-\alpha^4-2\alpha^2-1-16\delta_m^4)}{\delta_m\{(\alpha^2+1)^2-4\delta_m^2\alpha^2\}^2}\right.$$

$$(-)^m \sin k_m\delta_m L/\alpha\, e^{-m\pi/2\alpha Q_m}$$

$$+\frac{4\alpha^2 Q_m(8\alpha^2\delta_m^2+20\delta_m^2-\alpha^4-6\alpha^2-5-16\delta_m^4)}{\{(\alpha^2+1)^2-4\delta_m^2\alpha^2\}^2} \qquad 3.27$$

$$\left.\left\{(-)^m e^{-\frac{m\pi}{2\alpha Q_m}}\cos\frac{k_m\delta_m L}{\alpha}-1\right\}+\frac{2m\pi}{\alpha\{(\alpha^2+1)^2-4\delta_m^2\alpha^2\}}\right]\ ,$$

where $\alpha = \frac{v}{c}$ is the ratio of the velocity of the noise field to the phase velocity of waves on the string, $\delta_m = \frac{\omega_{1,m}}{\omega_m}$ is the ratio of the modified frequency to that of the undamped mode, and $Q_m = \frac{\omega_m}{\beta}$ is

the "quality factor" of the m^{th} mode. This expression is much simplified when one assumes that $\delta_m = 1$, which is equivalent to saying that $Q_m^2 \gg 1$. This assumption is good over most of the values of α but breaks down and leads to a singular response at $\alpha = 1$. However, with viscous dissipation an infinite response is impossible. If we take $\delta_m = 1$ above, we get

$$\langle \Phi_m^2 \rangle_{\delta_m=1} = \frac{16\pi^2\alpha^3 p}{k_n^5 L \rho_\ell^2 (\alpha^2-1)^2} \left[2Q_m + \frac{m\pi}{\alpha^3} - \left(2Q_m \cos\frac{m\pi}{\alpha} + \frac{\alpha^2-5}{\alpha^2-1}\sin\frac{m\pi}{\alpha}\right)(-)^m e^{-\frac{m\pi}{2\alpha Q_m}} \right]. \qquad 3.28$$

This has been calculated and plotted for $m=1$, $Q_1=10$; $m=4$, $Q_4=10$; $m=4$, $Q_4=40$ and is shown as the solid parts of the curves in Figure 3.09. The dashed part of the first curve was calculated from the exact expression for the $m=1$ mode and indicates the peak in the response of the system slightly below $\alpha = 1$, or when the phase velocity of waves on the string equals the flow velocity of the forcing field. The "coincidence effect" occurring at α slightly less than one is due to the effect of viscosity. It is also interesting to note that when $\alpha \to 0$, $\langle \Phi_m^2 \rangle$ does not vanish, since a static force field will produce in the string a deflection which will have a non-zero mean square value.

In the next example we shall introduce a correlation time or length for the source. It is interesting to note that a sort of correlation time or length is encountered for viscosity, the effect of which may be seen from equation 3.27. We shall define a length $L_o = c/\beta$ and call this the correlation length for viscosity. Essentially, this means that two points on the string separated by a distance greater than L_o have their

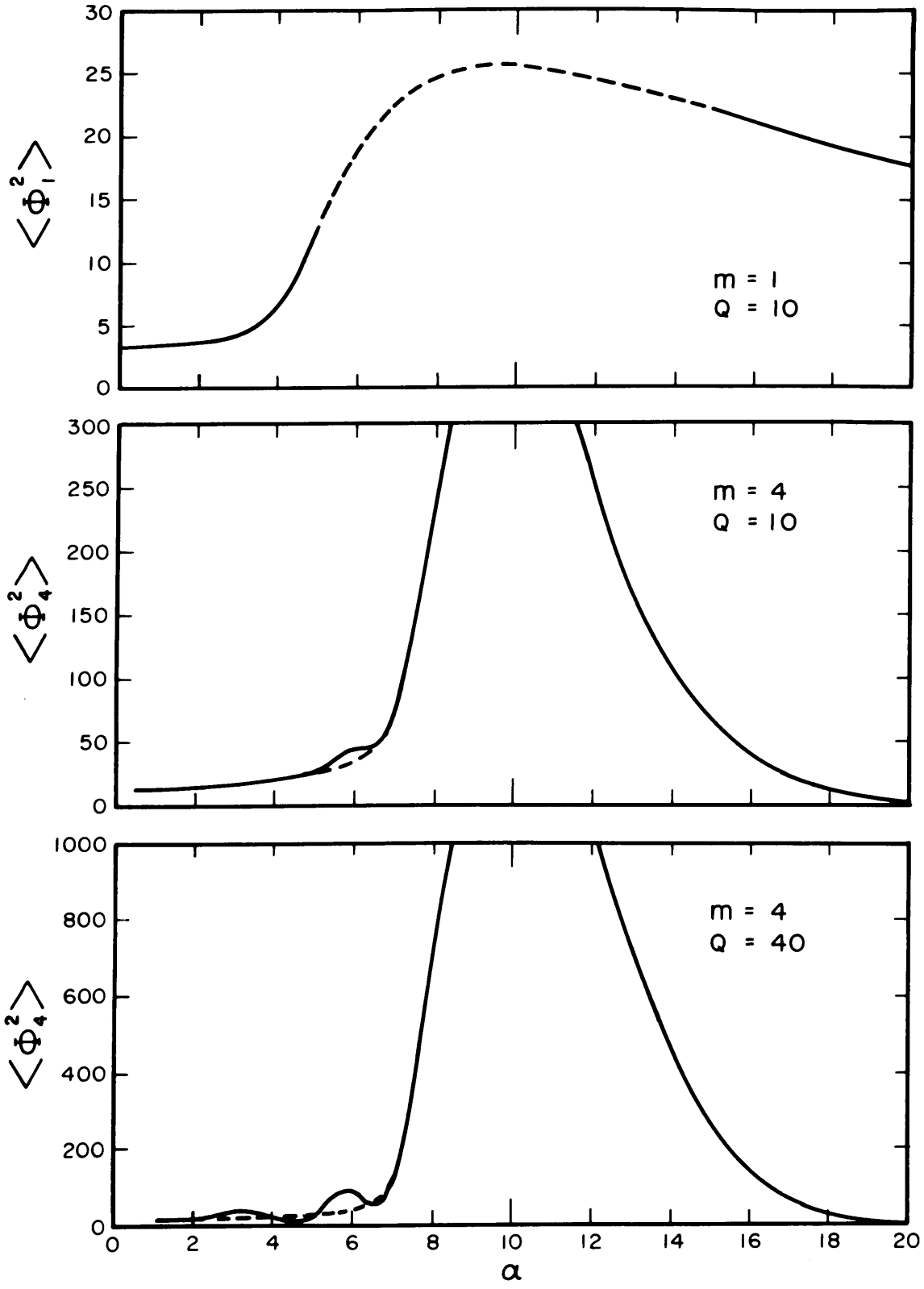


Figure 3.09 Mean Square Response to Moving Noise Field.

free motion uncorrelated since wavelets started at one point have been dissipated before they reach the second point. Of course the total motion at the two points will still be correlated because of the correlation of the source, but not nearly so much as if they were separated by a distance $\ll L_o$. For modes of wavelength greater than L_o, we might not expect very much of a coincidence effect as $\alpha \to 1$. To test this, let us consider the simple situation when $\omega_m = \frac{\beta}{2}$ and choose the first mode of the string, m =1. Then, $\omega_1 = \frac{\pi c}{L} = \frac{\beta}{2}$, or $L = 2\pi\frac{c}{\beta} = 2\pi L_o$, and $Q_1 = 1/2$. Hence, in this situation, we have the correlation length about one-sixth of the length of the string. Equation 3.27 becomes

$$\langle \Phi_1^2 \rangle = \frac{16\pi^2 D_c}{\omega_1^5 \rho^2 L} \cdot \frac{1}{(\alpha^2+1)^2} \left\{ \alpha^3 \frac{(\alpha^2+5)(e^{-\frac{\pi}{\alpha}}+1)}{\alpha^2+1} + \pi \right\} , \qquad 3.29$$

which is plotted in units of $\frac{16\pi^2 D_c}{\rho^2 L \omega_1^5}$ as a function of α in Figure 3.10. It is evident that the coincidence effect has disappeared as expected. Since we have let $\omega_{1,1}$ go to zero, one might object that this illustration is not conclusive -- that the effect could be due to the non-oscillatory nature of the mode which prevented coincidence from occurring. That such correlation lengths do affect coincidence, even if the motion is oscillatory, shall be seen in the next example and in Chapter **IV**. At present we must regard the result in Figure 3.08 as only a hint of the effect of correlation lengths.

Since it is experimentally very difficult to set up a noise field like that postulated in this example, we have not attempted to get experimental verification of Figure 3.09. A flowing turbulent field does

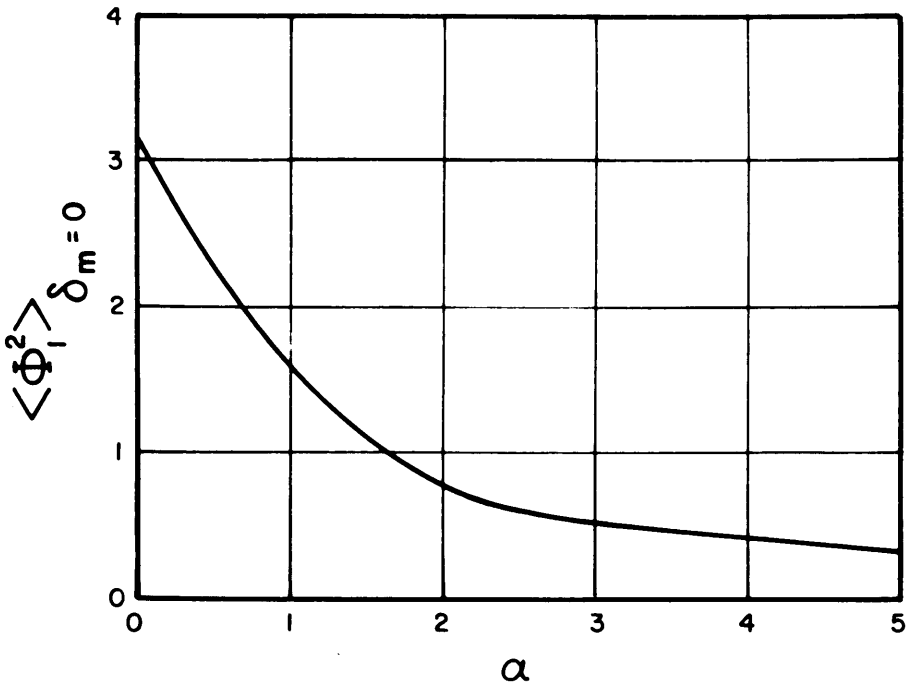


Figure 3.10 Mean Square Response for High Damping.

have properties which are capable of representation by a correlation function similar to that of equation 3.24, and in the next example we shall make an attempt to approach the representation in one dimension of the kind of force field applied to a string or ribbon by a flowing turbulent field.

3.5 EXAMPLE 4, AN APPROACH TO TURBULENCE FORCING

What we should like to do is to predict the mean square response of a finite ribbon to a field of turbulence flowing with a velocity U in the direction of the length of the ribbon. This is a situation we are able to set up experimentally and represents a problem which people in aerodynamics are concerned about: namely, the excitation of aerodynamic surfaces by turbulent flow and coincidence effects which occur when the flow velocity of the stream reaches the phase velocity of waves in the structure.

A. The Noise Field

In this section we wish, then, to leave the idealized moving field and attempt to make the noise field correspond more nearly to turbulence. Let us examine the previously assumed noise field and see in what manner it needs to be corrected.

1) The δ-function correlation: Using a δ-function correlation says, in effect, that the spectrum of frequencies and wavelengths exciting the string is perfectly flat. This has a tremendous simplifying effect on the integrals; and since we measure the response of a particular mode which usually occupies only a small part of the spectrum, this is

a satisfactory assumption as long as the actual source is sensibly flat over the spectrum of the mode.

2) The correlation period: Thus far we have assumed that if we measured, as a function of time, the source at two points separated by a distance σ, then we could obtain perfect correlation if we only delayed one time function by the time σ/v. This is a result of assuming that the field does not change as it flows along, but in turbulence there are three processes at work changing the field. These we shall call internal flow, turbulence production, and decay. Internal flow is just a result of the existence of pressure gradients in the fluid, and is predicted, as is the decay, by the Navier-Stokes equation. The field of turbulence is constantly replenished by the creation of vortices by disruptions in the boundary layer. This creates a loss of correlation by the introduction of random components of flow between the measuring points. Also there is the decay of motion due to the viscous terms in the Navier-Stokes equation, which transfer energy from the fluid motion to heat. These processes tend to decrease the amount of correlation which one will obtain after a given time delay. It is reasonable and useful to assume that they cause a loss of correlation by the factor $e^{-|\tau|/\tau}$, where τ is a sort of lifetime for the state of turbulence to be determined by experiment.

3) The dependence of the amplitude of fluctuations on flow velocity: In addition, we shall assume that the strength of the fluctuations depends on the flow velocities, as one may easily demonstrate by experiment. In fact, the dependence seems to be linear, but we shall leave it general for the moment as a function $g(\alpha)$. Hence the correlation will go as

$g_0^2(\alpha)$.

Taking account of these modifications, the source correlation is

$$\left\langle f(x_o - vt_o) f(x_o' - vt_o') \right\rangle_{ave} = D\, g_0^2(\alpha)\, \delta(\sigma - v\zeta) e^{-|\zeta|/\tau} \quad . \qquad 3.31$$

B. Calculation of the Mean Square Response

Proceeding as we did in the previous example but using 3.31, the mean square response of the m^{th} mode is

$$\langle \Phi_m^2 \rangle = \frac{4\pi^2 D g_0^2(\alpha)}{\rho_\ell^2 \omega_{1,m}^2} \iint_0^L dx_o dx_o' \psi_m(x_o) \psi_m(x_o') \int_{-\infty}^{2t} d\mu\, e^{-\frac{\beta}{2}(2t-\mu)}$$

$$\cdot \int_{\mu-2t}^{2t-\mu} d\zeta\, e^{-|\zeta|/\tau} \left\{ \cos \omega_{1,m}\zeta - \cos \omega_{1,m}(2t-\mu) \right\} \delta(\sigma - v\zeta). \qquad 3.32$$

Integrating over ζ and changing the variables x_o, x_o' to ρ, σ according to the transformation 3.11, one has

$$\langle \Phi_m^2 \rangle = \frac{8\pi^2 D g_0^2}{v \omega_{1,m} \omega_m^2 \rho_\ell^2 L} \int_0^L d\rho \int_0^\rho d\sigma\, e^{-(\frac{1}{\tau}+\frac{\beta}{2})\frac{\sigma}{v}} \Bigg[-\cos k_m \rho \left(\frac{2\omega_{1,m}}{\beta} \cos \omega_{1,m}\frac{\sigma}{v} \right.$$

$$\left. + \sin \omega_{1,m}\frac{\sigma}{v} \right) + \frac{1}{2} \left\{ \sin k_m \left(1 + \frac{\delta_m}{\alpha}\right)\sigma - \sin k_m \left(1 - \frac{\delta_m}{\alpha}\right)\sigma \right\}$$

$$+ \frac{\omega_{1,m}}{\beta} \left\{ \cos k_m \left(1 + \frac{\delta_m}{\alpha}\right)\sigma + \cos k_m \left(1 - \frac{\delta_m}{\alpha}\right)\sigma \right\} \Bigg] \quad . \qquad 3.325$$

It would be very laborious to evaluate this exactly. The integral over σ from $0 \to \rho$ is allowed to go to $+\infty$ since most of the contribution is

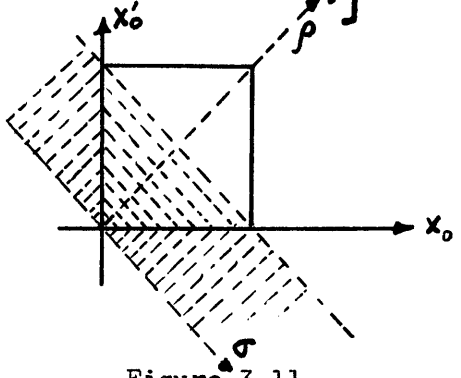


Figure 3.11

near the line $\sigma = 0$. This means that in order to expect our results to be valid, we must have $c\tau \ll L$. In the intermediate situations, where the correlation length is a half or a third the length of the string, one expects to find features like those in the solution of the last section and also properties based on the assumption that the correlation length is small compared to the length of the string. In particular, it seems like the small sub-peaks, which appear in Figure 3.07 between $\alpha = 0$ and $\alpha = 1$ for $m = 4$, are a result of the integration limit of ρ on σ, which we neglect here. With this approximation, the integrals are quite straightforward and one obtains

$$\langle \Phi_m^2 \rangle = \frac{16\pi^2 g^2(\alpha) D m \lambda Q_m}{\rho_\ell^2 c^4} \left[\frac{(Q_m + m\pi\lambda)^2 + (m\pi\lambda Q_m)^2 \left(\alpha^2 + 1 + \frac{m\pi\lambda}{Q_m}\right)}{(Q_m + m\pi\lambda)^2 + 2Q_m(m\pi\lambda)^2(\alpha^2+1)(Q_m + m\pi\lambda) + (m\pi\lambda)^4 \left\{ (\alpha^2-1)^2 Q_m^2 + \alpha^2 \right\}} \right] \qquad 3.33$$

where $\lambda = {}^{c\tau}/_{L}$, the ratio of the correlation length to the length of the string. This function has interesting properties which we may investigate by plotting it for various values of the parameters. Let us assume that λ is 0.1 and examine the effect of coincidence between wave and flow velocities by plotting the bracketed expression in 3.33 as a function of α for various values of m. The results are as shown in Figure 3.12. Coincidence is hardly visible for $m = 1,2$, but becomes increasingly manifest as the mode number becomes higher and the wavelength shorter. This is what we expected on the basis of the viscosity correlation length in the previous example.

If we set $g(\alpha) = \alpha$ -- as is suggested by experimental results -- and include $m\lambda Q_m$, then we have the results for $\langle \Phi_m^2 \rangle$ in units of $\frac{16\pi^2 D}{\rho_\ell^2 c^4}$, which should suggest a functional dependence for the experimental

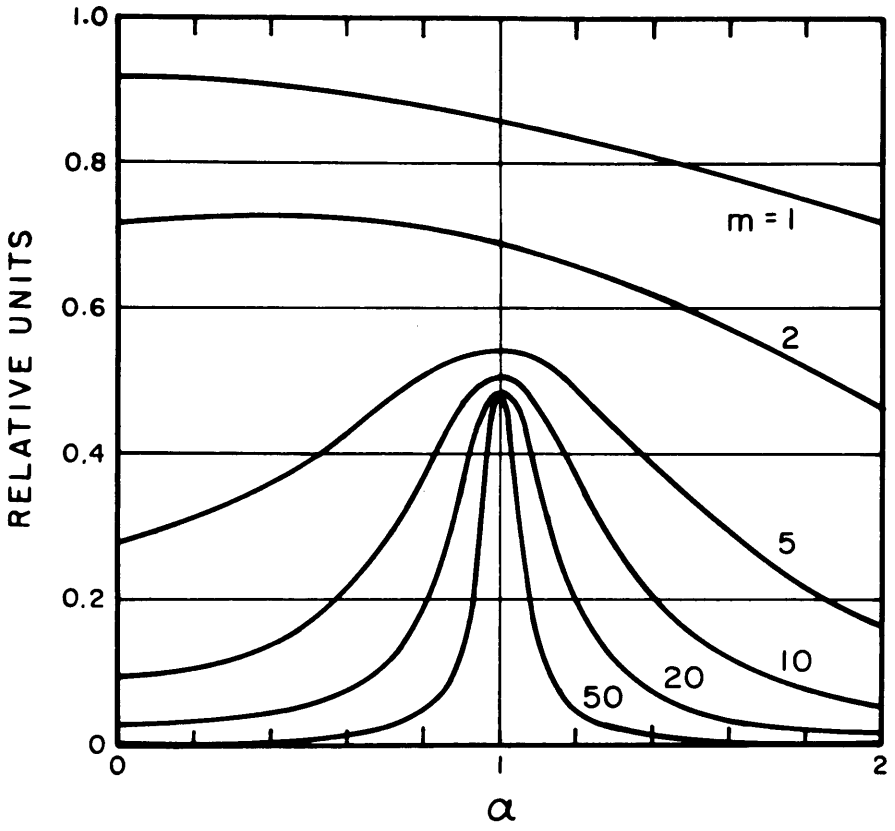


Figure 3.12 Coincidence Factor Versus Flow Velocity.

results in the next chapter. These curves are plotted in Figure 3.13.

We notice from the expression 3.325 that $\frac{1}{\tau}$ and $\frac{\beta}{2}$ enter into the exponential in the same manner. It is basically this exponential which determines whether or not the coincidence effect occurs. If we multiply $\frac{1}{\tau}$ and $\frac{\beta}{2}$ by $\frac{1}{c}$, we have $\frac{1}{\ell}$ and $\frac{1}{2L_o}$, respectively, which tell us that the correlation lengths of the source and of the viscous string both have a similar effect on the coincidence phenomenon. However, since $L_o = 3.18L$ in this example, there is no observable effect due to the viscosity. In Chapter V when we consider the infinite string, the viscous correlation length will be seen to have an effect very similar to that produced by the source correlation length $c\tau$ in this example.

3.6 CONCLUSION

In the next chapter we shall study experimentally the excitation of finite strings and bars by turbulent flow. We shall expect the former to bear out somewhat our work of Section 3.5, while the latter is included for its experimental interest. That is, when the bar mode number is high enough so that the end effects are negligible, the excitation versus flow velocity should be very similar to that of the string. In addition we wish to measure the properties of a flowing turbulent field as a random noise field and evaluate some of the quantities introduced in this chapter as parameters, namely $\xi(\alpha)$ and τ.

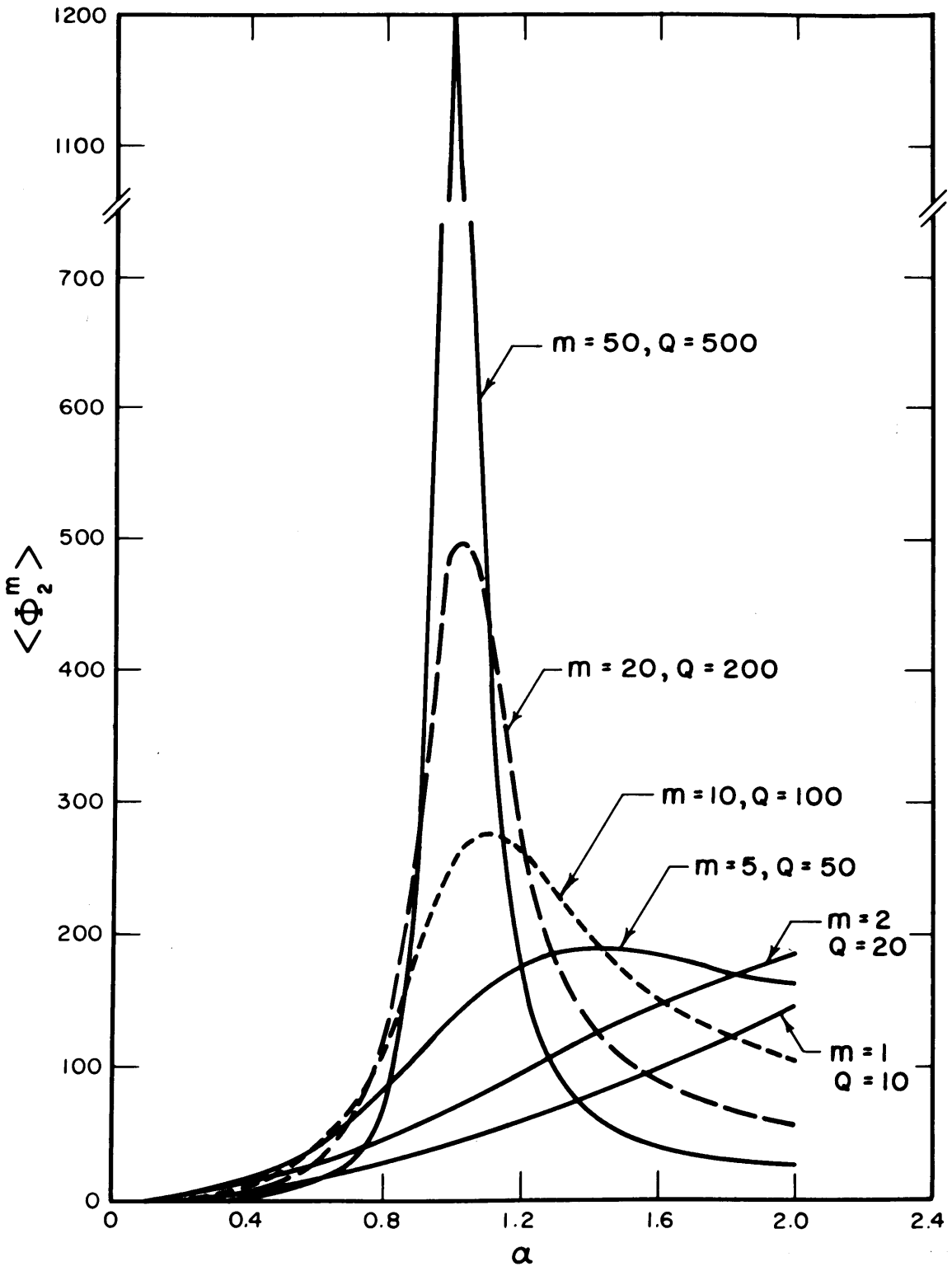


Figure 3.13 Response of Finite String to Moving and Changing Noise Field.

IV EXCITATION OF FINITE STRINGS AND BARS BY FLOWING TURBULENCE

4.1 INTRODUCTION

In the preceding chapter we spent a great deal of time calculating the mean square displacement of strings which are excited by stationary and moving noise fields. In the case of the moving noise field, in section 3.5 we made several intuitive assumptions concerning its correlation function with the intention of approximating a field of turbulent flow. With these assumptions the excitation of modes of the string was predicted. The first part of this chapter is concerned with testing the validity of our hypothesized noise field. Subsequently, we experimentally set up a finite ribbon in a pipe carrying turbulent flow and compare the mean square deflection of various modes of the string with that predicted by the theory.

Finally, a thin elastic bar is set up in an arrangement similar to that of the ribbon in the preceding paragraph. Its motion as a function of frequency and flow velocity is likewise studied and the similarities with and differences from the string are noted.

4.2 TURBULENCE AS A NOISE FIELD

We shall consider the forcing effect in turbulent flow to lie in the pressure fluctuations which it contains. At a boundary of a system containing flow, there are two forces due to the passage of the fluid -- a tangential force due to viscous shear and a normal force due to the pressure. If the boundary can move in a direction normal to its surface but not tangentially, the pressure fluctuations will excite it into vi-

bration. A ribbon of finite width and length, fastened at its ends, with turbulent flow passing along its length may be considered such a boundary. We are interested, then, in the correlation field of the pressure fluctuations in turbulent flow. In terms of our assumptions concerning the noise field in Section 3.5, we are interested in the spectrum of the pressure fluctuations, the correlation decay constant τ, and the dependence of the fluctuations on the flow velocity.

All of the work done with strings and bars was done in rectangular brass tubing, one-quarter by one-half inches inside dimensions. The walls were one-sixteenth inches thick, and for all practical purposes would be considered rigid. Air flow was introduced at one end of the tube. There were no precautions taken to obtain a smooth entry flow pattern, and the fluid motion was fully turbulent by the time it reached the experimental apparatus.

A. Mean Square Pressure

For the measurement of the mean square pressure fluctuation a very small hole (.020 inches in diameter) was drilled in the side of the tubing. An Altec 29-B microphone was then mounted on the side of the tube to pick up the pressure fluctuations as they were transmitted by the small hole. A photograph of this set-up is shown in Figure 4.01. In the photograph there are shown several microphone mounts, which were used for the work described in the next paragraph. The results of this experiment are shown in Figure 4.02, which shows two fairly linear regions. This is the reason for the assumption of a linear dependence in Chapter III. Further interpretation is deferred to the following paragraph.

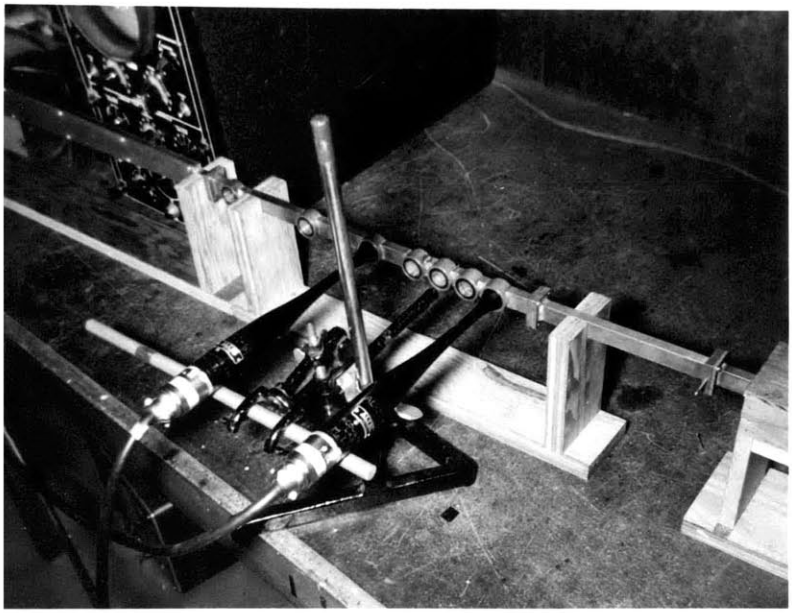


Figure 4.01. Photograph of Pressure Measuring Experiment.

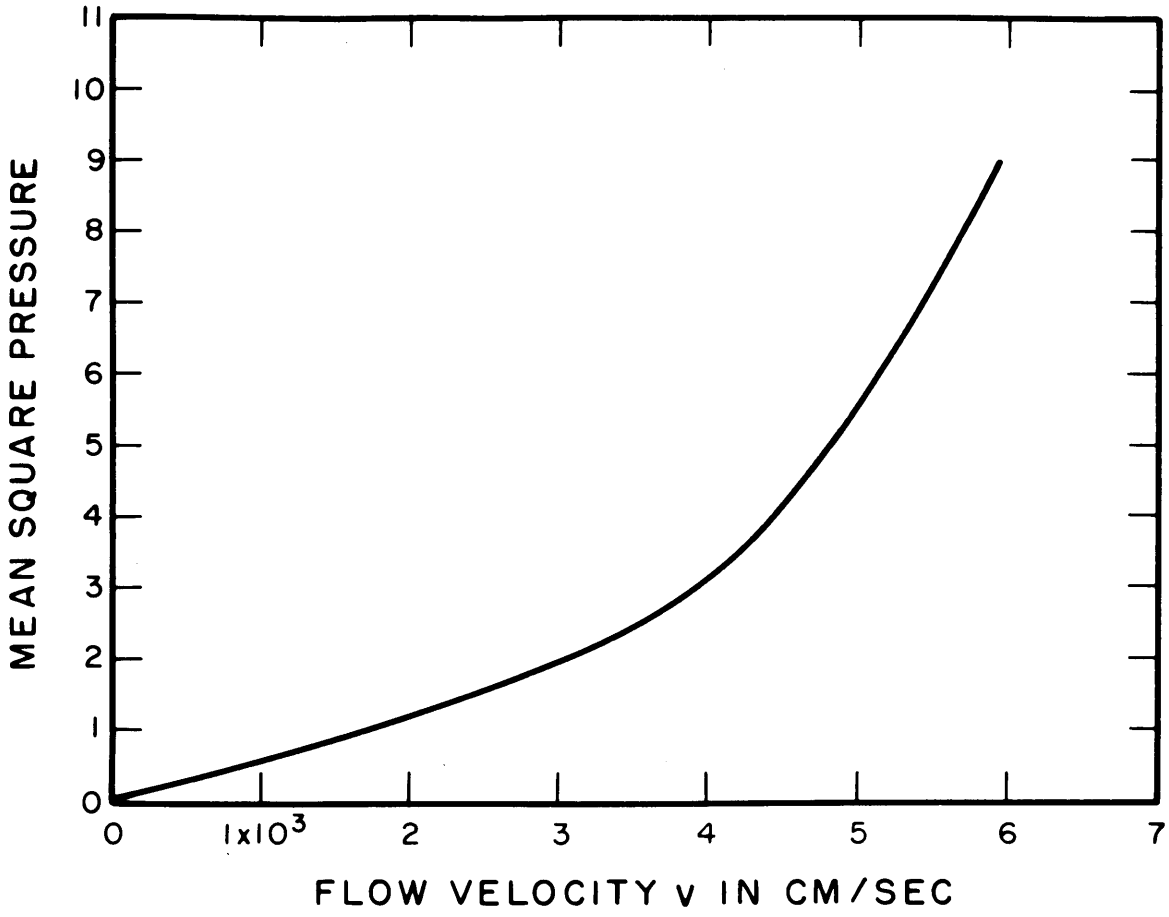


Figure 4.02 Dependence of Pressure Fluctuations on Flow Velocity.

B. Pressure Correlation

In order to measure the correlation function of the pressure fluctuations in space and time, one must make the rather reasonable assumptions that the noise field is spatially homogeneous in the direction of flow, and the fluctuations at a point produce a stationary time function. One then uses two microphones, separated by some known distance, and cross correlates the signals coming from the two microphones as a function of time delay. For each separation the maximum correlation is plotted in Figure 4.03 on a semi-log scale. The minimum separation possible for the set-up shown in Figure 4.01 is five-eighths inches, and the maximum is twelve and one-eighth inches. The correlation seems to have two very distinct decay constants, a steep one for small separation and a long slow decay for large separations. The flow velocity in this measurement was 2500 cm./sec., which is about 1000 inches/sec.

A hint at the reason for these two decays was obtained when it was noted that the peak correlation at 12 $\frac{1}{8}$ inches occurred at a 1 millisecond (msec.) time delay. The flow traveling at 1000 inches per second would take 12 msec. to traverse the distance so that the pressure fluctuations being correlated at this separation were not travelling with the flow. Since sound travels about 1000 feet/second, the time delay of one msec. leads one to believe that the pressure fluctuations are acoustic. The initial drop-off is due to the hydrodynamic fluctuations which decay rapidly but travel with the flow.

It is the fluctuations that travel with the flow which we expect must excite the ribbon and bar. One reason is that acoustic plane waves travel much too fast to achieve coincidence with the waves on the ribbon.

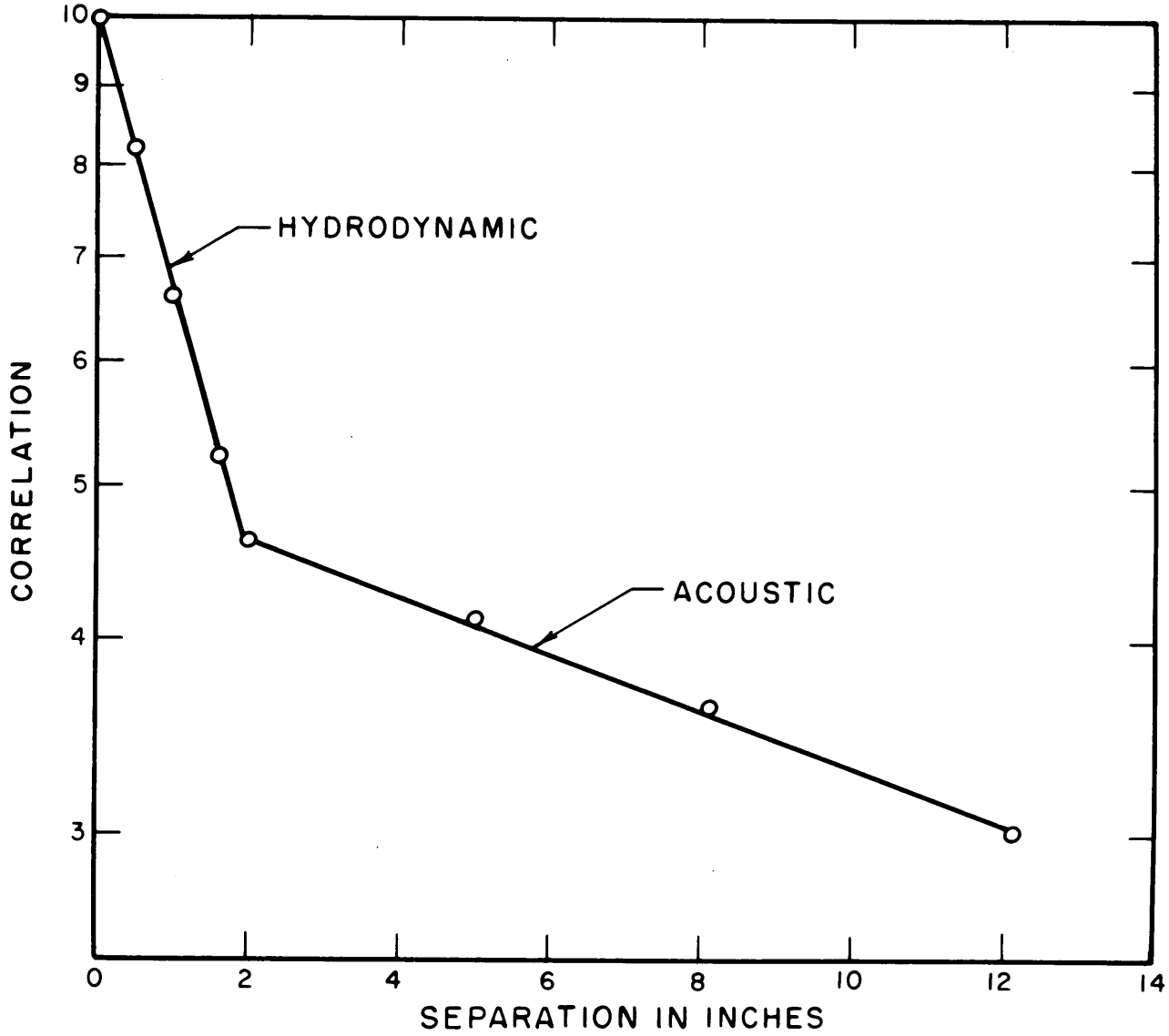


Figure 4.03 Pressure Correlation Versus Separation.

The other is that an acoustic plane wave would push on both sides of the ribbon and bar equally, thus forcing it very little, if at all.

Velocity correlation measurements* taken perpendicular to the flow in a tube of similar dimensions to the rectangular one above indicate that correlation as a function of separation extends only about one or two millimeters across the stream, which may be interpreted as a mean eddy size for the turbulence.[27] If one takes pressure correlation measurements around the circumference of a tube which contains turbulent flow moving along the tube's axis, the correlation function has a peak about one or two millimeters wide, leveling off to a constant value as one separates the probes farther apart. This constant value is about fifty per cent of the correlation at zero separation. The fifty per cent asymptotic correlation is interpreted as being due to plane waves of acoustic noise in the tube, while the width of one to two mm. enables us to interpret the other fifty per cent as hydrodynamic fluctuations. It is clear that direct measurement of the pressure fluctuations is not a satisfactory way to measure the property of the hydrodynamic fluctuation field which travels with the average flow velocity of the stream.

C. Velocity Correlation Functions*

Since we are concerned with the pressure fluctuations which travel with the mean velocity of the flow, we should like to be able to separate out these hydrodynamic pressures, as we have called them, from the acoustic pressures, or aerodynamic noise. From elementary considerations it is apparent that measurements of the velocity fluctuations should do this for

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*The experimental work in this section was done with the help of Stuart C. Baker.

us. The relation between the acoustic pressure p_a and the corresponding particle velocity u_a is for plane waves

$$p_a = \rho_f c u_a \quad ,$$

where ρ_f is the fluid density and c is the sound velocity. From Bernoulli's law, the hydrodynamic pressure p_h and velocity u_h are related by

$$p_h = \frac{1}{2} \rho_f u_h^2 \quad .$$

Since p_h and p_a were observed to be of the same order of magnitude, then we must have

$$c u_a \simeq u_h^2 \quad , \text{ or } \quad \frac{u_a}{u_h} \simeq \frac{u_h}{c} \quad .$$

That is, the hydrodynamic velocity forms a geometric mean between u_a and c; and since c is ordinarily very much larger than u_h, it follows that u_h is larger than u_a by the same factor. As expected, all trace of the acoustic motion disappeared when velocity correlations were used. In particular, with the pressure correlation we had particular difficulty with the setting up of acoustic standing waves in the pipe. When velocity measurements were taken, this trouble vanished.

The velocity cross correlations were taken with the set-up shown in Figure 4.04. The velocity fluctuations were picked up by two hot wires entering the tube from opposite sides. The hot wires were mounted so that they were sensitive to velocity fluctuations in the direction of the flow. The position of one of them was adjustable so that it might be moved cross stream or downstream from the other. The two hot wires were

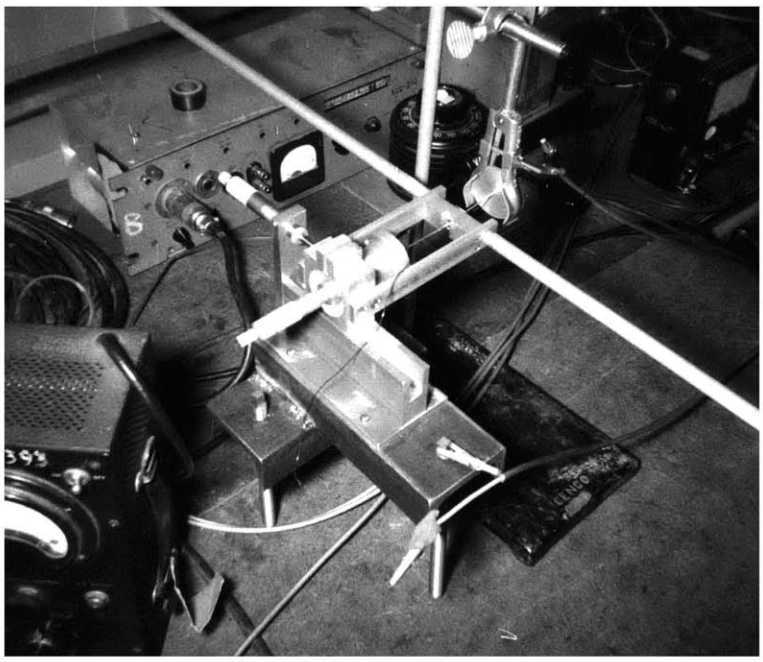

Figure 4.04. Photograph of Velocity Correlation Experiment.

adjusted for zero separation, and a cross correlation of the signals coming from them was taken as a function of time delay. As one would expect, the maximum correlation occurred at zero delay. The movable probe was then moved downstream, and cross correlations were taken at 1, 2, 5, 10, 20 mm. separation for two flow velocities -- 1800 and 3600 cm./sec. In all cases the peak correlation as a function of delay occurred when the delay ζ was equal to the separation σ divided by the flow velocity v, i.e.,

$$\zeta_{max} = \frac{\sigma}{v} \quad .$$

This is in agreement with the concept of a moving noise field which was represented by $\delta(\sigma - v\zeta)$, which has its maximum when $\zeta = \sigma/v$. Since the turbulence, as well as our instruments, had a finite spectrum, a δ-function was not obtained, but a very sharp correlation maximum was obtained having half width $\sim \frac{1}{10}$ msec. -- indicating a spectrum up to about 10 kc.

In Figure 4.05 we have plotted on semi-log paper the maxima of the correlation functions above as a function of delay (or separation) for the two flow velocities used -- 1800 and 3600 cm./sec. Aside from an initial steep decay, the plot is fairly linear, indicating that an assumption of exponential decay of the correlation function is quite adequate in describing the turbulence correlation function. The initial steep decay near $\zeta = 0$ is thought to be due to turbulence created by the hot wires themselves.

It is seen from 4.05 that the $\frac{1}{e}$ th delay for 1800 cm./sec. is 1.30 msec., while for 3600 cm./sec. it is 1.0 msec. From these values one may

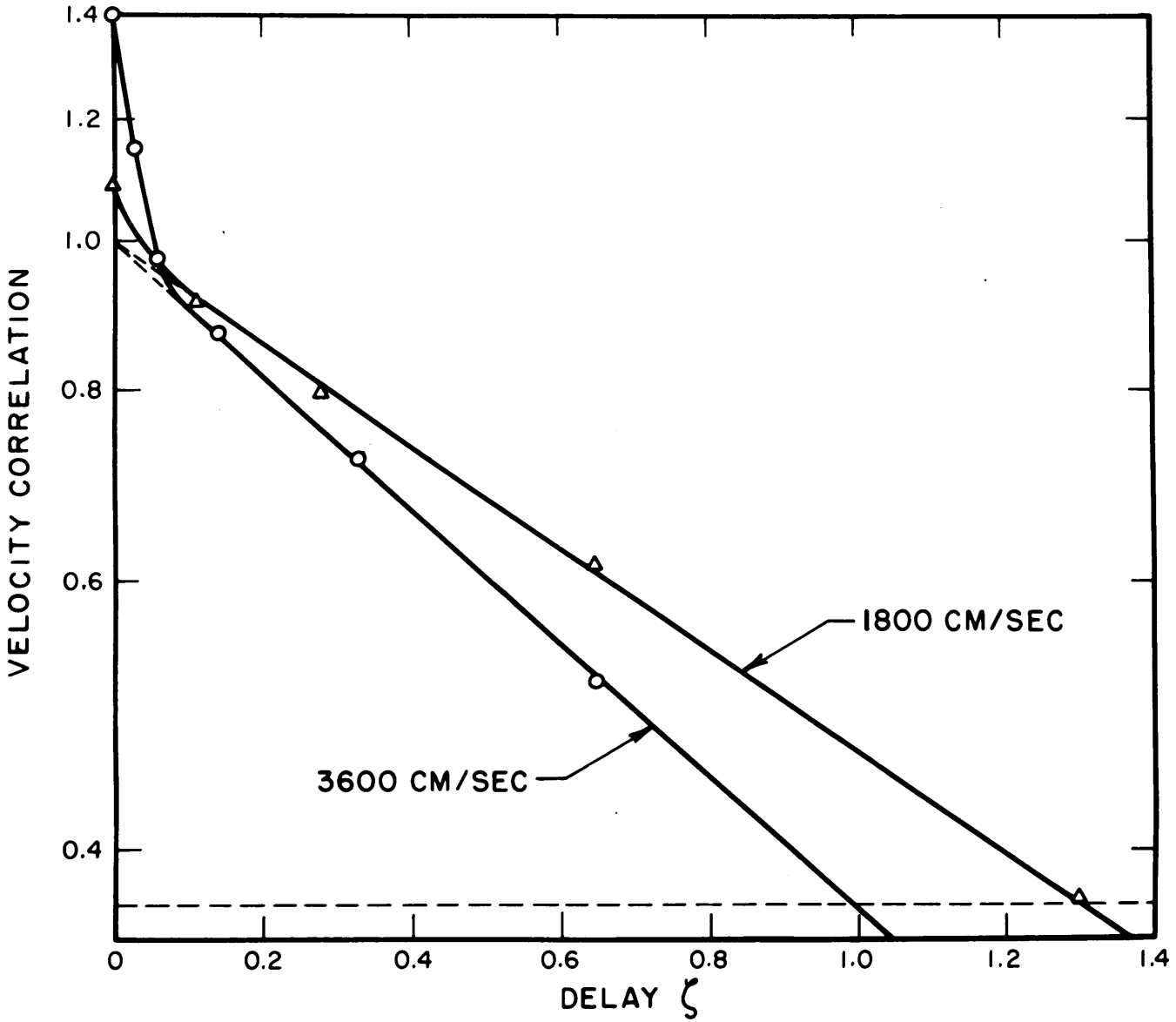

Figure 4.05 Velocity Correlation Versus Time Delay.

obtain a linear expression for the $\frac{1}{e}^{th}$ delay as a function of velocity --

$$\zeta_v = -\frac{10^{-3}}{6} v + 1.6 \quad ,$$

where ζ_v is in msec. and v is in cm./sec. Since the pressure goes as u^2, the $\frac{1}{e}^{th}$ delay for pressure correlations would be just half this value

$$\zeta_p = -\frac{10^{-3}}{12} v + 0.8 \equiv \tau \quad ,$$

as defined in Section 3.5. In Chapters III and V we assume that the critical delay is independent of the flow velocity. We see from this that such an assumption is not quite accurate.

The hot wires have an upper frequency cutoff at about 7000 cps. The turbulence spectrum is essentially flat below this frequency. Hence, the spectrum of the turbulence cannot be measured with the hot wire. In the next section we shall analyze the response of the finite ribbon for some of its modes. From the measurements here we can be sure that the spectrum will be sensibly constant for the modes chosen.

4.3 THE RESPONSE OF A FINITE RIBBON TO TURBULENT FLOW

As mentioned above, a steel ribbon was used to simulate a string in the experiments. The ribbon had a width of three-sixteenths inches and a thickness of .01 inches. The experimental set-up is shown in the photographs of Figure 4.06. The air supply passes through a "Flowrater" meter which allows the air flow to be adjusted and measured. The air is then passed through a $\frac{1}{2}" \times \frac{1}{4}"$ rectangular tube in which the ribbon is mounted. Since the Reynolds number for the flow is well above the critical value, the flow is fully turbulent. As it passes along the ribbon, the

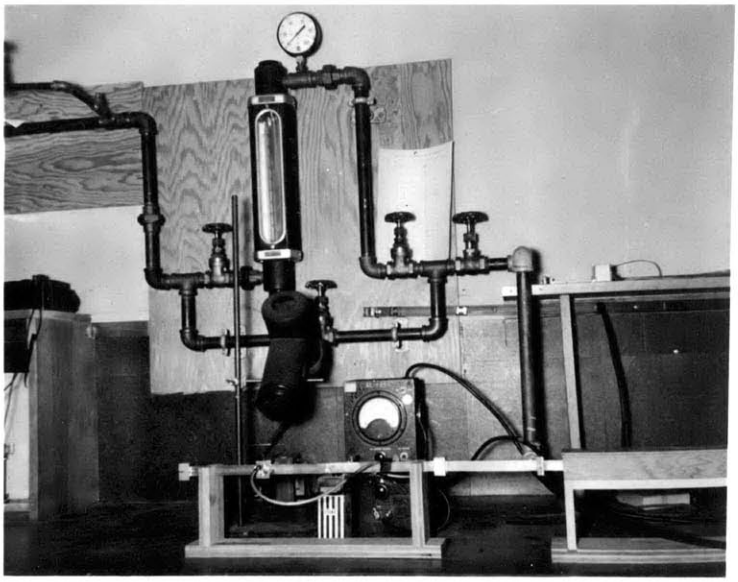

Figure 4.06 Photograph of Ribbon Excitation Experiment.

ribbon is excited, and its motion is then detected by a photoelectric device.

The detector is shown by the diagram in Figure 4.07. A collimated light beam enters the tube through a small hole in its side. The light is then reflected by the steel ribbon and is partially transmitted out of another hole on the other side of the tube. The part of the beam which is allowed to escape enters a photoelectric cell. In the normal position of the ribbon about one-third to one-half of the light escapes to the cell. Then as the ribbon vibrates, the light beam is modulated, and a signal is obtained from the photo cell which corresponds to the motion of the ribbon.

This signal, of course, represents the total motion of the string, and we would like to examine the response of each mode separately. We can do this by utilising the natural frequency selective properties of the modes. That is, if we consider a mode with a natural frequency f, most of the motion in this mode will have frequencies very close to f if the Q has a reasonable value. Accordingly, in the experiment we have used a one-third octave band analyzer to select out the motion of the various modes. We obtain the mean square by squaring the output of the filter and then integrating (time averaging). By using a long enough integration time, the fluctuations in the signal may be smoothed adequately for recording.

The length of the ribbon was 44 cm. The tension was adjusted by a screw so that the fundamental resonance was 50 cycles. This gave a wave velocity of 4400 cm./sec. Third octave bands centered at 50, 100, 250, 500, 1000 cps were chosen to duplicate the modes plotted in Figure 3.13,

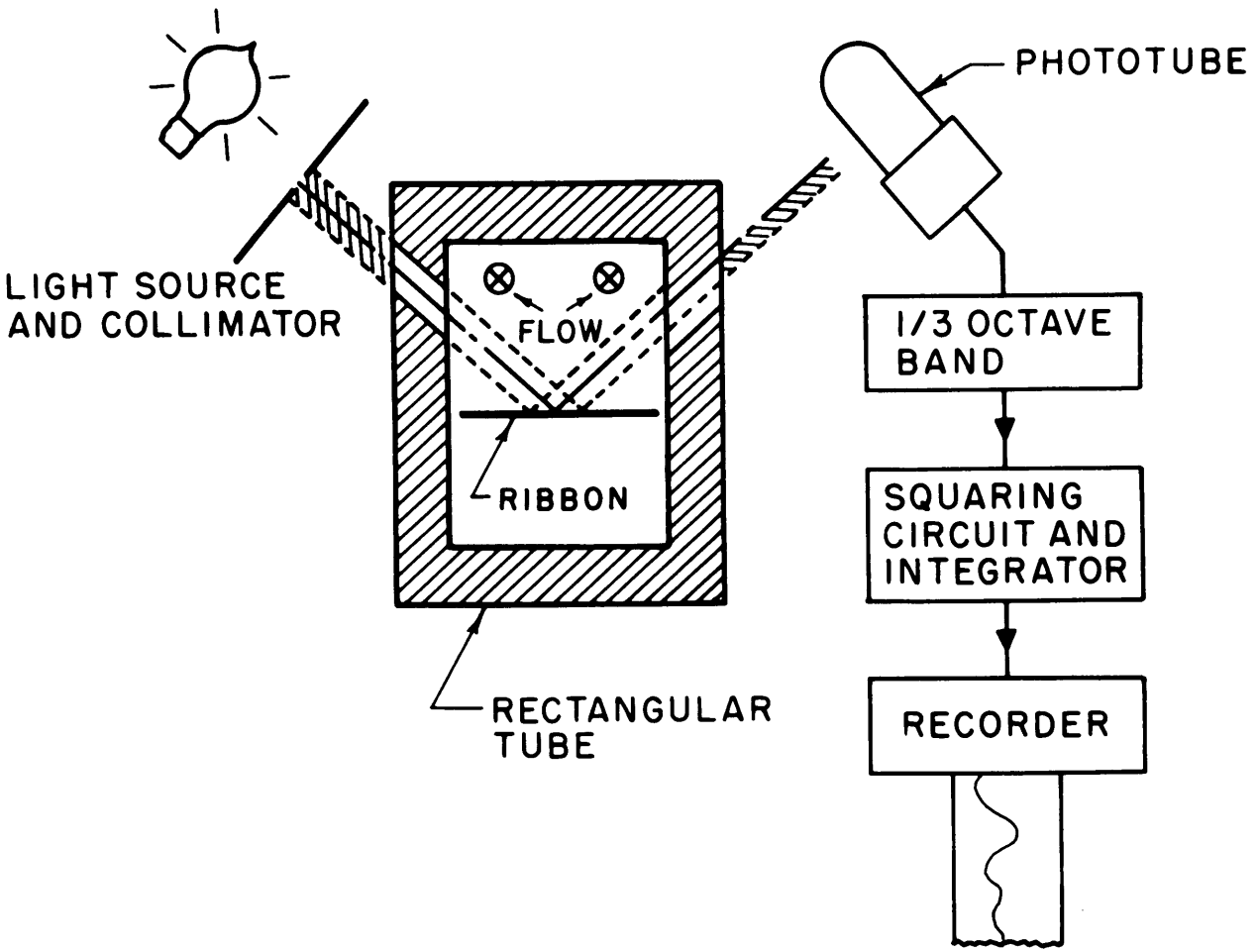

Figure 4.07 Diagram of Photoelectric Detector.

that is, m = 1, 2, 5, 10, 20. The mean square displacement for these modes as a function of the flow velocity for the various modes is indicated in Figure 4.08. It is interesting to compare these with the theoretical results of Section 3.5. As the theory predicts, we observe that there are no coincidence effects for m = 1,2. For m = 5, 10, 20 a coincidence effect is in evidence with a response peak at 3600 cm./sec. for m = 5, 10, and 4500 cm./sec. for m = 20. The theory predicted this peak to occur at 4400 cm./sec. for all modes.

One possibility of explaining the discrepancy between the flow velocity and wave velocity lies in the assumed length of the ribbon. Since the supporting ends for the ribbon are less flexible than the center, due to the method of attaching the ribbon to the metal rods which support it, the effective length of the ribbon may be somewhat shorter than the 44 cm. assumed. The upward shift of frequency for m = 20 may be due to the flow coincidence (if 44 cm. length is correct); but part of the increase in phase velocity is probably due to the action of the bending stiffness of the ribbon coming into play. For m = 50 the coincidence effect has vanished, probably because of the viscosity coefficient increasing with frequency.

In order to see whether the peak response continued to bear the same relationship to the phase velocity when the latter was varied, the fundamental frequency of the ribbon was lowered to forty cycles. The phase velocity was then 3520 cm./sec. The mode m = 5 was chosen to check the response. The result of this experiment is shown in Figure 4.09. The peak response is about 3000 cm./sec., which is again lower than the phase velocity.

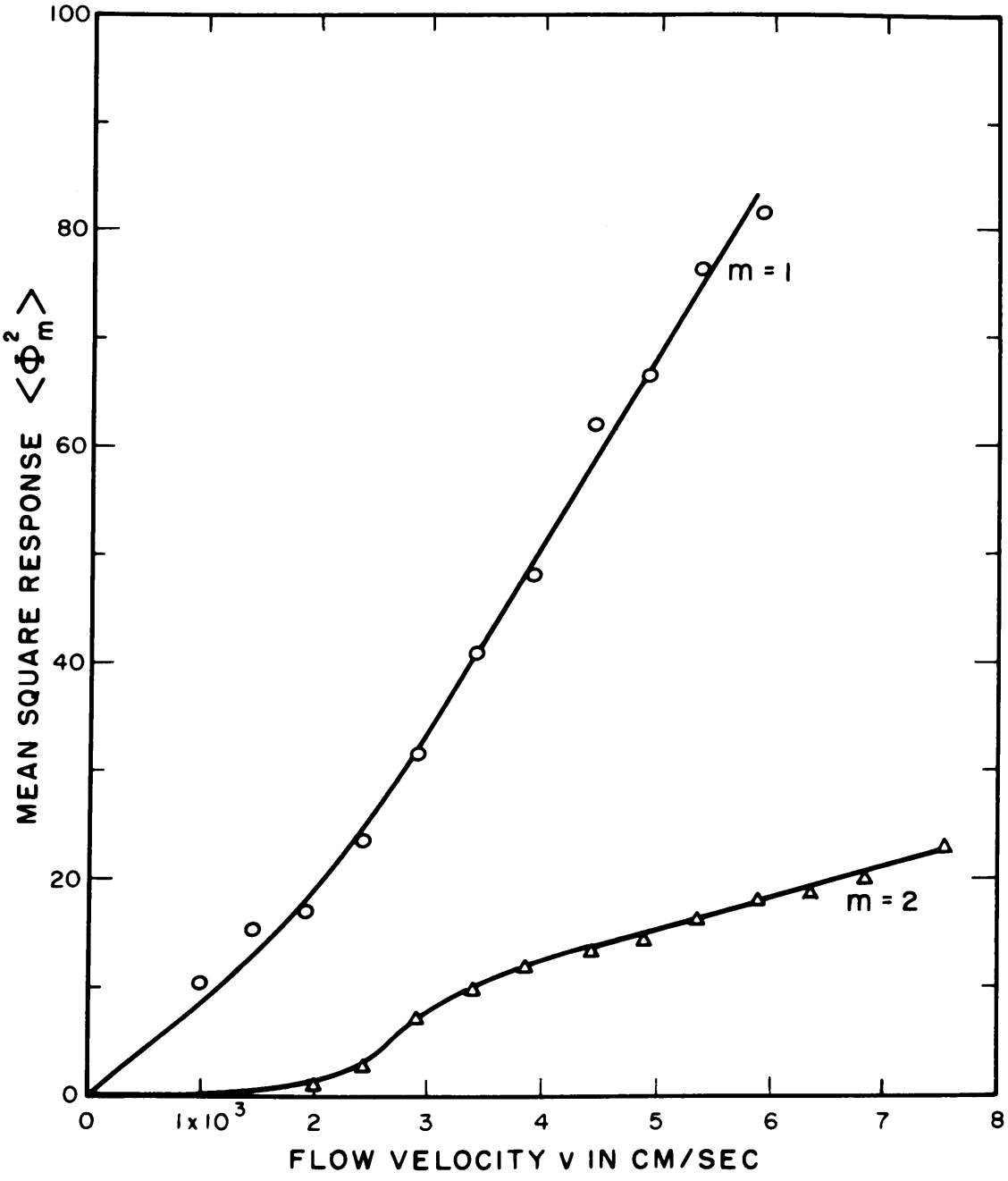

Figure 408a Response of First and Second Modes of Ribbon to Turbulence Versus Flow Velocity for 4400 cm./sec. Phase Velocity.

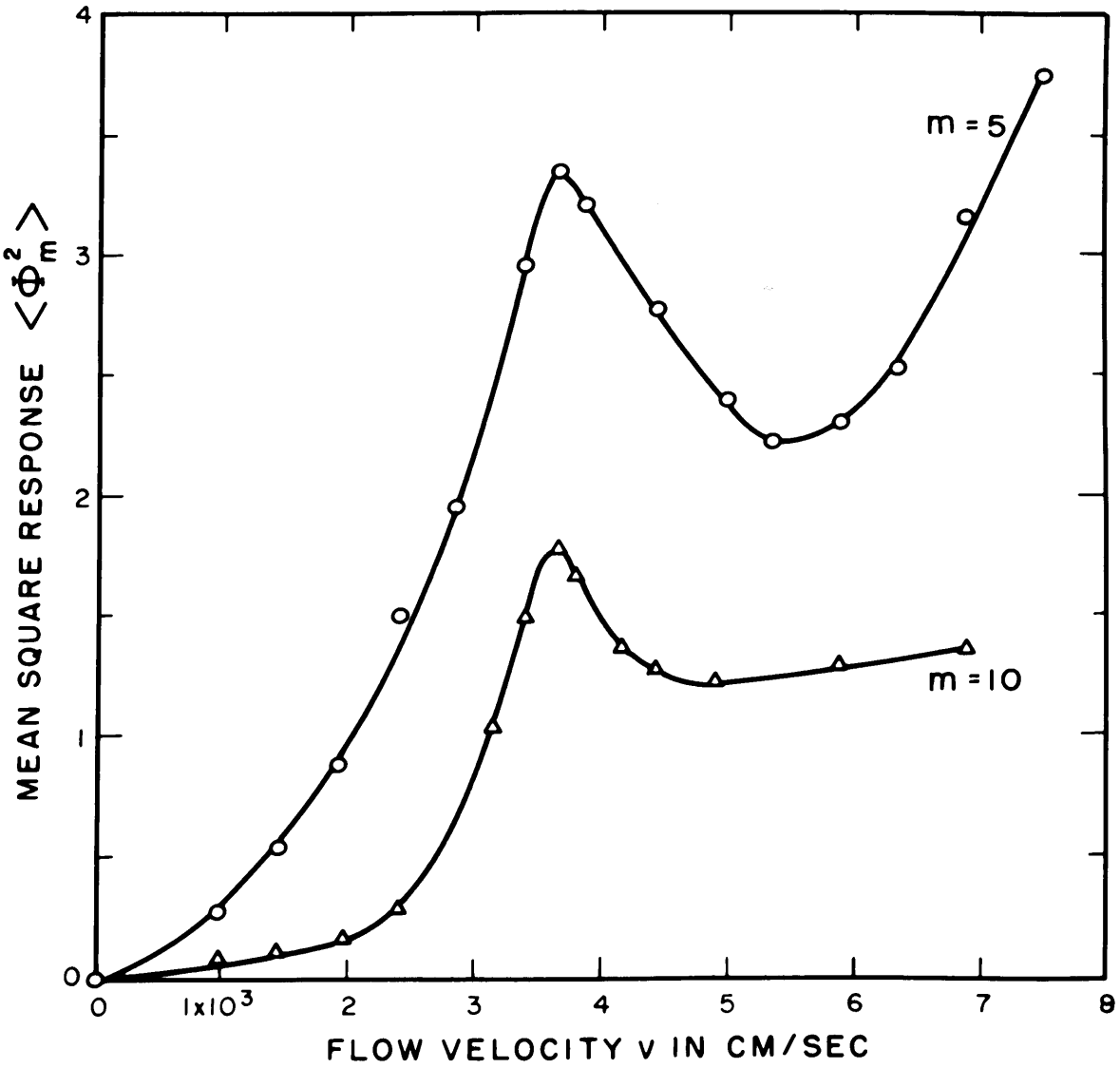

Figure 408b Response of Fifth and Tenth Modes of Ribbon to Turbulence Versus Flow Velocity for 4400 cm./sec. Phase Velocity.

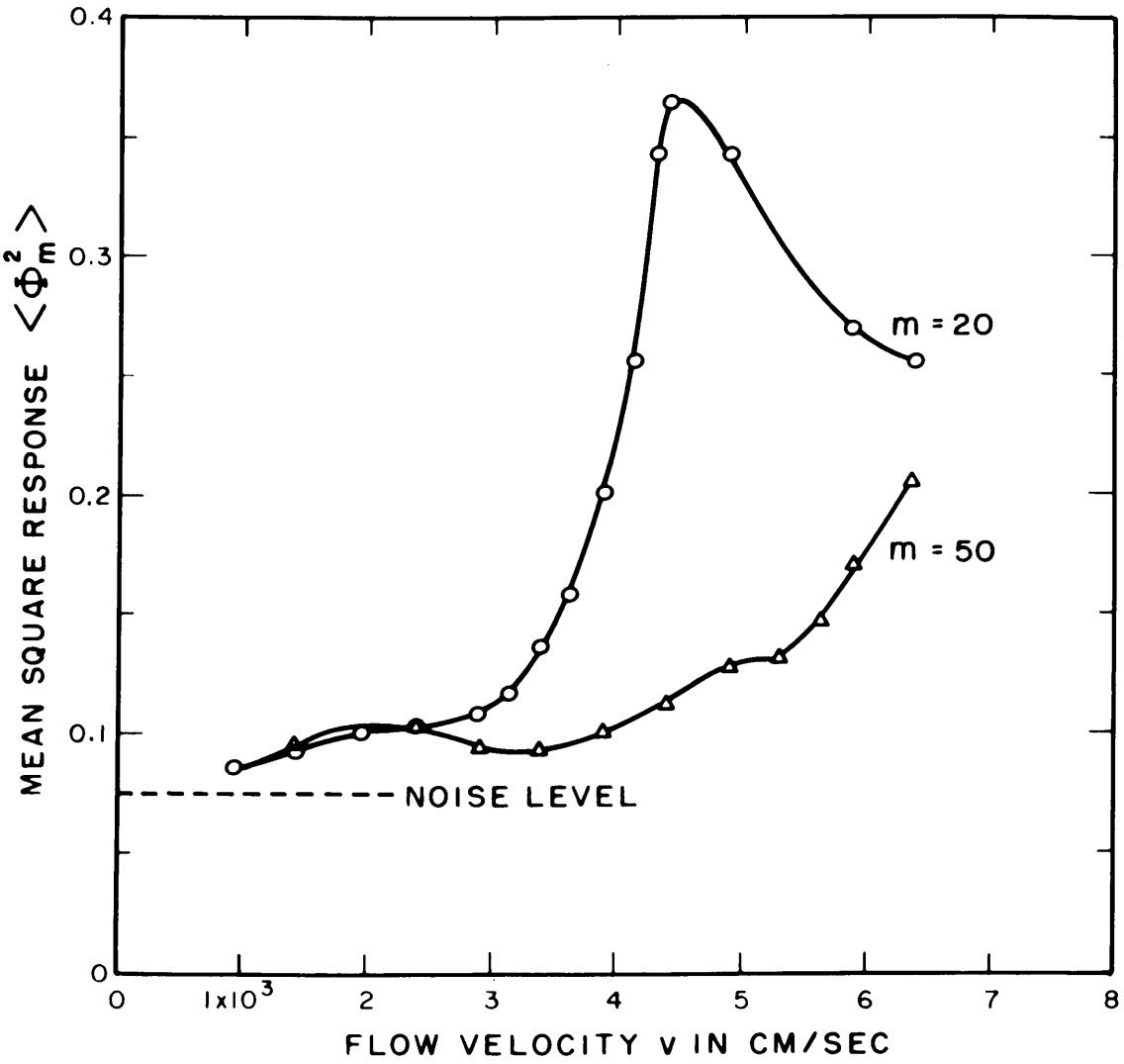

Figure 408c Response of Twentieth and Fiftieth Modes of Ribbon to Turbulence Versus Flow Velocity for 4400 cm./sec. Phase Velocity.

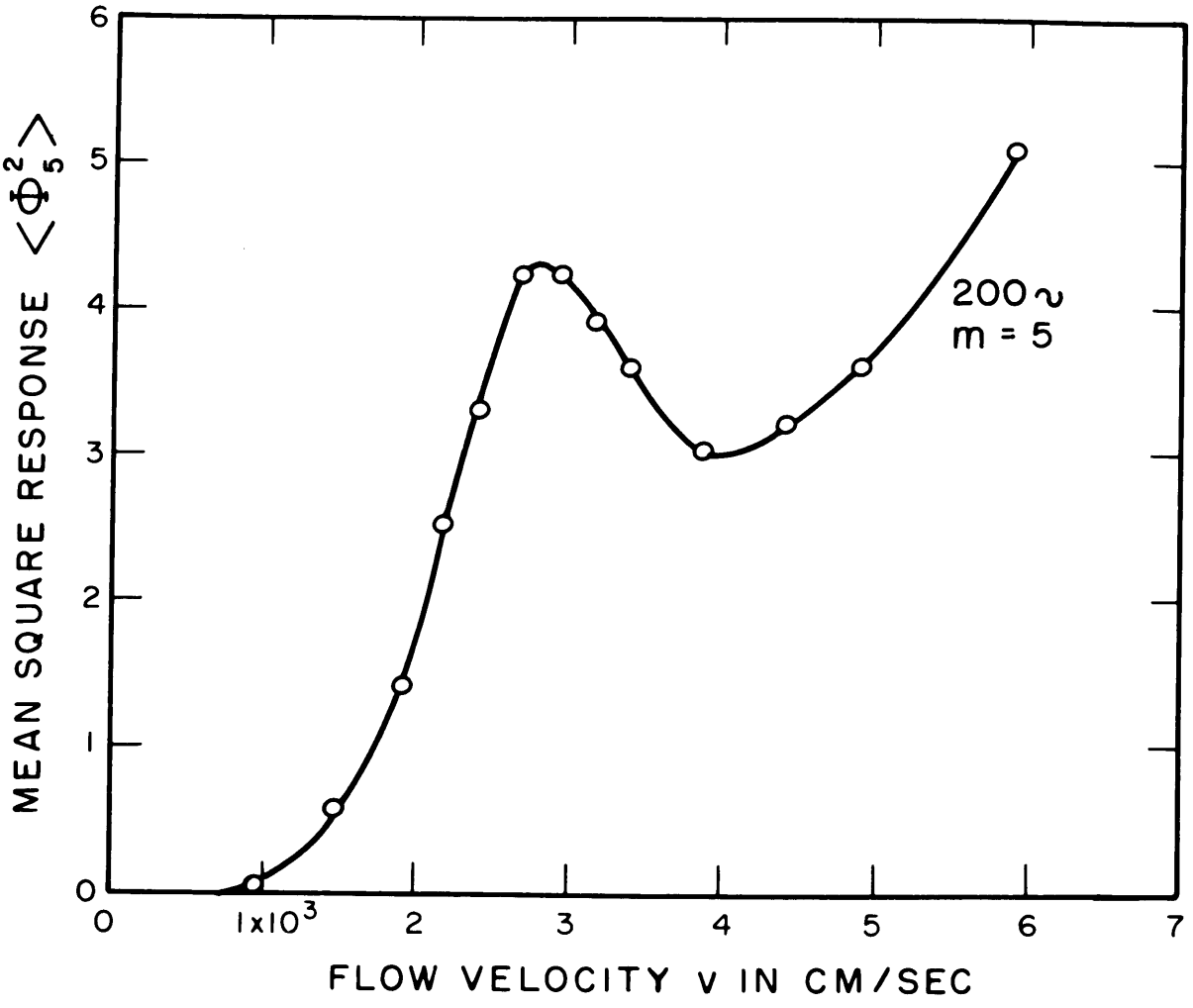

Figure 4.09 Response of Ribbon to Turbulent Flow Versus Flow Velocity for 3520 cm./sec. Phase Velocity.

The uncertainty in the accuracy of the flow meter at these flows is about ± 100 cm./sec. This, coupled with the uncertain length of the ribbon, could account for the discrepancy. Another possibility is unequal dividing of the flow above and below the ribbon so that the flow might be faster than the computed value on one side, slower on the other. When the faster flow reached coincidence, the peak would be observed although the average flow had not attained the phase velocity.

In an attempt to evaluate the effect of flow on the "Q" value of the ribbon for the fundamental mode, a small electromagnet was connected to an audio-oscillator and used to excite the steel ribbon. The half power points of the response were located, and the Q value was calculated for several values of flow. The results are shown in Figure 4.10. It is striking to note the rapid increase in the viscous coefficient (proportional to 1/Q) as the flow is increased. This may partially account for the lack of coincidence effect for the mode m = 50 above. In any case the effect of the flow on viscous dissipation is very striking, and it might be worthwhile to follow this up by careful experiments and some theoretical work.

One may say that in general the effects of coincidence and dependence of mode excitation on the correlation length of the source are experimentally verifiable. Certain details do not check because of deviations in experiment from the assumed properties of the system.

4.4 THE RESPONSE OF A FINITE BAR TO TURBULENT FLOW

As an experimental continuation of the work on the string, it was thought feasible to repeat the experiment above for another physical

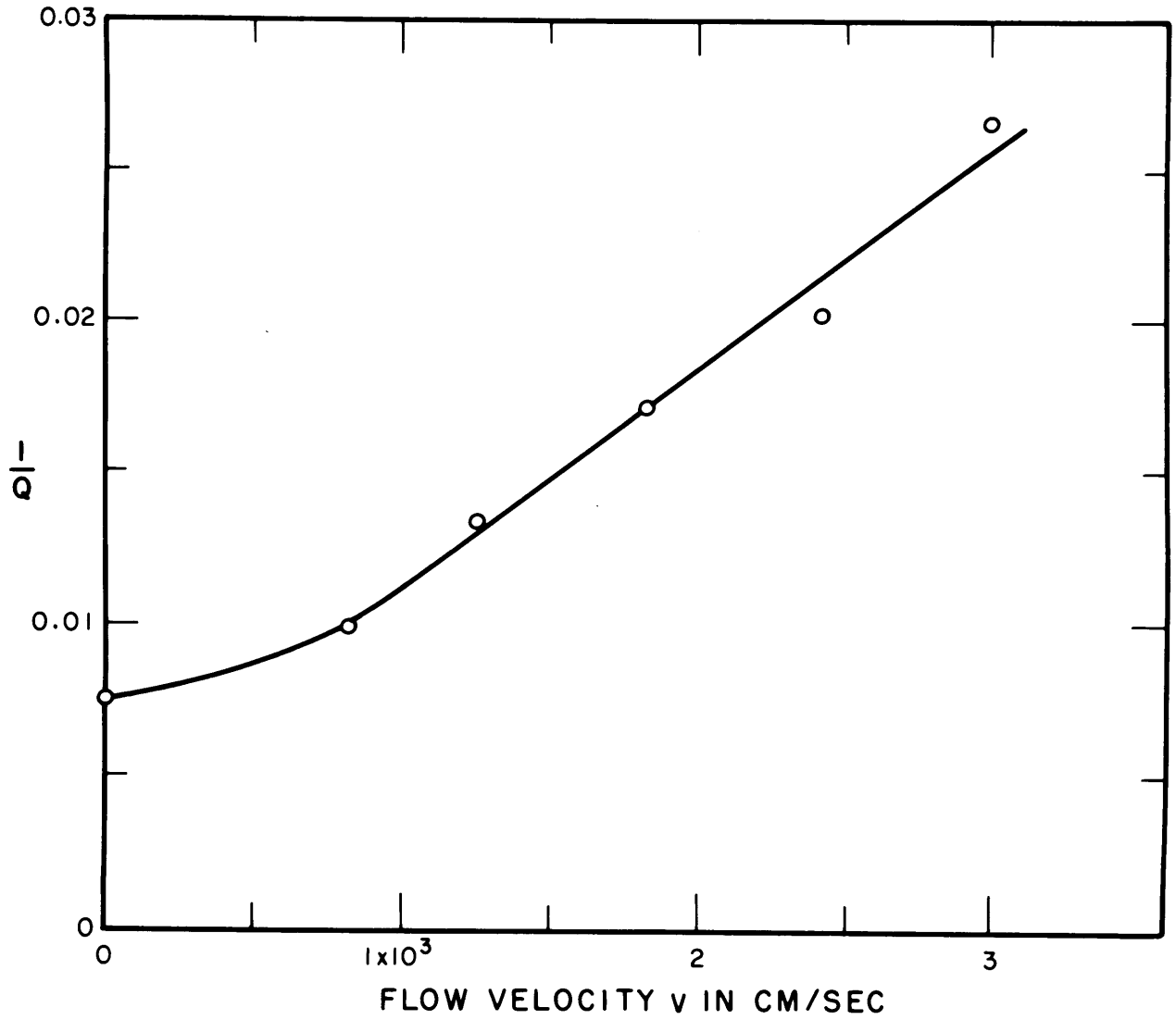

Figure 4.10 Variation of Q Factor with Flow Velocity.

system. For this purpose the finite bar was chosen as a system which has important differences from the string but becomes more like a string as the wavelengths become shorter and the end effects become less important.

The experimental set-up for the bar excitation is shown in the photograph of Figure 4.11. The flow set-up is the same as before, with the bar mounted in the rectangular tube. At the end of the bar can be seen the phonograph cartridge which was used to detect its vibration. The bar is nineteen inches long, 15/32 inches wide, and one-sixteenth inches thick. It is made of brass, and as such the flexural velocity is 3900 cm./sec. With these dimensions the fourth mode is 250 cps, and the bar is supported by pins at the outermost nodes for this mode.

The scheme of the measurements is diagramed in Figure 4.12. The output of the phono pickup is fed to a third octave band filter to select the mode, and the mean square is taken as for the ribbon. The results of this experiment are shown in Figure 4.13. The coincidence effect is again missing for the lower frequencies but comes in plainly for the higher ones.

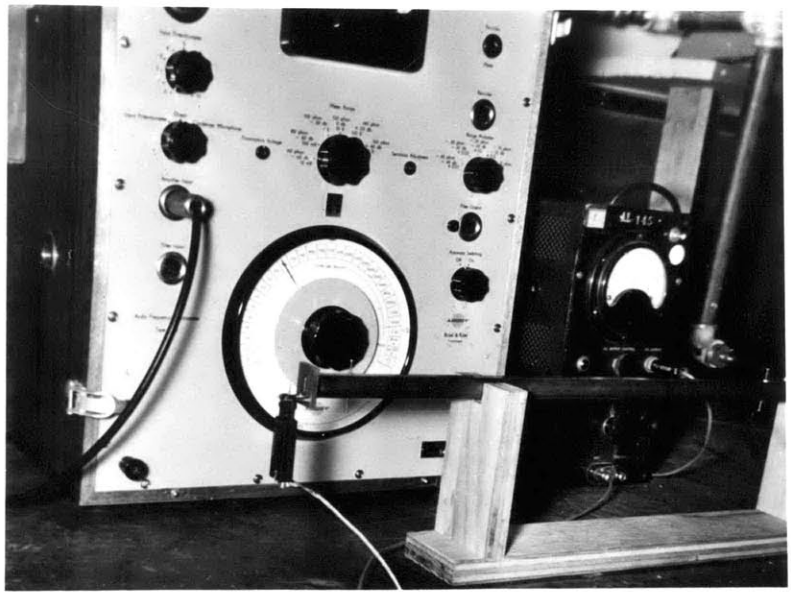

Figure 4.11. Photograph of Bar Excitation Experiment.

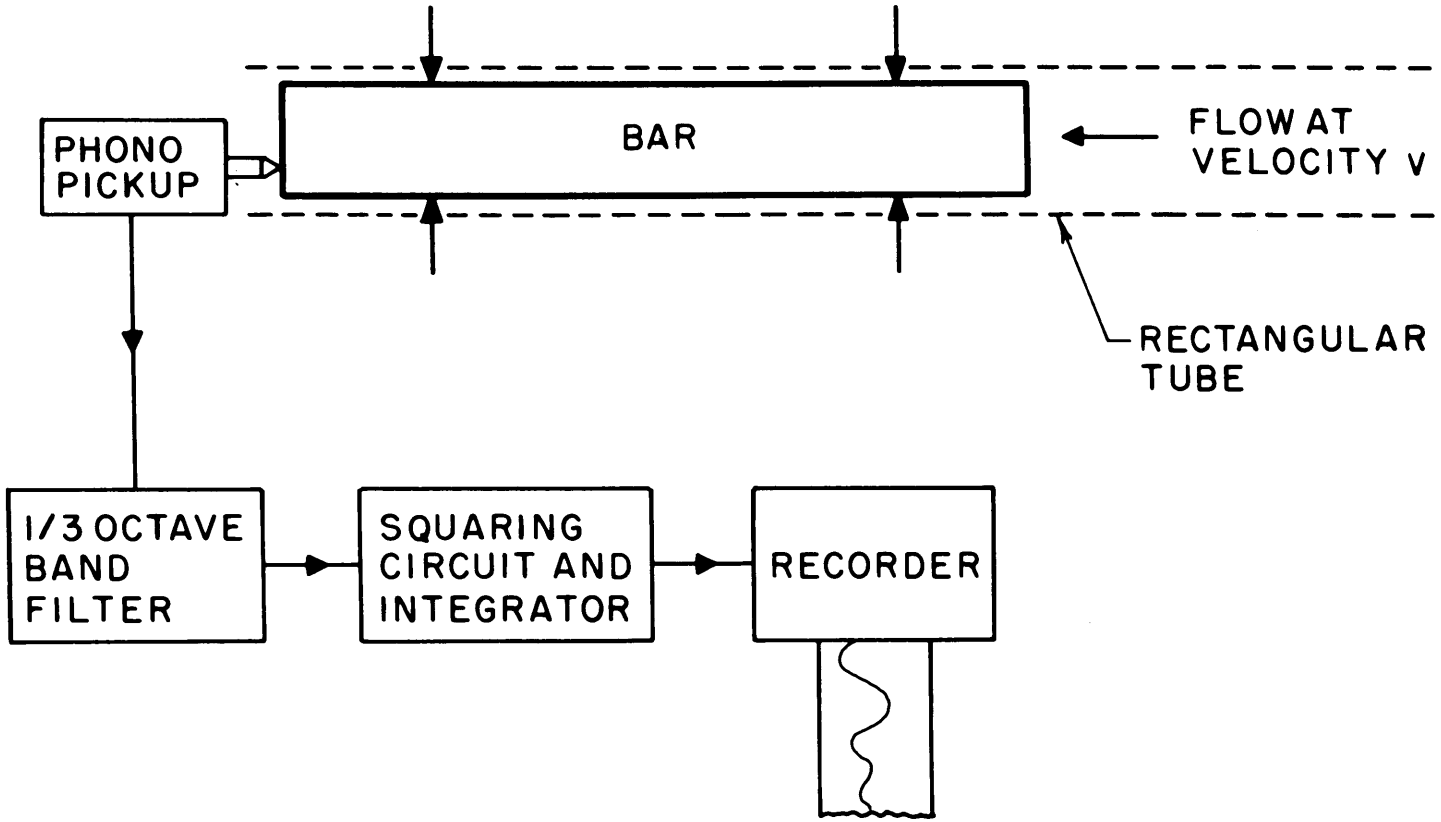

Figure 4.12 Diagram of Bar and Detection Equipment.

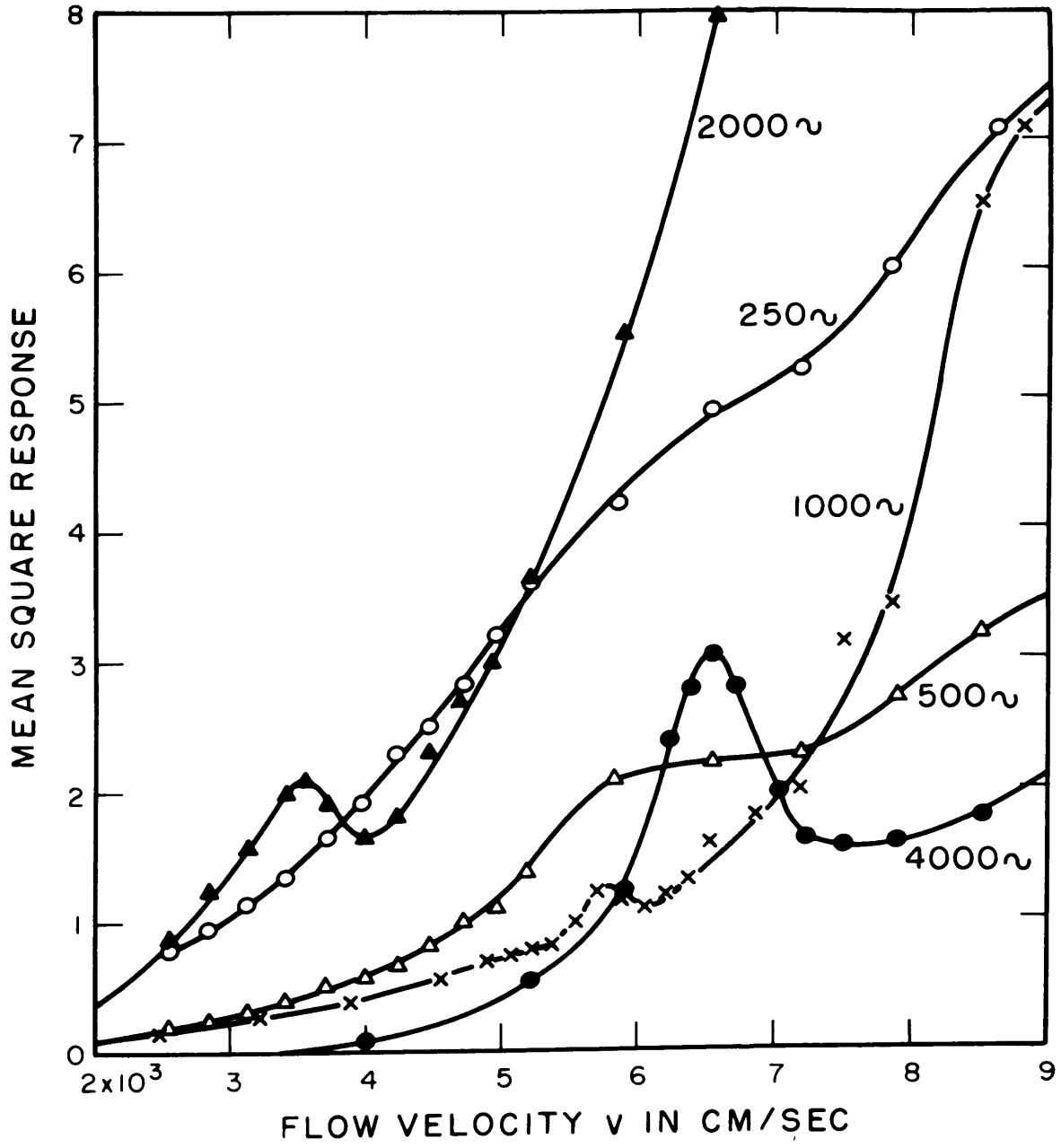

Figure 4.13 Response of Bar to Turbulent Flow Versus Flow Velocity.

V EXCITATION OF INFINITE STRINGS

5.1 INTRODUCTION

In this chapter we shall study the response of infinite, damped strings to noise fields having certain correlation properties. The source correlations will be the same as in Examples 3 and 4 in Chapter III. We shall do this for two reasons. First, we want to extend the methods of analysis from the finite domain to the infinite and obtain some of the interesting results of correlation analysis. Second, the results for finite strings have some interesting similarities to and differences from those for the infinite case. We shall point these out as the examples are discussed. We shall attempt no experimental verification of the results in this chapter, partly because of the difficulty of getting the proper conditions and partly because the essential results are not different from those of the preceding chapter.

5.2 EXAMPLE 1, EXCITATION BY A PURELY RANDOM AND MOVING NOISE FIELD

In this section we shall consider an infinite string excited by a moving noise field having the correlation properties assumed in Section 3.4. We shall then examine the mean square energy density in the string and the spectral distribution of the energy.

A. The Infinite, Damped String

We now require the Green's function for the infinite string in a viscous medium. If we write down the equation, it must satisfy

$$c^2 \frac{\partial^2 g}{\partial x^2} - \frac{\partial^2 g}{\partial t^2} - \beta \frac{\partial g}{\partial t} = -4\pi\delta(x-x_o)\delta(t-t_o) \quad . \qquad 3.05$$

If we expand this in a double Fourier integral

$$g = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} dk d\omega \, \bar{g}(k,\omega) \, e^{ik(x-\omega t)} \quad , \qquad 5.01$$

then $\bar{g}$ is given by

$$\bar{g} = 4\pi \frac{e^{-i(kx_o - \omega t_o)}}{k^2c^2 - \omega^2 - i\omega\beta} \quad .$$

Hence,

$$g(x,t \mid x_o,t_o) = \frac{1}{\pi} \iint_{-\infty}^{\infty} dk d\omega \, \frac{e^{i\{k(x-x_o)-\omega(t-t_o)\}}}{(\omega + i\beta/2 - k_o c)(\omega + i\beta/2 + k_o c)} \quad ,$$

when $k_o = \sqrt{k^2 - \beta^2/4c^2}$.

If we now integrate this expression over ω, taking care to choose the path so that the integrand is zero as we close the "loop," then we have by the Cauchy integral theorem,

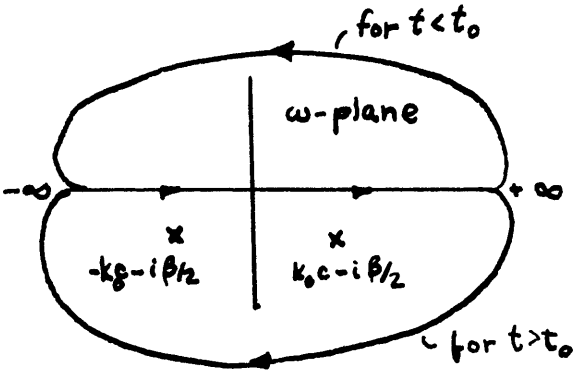

Figure 5.01

$$g = \frac{1}{c}(-i) \int_{-\infty}^{\infty} dk \left[\frac{e^{i\{k(x-x_o)-(-i\frac{\beta}{2} + k_o c)(t-t_o)\}}}{k_o} \right.$$

$$\left. + \frac{e^{i\{k(x-x_o)-(-i\frac{\beta}{2} - k_o c)(t-t_o)\}}}{-k_o} \right] \quad , \text{ for } t > t_o \quad .$$

There are no poles in the upper half plane, so $g = 0$ for $t < t_o$, as would be expected from considerations of causality.

$$g(x,t \mid x_o,t_o) = \begin{cases} -\frac{2}{c}\int_{-\infty}^{\infty} dk \cdot \frac{1}{k_o}\, e^{ik(x-x_o)-\frac{\beta}{2}(t-t_o)} \sin k_o c(t-t_o) & ; (t > t_o) \\ 0 & (t < t_o) \end{cases} \qquad 5.02$$

Since later on we shall want to have the spectrum of the response, let us leave the Green's function in this integral form. For the infinite string the mean energy is just $\frac{1}{2}\rho_\ell\langle u^2\rangle$, and so we are really interested in the velocity response of the source rather than the displacement response. If we define $\Gamma = \frac{\partial g}{\partial t}$, then we have

$$\Gamma(x,t \mid x_o,t_o) = \begin{cases} \frac{2}{c}\int_{-\infty}^{\infty} dk \frac{1}{k_o}\, e^{ik(x-x_o)-\frac{\beta}{2}(t-t_o)}\left[\frac{\beta}{2}\sin k_o c(t-t_o) \right. \\ \qquad \left. -kc\cos k_o c(t-t_o)\right] & ; (t > t_o) \\ 0 & (t < t_o) \end{cases} \qquad 5.03$$

The mean square velocity, where $u = \frac{\partial \phi}{\partial t}$, is

$$\langle u^2(x,t)\rangle = \int_{-\infty}^{t}\int dt_o dt_o' \int_{-\infty}^{\infty}\int dx_o dx_o' \Gamma(x,t \mid x_o,t_o)\Gamma(x',t' \mid x_o',t_o') \langle f(x_o,t_o)\, f(x_o',t_o')\rangle_{ave} . \qquad 5.04$$

B. The Source Correlation

As before, we shall picture a stationary force field, purely random in the x dimension and then dragged along with a velocity v to produce the source correlation

$$\langle f(x_o - vt_o) f(x_o' - vt_o')\rangle_{ave} = D\,\delta(\sigma - v\zeta) . \qquad 5.05$$

This will now be placed in the expression 5.04 and the energy of motion calculated.

C. Calculation of the Energy

$$\langle u^2(x,t)\rangle = \frac{D}{c^2}\int_{-\infty}^{2t} d\mu\, e^{-\frac{\beta}{2}(2t-\mu)} \int_{2t-\mu}^{\mu-2t} d\zeta \iint_{-\infty}^{\infty} d\rho\, d\sigma \iint_{-\infty}^{\infty} \frac{dk\,dk'}{k_o k_o'}$$

$$\cdot e^{ik(x-x_o)+ik'(x-x_o')} \left\{ \frac{\beta^2}{4} \sin k_o c(t-t_o) \sin k_o' c(t-t_o') \right. \qquad 5.06$$

$$- \frac{\beta k_o c}{2} \sin k_o' c(t-t_o') \cos k_o c(t-t_o) - \frac{\beta k_o c}{2} \sin k_o c(t-t_o)$$

$$\left. \cdot \cos k_o' c(t-t_o') + k_o^2 c^2 \cos k_o c(t-t_o) \cos k_o' c(t-t_o') \right\} \delta(\sigma - v\zeta) .$$

Picking out the terms depending on ρ, one has

$$\int_{-\infty}^{\infty} d\rho\, e^{-\frac{\rho}{2}(k+k')} = 4\pi\delta(k+k') .$$

Integrating over k' and ζ, one obtains easily

$$\langle u^2\rangle = \frac{4\pi D}{vc^2} \int_{-\infty}^{\infty} d\sigma \int_{|\sigma|/v}^{\infty} d\gamma\, e^{-\beta\gamma/2} \int_{-\infty}^{\infty} \frac{dk}{k_o^2} e^{-ik\sigma} \left\{ \left(\frac{k^2c^2}{2} - \frac{\beta^2}{4}\right) \cos k_o c\gamma \right.$$

$$\left. - \frac{\beta k_o c}{2} \sin k_o c\gamma + \frac{k^2c^2}{2} \cos k_o \frac{\sigma}{\alpha} \right\}, \qquad 5.07$$

where we have again put $\gamma = 2t-\mu$ and $\alpha = v/c$. The integration over γ is involved but has a simple result.

$$\langle u^2\rangle = \frac{4\pi D}{\alpha c^2} \int_{-\infty}^{\infty} d\sigma\, e^{-\beta|\sigma|/2v} \int_{-\infty}^{\infty} dk\, e^{-ik\sigma} \left(\frac{c}{\beta} \cos k_o \frac{\sigma}{\alpha} - \frac{1}{2k_o} \sin k_o \frac{|\sigma|}{\alpha} \right) . \qquad 5.08$$

This expression must be integrated over positive and negative values of σ independently and then the two integrals added. With this done, the result is

$$\langle u^2 \rangle = \frac{8\pi D}{c^2} \cdot \frac{\alpha}{(1-\alpha^2)^2} \int_{-\infty}^{\infty} \frac{dk}{k^2 + A^2} \quad , \qquad 5.09$$

where $A = \frac{\beta}{c} \frac{\alpha}{|1-\alpha^2|}$. This result is interesting because it tells us how the energy is spread over wave numbers. First of all, let us define a correlation length for viscosity as we did in Chapter III by

$$L_o \equiv \frac{c}{\beta} \quad . \qquad 5.10$$

If we assume that the viscosity is the same as before, then $L_o = 2\pi L$, where L was the length of the finite string. This artifice enables us to compare results for the same wavelengths on the finite and infinite strings. We shall then define the variable $\xi \equiv \frac{b}{L}$, where b is the wavelength $= \frac{2\pi}{k}$. The values $\xi = \frac{2}{m}$ (m = 1, 2, ---) represent values of ξ which are equivalent to the eigenmode wavelengths for the finite string. If we do this, then the spectrum in terms of wavelengths is

$$\Psi(\xi) = 80\pi c^2 L_o \frac{\alpha^{-1}}{400(\frac{1}{\alpha^2} - 1) + \xi^2} \quad , \qquad 5.11$$

such that

$$\int_{-\infty}^{\infty} \Psi(\xi)\, d\xi = \langle u^2 \rangle \quad .$$

In this equation, we have been able to evaluate L_o (or β) by assuming the viscosity mentioned above. When we do this, we know that Q_1 (from Chapter III) $= 10 = \frac{\pi c}{\beta L}$, or $\frac{L_o}{L} = \frac{10}{\pi}$. The spectrum Ψ is plotted in Figure 5.02 in units of 80 $c^2 L_o$. There are several interesting features of these curves which one may notice before going on to the calculation of $\langle u^2 \rangle$.

The first interesting property is the wavelength cutoff (the point

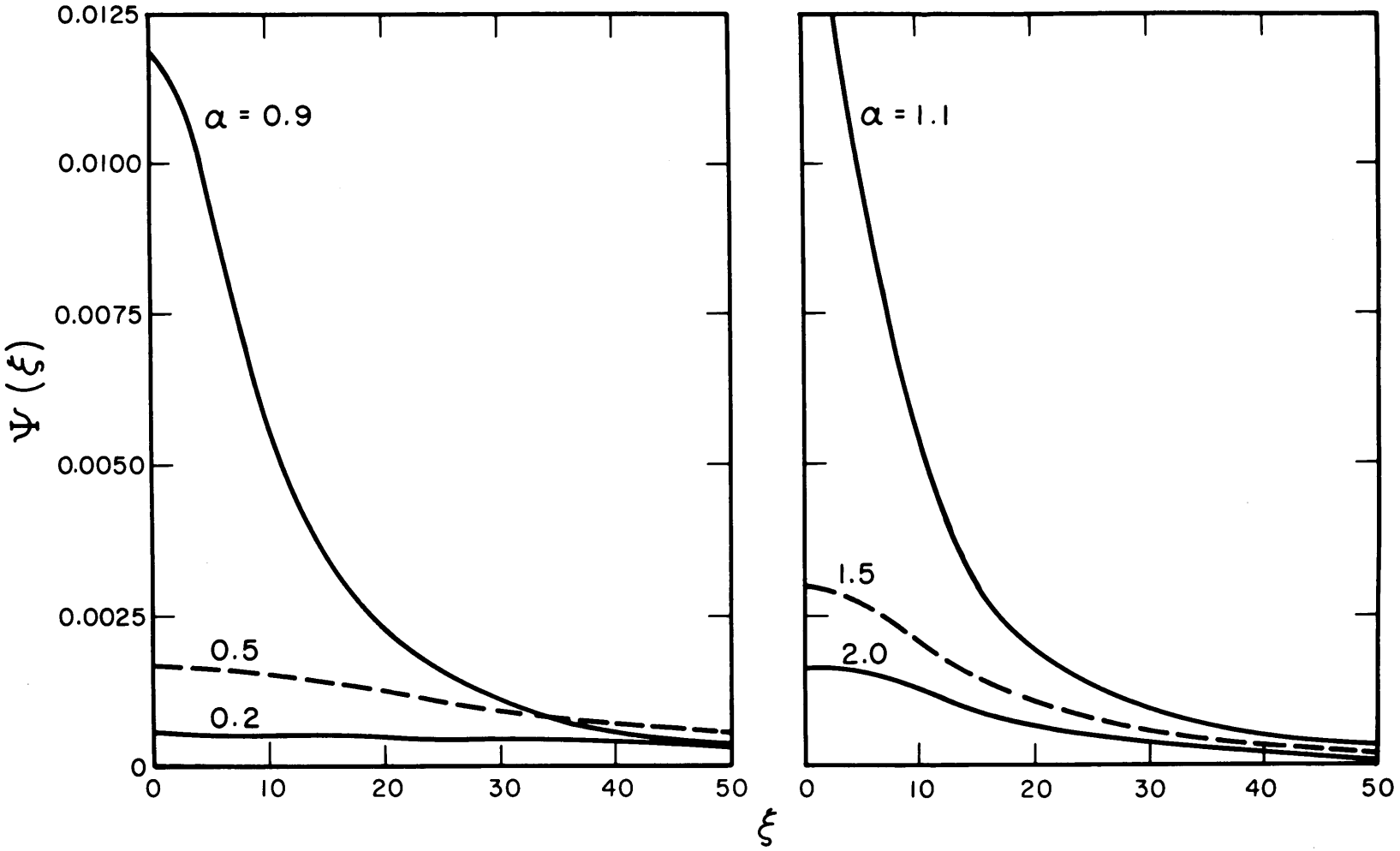

Figure 5.02 Spectral Response to Purely Random Moving Noise Field.

where the spectrum is $^1/2$ its value at $\xi = 0$), given by

$$\xi_{\text{cutoff}} = \frac{2\pi L_o \sqrt{\frac{1}{\alpha^2} - 1}}{L} \quad . \qquad 5.12$$

Here the viscosity correlation length, along with the flow velocity, governs the spectral excitation of the string such that for wavelengths $>> L\,\xi_{\text{cutoff}}$ one does not get excitation, but for $b << L\,\xi_{\text{cutoff}}$ the excitation is large. This cutoff phenomenon is more pronounced as one approaches $\alpha = 1$. The diverging result at $\alpha = 1$, namely $\Psi(\xi) = \frac{1}{\xi^2}$, is limited by the finite bandwidth of the source. Actually, any source has a finite bandwidth and the finite values of $\Psi(\xi)$ at $\xi = 0$ are idealized. This explains the reason why, experimentally,* coincidence for turbulence becomes sharper as the mode number is increased.

If we now select various values of ξ and plot the spectrum Ψ as a function of α, we get the curves in Figure 5.03. Since a value of $\xi = 3.18$ is the correlation lengthh due to viscosity ($3.18 = \frac{L_o}{L} = \frac{10}{\pi}$), we expect that for ξ values much smaller than this the coincidence effect will be strong, and for those larger the effect will diminish. In general, for short wavelengths the coincidence effect is strong, and for long wavelengths it is weak. Thus, after deferring the question in Chapter III, we see that the length $L_o = \frac{c}{\beta}$ does behave as a correlation length in much the same way as does the characteristic length of the source $\ell = c\tau$.

In order to calculate $\langle u^2 \rangle$, we now integrate the expression $\Psi(\xi)$ from $-\infty \to +\infty$. That is, we perform the integration 4.09. By the Cauchy integral theorem this is

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*See Figure 4.08.

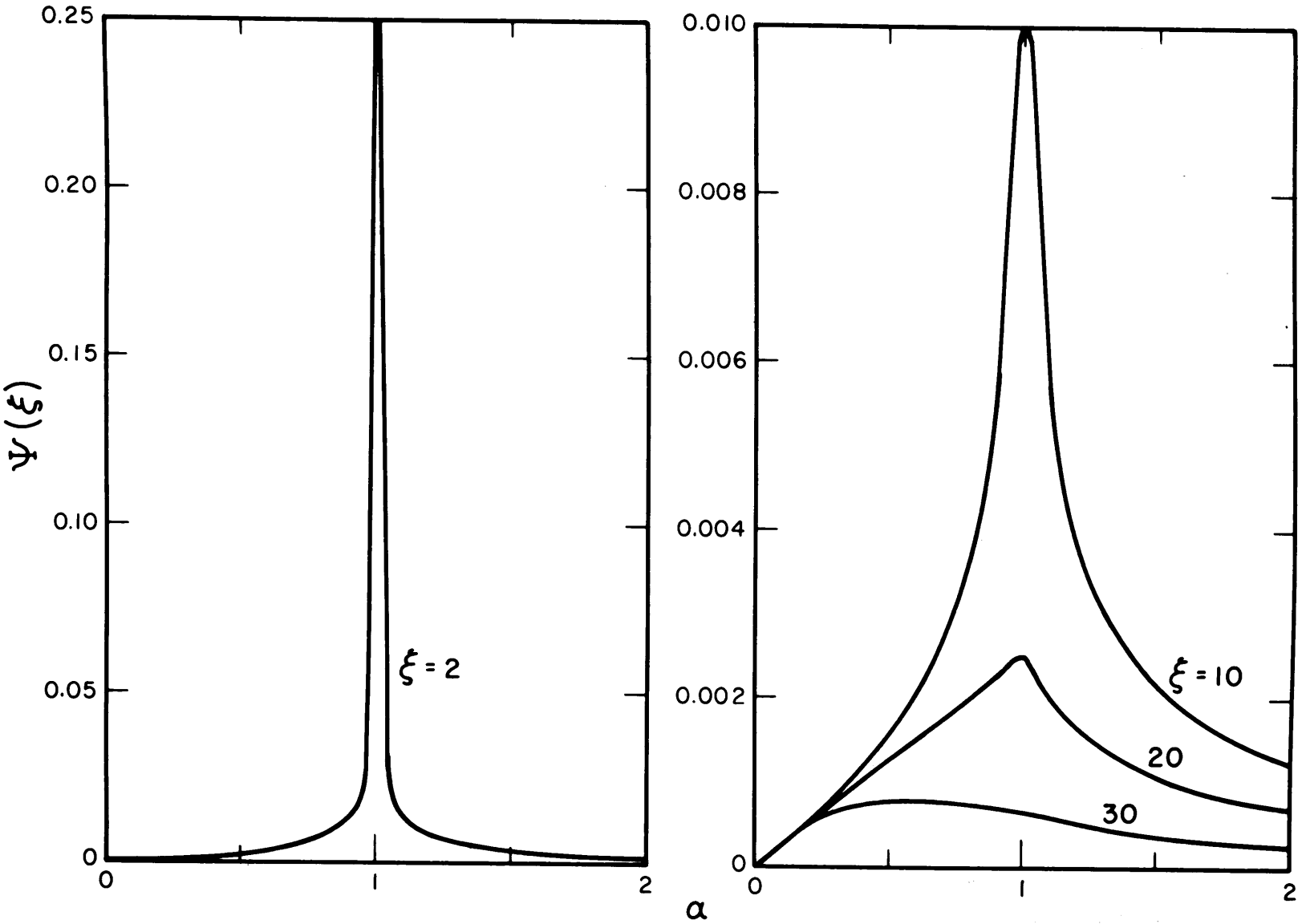

Figure 5.03 Variation of the Coincidence Effect with Wavelength.

$$\langle u^2 \rangle = \frac{8\pi D}{c^2} \cdot \frac{\alpha^2}{(1-\alpha^2)^2} (2\pi i) \frac{1}{2iA}$$

$$= \frac{8\pi^2 DL_o}{c^2} \frac{\alpha}{|1-\alpha^2|} \quad . \qquad 5.13$$

Since this is such a simple function, we can point out its features without a graph. It starts from zero at $\alpha = 0$ and has a singularity at $\alpha = 1$. It then trails off slowly for $\alpha >> 1$ with $\langle u^2 \rangle = 0$ as an asymptote. Since $\Psi(\xi, \alpha = 0) \equiv 0$ and $\Psi(\xi, \alpha = 1) = \frac{1}{\xi^2}$, the first integral over ξ vanishes and the second diverges as we stated above.

We now wish to consider the response of the infinite string to the moving and changing field of Example 4 of Chapter III.

5.3 EXAMPLE 2, THE RESPONSE TO A MOVING AND CHANGING NOISE FIELD

Let us now use the source correlation function of Example 4 in Chapter III to obtain some idea of the response of an infinite string to turbulent flow in the direction of its length. We are now particularly interested to see how the correlation length $\ell = c\tau$ for the source affects the excitation spectrum of the string. We expect that the effect will not be dissimilar to that produced by the viscous correlation length L_o in the preceding section.

A. The Noise Field

As in Chapter III we assume that the correlation function is

$$\langle f(x_o - vt_o) f(x_o' - vt_o') \rangle = D\,\delta(\sigma - v\zeta)\, e^{-|\zeta|/\tau} \quad ,$$

where we have set the dependence of fluctuation strength on flow velocity

equal to one. The integration over ρ , σ is now from $-\infty \to +\infty$ since we are concerned with an infinite string.

B. Calculation of the Average Energy Density

We start with the relation

$$\langle u^2 \rangle = \iint_{-\infty}^{t} dt_o dt_o' \iint_{-\infty}^{\infty} dx_o dx_o' \Gamma(x,t \mid x_o,t_o)\Gamma(x',t' \mid x_o',t_o') \cdot D\delta(\sigma - v\zeta)\, e^{-|\zeta|/\tau} .$$

Since the first integration is over ζ , the exponential becomes $e^{-|\sigma|/v\tau}$, and since the integration over σ does not occur until very late in the calculation, we may omit the analysis (which appears in the previous example) up to the σ integration. Referring to equation 5.08, we get

$$\langle u^2 \rangle = \frac{4\pi D}{\alpha c^2} \int_{-\infty}^{\infty} d\sigma\, e^{-(\frac{\beta}{2} + \frac{1}{\tau})\frac{|\sigma|}{v}} \int_{-\infty}^{\infty} dk e^{-ik\sigma} \left(\frac{c}{\beta} \cos k_o \frac{\sigma}{\alpha} - \frac{1}{2k_o} \sin k_o \frac{|\sigma|}{\alpha}\right) .$$

If we again define $c\tau = \ell$, the correlation length for the source, and $c/\beta = L_o$, the viscous correlation length for the string, we see the almost equivalent roles they play in the exponential term. The primary difference is that β also affects k_o, the wave number modified by viscosity ($k^2 - \beta^2/4c^2)^{1/2}$. For the part of the integral where $\sigma > 0$, the result is

$$\frac{4\pi D}{\alpha c^2} \int_{-\infty}^{\infty} dk \frac{\frac{1}{\alpha\beta\tau} + ik\frac{c}{\beta}}{k^2(\frac{1}{\alpha^2} - 1) + \frac{1}{v\tau}(\frac{1}{v\tau} + \frac{\beta}{v}) + \frac{ik}{v}(\beta + \frac{2}{\tau})} ,$$

and for $\sigma < 0$ it is

$$\frac{4\pi D}{\alpha c^2} \int_{-\infty}^{\infty} dk \frac{\frac{1}{\alpha\beta\tau} - ik\frac{c}{\beta}}{k^2(\frac{1}{\alpha^2} - 1) + \frac{1}{v\tau}(\frac{1}{v\tau} + \frac{\beta}{v}) - \frac{ik}{v}(\beta + \frac{2}{\tau})} .$$

The result is then the sum of these, or

$$\langle u^2 \rangle = \frac{8\pi D}{c^2} \int_{-\infty}^{\infty} dk \frac{k^2 \left\{ 1 + \frac{L_o}{\ell}\left(\frac{1}{\alpha^2}+1\right) + \frac{1}{\alpha^2}\frac{L_o}{\ell^2}\left(\frac{1}{\ell}+\frac{1}{L_o}\right) \right.}{(k^2+k_1^2)(k^2+k_2^2)} \cdot \frac{\alpha^2}{|1-\alpha^2|^2} ,$$

where

$$k_{\substack{1\\2}}^2 = \frac{\alpha^2}{|1-\alpha^2|} \cdot \left[\left\{ \frac{1}{\ell}\left(\frac{1}{\alpha^2}+1\right)\left(\frac{1}{\ell}+\frac{1}{L_o}\right) + \frac{1}{2L_o^2} \right\} \pm \sqrt{\frac{1}{\alpha^2\ell^2}\left(\frac{1}{\alpha^2}+2\right) \cdot \left(\frac{1}{\ell}+\frac{1}{L_o}\right)^2 + \frac{1}{\ell L_o^2}\left(\frac{1}{\alpha^2}+1\right)\left(\frac{1}{\ell}+\frac{1}{L_o}\right) + \frac{\alpha^4}{4L_o^4}} \right] .$$

These expressions give us the excitation of the string in wave number. If we set $k = \frac{2\pi}{b}$, $L_o = \frac{10}{\pi}L$, $\ell = \lambda L$, and $\xi = \frac{b}{L}$, we can obtain a spectrum to be compared with that in the previous example.

If we make these substitutions, the spectrum $\Psi(\xi)$, such that

$$\langle u^2 \rangle = \frac{8\pi D L^2}{c^2} \int_{-\infty}^{\infty} d\xi \Psi(\xi) , \qquad 5.17$$

is

$$\Psi(\xi) = \frac{\alpha^2}{|1-\alpha^2|^2} \cdot \frac{4\pi^2 \left\{ 1 + \frac{10}{\pi\lambda}\left(\frac{1}{\alpha^2}+1\right) \right\} + \frac{\xi^2}{\alpha^2} \cdot \frac{10}{\pi\lambda^2}\left(\frac{1}{\lambda}+\frac{\pi}{10}\right)}{(4\pi^2+\xi^2 k_1^2 L^2)(4\pi^2+\xi^2 k_2^2 L^2)} . \qquad 5.18$$

This is plotted in Figure 5.04 as a function of ξ for various values of α, for $\lambda = 0.1$, as before, and $L_o = {}^{10}/_{\pi} L$. It is interesting to note that the shortening of the correlation due to the decay constant has a very profound effect, particularly for the curves near the value $\alpha = 1$. In general one may say that the effect of finite correlation distances (or times) in the noise field is to produce an excitation cutoff, particularly near the condition where the flow velocity coincides with the phase velocity.

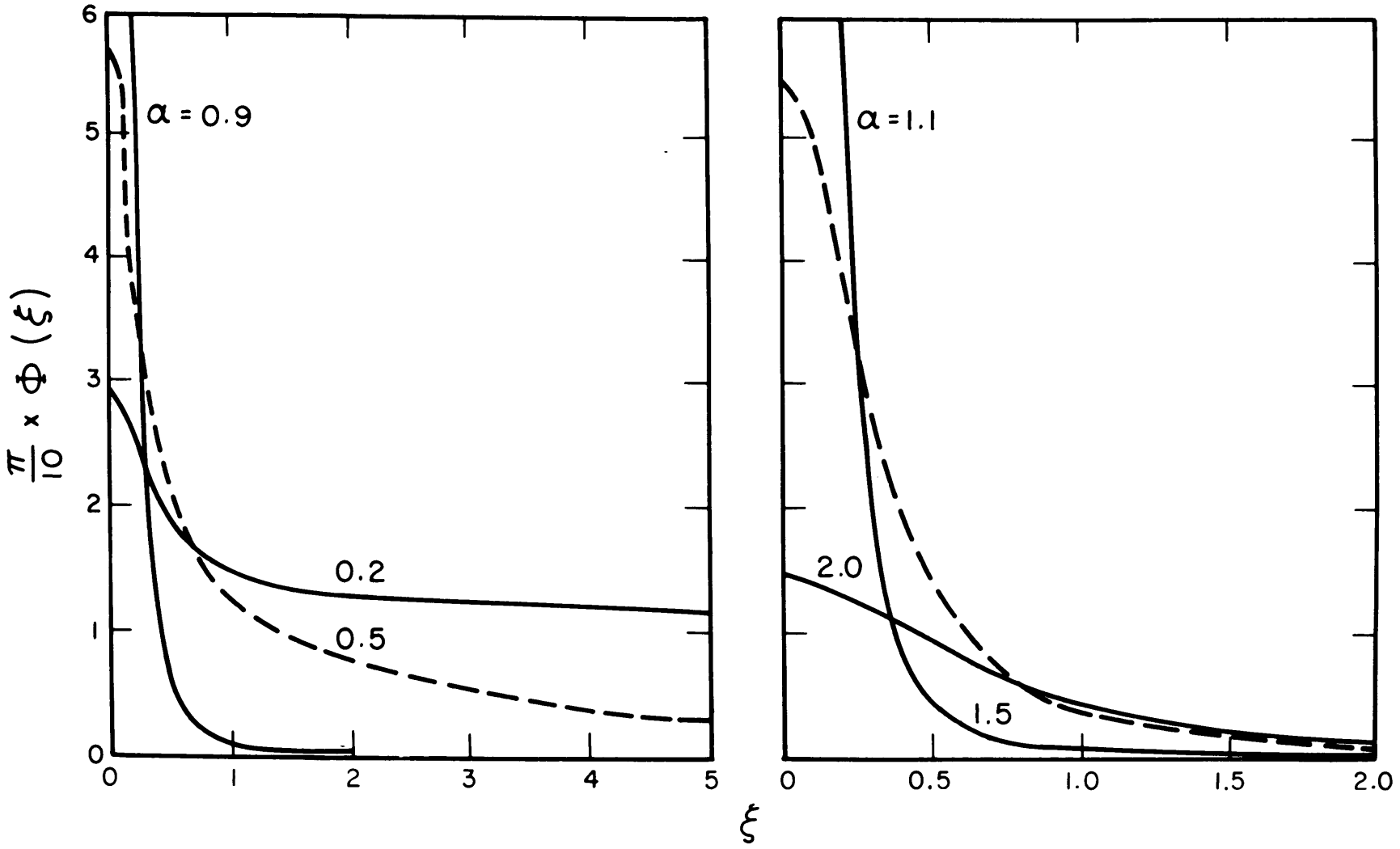

Figure 5.04 Spectral Response to Moving and Changing Noise Field.

Let us now proceed to integrate 5.17 so as to obtain the kinetic energy density, which is the only energy in the infinite string. By Cauchy's integral theorem, the integral is

$$\langle u^2 \rangle = 8\pi^2 c^2 \alpha^2 (2\pi i) \left[\frac{k_1^2 \left\{ 1 + \frac{L_o}{\ell}\left(\frac{1}{\alpha^2}+1\right)\right\} + \frac{1}{\alpha^2}\,\frac{L_o}{\ell^2}\left(\frac{1}{\ell}+\frac{1}{L_o}\right)}{2ik_1(k_1^2+k_2^2)} \right.$$

$$\left. + \frac{k_2^2 \left\{ 1 + \frac{L_o}{\ell}\left(\frac{1}{\alpha^2}+1\right)\right\} + \frac{1}{\alpha^2}\,\frac{L_o}{\ell^2}\left(\frac{1}{\ell}+\frac{1}{L_o}\right)}{2ik_2(k_1^2+k_2^2)} \right] .$$

By using 5.16 this becomes

$$\langle u^2 \rangle = \frac{8\pi^2 c^2 (k_1+k_2)}{\left\{ \frac{2}{\ell}\left(\frac{1}{\alpha^2}+1\right)\left(\frac{1}{\ell}+\frac{1}{L_o}\right)+\frac{1}{L_o^2}\right\}} \left\{ 1+\frac{L_o}{\ell}\left(\frac{2}{\alpha^2}-1+\alpha^2\right)\right\} .$$

This is plotted in units of $8\pi^2 c^2$ in Figure 5.05 for the same values of $\lambda = \ell/L$ and L_o as in Figure 5.04. Again we get the singular behavior at $\alpha = 1$. A source with a finite spectrum would modify this behavior a good deal, depending on how much high frequency component it had.

5.4 CONCLUSION

The infinite string is of particular interest since it tells us the excitation pattern of the source uncomplicated by boundary effects. For the higher modes it seems reasonable that the finite string should be excited very much to the same degree as an infinite string would be over the same length. This also indicates to us the advisability of carrying out theoretical analysis for more complicated systems assuming infinite

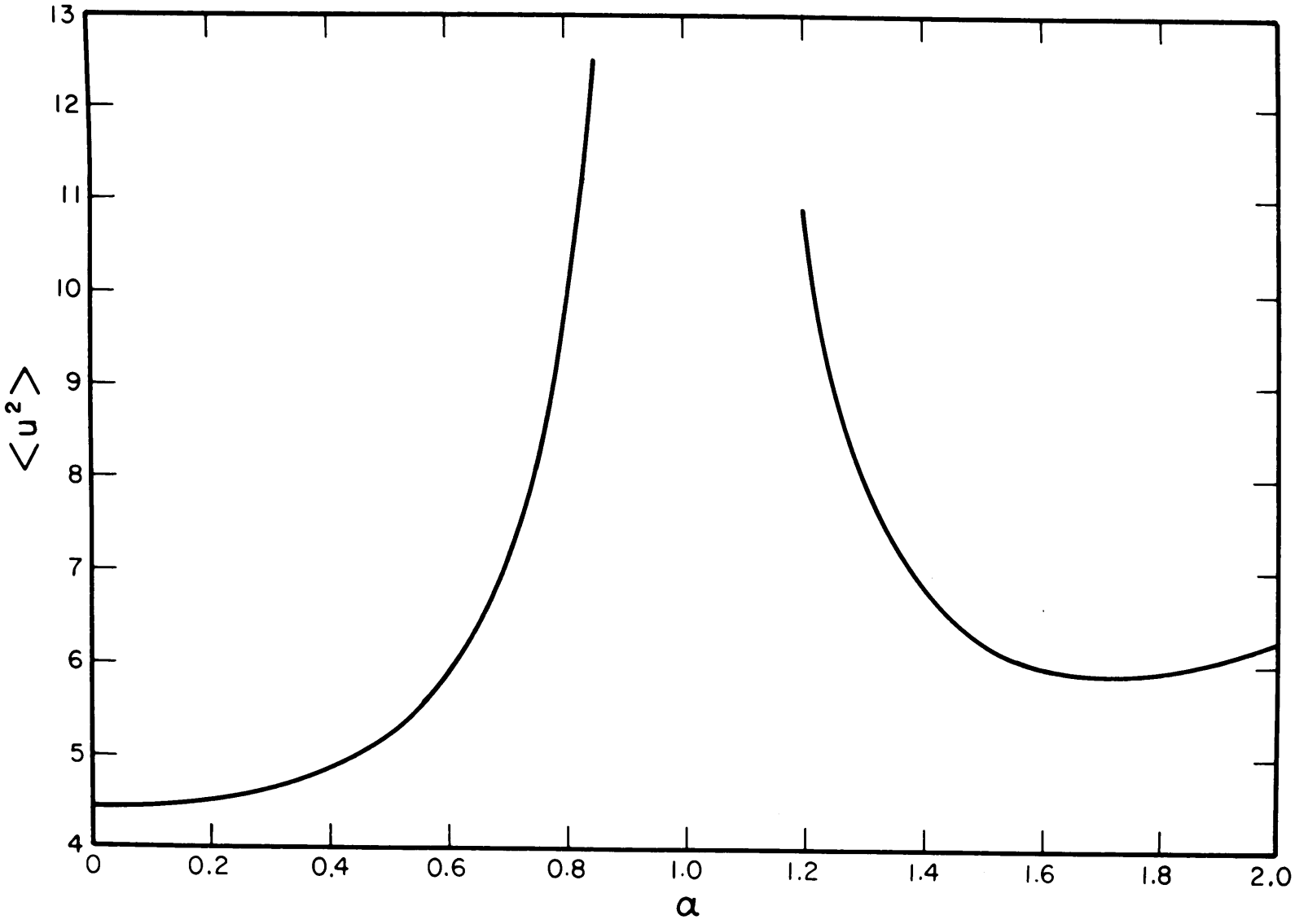

Figure 5.05 Energy Density Versus Flow Velocity.

extent, particularly when the dimensions of the actual object are long compared to the correlation lengths of the source.

With this chapter we complete the examples which illustrate the response of strings to random noise fields and now consider the problem of the creation of the random noise field. We shall attempt to form, by a random superposition of elementary sources, the noise field correlations which we assumed in the two examples in this chapter.

VI THE EDDY PROBLEM

6.1 INTRODUCTION

Thus far in our work we have assumed that a noise field exists in space, which we are able to represent by its correlation function. In many problems it is not convenient to measure the correlation of the source directly, and one would like to infer its properties from a knowledge of the elementary processes which create the total source noise field. There are many situations in which a noise field may be thought of as being made up of a random superposition of elementary sources. One familiar example is the "frying" of a teapot before it begins to boil, where the elementary source is created by the impulsive collapse of vapor bubbles when they emerge from a super-heated region into a cold environment. One feels (quite accurately) that a knowledge of the spectrum of the elementary process and the distribution of events should go a long way in predicting the source correlation field. It should be added that any correlation between the events of the elementary processes must also be considered.

Although in this chapter it will be assumed that the elementary events take place independently of one another, it should be borne in mind that in many important processes this is not the case. Consider, for example, the excitation of a lecture room by applause. If it is a large room, I have observed that people seem to applaud independently of each other -- at least the correlation would be rather short range compared to the dimensions of the room. On the other hand, in a small room the

applause is so well correlated that the excitation seems to be nearly periodic. An analysis assuming independent source events would almost certainly be in error for this situation.

In a well known paper,[15] Rice has analyzed thoroughly the statistical properties of noise currents when they are caused by the superposition of a large number of elementary pulses. These pulses are thought to occur randomly in time; and by assuming a probability distribution for their occurrence, such quantities as the signal spectrum, correlation function, and distribution may be calculated.

Extending Rice's treatment of the "shot effect" to the space dimensions as well, we shall assume that our noise field is a superposition of elementary sources and obtain the spectrum, correlation function, and the first order probability distribution function in terms of the properties of the elementary sources. These elementary sources will be termed "eddies" in deference to the central position of the turbulence problem in this thesis.

These results will then be applied to some simple distributions of elementary sources in order to create some of the source correlations assumed in Chapters III and V. In addition, an attempt will be made to approximate the turbulent field by a random superposition of vortex elements.

6.2 CALCULATION OF $W_1(f,\vec{r},t)$

Let a particular eddy, say the i^{th} one, be denoted by

$$b_i\, a(\vec{r} - \vec{y}_i, t - \tau_i) ,$$

where $\vec{y}_i, \tau_i$ is the point in space and time at which the eddy is created (or attains some specified value). We shall require that $\underline{a}$ possess a Fourier transform in time and space. That is, we insist that

$$\lim_{V,T\to\infty} \int_0^T dt \int_V d\vec{r}\; a^2(\vec{r},t) \qquad 6.01$$

exists and is non-zero.

The amplitudes b_i are independent random variants and are governed by a probability distribution $w(b)$. We assume that $a(\vec{r},t)$ is of such a form that

$$\int_{\text{all } b} w(b)db = 1 , \qquad 6.02$$

since this can always be accomplished by normalization.

The probability distribution which governs the occurrence of eddies is often easily inferred from the physical situation. We shall place no particular restrictions on its space and time dependence at the moment, but shall merely represent it by $P(\vec{y},\tau)$. We shall define it by saying that, if in the period $0,T$ and the region V, a single eddy occurs, $P(\vec{y},\tau)d\vec{y}d\tau$ is the probability that it will occur in the region and time $d\vec{y},d\tau$. This places the restriction

$$\int_0^T d\tau \int_V d\vec{y}\; P(\vec{y},\tau) = 1 . \qquad 6.03$$

The forcing function f is thought to be a superposition of the eddies, that is,

$$f^{(i)}(\vec{r},t) = \sum_i b_i a(\vec{r}-\vec{y}_i, t-\tau_i). \qquad 6.04$$

This notation requires some explanation. The superscript (i) on f means that this is the noise field resulting from a particular set of random parameters $b_i, \vec{y}_i, \tau_i$. From this summation of independent random variants, we should like to calculate the first order probability distribution for f, $W_1(f,\vec{r},t)$. We first select out all those situations which have a certain number, K, of eddies from $0 \to T$. Then we can write

$$W_1(f,\vec{r},t) = \sum_{K=0}^{\infty} (\text{Probability of K arrivals}) \cdot (\text{Probability that} \qquad 6.05$$

if there are K arrivals, f will lie between f and f df).

If there are K arrivals, the source function is

$$f_K(\vec{r},t) = \sum_{j=1}^{K} b_j a(\vec{r}-\vec{y}_j, t-\tau_j) \; . \qquad 6.045$$

Since each term is an independent random variable, the sum has the distribution,[28]

$$df\, W(f_K) = \frac{df}{2\pi} \int_{-\infty}^{\infty} e^{-iuf} \prod_{j=1}^{K} \left\langle e^{ib_j a(\vec{\sigma}_j, \zeta_j)} \right\rangle_{ave} du \; , \qquad 6.06$$

where $\vec{\sigma}_j = \vec{r}-\vec{y}_j$, $\zeta_j = t-\tau_j$. Since the distributions for $\vec{y}, \tau$ and b are known, these averages may be calculated. The distributions for each value of j are the same. Hence we may write

$$df\, W(f_K) = \frac{df}{2\pi} \int_{-\infty}^{\infty} du\, e^{-iuf} \left(\int_0^T d\tau \int_V d\vec{y} \int dbw(b)\, P(\vec{y},\tau) \exp\left\{ iuba(\vec{\sigma},\zeta) \right\} \right)^K \; . \; 6.07$$

We now need to know the probability that K, and only K, eddies will occur in the region 0,T; V. This is obtained from Poisson's law of small probabilities. The probability that the eddy will occur in the region

$d\vec{y}, d\tau$ is extremely small, and under such conditions the distribution of the number of eddies over a given region of space and time is

$$p(K) = \frac{\nu^K}{K!} e^{-\nu} \quad , \tag{6.08}$$

where ν is the average number of eddies occurring in V from $0 \to T$. This is known as Poisson's distribution[29] and has many convenient properties (one being that ν equals the mean as well as all the semi-invariants of the distribution). Placing 6.08 and 6.07 in 6.05 and summing gives

$$W_1(f,\vec{r},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} du \exp \left[-iuf + \nu \int_0^T d\tau \int_V d\vec{y} \int dbw(b)\, P(\vec{y},\tau) \cdot (\exp\{iuba\} - 1) \right] \tag{6.09}$$

by virtue of the Taylor expansion of the exponential and equations 6.02, 6.03.

We are interested in seeing just what the distribution W_1 is for given values of a and ν. To do this we introduce the semi-invariants of a distribution as the coefficients in the expansion

$$\ln \left\langle e^{ifu} \right\rangle_{ave} = \sum_{m=1}^{\infty} \frac{\lambda_m}{m!} (iu)^m \quad , \tag{6.10}$$

where λ_m is the m^{th} semi-invariant.[30] From 6.09 we have

$$\ln \left\langle e^{ifu} \right\rangle_{ave} = \nu \int_0^T d\tau \int_V d\vec{y} \int dbw(b) P(\vec{y},\tau) \ (\exp\{iuba\} - 1) \quad , \tag{6.11}$$

And the expansion of the expression of the r.h.s. of 6.11 is

$$\sum_{m=1}^{\infty} \frac{(iu)^m}{m!} \nu \int_0^T d\tau \int_V d\vec{y} \int dbw(b)\, P(\vec{y},\tau) \cdot b^m a^m (\vec{\sigma},\zeta) \quad . \tag{6.115}$$

Since $a(\vec{\sigma},\zeta)$ may be considered localized in extent in space and time (because the Fourier transform exists), let the temporal extent be given by Δ and the spatial extent by $\vec{\delta}$. Within Δ of the time limits 0 and T and within $\vec{\delta}$ of the surfaces of V, the integrations above over τ and $\vec{y}$ may be replaced by infinite integrals. Hence, by identification, the m^{th} semi-invariant of 6.09 is

$$\lambda_m = \nu \int_{-\infty}^{\infty} d\tau \int d\vec{y} \int dbw(b)\, P(\vec{y},\tau)\; b^m\, a^m(\vec{\sigma},\zeta) \quad . \tag{6.12}$$

By placing 6.11 into 6.09 and integrating term by term, one obtains Edgeworth's series,

$$W_1(f,\vec{r},t) \simeq \sigma^{-1}\varphi^{(o)}(x) - \frac{\lambda_3\sigma^{-4}}{3!}\varphi^{(3)}(x) + \left[\frac{\lambda_4\sigma^{-5}}{4!}\varphi^{(4)}(x) + \frac{\lambda_3^2\sigma^{-7}}{72}\varphi^{(6)}(x)\right] + \cdots \quad , \tag{6.13}$$

where $x=\frac{f-\lambda_1}{\sigma} = \frac{f-\langle f\rangle_{ave}}{\sigma}$, $\lambda_2 = \sigma^2$, and $\varphi^{(n)}(x) = \frac{1}{\sqrt{2\pi}}\frac{d^n}{dx^n}e^{-x^2/2}$,

which is the n^{th} eigenfunction of the harmonic oscillator. The terms in Edgeworth's series are in decreasing powers of ν in the order $\nu^{-1/2}$, ν^{-1}, and the term in square brackets goes as $\nu^{-3/2}$. Hence, as the frequency of occurrence of eddies increases, the distribution of f approaches a normal distribution, provided that the semi-invariants λ_m converge. From equation 6.12 it is evident that if we use a point source for $a(\vec{\sigma},\zeta)$, the λ's above m=1 do not converge; hence $\sigma\to\infty$ and the distribution $W(f)$ does not exist. There is nothing catastrophic about this. In many problems we are not concerned about the distribution, and in nature δ-functions do not exist anyway. We shall merely not concern

ourselves about $W_1(f,\vec{r},t)$ when we use δ -functions as elementary sources.

6.3 THE CORRELATION FUNCTION $\left\langle f(\vec{r}_o,t_o)\, f(\vec{r}'_o,t'_o)\right\rangle_{ave}$

On the face of it, the last sentence in the preceding paragraph would seem to destroy the theoretical basis for the examples of Chapters III and V. That is, the basic equations which were used were derived on the basis of the validity of equations like 2.115, which use $W_1(f)$, $W_2(ff')$, etc. However, the examples used correlation functions like $\delta(\sigma - v\zeta)$, which as we shall see, may be obtained by a superposition of δ-functions, in which case the W_n's do not exist. We shall avoid this difficulty, however, by averaging over the parameters b_i, $\vec{y}_i$, and τ_i since this is an equally valid procedure for ensemble averaging. That is, any particular ensemble member $f^{(i)}(\vec{r},t)$ may be considered as defined by a <u>particular</u> set of parameters $b_i, \vec{y}_i, \tau_i$ and the ensemble average performed by averaging over these parameters.

According to this the product $f_K^{(i)}(\vec{r}_o,t_o)\, f_K^{(i)}(\vec{r}'_o,t'_o)$ is given by

$$f_K(\vec{r}_o,t_o)\, f_K(\vec{r}'_o,t'_o) = \sum_{i,j=1}^{K} b_i b_j\, a(\vec{\sigma}_i,\zeta_i)\, a(\vec{\sigma}'_j,\zeta'_j) \quad , \qquad 6.14$$

where $\vec{\sigma}'_j = \vec{r}'_o - \vec{y}_j$, $\zeta'_j = t'_o - \tau_j$.

We now average over the total number of eddies, i.e., we form the sum

$$\sum_{K=0}^{\infty} p(K)\, f_K^{(i)}(\vec{r}_o,t_o)\, f_K^{(i)}(\vec{r}'_o,t_o) = \sum_{K=0} \frac{\nu^K}{K!} e^{-\nu} \cdot$$

$$\cdot \sum_{i,j=1}^{K} b_i b_j a(\vec{\sigma}_i,\zeta_i)\, a(\sigma'_j,\zeta'_j) \quad .$$

There are K terms for which i=j, and there are K(K-1) terms for which

$i \neq j$. Integrating over the parameters $b, \vec{y}, \tau$, one gets

$$\left\langle f(\vec{r}_o,t_o)\, f(\vec{r}_o',t_o')\right\rangle_{ave} = e^{-\nu} \sum_{K=0}^{\infty} \frac{\nu^K}{K!} \Bigg\{ K \int_0^T d\tau_1 \cdots \int_0^T d\tau_K \int_V d\vec{y}_1 \cdots$$

$$\int_V dy_K \int db_1 \cdots \int db_K \cdot P(\vec{y}_1,\tau_1) \cdots P(\vec{y}_K,\tau_K)\, w(b_1) \cdots$$

$$w(b_K)\, a(\vec{\sigma}_i,\zeta_i)\, a(\vec{\sigma}_i',\zeta_i') + K(K-1) \int_0^T d\tau_1 \cdots \int_0^T d\tau_K \int d\vec{y}_1 \cdots$$

$$\int_V d\vec{y}_K \int db_1 \cdots \int db_K \cdot P(\vec{y}_1,\tau_1) \cdots P(\vec{y}_K,\tau_K)\, w(b_1) \cdots$$

$$w(b_K)\, a(\vec{\sigma}_i,\zeta_i)\, a(\vec{\sigma}_j',\zeta_j') \Bigg\} \quad .$$

The first term on the right is

$$\int_0^T d\tau \int_V d\vec{y} \int db\; w(b) b^2 a(\vec{r}_o-\vec{y},t_o-\tau)\, a(\vec{r}_o'-\vec{y},t_o'-\tau) \quad ,$$

which will be denoted $\overline{b^2} \left\{ a(\vec{r}_o,t_o)\, a(\vec{r}_o',t_o') \right\}$. The second term is

$$\int_0^T d\tau \int_0^T d\theta \int_V d\vec{y} \int_V d\vec{z} \int db \int dc\; w(b) w(c)\, bc\, a(\vec{r}_o-\vec{y},t_o-\tau) \cdot a(\vec{r}_o'-\vec{z},t_o'-\theta) \quad ,$$

which may properly be written

$$(\overline{b})^2 \left\{ a(\vec{r}_o,t_o) \right\} \left\{ a(\vec{r}_o',t_o') \right\} \quad .$$

This second term represents the steady part of the force. If we do not wish to work with "d.c." forces, we may choose $w(b)$ such that $\overline{b}=0$. In any case, we have

$$\left\langle f(\vec{r}_o,t_o)\, f(\vec{r}_o',t_o')\right\rangle_{ave} = \overline{b^2}\nu \left\{ a(\vec{r}_o,t_o)\, a(\vec{r}_o',t_o') \right\} + \overline{b}^2 \nu^2 \cdot \left\{ a(\vec{r}_o,t_o) \right\} \left\{ a(\vec{r}_o',t_o') \right\} \quad , \qquad 6.15$$

since $\sum_{0}^{\infty} \frac{\nu^K}{K!} e^{-\nu} \cdot K = \nu$, and $\sum_{0}^{\infty} \frac{\nu^K}{K!} e^{-\nu}(K^2-K) = \nu^2$.

The equations for the coefficients can be obtained from the properties of the Poisson distribution, which were stated above. If we set

$$\langle K \rangle_{ave} \equiv m_1 \ , \ \langle K^2 \rangle_{ave} \equiv m_2 \ ,$$

then, by straightforward expansion of the probability functions, one can show that

$$m_1 = \lambda_1$$

$$m_2 = \lambda_2 + \lambda_1 m_1 = \lambda_2 + m_1^2 \ .$$

It was stated that for the Poisson distribution all the semi-invariants were equal to ν. Hence

$$m_1 = \langle K \rangle_{ave} = \nu$$

and

$$\langle K^2-K \rangle_{ave} = \langle K^2 \rangle_{ave} - \langle K \rangle_{ave} = \nu + \nu^2 - \nu = \nu^2 \ .$$

This gives us, then, a way of calculating the correlation function for f when we know a and the distribution of its occurrence. Since we have the correlation function, we shall not derive the spectrum of f directly. If the probability $P(\vec{r},t)$ is a constant, the noise field is stationary and homogeneous, and the relations between the spectrum and the correlation function in Section 2.3 - D, under Fourier Representations, must be used. If $P(\vec{r},t)$ is truly a function of $\vec{r}$ and/or t, the source

will be non-homogeneous and/or non-stationary. In this case the relationships in the same section under non-stationary processes (2.3 - E) may be used.

6.4 THE CALCULATION OF SOURCE CORRELATIONS

We shall now attempt to combine elementary point sources in such a manner as to produce the source correlations for the moving noise field. Let us first picture a string of length $2X$ with its ends at $+X$, $-X$. We shall assume that at a time t_a there are point sources created with equal probability along the length of the string, and they immediately begin to move to the right with a velocity v. The elementary source is given by

$$a(x-y_i, t-\tau_i) = \begin{cases} \delta\{x-y_i - v(t-\tau_i)\} & (t > \tau_i) \\ 0 & (t < \tau_i) \end{cases} . \qquad 6.16$$

We shall assume that the amplitudes b_i have a distribution like

$$w(b) = \frac{1}{2}\left\{\delta(b-1) + \delta(b+1)\right\} , \qquad 6.17$$

in which case

$$\overline{b} = \int w(b)\, b\, db = \frac{1}{2}(1-1) = 0$$

$$\overline{b^2} = \int w(b)\, b^2 db = \frac{1}{2}(1+1) = 1 ,$$

and

$$\int w(b)\, db = \frac{1}{2}(1+1) = 1 \quad , \text{ as required by 6.02.}$$

The probability $P(y, \tau)$ becomes

$$P(y,\tau) = \frac{1}{2X}\ \delta\ (\tau - t_a)\ . \qquad 6.18$$

Then

$$\left\langle f(x_o,t_o)\ f(x_o',t_o')\right\rangle_{ave} = \frac{\nu}{2X}\int_0^T d\tau\,\delta(\tau - t_a)\int_{-X}^{X} dy\,\delta\{x_o - y - v(t_o - \tau)\}$$

$$\delta\{x_o' - y - v(t_o' - \tau)\}$$

$$= \begin{cases} \dfrac{\nu}{2X}\displaystyle\int_{-X}^{X} dy\,\delta\{x_o - y - v\ (t_o - t_a)\}\,\delta\{x'_o - y - v(t_o' - t_o)\} & (t_o, t_o' > t_a) \\ 0 & \text{either } t_o \text{ or } t_o' < t_a \end{cases} \qquad 6.19$$

For the upper possibility we let $X \to \infty$ and define

$$n = \lim_{X\to\infty}\ \frac{\nu}{2X}\ .$$

Since ν is the average of the total number of eddies created, n is the average number of eddies per unit length. Integrating over y and letting the creation time $t_a \to -\infty$, we have, using the transformation defined in equation 3.11,

$$\left\langle f(x_o,t_o)\ f(x_o',t_o')\right\rangle_{ave} = n\delta(\sigma - v\zeta)\ , \qquad 6.19$$

which is the correlation function for Sections 3.4 and 5.2.

Let us now proceed to another situation. Suppose we assume that eddies are created randomly in time and space uniformly along the string and uniformly in time. We assume that they are created with magnitudes (or strength) equal to ± 1 and that they immediately move with a velocity

v to the right. As they move, they decay according to an exponential law. Hence for an eddy created at τ_i, y_i we have

$$b_i a(x_o - y_i, t_o - \tau_i) = \frac{b_i}{\theta} \delta(x_o - y_i - v(t_o - \tau_i))\, u(t_o - \tau_i)\, e^{-(t_o - \tau_i)/\theta} , \quad 6.20$$

where the decay constant θ is included as a coefficient to make the total integrated force of a single eddy independent of θ; and $u(t_o - \tau_i)$ is the unit step function which is zero for $t_o < \tau_i$ and equal to one for $t_o > \tau_i$. Again we have

$$\overline{b} = 0, \quad \overline{b^2} = 1 .$$

If we assume that the string extends from -X to X and the time interval is from -T to T, then the probability $P(y, \tau)$ is just $\frac{1}{4XT}$. Accordingly, we have

$$\left\langle f(x_o, t_o)\, f(x_o', t_o') \right\rangle_{ave} = \frac{\nu}{\theta^2 4XT} \int_{-T}^{T} d\tau \int_{-X}^{X} dy\, u(t_o - \tau)\, u(t_o' - \tau) \cdot \delta\{x_o - y - v(t_o - \tau)\}\, \delta\{x_o' - y - v(t_o' - \tau)\}\, e^{\frac{-(t_o + t_o') + 2\tau}{\theta}} . \quad 6.21$$

If we let the space and time limits extend to $\pm\infty$, then we define

$$\lim_{X;T \to \infty} \frac{\nu}{4XT} = m ,$$

where m is the average number of eddies created per second per unit length. We now integrate over y and obtain

$$\left\langle ff' \right\rangle_{ave} = \frac{m}{\theta^2} \int_{-\infty}^{\infty} d\tau\, u(t_o - \tau)\, u(t_o' - \tau)\, \delta(\sigma - v\zeta)\, e^{\frac{-(t_o + t_o') + 2\tau}{\theta}} .$$

In the case that $t_o > t_o'$, or $\zeta > 0$, we have

$$\langle ff' \rangle_{ave} = \frac{m}{\theta^2}\, e^{-(t_o + t_o')/\theta}\, \delta(\sigma - v\zeta) \int_{-\infty}^{t_o'} d\tau\, e^{2\tau/\theta}$$

$$= \frac{m}{2\theta}\, \delta(\sigma - v\zeta)\, e^{-\zeta/\theta} \quad .$$

And similarly, if $t_o < t_o'$ or $\underline{\zeta < 0}$, we obtain

$$\langle ff' \rangle_{ave} = \frac{m}{2\theta}\, \delta(\sigma - v\zeta)\, e^{+\zeta/\theta} \quad .$$

Hence we obtain the correlation function

$$\left\langle f(x_o, t_o)\, f(x_o', t_o') \right\rangle_{ave} = \frac{m}{2\theta}\, \delta(\sigma - v\zeta)\, e^{-|\zeta|/\theta} \quad , \qquad 6.22$$

which is identical to that used in Sections 4.5 and 5.3.

We have illustrated how a knowledge of the elementary source may be used to form the correlation function of the total noise field. It would be interesting to see if one could obtain a representation of the turbulent field by such a procedure.

6.5 THE SPECIFICATION OF TURBULENCE

The problem of the description of turbulent flow has been actively dealt with during the past few decades, with the names of A. N. Kolmogaroff, W. Heisenberg, A. A. Townsend, G. I. Taylor, and O. Reynolds in prominence. The problem is complicated primarily by the non-linear character of the Navier-Stokes equation, namely

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{u} \quad , \qquad 6.23$$

where $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{u} \cdot \nabla$ is the rate of change in time of any quantity as we move along with the flow; $\vec{u}$ is the flow velocity as a function of

$\vec{r}$ and t; p is the pressure; ρ is the fluid density; and ν is the kinematic viscosity. The second equation governing the flow is the equation of conservation of mass, commonly referred to as the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0. \qquad 6.24$$

If the flow is incompressible, the density is constant as the fluid moves along,

$$\frac{D\rho}{Dt} = 0 \ ,$$

which is equivalent to

$$\text{div } \vec{u} = 0. \qquad 6.25$$

This situation would seem to be primarily concerned with the motion of incompressible fluids and to be inaccurate when applied to compressible fluids. However, we can always write any vector $\vec{u}$ like

$$\vec{u} = - \text{grad } \phi + \text{curl } \vec{A} \equiv \vec{u}_1 + \vec{u}_2 \ . \qquad 6.26$$

Now $\vec{u}_1$ is the acoustic part of the motion, while $\vec{u}_2$ represents the "hydrodynamic," which is vortical. If we take the curl of the Navier-Stokes equation, we get

$$\frac{D\vec{\omega}}{Dt} = \nu \nabla^2 \vec{\omega} \ , \qquad 6.27$$

where $\vec{\omega} = \text{curl } \vec{u} = \text{curl curl } \vec{A}$. This is not to imply that the $\vec{u}_1$ does not contribute to the random pressure and velocity fluctuations in turbulent flow, but these will propagate with the velocity of sound and not with the average flow velocity, as we saw in Chapter IV. The fluctuations

due to $\vec{u}_1$ are commonly called aerodynamic noise. Hence, if we are primarily concerned with the fluctuations which travel with the flow, we shall concentrate on the "incompressible" part of compressible fluid flow.

$\vec{\omega}$ is called the vorticity and is related to the density of angular momentum in the flow.[31] When the fluctuations are small enough,* one may ignore the second order terms like $\vec{u}\cdot\nabla\vec{\omega}$ and write

$$\frac{1}{\nu}\frac{\partial\vec{\omega}}{\partial t} - \nabla^2\vec{\omega} = 0 \quad , \tag{6.28}$$

which is a standard diffusion equation. It is well known[31] that vorticity may be introduced only at the boundaries of the medium and that it then diffuses into the fluid. The diffusion operator is a statement of the conservation of the quantity which it operates on, indicating, as is well known, that vorticity is conserved in any unbounded region of fluid.

If we then associate vorticity with the hydrodynamic part of the turbulence and assume the correctness of the linearized equation 6.28, then we may obtain the elementary solutions which are the impulse Green's functions for the diffusion equation. That is, if we have the operator $\mathcal{L}$ in Chapter II become

$$\mathcal{L} \equiv \nabla^2 - \frac{1}{\nu}\frac{\partial}{\partial t} \quad , \tag{6.29}$$

then the Green's function is[32]

$$\mathcal{G}(\vec{r},t \mid \vec{r}_o,t_o) = 4\pi\nu\left(\frac{1}{2\sqrt{\pi\nu(t-t_o)}}\right)^n e^{-|\vec{r}-\vec{r}_o|^2/4\nu(t-t_o)} \cdot u(t-t_o) \quad . \tag{6.30}$$

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*As usual, "small enough" is not an adequate phrase, but the consideration of scale length and decay times is too involved to be dealt with here.

Again, $\vec{r}_o, t_o$ is the source point, $u(t-t_o)$ is the unit step function which is zero for $t < t_o$ and + 1 for $t > t_o$, and n is the number of dimensions equal to 1,2, or 3. In order to save ourselves the complications of dyadic Green's functions, let us assume that we are concerned with two-dimensional flow in the x-y plane, in which case n = 2 and the single non-vanishing component of the vorticity is ω_z. By our formalism from Chapter II we may immediately write the expression for the vorticity correlation function,

$$\left\langle \omega_z(\vec{r},t)\ \omega_z(\vec{r}',t') \right\rangle_{ave} = \int_{-\infty}^{t}\int_{-\infty}^{t'} dt_o dt_o' \iint dv_o dv_o' \left\{ \frac{1}{(t-t_o)(t-t_o')} \right\}^{\frac{n}{2}} \cdot e^{-|\vec{r}-\vec{r}_o|^2/4\nu(t-t_o)} \, e^{-|\vec{r}'-\vec{r}_o'|^2/4\nu(t'-t_o')} \cdot \left\langle \vec{f}(\vec{r}_o,t_o)\ f(\vec{r}_o',t_o') \right\rangle_{ave} \quad . \tag{6.31}$$

This formula would be appropriate for use in situations where the vorticity is created in more or less open regions, as in the widely used situation of turbulence created by passing flow through a wire gauze. Should the turbulence be created on boundaries and there be no volume sources, one can use a similar expression replacing the volume integrals by surface integrals (or line integrals for two-dimensional flow). For a region "driven" from its boundaries in two dimensions the solution may be written

$$\omega_z(\vec{r},t) = \frac{1}{4\pi}\int_{-\infty}^{t} dt_o \int_C d\ell_o \left[g \frac{\partial \omega_z}{\partial n_o} - \omega_z \frac{\partial g}{\partial n_o} \right] \quad , \tag{6.32}$$

where g is now the appropriate Green's function for the bounded domain. The integration is a contour integral around the surface profiles in

the x-y plane whence $d\ell_o = \sqrt{(dx_o)^2 + (dy_o)^2}$, and $\frac{\partial}{\partial n_o}$ means the derivative with respect to the outward normal of these surfaces. Since it is the production of vorticity itself which we specify along the boundary, we choose ξ to be zero on the boundary. This determines ξ' for the domain and one may then write

$$\langle \omega_z(\vec{r},t)\,\omega_z(\vec{r}',t') \rangle_{ave} = \frac{1}{16\pi^2} \int_{-\infty}^{t}\int_{-\infty}^{t'} dt_o dt_o' \iint_C d\ell_o d\ell_o' \qquad 6.33$$

$$\frac{\partial \xi_\delta(\vec{r},t \mid \vec{r}_o,t_o)}{\partial n_o} \; \frac{\partial \xi(\vec{r}',t' \mid \vec{r}_o',t_o')}{\partial n_o'} \cdot \langle \omega_z(\vec{r}_o{}^s,t_o)\,\omega_z(\vec{r}_o'{}^s,t_o') \rangle_{ave} \;,$$

where we have averaged the product of two solutions over the ensemble. The superscript ^{s} refers to values on the surface of the domain. This kind of calculation could predict the vorticity correlation in flow between two parallel planes, and we would expect it to give reasonable results as long as the turbulent components did not become too large compared to the average flow. If there is an average flow, we may either transform the $\vec{r},t,\vec{r}',t'$ system above to a moving frame of reference, or we might replace the operator $\mathcal{L}$ above with

$$\mathcal{L} \equiv \nabla^2 - \frac{1}{\nu}\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right) \;, \qquad 6.34$$

where v is the average flow velocity.

The method above has the advantage of structural unity with the procedures set forth in Chapter II. However, the integrals to be evaluated are extremely involved for even the simplest kind of vorticity source assumptions. We shall then proceed to a treatment of turbulence correlation functions which uses the results of work done by G. I. Taylor on the decay of a single eddy.[33]

6.6 THE SUPERPOSITION OF DECAYING EDDIES

Taylor's analysis of the single decaying eddy assumes a flow pattern in the x-y plane. The eddy is centered at $r \equiv \sqrt{x^2+y^2} = 0$ and has only tangential velocity u_ϕ. He considers the forces operating on a cylinder of the fluid, and from the equations of rotary motion he derives the following differential equation:

$$\frac{d^2\vartheta}{dr^2} + \frac{1}{r}\frac{d\vartheta}{dr} = \frac{1}{\nu}\frac{d\vartheta}{dt} \quad , \tag{6.35}$$

where $u_\phi = \frac{d\vartheta}{dr}$. It is not hard to show from the definition of ϑ and the condition of incompressibility that ϑ is just the negative of the single non-vanishing component of the vector potential for the flow. That is,

$$\vec{A} = -\vec{k}\,\vartheta \quad , \tag{6.36}$$

where $\vec{k}$ is the unit vector in the z-direction. The solution to 6.35 is

$$\vartheta = \frac{K}{t}\, e^{-\eta^2} \quad , \tag{6.37}$$

where $\eta = \sqrt{\frac{r}{4\nu t}}$. When this is differentiated, the velocity u_ϕ is

$$u_\phi = Bt^{-3/2}\,\eta\, e^{-\eta^2} \tag{6.38}$$

where $B = -\frac{K}{\sqrt{\nu}}$.

This is Taylor's result. It is easy to show that this velocity distribution will not satisfy the vorticity equation 6.27. This need not concern us, however, since many approximate solutions which give very good results for asymptotic regions do not satisfy the fundamental exact equations. A good example of this is the solution of the equation

of motion for bending waves on a thin rod which does not satisfy the equations of equilibrium for an elastic solid.

Since we believe that it is primarily the hydrodynamic pressure fluctuations which excite the ribbon in Chapter IV, we would like to know the pressure field resulting from the velocity distribution in 6.38. Accordingly, we write the component Navier-Stokes equation in the radial direction,

$$\vec{a}_r \cdot \left[\rho \frac{\partial \vec{a}_\phi u_\phi}{\partial t} - \rho u_\phi \frac{1}{r} \frac{\partial}{\partial \phi} (u_\phi \vec{a}_\phi) \right] = - \frac{\partial p}{\partial r} + \vec{a}_r \cdot \nu \nabla^2 \vec{u} \quad . \qquad 6.39$$

The first term on the l.h.s. is zero, and in the second term we use

$$\frac{\partial \vec{a}_\phi}{\partial \phi} = - \vec{a}_r \ ,$$

giving us

$$\frac{\partial p}{\partial r} = \frac{u_\phi^2}{r} \quad . \qquad 6.40$$

Since we know u_ϕ , we can integrate the r.h.s. of 6.40 and obtain for p

$$p(r,t) = \frac{\rho B^2}{4t^3} \ e^{-2\eta^2} \quad . \qquad 6.41$$

Thus, the pressure is a Gaussian having amplitude and a spread dependent on its age t. If this pressure distribution travels along the ribbon of Chapter IV, the total pressure exerted on the ribbon may be taken to be

$$f(t) = \int_0^\infty 2\pi r p(r) \, dr = \frac{\pi \rho \nu B^2}{2t^2}$$

as long as the spread of the eddy is less than the width of the ribbon w, or

$$w \geq \sqrt{\nu t} \quad .$$

If $t = \frac{w^2}{\nu}$ represents a time over which the total pressure of the eddy has decayed considerably, we can take this pressure pulse to be a δ-function. In such case, for the one-dimensional analysis of the ribbon, the eddy would be assumed to be $\frac{\pi \nu \rho B^2}{2t^2} \delta(x)$. Since the eddy strength goes to infinity as $t \to 0$, we shall assume that the eddy is created at some time Θ. This is reasonable for the Taylor analysis, which would not be expected to be valid for t near zero since this means very high values of u_ϕ, and no linear theory could be adequate. We choose the amplitude B such that the elementary excitation is

$$a(x,t) = \frac{1}{(t+\Theta)^2} \delta(x - vt) \quad , \qquad 6.42$$

where the eddy immediately begins to move to the right after it is created. We may now calculate the one-dimensional pressure correlation field using equation 6.15. Because we are only interested in the fluctuations in the pressure and not in the average "d.c." terms, we shall only calculate the first term on the r.h.s. of equation 6.15. Calling the creation place y_i and the time τ_i, we shall assume that the eddies are created uniformly in time and space, that is

$$P(y, \tau) = \frac{1}{4XT} \quad .$$

In such a case we have

$$\lim_{X,T \to \infty} \nu \left\{ a(x,t)a(x',t') \right\} = m \int_{-\infty}^{\infty} dy \int_{-\infty}^{\tau^+} d\tau \frac{\delta\{x-y-v(t-\tau)\}\delta\{x'-y-v(t'-\tau)\}}{(t+\Theta-\tau)^2(t'+\Theta-\tau)^2} \qquad 6.43$$

where

$\tau^+ = t$ if $t < t'$

$\tau^+ = t'$ if $t > t'$, and $m = \lim_{X,T\to\infty} \frac{\nu}{4XT}$.

Integrating over y yields

$$\int_{-\infty}^{\tau^+} d\tau \; \delta(\sigma - v\zeta) \; \frac{1}{(t+\Theta-\tau)^2 \, (t'-\Theta-\tau)^2} \quad ,$$

where $\sigma = x-x'$, $\zeta = t-t'$. This integration is straightforward but rather involved. The result is

$$\nu\left\{a(x,t)a(x',t')\right\} = \begin{cases} \frac{1}{\zeta}\left[\frac{1}{\Theta(\Theta-\zeta)} + \frac{2}{\zeta}\left\{\frac{1}{\Theta} - \frac{1}{\Theta-\zeta}\ln\frac{\Theta-\zeta}{\Theta}\right\}\right]\delta(\sigma-v\zeta) \; ; \; (\zeta<0) \\ \frac{1}{\zeta}\left[\frac{1}{\Theta(\Theta+\zeta)} + \frac{2}{\zeta}\left\{\frac{1}{\Theta+\zeta} + \frac{1}{\Theta}\ln\frac{\Theta+\zeta}{\Theta}\right\}\right]\delta(\sigma-v\zeta) \; ; \; (\zeta>0) \end{cases} \qquad 6.44$$

The use of this expression as a source correlation would complicate the integrations substantially. However, one could attempt to compare the hydrodynamic pressure correlations in a turbulent field with the dependence as predicted by 6.44.

BIOGRAPHICAL NOTE

Richard H. Lyon was born August 24, 1929, in Evansville, Indiana, to Gertrude and Chester Lyon. He attended grade school and high school in Evansville. After graduating from high school in 1947, he worked for a year in the local refrigerator plants.

He entered Evansville College in September, 1948, as a physics major and was active in the local chapter of Sigma Pi Sigma, honorary physics society. Much of his spare time throughout college was taken up with radio servicing and audio amplifier design. He graduated from Evansville College with an A. B. degree in 1952.

Coming to M. I. T.'s Acoustics Laboratory in July of 1952 as a research assistant in physics, he worked on the theory of damped acoustic resonators with Professor K. U. Ingard. Under the direction of Dr. T. F. Hueter and Professor R. D. Fay, he then began working on the excitation of infinite elastic plates by localized forces.

In July, 1954, he was awarded the Owens-Corning Fellowship in Acoustics and began working on his Ph. D. thesis, "The Excitation of Continuous Systems by Random Noise Fields."

He was married January 29, 1955, to Joy Hallum of Dayton, Ohio.

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