Instabilities in Interstellar Space

by

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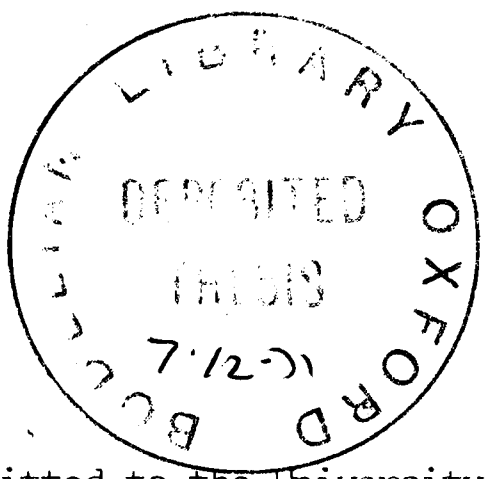

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ABSTRACT

This thesis is a partial investigation of instabilities in the interstellar gas which are driven by a coupling between the ambient radiationfield and the gas, and which do not arise when this coupling is missed out. The modes of couplings considered are, firstly, the attenuation of the radiation with the concommitant effects on the temperature, density and composition of the gas, in various combinations. Secondly, velocity dependent effects are examined in various circumstances and thirdly, radiation pressure, not included in the other two, is looked at in the simple case in which temperature and compositional changes are excluded.

The explanation of why these instabilities may be of interest, and an outline of the extent to which similar instabilities have been investigated, is given in Chapter 1. Chapter 2 gives details of the basic equations used in the case in which the absorption line shape is ignored.

Many of the equations are used in the other chapters. The equations are linearised in perturbations of the density, temperature, radiation field and composition, and the resulting dispersion relationship is found for a harmonic perturbation. Because of the attenuation term in the radiative transfer equation, the polynomial has complex coefficients.

In Chapter 3 we investigate the properties of the roots of a complex polynomial by an extension of Routh's methods, and derive a set of criteria to determine the number of roots which have positive real part. These roots correspond to exponentially growing

perturbations, or, in other words, they correspond to instabilities. Later in the chapter we apply these methods to Field's dispersion relationship for thermal instabilities and derive many of his conclusions in a fairly simple way. By a slight extension the method yields estimates of the growth times of the instabilities. Some related situations are also examined in a similar way.

After the detour of Chapter 3, Chapter 4 gives details of some models of the heating and cooling of the interstellar gas as well as of the reactions to be considered, namely the formation and destruction of H_2 and of carbon ions. Some of the limitations of the models are also discussed and the roots of the dispersion relation are given for different values of the parameters. New instabilities do appear; for H_2 their timescales of growth are rather too long to be of interest; for carbon no short timescale instabilities are discovered. Chapter 5 gives similar details for a system of pure hydrogen gas which may be of interest in studies of the formation of the first generation of stars. In Chapter 6 there is a criticism of an earlier work by Schatzman on a similar subject, in which it is shown that his analysis was wrong.

Chapter 7 deals with a new possibility, namely that, as the gas moves, photons will be seen to be shifted in frequency and so the molecules will be exposed to a new set of destructive photons at frequencies which have not been selectively absorbed in the unperturbed gas. First the simplest case, that in which the temperature is unperturbed, is treated analytically. The attenuation of the radiation field is not considered. The effectiveness of this doppler-induced effect depends upon both the absorption profile

and the radiation spectrum; these factors as well as temperature perturbations are included next. Both line absorption and continuum absorption are considered. The former is used to investigate the stability of the interstellar gas and of pure hydrogen gas, where hydrogen molecules are dissociated by line absorption; the latter is used in connection with HII regions and also the interstellar gas where the photodissociated species are hydrogen atoms and neutral carbon respectively.

Radiation pressure was not included in the previous chapters but in Chapter 8 a modified version of Field's theory of instabilities driven by radiation pressure is presented. The new feature is that the frequency dependence of the absorption coefficient is included in the equations and this, in the case of a flat radiation spectrum, leads to an exact cancellation of the dominant term in Field's equation. Several restrictive features of Field's conclusions are thus modified and seem to make this instability rather more useful in the study of instabilities in the interstellar gas than it appeared in Field's work.

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CHAPTER 1

INTRODUCTION

Instabilities of various kinds in large gas clouds have been of interest to astrophysicists for some time. The explanation for this interest is in part the fact that the distribution of matter in the universe is not smooth but clumpy, clumpy on the scale of megaparsecs in the form of galaxies and on the scale of parsecs in the form of stars and discrete clouds in the galaxies. These clumps do not seem to be explicable by arguments about statistical fluctuations. Astrophysicists would like to explain these inhomogeneities in a way that is simple enough to understand and to calculate, and that is also not too 'ad hoc'. One way to start is to assume everything is uniform and steady and then see what develops, introducing a few simplifying assumptions along the way. An alternative is to assume that initially there is chaos which is uniform in some sense. Both of these have the advantage that one can argue that the initial conditions chosen are not too 'ad hoc' in that steadiness and chaos are in some way objectively defined starting points and not ones chosen subjectively to explain the particular phenomenon being discussed. Also one would hope that the final state would to a large extent be independent of the initial state.

One method of investigating instabilities in a uniform, steady, unbounded gas involves linearising the equations of motion, mass and energy with respect to small variations in some of the physical quantities superimposed upon the steady uniform initial state. These perturbations are then assumed to have harmonic space and time variations $\exp(\beta t + i\underline{k}.\underline{r})$ so that, because the equations are linearised,

the condition for consistency usually becomes $P(k, \beta) = 0$, where P is a polynomial in k and β whose coefficients are functions of the initial conditions. Regarding k as a real parameter one might be able to solve the equation for β and find the conditions which allow $Re(\beta)$ to be positive, corresponding to the exponential factor increasing in time, that is, corresponding to an instability. It is in fact not always necessary to solve for β explicitly because there are often easily stated criteria which give the number of roots with positive real part for a given polynomial. However, these latter methods do not, without further work, give one the size of $Re(\beta)$, which is of considerable interest since the timescale for growth of the perturbation is $1/Re(\beta)$. The timescale is critically important for if this is too long the existence of the instability may be insignificant; too long in this context means longer than the time-scales of those processes which had been neglected in the initial set of equations. Examples of the points mentioned above will be given in the following chapters.

The treatment of non-uniform situations is rather different. One method involves assuming that β and the various amplitudes are dependent on the position in the gas in a continuous and slowly varying manner. A second method involves finding an exact solution to the full equations and boundary conditions, then linearising the equations and solving them for the spatial dependence of the perturbation exactly, against this initial exact solution of the exact equations. The time constant is obtained from the consistency condition that a non-zero solution exist. An example of the latter method is given in Chapter 6.

When one is interested in perturbations that are not infinitesimal then the equations must not be linearised but must be solved in some other way. Numerical solutions have been found in some cases. These calculations are of importance in order to determine how far the instabilities which develop in the infinitesimal regime extend into observable consequences.

Jeans (1929) gave an approximate treatment for a self-gravitating, uniform, stationary gas cloud, using the first method outlined above. It was approximate in the sense that the initial conditions did not satisfy the original equations, however, later examinations of geometries which are soluble have given similar results. The details are given in Spitzer (1968). Jeans obtained a minimum wavelength λ_J required for the instability to develop and by taking this as the radius of a spherical region was led to the conclusion that the minimum mass of gas which could collapse in this fashion was $150(T^3/n)^{\frac{1}{2}}M_\odot$, where T is the gas temperature in Kelvin, n the number of particles per cc, and $M_\odot$ the solar mass. The time-scale of collapse is of order $5.10^7/n^{\frac{1}{2}}$yrs for a gas of molecular weight about unity, and this is independent of temperature, being just the free-fall timescale. The Jeans instability does not, by itself, appear to be able to explain the size or the rate of formation of many of the objects seen. Other physical processes must be invoked if this approach is to be continued.

Field (1965) considered instabilities of a different kind, based on the thermal properties of a gas. In the steady state the gas has a certain density, and the temperature is one which allows a balance between the heating and cooling processes in the gas. This

stationary state is perturbed as outlined previously and conditions for the existence of instabilities were given as were the timescales obtained for particular models. He neglected the gravitational terms in his equations because he was interested in those instabilities which had shorter timescales than the gravitational one. Kegel and Traving (1976) have made the extension to cover both thermal and gravitational effects and they show the way in which the two types merge.

Other authors had previously suggested that thermal instabilities were of importance in the formation of solar prominences, interstellar clouds, condensations in planetary nebulae and also of galaxies. Field lists these and investigates them more thoroughly. Infinitesimal perturbations in the interstellar medium, $n = 7.10^{-2}$ cm^{-3}, $T = 10^4$ ^oK, have growth times of about 10^9 yrs.

More recently Field (1971) has considered the effect of radiation pressure on a gas and general dust mixture, taking into account relativistic effects which alter the angular distribution of the radiation in the frame of a moving particle. Only the dust interacts with the radiation but it also collides with the gas. The consistency condition becomes an integral one leading to a rather complex non-polynomial function. However, Field managed to obtain some approximate roots of the equation. He found that instabilities occur only when the radiation pressure exceeds the gas pressure and on wavelengths longer than some critical value. In the solar neighbourhood he finds this instability would have a timescale of about $\frac{10^8}{\sqrt{n}}$ yrs, and this timescale is greater than the free-fall timescale ($5\ \frac{10^7}{\sqrt{n}}$ yrs).

The timescale varies as $\rho_r^{-\frac{1}{2}}$ where ρ_r is the radiation energy density so that in brighter regions the timescale decreases. Also the critical mass of a region which would just collapse because of radiation pressure varies as $(p_{r_o})^{-3/2}$ where p_{r_o} is the radiation pressure. Near the Sun this mass is about 310^6 $M_\odot$ if n = 10 cm^{-3}.

Gerola and Schwartz (1976) have performed a similar calculation with more limited scope, but they include photodesorption from the surface of interstellar grains which gives a jet like effect on the grain and thus a greater momentum transfer. They find a critical density of $10^4\varepsilon$ cm^{-3} in the solar neighbourhood where ε is the yield per photon from a fully covered surface. Below this density the growth time is $\frac{10^7}{n}$ years; above it the growth time is $\frac{510^2}{\varepsilon}$ e^τ years where τ is the opacity. The photoelectric effect can play a similar role leading to a time of growth for optically thin clouds of 310^5 - 310^6 years depending upon the photoelectric yield taken in the two cases as either 0.1 or 0.01. If $\varepsilon \geqslant 10^{-4}$ the times are less than the average cloud lifetime and so may lead to significant segregation of dust from gas, moreover these times depend very little on k, the wavenumber of the perturbation, so that the perturbations grow equally at all length scales.

Instabilities in gases due to radiation pressure have also been studied in relation to models of the atmosphere of quasars. Mestel et al. (1976) found that if the force of the radiation per unit mass was density dependent then instabilities could grow in a smooth model of quasar winds. However, in the most interesting case of winds of relativistic velocity this was not the case.

In their work they did not consider the modulation of the radiation field resulting from the waves in the gas. This followed Hearn (1972) who applied similar methods to models of the chromosphere of a very hot star. He showed that under these conditions the modulation of the radiation field resulting from re-radiation of the energy obsorbed by the waves would be less than 1 in 10^5 with reasonable assumptions, and because of its smallness he neglected this effect.

The composition of the gas may also vary and Stein et al. (1972) considered the growth of perturbations in a cooling hydrogen plasma in which recombination was occurring. This was generalised by Yoneyama (1973) who considered the linearised equations of change in composition as well as of conservation of mass, momentum and energy. He found a new type of instability which he termed thermal-reactive and which may occur in systems stable according to the Field criteria. On the other hand some systems which would be unstable according to the Field criterion may be stabilized by chemical reactions. For example, if a particular system is unstable without chemical reactions then inclusion of reactions may cause the coolants to be sufficiently depleted as the temperature falls, or to be increased in concentration as the temperature rises, to force the system back to the initial steady temperature, eliminating the instability. Yoneyama studied four reactions in gases; the recombination of hydrogen in a pure hydrogen plasma, which stabilises the system for sufficiently high temperatures, recombination of oxygen in a high temperature gas, the formation of hydrogen molecules at high gas temperatures and the accretion of hydrogen molecules onto grains at low temperatures.

He considered each situation separately starting with a steady initial state and for some reactions he considered two destruction mechanisms, namely collisional destruction and photodestruction but treated each case separately. He investigated the timescale only for the last reaction and found it to be $\frac{410^8}{n}$ years both for the very long and very short, but unspecified, wavelengths. This is just the timescale for the reaction. Also he states that the third reaction, that of grain catalysed formation of hydrogen molecules, only leads to instabilities when H_2 is the dominant coolant and the density and temperature are high so collisional dissociation is the main destruction mechanism.

Stein et al's calculation mentioned above was extended into the non-linear regime by numerical methods. Starting with an initial situation of $n = 0.3\ cm^{-3}$, fractional ionisation = 0.05, $T = 8200^{o}k$ the gas is allowed to cool uniformly and then the development of a spherically symmetric perturbation $\propto \frac{\sin kr}{kr}$, $k = \frac{\pi}{3}\ (pc)^{-1}$ is followed numerically. A very dense, cool core is formed ($n = 10^{11}$, $T = 110^{o}k$) which is similar to the initial conditions used by Larson (1969) in calculations on protostar formation. One dimensional perturbations in this same system lead only to a 100 fold increase in density in 10^6 yrs. In both cases the initial fractional perturbation in the parameters is about 10%.

Glassgold and Langer (1976) have studied in great detail the reactions of formation and destruction of Carbon Monoxide (CO) and also of water (H_2O) molecules and have used these in calculations of thermochemical instabilities in the same way as Yoneyama. They

found timescales for these instabilities of about 4×10^5 yrs for $k >> |\sigma|/c = \frac{1}{4 10^5 \; 3 10^7 \; 3 10^{10}} = 2 10^{-24}$ cm^{-1} corresponding to $\lambda << 3 10^{24}$cm and timescales $9 10^5 - 7 10^6$ yrs as k^{-1} varies from 0.1 to 1 pc, assuming densities 100 - 1000 cm^{-3} and temperatures about 20°K. At the end of their paper they mention briefly that there is an instability associated with the attenuation of the radiation field which they included in the following approximate manner. The radiation field is attenuated going into the cloud by the molecules already present and this gives rise to additional terms in the density derivatives. For example the photodestruction rate of CO is $a e^{-\tau_{gr}(CO)}$ so its density derivative is proportional to $\frac{\tau_{gr}}{n}(CO) \times a e^{-\tau_{gr}(CO)}$ For $n = 100$ cm^{-3} there is an unstable mode for $40 < T < 80^{\circ}$k and for $1 \lesssim A_v \lesssim 3$ with timescale $\approx 2 10^5$yrs and lengthscale < 0.1 pc. This instability disappears both for large and for small optical depths. A_v is a measure of the total extinction of radiation at visible wavelengths and is defined for example in Spitzer's book (Spitzer 1968).

A broader, if somewhat less detailed, theory was put forward by Reddish (1975) who considered the formation of H_2 molecules on grains associated with the gas. Laboratory experiments had shown that the efficiency of conversion of atoms to molecules was increased by the presence of layers of H_2 on the grain surface. Reddish wrote

$$\frac{d n_{H_2}}{dt} = \gamma \pi a^2 v_H n_g n_H n_{H_2} \qquad (R)$$

where n_g, n_H, n_{H_2} are the number densities of grains, hydrogen atoms and hydrogen molecules respectively, v_H is the thermal velocity of H atoms, γ is the usual efficiency factor - taken as approximately unity

and πa^2 is the surface area of each grain. The extra factor n_{H_2} is supposed to represent the effect just mentioned. However, this cannot be correct for $n_{H_2} > 1$ since the maximum rate of molecule formation is the rate of incidence of H atoms onto grains. In fact this n_{H_2} represents the gas density of H_2 whereas the experiment relates to the amount of molecules on the grains so that a factor $P(n_{H_2})$ should be used where $\gamma P(n_{H_2}) \leq 1$, and $P(0) = 1$.

Reddish suggested that regions within 1 mag of extinction of thermal radiation are related in grain temperature and so spheres of this radius are the regions which will collapse if anything is because of the great dependence of reaction efficiency on grain temperature at least near a certain critical temperature. Conversion of atoms to molecules leads to a reduction in number density and hence of pressure and so the region will be compressed by the surrounding regions. The time for this collapse is given by (R) as about $\frac{n_{H_2}}{(dn_{H_2}/dt)}$, which may be compared with the other timescales of interest and a scenario of a self-regulating star formating system is outlined by Reddish.

Earlier Reddish and Wickramasinghe (1969) had considered the freezing of H_2 onto grains which again leads to a reduction in pressure and hence to compression by the surrounding gas. Various authors had later argued against this freezing process (Greenberg 1969). However, the form of (R) allows many of the details of Reddish and Wickramasinghe's calculation to be carried over and in fact many observed features of galaxy and star evolution are apparently explained.

When the more reasonable factor of $P(n_{H_2})$ is used the form of the timescales are changed and the model can no longer explain star formation as satisfactorily as before. A similar modification has in fact been made by Reddish (1974). However, he did not take the thermal properties of the medium into account. Yoneyama concluded that only in those situations in which H_2 is the main coolant will his thermal-reactive instability occur, and he consequently gave details only of high temperature cases with T between 10^3 and 10^4k. In the low temperature regime he considered only accretion onto grains. As mentioned previously only in this last case did he give any timescales.

A further variable of importance, which was hinted at by Glassgold and Langer (1976), is the radiation field; these authors, however, only consider its attenuation by the cloud as leading to an extra term in the density derivative. Reddish mentioned a paper by Schatzman (1958) which he said showed the importance of the coupling between the hydrodynamics and the radiation field. In that paper Schatzman performed a one-dimensional stability analysis on a finite slab of hydrogen gas ionised by radiation incident on both sides of the slab. He used the second method described at the beginning of this chapter for this non-uniform situation. He assumed further that the temperature was constant and also that the system could be marginally stable, that is that both the real and the imaginary parts of the time constant could be zero. It then remained to find an optical depth of the plasma which made the equations consistent and he claimed to have done this. However, as is shown in Chapter 6 there is a mistake in Schatzman's working - which was not given in

detail - and his results are invalid*. This does not invalidate the point that there may be a link between the radiation and the hydrodynamics, and more important, there may be important instabilities linking these two to the thermal properties of the medium.

It is the purpose of this thesis to investigate these last mentioned possible instabilities in a simple way. The method is, however, no more simple than that of many other investigations into the important subject of the nature and timescales of the growth of instabilities in gas clouds.

Numerous applications and questions spring to mind. Could these instabilities aid Reddish's scenario by decreasing the timescale in the required way? Given this then his scheme would have a firmer basis. Is the method useful for investigating the stability of H and H_2 globules in HII regions (Dyson 1968)? On a larger scale could similar instabilities be helpful in explaining galaxy formation, applying to not only hydrogen ions, atoms and molecules but also to other species? This raises a further point. What are the limits to the approximations used especially in the treatment of the radiative transport of the molecule destroying radiation and also of the cooling radiation? This last point has been discussed by Le Guet (1973) in the case of a two level atom in L.T.E. The Eddington approximation was used in the equations of radiative transfer. Her investigation concerned the rate of decay of temperature fluctuations in an initially uniform medium.

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* We are grateful to Professor Schatzman for confirming to us that his results are, indeed, invalid.

She used her results to suggest that the effective diffusion time of cooling radiation through a medium was much increased over that expected from simply scattering because the photons are not only scattered but also trapped for some characteristic time before re-emission.

As mentioned in connection with Yoneyama's work, perhaps inclusion of other factors, such as chemical reactions and attenuation of the radiation field, leads to the result that the thermal instabilities Field considered and the instabilities discussed by Yoneyama may be damped out.

In summary then, numerous unstable modes have been investigated; slowly more and more processes are being included, especially when the known instabilities are thought to be inadequate for some particular purpose. Indeed it is necessary to extend our considerations because it is incorrect to neglect known processes when the various parameters are intimately and subtly linked. The gradual building up of the number of variables will, hopefully, give one a feeling for the relative importance of the various possible instabilities. Furthermore it is useful to have some general algebraic methods for determining the conditions under which instabilities grow faster than, say, the free fall timescale, otherwise it is necessary to solve the dispersion relationships either analytically, for very simple cases, or numerically in a hit or miss fashion.

This thesis aims at both of these objectives, firstly by looking at ways of including the radiation field and secondly by developing Routh's algebraic methods. These latter methods are applied in many situations but are a little cumbersome to use in some of the other special applications.

CHAPTER 2

THE EQUATIONS, INCLUDING A SIMPLE TREATMENT OF THE RADIATIVE TRANSFER PROBLEM

2.1 Derivation of Equations

Consider an extended, uniform, stationary gas cloud inside which is a uniform radiation field. Let n_i be the total gaseous number density of element i, $n_1 x_j$ the number density of species j and α_{ij} the number of atoms of element i in species j where element 1 has been chosen arbitrarily as the reference for abundances. Let the gas have kinetic temperature T and pressure P. It will be noted that $n_j = n_1 \sum_\ell \alpha_{j\ell} x_\ell$ and $\frac{n_j}{n_1}$ is constant for all j.

The equation of number conservation for element i is

$$\frac{dn_i}{dt} + n_i \nabla \cdot \underset{\sim}{v} = 0 \qquad \left(\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \underset{\sim}{v} \cdot \nabla\right) \tag{2.1}$$

assuming a common velocity $\underset{\sim}{v}$ for all species. The equation of motion is

$$\left\{ \sum_i n_i m_i \right\} \frac{d\underset{\sim}{v}}{dt} + \nabla P = 0 \tag{2.2}$$

m_i is the mass of element i,

$$P = n_1 k_B T \sum_j x_j \tag{2.3}$$

Gravitational terms have been ignored since the instabilities of interest will be those with timescales much less than the

gravitational one. Field (1971) and Gerola and Schwartz (1976) have considered the effect interactions with the radiation field have directly on the gas dynamics. It is neglected here, but discussed in Chapter 8.

For each species j, and considering i's such that $\alpha_{ij} \neq 0$

$$\frac{\partial}{\partial t}(n_1 x_j) + \nabla.(\underset{\sim}{v}\, n_1 x_j) = \text{Formation rate of j per unit volume minus destruction rate per unit volume}$$

$$= n_1\left\{\frac{\partial x_j}{\partial t} + \nabla\cdot(\underset{\sim}{v}\, x_j)\right\} + x_j\left\{\frac{\partial n_1}{\partial t} + \underset{\sim}{v}\cdot\nabla n_1\right\}$$

$$= n_1\left\{\frac{\partial x_j}{\partial t} + \underset{\sim}{v}\cdot\nabla x_j + x_j\nabla\cdot\underset{\sim}{v}\right\} + x_j\left\{-n_1\nabla\cdot\underset{\sim}{v}\right\}.$$

$$= n_1\left\{\frac{\partial x_j}{\partial t} + \underset{\sim}{v}\cdot\nabla x_j\right\} = n_1\frac{dx_j}{dt}$$

$$\therefore \quad \frac{dx_j}{dt} = -X_j = \frac{1}{n_1}\left\{\begin{array}{l}\text{Formation rate minus destruction}\\ \text{rate per unit volume of j}\end{array}\right\} \tag{2.4}$$

The energy equation is

$$-\frac{k_T}{n_1\mu}\nabla^2 T + \frac{dU}{dt} - \frac{P}{n_1^2\mu^2}\frac{d}{dt}(n_1\mu) + L = 0 \tag{2.5}$$

where

$$\mu = \frac{1}{n_1}\sum_i n_i m_i \tag{2.6}$$

$$U = \frac{1}{\mu}\sum_j x_j \left\{ \frac{k_B T}{\gamma_j - 1} + \chi_j \right\}$$

= internal energy per unit mass of gas (2.7)

γ_j = ratio of specific heats for species j

χ_j = that part of the heat of formation of species j, per molecule, which is converted into thermal energy of the gas

L = rate of heat loss per unit mass of gas. The system is assumed to be optically thin to the cooling radiation

K_T = coefficient of thermal conductivity of the gas.

If $J(\nu, z, t)$ is the flux density of photons of frequency ν, position z at time t, the equation of radiative transfer is

$$\frac{1}{c}\frac{\partial J}{\partial t} + \frac{\partial J}{\partial z} + \sum_j n_1 x_j \sigma_j^{(1)}(\nu) J = S(\nu, z) \qquad (2.8)$$

for a one dimensional system with monodirectional photon flux. Here $\sigma_j^{(1)}(\nu)$ is the frequency dependent extinction cross section of species j, S(ν, z) is the source term which maintains the steady-state field. Scattering has been ignored.

One can write

$$\frac{1}{c}\frac{\partial}{\partial t} I_k + \frac{\partial}{\partial z} I_k + n_1 x_k \sigma_k^{(1)} I_k + R_k I_k = S_k \qquad (2.9)$$

where

$$I_K = \int_{\Delta\nu_K} J \, d\nu \quad ; \quad S_K = \int_{\Delta\nu_K} S(\nu, z) \, d\nu$$

$$R_K = \frac{1}{I_K} \int_{\Delta\nu_K} n_1 \sum_{J \neq K} x_j \sigma_j^{(1)}(\nu) J \, d\nu \qquad (2.10)$$

$$\sigma_K^{(1)} = \frac{1}{I_K} \int_{\Delta\nu_K} \sigma_K^{(1)}(\nu) J \, d\nu$$

$\Delta\nu_K$ is the domain of frequencies within which $x_K \, \sigma_K^{(1)}(\nu)$ is not negligible compared to the other contributions to the extinction. This definition is vague but will be taken to be precisely defined in the models below, and furthermore the $\Delta\nu_K$ are assumed not to overlap for different k's. It will be noted that σ_K and R_K depend on the spectral distribution , and so will in general depend on z and t.

2.2 The Linearised Equations

Initially the system is assumed to be in a uniform steady state. $\underset{\sim}{v} = 0$ and all derivatives are zero. For simplicity let R_j be proportional to n_1 for all j so neglecting changes in abundances of other species but including the effects of changing the density, and then vary I_j, T, n_1, x_j for all j. For example n_1 becomes $n_1 + \delta n_1$. $\frac{n_j}{n_1}$ is constant. If these variations are sufficiently small one need consider only first order terms so that for example L changes from zero in the steady state to

$$L = \sum_j L_{x_j} \delta x_j + L_T \delta T + L_{n_1} \delta n_1 + \sum_J L_{I_j} \delta I_j \qquad (2.11)$$

where

$$L_T = \left(\frac{\partial L}{\partial T}\right)_{x_1, x_2, x_3, \ldots, n_1, I_1, I_2 \ldots} \qquad \text{etc.}$$

Since $\frac{n_\kappa}{n_1}$ = constant = $\sum_j \alpha_{\kappa j} x_j$ it follows that

$$\therefore \qquad \sum_j \alpha_{\kappa j} \delta x_j = 0. \tag{2.12}$$

Assuming all perturbations vary as $\exp\{\beta t + ikz\}$ the linearized equations become:

$$\beta \delta n_1 + ik n_1 \delta v = 0 \tag{2.13}$$

$$n_1 \mu \beta \delta v + ik \delta P = 0 \tag{2.14}$$

$$-\frac{\delta P}{P} + \frac{\delta T}{T} + \frac{\delta n_1}{n_1} + \frac{\sum_j \delta x_j}{\sum_j x_j} = 0 \tag{2.15}$$

$$\left(\beta + X_{j,x_j}\right)\delta x_j + \sum_{\kappa \neq j} X_{j,x_\kappa} \delta x_\kappa + X_{j,n_1} \delta n_1 \tag{2.16}$$

$$+ X_{j,T} \delta T + \sum_\kappa X_{j,I_\kappa} \delta I_\kappa = 0$$

$$\left(L_{n_1} - \frac{\beta P}{\mu n_1^2}\right)\delta n_1 + \left(L_T + \frac{1}{\mu}\sum_j \frac{\beta k_B}{\gamma_j - 1} x_j + \frac{K_T K^2}{n_1 \mu}\right)\delta T \tag{2.17}$$

$$+ \sum_\kappa \left(L_{x_\kappa} + \frac{1}{\mu}\left(\frac{k_B T}{\gamma_\kappa - 1} + X_\kappa\right)\beta\right)\delta x_\kappa + \sum_\kappa L_{I_\kappa} \delta I_\kappa = 0$$

$$\left\{\frac{\beta}{c} + iK + n_1 x_\kappa \sigma_\kappa^{(1)} + R_\kappa\right\}\delta I_\kappa + \left(x_\kappa \sigma_\kappa^{(1)} + \frac{R_\kappa}{n_1}\right) I_\kappa \delta n_1 \tag{2.18}$$

$$+ n_1 \sigma_\kappa^{(1)} I_\kappa \delta x_\kappa = 0$$

$$\kappa = 1, 2, 3 \ldots .$$

$$\sum_j \alpha_{kj} \delta x_j = 0 \qquad (2.19)$$

The necessary condition for the above linear equations to be consistent, with non-zero δn_1, δT etc., is that the determinant of the coefficients is zero. This provides a relationship between k and β in the form of a complex polynomial equation in β with parameter k. If only one set of x_ℓ, I_ℓ is varied the equation is a quintic, if r sets of x_ℓ, I_ℓ are varied the equation is of degree (3 + 2r).

Specific models are described below and the values of those $(\mathrm{Re}(\beta))^{-1} > 0$ are given for a range of values of k. The timescale of growth, that is the time in which the perturbation grows in amplitude by a factor e, is $1/\mathrm{Re}(\beta)$. By a simple extension of Routh's criteria one can discover whether the polynomial above has any roots with $\mathrm{Re}(\beta) > 0$, or in fact $\mathrm{Re}(\beta) > \beta_c$ where $1/\beta_c$ is some critical timescale, such as the free-fall timescale.

If only one reaction is considered, namely $H + H \xrightarrow{\text{grain}} H_2$ the equations are

$$\left(\mu\beta^2 + \frac{K^2 P}{n}\right)\delta n + \frac{K^2 P}{T}\delta T + \frac{K^2 P}{M}\delta x = 0$$

$$X_n \delta n + X_T \delta T + (X_x + \beta)\delta x + X_I \delta I = 0$$

$$\left(L_n - \frac{P\beta}{\mu n^2}\right)\delta n + (L'_T + AR(\beta))\delta T + (L_x + BRT\beta)\delta x + L_I \delta I = 0$$

$$q_I \delta n \qquad - \frac{n\sigma^{(1)}}{2} I \delta x + \left(\frac{\beta}{c} + nq + ik\right) \delta I = 0$$

bearing in mind that $x_{H_2} = \frac{(1-x)}{2}$ and the determinantal condition can be written

$$\begin{vmatrix} \frac{\mu\beta^2}{k^2 p} + \frac{1}{n} & \frac{1}{T} & \frac{1}{M} & 0 \\ X_n & X_T & X_x + \beta & X_I \\ L_n - \frac{p\beta}{\mu n^2} & L_T' + ARB & L_x + BRT\beta & L_I \\ q_I & 0 & -\frac{n\sigma^{(1)}}{2} I & \frac{\beta}{c} + inq + ik \end{vmatrix} = 0 \tag{2.20}$$

$$q = \sigma^{(1)}\frac{(1-x)}{2} + \frac{R}{n_1} n_{H_2} \quad , \quad L_T' = L_T + \frac{X_I}{n\mu} k^2$$

$$n_H = xn \quad , \quad n_{H_2} = \frac{(1-x)}{2} n \quad , \quad n = n_1$$

R = gas constant.

and the various subscripts have been dropped. A, M and B can be defined by comparison with the equations given above.

$$A = \frac{k_B}{R} \frac{1}{\mu} \sum_j \frac{x_j}{\gamma_j - 1} \tag{2.21}$$

$$B = \frac{k_B}{R} \frac{1}{\mu} \sum_j \left(\frac{1}{\gamma_j - 1} + \frac{\chi_j}{\alpha_{ij} k_B T} \right) \frac{\partial x_j}{\partial x_H} \tag{2.22}$$

$\alpha_{ij} \neq 0$

$$M = \sum_j x_j \tag{2.23}$$

The expansion may be written:

$$\beta^5\left(\frac{F}{c}\right) - \beta^4\left(\frac{n\mu}{P_c}Z - (nq+ik)F\right)$$

$$- \beta^3\left(\frac{1}{c}\left\{\frac{n\mu}{P}L_T^x - K^2\theta\right\} - \frac{n\sigma}{2}I X_I F + (nq+ik)\frac{n\mu}{P}Z\right)$$

$$- \beta^2\left[\frac{K^2}{c}\left\{Z + \frac{n}{M}\left(ARX_n + \frac{P}{\mu n^2}X_T\right) - \frac{n}{T}\left(BRTX_n - L_n + \frac{PX_x}{\mu n^2}\right)\right\}\right.$$

$$\left. + \frac{n^2\mu\sigma}{2P}I L_I^I + (nq+ik)\left\{\frac{n\mu}{P}L_T^x - K^2\theta\right\}\right]$$

$$- \beta K^2\left[\frac{1}{c}\left\{L_T^x - \frac{n}{T}L_n^x + \frac{n}{M}L_n^T\right\} - \frac{n\sigma}{2}I X_I\theta - \frac{nqI}{T}\left\{L_I + \frac{ART}{M}X_I - BRTX_I\right\}\right.$$

$$\left. + (nq+ik)\left\{Z + \frac{n}{M}\left\{ARX_n + \frac{P}{\mu n^2}X_T\right\} - \frac{n}{T}\left\{BRTX_n - L_n + \frac{P}{\mu n^2}X_x\right\}\right\}\right]$$

$$- K^2\left[\frac{n\sigma}{2}I\left(L_T^I - \frac{n}{T}L_n^I\right) - \frac{nqI}{T}\left\{\frac{X_T}{M}\left(TL_n' - ML_x\right) - \frac{L_I}{M}\left(TX_T - MX_x\right)\right\}\right.$$

$$\left. + (nq+ik)\left\{L_T^x - \frac{n}{T}L_n^x + \frac{n}{M}L_n^T\right\}\right]$$

$$= 0$$

(2.24)

$$Z = BRTX_T - L_T' - ARX_x$$

$$\Theta = AR + \frac{P}{\mu T_n}$$

$$F = \frac{AR_n\mu}{P}$$

$$L_T^x = L_x X_T - L_T' X_x \qquad (2.25)$$

$$L_n^T = L_T' X_n - L_n X_T$$

etc

We note again that we have assumed that there is a mono-directional flux of photons.

CHAPTER 3

CRITERIA FOR DETERMINING THE NATURE OF THE ROOTS OF COMPLEX POLYNOMIALS

3.1 Uses for such criteria

In the previous chapter it was shown that if a uniform system in a steady state were to be perturbed infinitesimally and the space and time dependence of the perturbations were assumed to be of the form $\exp\{ikz + \beta t\}$, then a polynomial equation relating β to k and various parameters (including derivatives of the thermodynamic functions) of the system could be obtained. For the particular case studied the polynomial was of fifth or higher degree and the coefficients were complex. It would be of considerable use to have an analytic criterion to tell whether or not the system were unstable, that is whether or not there were roots for β which had positive real part. Field (1965) for example provided a criterion for thermal instabilities depending upon whether any of a set of three test functions were negative. He derived these conditions by considering the explicit expression for the roots of a cubic equation; the dispersion relation he derived was a cubic equation with real coefficients. Had the equation been complex such a method would have become much more difficult because of the mixing of real and imaginary parts on multiplication of complex numbers, and also the greater difficulty in algebraically finding roots of complex numbers. For a quartic equation, real or complex, an analytic solution is also available but is of considerably greater complexity while for quintic and higher equations no general analytic solution is possible. It is therefore perhaps a little surprising that a criterion similar in

form to Field's may be obtained for any polynomial, real or complex. In fact Field's criterion can be quickly derived using more general methods.

The method for obtaining these criteria, which give also the number of roots with strictly positive (>0) real parts and the number of roots with zero real part, is due to Routh (1892). Routh derived a set of rules to find the information just mentioned for polynomials with real coefficients which consisted of considering the signs of various test functions of the coefficients of the polynomial. Hurwitz (1895) used a different method, following work of Hermite, and obtained equivalent criteria to Routh's but gave the test functions, whose signs are of interest, in a convenient determinantal form. Frazer and Duncan (1929) gave the same determinantal expressions independent of Hurwitz. They all considered real polynomials.

This chapter contains an application of Routh's method to complex polynomials and provides determinantal forms for the test functions. A further extension to Routh's work contained here lies in the possibility that one can determine whether there are any roots which have real part greater than some selected value. The importance of this is that often the equations one starts with are approximate in the sense that known physical processes are ignored because they are expected to produce effects on a timescale too long to be of interest. For example Field derived his criterion for thermal instability neglecting gravitational processes.

If, in a particular case, the timescales of the instabilities were longer than the free-fall timescale (τ_g) and if the system

considered were graviationally bound, one would expect that gravitational collapse would occur before the instabilities had had time to grow appreciably. The gravitational effects would have to be included in the equations properly to describe any such long time-scale instabilities. Alternatively one may be specifically interested in instabilities growing on timescales shorter than τ_g. The extension to Routh's work mentioned would allow the statement of criteria for there to be instabilities growing on timescales shorter than τ_g, because the timescale of growth (τ) corresponding to a root β is given by

$$\tau = \frac{1}{Re(\beta)} \qquad (3.1)$$

so demanding that

$$Re(\beta) > \frac{1}{\tau_g} \quad \text{implies}$$

$$\tau < \tau_g .$$

Section 2 gives an outline of Routh's method and states the expression for the test functions if none of them are zero. Details of the proof are in Appendix 1 as are the methods for dealing with the cases which have some test functions zero, some applications are included. Also in Section 2 is the modification last mentioned as well as a related one. In Section 3 the methods are applied to Field's equation and also equations related to that derived in Chapter 2.

3.2 Derivation of the criteria

Consider the general polynomial of degree n

$$f(z) = P(z) + iQ(z)$$
$$= a_o z^n + (a_1 + ib_1)z^{n-1} + (a_2 + ib_2)z^{n-2} + ...$$
$$+ (a_n + ib_n) \qquad (3.2)$$

where $z = x + iy$, x and y are real; a_i, b_i are real for all i. A theorem due to Cauchy states that if, as z travels around a closed contour in the complex plane in a positive direction, the argument of f(z) increases by $2\pi m$, then there must be 'm' roots of the equation $f(z) = 0$ lying within the contour, assuming no roots lie on the contour. Applying this theorem to a contour enclosing the entire right-half plane the problem of finding the number of roots of

$$f(z) = 0 \qquad (3.3)$$

which have real part greater than zero is solved if the increase in the argument of f(z) can be calculated; Routh provided a method for evaluating this quantity.

The contour chosen is one consisting of the y axis from +R to -R, closed by a semicircle in the right half plane, and R is made infinite so that the entire half plane is included.

On the semicircle the leading term dominates when R is large so that in polar coordinates

$$f = P + iQ = a R^n(\cos(\theta n) + i \sin(\theta n))$$

so as θ goes from $-\pi/2$ to $+\pi/2$ the argument of f increases by πn.

The decrease in phase of f(z) will be given by the number of sign changes from minus to plus less the number of sign changes from plus to minus of (-P/Q). This number can be obtained using a theorem due to Sturm which is proved in Appendix 2.

Along the y axis, where z = iy

$$P = (-1)^{n/2}\{a_o y^n + b_1 y^{n-1} - a_2 y^{n-2} - b_3 y^{n-3} + a_4 y^{-4} \ldots . \}$$

$$Q = -(-1)^{n/2}\{a_1 y^{n-1} + b_2 y^{n-2} - a_3 y^{n-3} - b_4 y^{n-4} + a_5 y^{n-5} + \ldots\}$$

(3.4)

if n is even, or if n is odd consider i.f(z), and replace $(-1)^{n/2}$ by $(-1)^{(\frac{n+1}{2})}$

$$f_o(y) = (-1)^{n/2} P(y)$$

$$f_1(y) = -(-1)^{n/2} Q(y)$$

$$f_2(y) = -\text{Rem}(f_o/f_1) = \text{remainder after dividing } f_o \text{ by } f_1$$

$$f_3(y) = -\text{Rem}(f_1/f_2)$$

until $f_r(y) = \text{const}$

$$f_{r+1}(y) = 0 \qquad (3.5)$$

If this sequence of functions is evaluated at the end points y = +R and y = -R the number required is the difference in the number of sign changes going along the sequence at the two end points. Zeros are excluded in the counting. Since R becomes infinite only the leading terms need be considered. Calling the leading coefficients a_o, a_1, c_2, c_3 etc. at y = +R then when y = -R the coefficients become a_o, $-a_1$, c_2, $-c_3$.... for even n and $-a_o$, a_1, $-c_2$, c_3.... for odd n, assuming none of the c_i's are zero so that r = n. The case r ≠ n is treated later.

If the first sequence has k changes of sign then the second and third sequences have (n - k) changes in sign, as may easily be proved by induction. The difference in the number of sign changes at the two endpoints as $R \to \infty$ is then 2k - n. It follows then that the increase in the argument of f(z) travelling around the contour is $\pi(2k - n + n)$ so that there are $\frac{1}{2\pi} \times \pi(2k) = k$ roots with positive real part, assuming that r = n, i.e. that none of the c_j's are zero and that no roots lie on the axis. In this case determinantal expression for the c_j will be

$$c_j = (-1)^{j(j-1)/2} \begin{vmatrix} a_1 & a_0 & 0 & \cdot & \\ -b_2 & -b_1 & a_1 & a_0 & 0 \\ a_3 & a_2 & b_2 & b_1 & a_1 \\ -b_4 & -b_3 & a_3 & a_2 & -b_2 & -b_1 \\ \vdots & & & & & \\ a_{2j-1} & a_{2j-2} & b_{2j-2} & & & \end{vmatrix} \qquad (3.6)$$

The proof of this equality is given in Appendix 1.

The timescale (τ) of growth of an instability is

$$\tau = \frac{1}{\mathrm{Re}(z_i)} \quad \text{for root } z_i.$$

If one is interested in timescales less than some τ_c

$$\tau_c = \frac{1}{A} \qquad A \text{ is real} \qquad (3.7)$$

then one can change the variable to $\beta = z - A$. $\mathrm{Re}(\beta) > 0$ corresponds to $\mathrm{Re}(z) > A$, so

$$\frac{1}{\mathrm{Re}(z)} < \frac{1}{A}.$$

The polynomial f becomes

$$a_0(\beta+A)^n + (a_1+ib_1)(\beta+A)^{n-1} + (a_2+ib_2)(\beta+A)^{n-2} +$$

$$= a_0\left(\beta^n + n\beta^{n-1}A + \frac{n(n-1)}{2}\beta^{n-2}A^2 + \cdots\right)$$

$$+(a_1+ib_1)\left(\beta^{n-1} + (n-1)\beta^{n-2}A + (n-1)\frac{(n-2)}{2}\beta^{n-3}A^2 + \cdots\right)$$

$$+ (a_2+ib_2)\left(\beta^{n-2} + (n-2)\beta^{n-3}A + (n-2)\frac{(n-3)}{2}\beta^{n-4}A^2 + \cdots\right)$$

$$+ \quad - - - - -$$

$$= a_0\beta^n + \beta^{n-1}\left\{(a_1 + nAa_0) + ib_1\right\} +$$

$$\beta^{n-2}\left\{(a_2 + (n-1)Aa_1 + \frac{n(n-1)}{2}A^2a_0) + i(b_2 + (n-1)Ab_1)\right\}$$

$$+ \beta^{n-3}\left\{(a_3 + (n-2)Aa_2 + (n-2)\frac{(n-1)}{2}A^2a_1 + (n-2)\frac{(n-1)}{3\times 2}nA^3a_0)\right.$$

$$\left. + i(b_3 + (n-2)Ab_2 + (n-2)\frac{(n-1)}{2}A^2b_1 \cdots)\right\}$$

$$+ \quad - - - - - -$$

$$= a_0\beta^n + (a_1'+ib_1')\beta^{n-1} + (a_2'+ib_2')\beta^{n-2} + \cdots \qquad (3.8)$$

In simple cases onemay use the a_o, a_1', b_1', etc. in the determinants of the c_i. These may be viewed in one of two ways. Either A can be fixed and conditions may be found which make $Re(\beta) > 0$ for roots of the polynomial, or one can regard the sequence as a method for determining A because each c_i becomes a real polynomial in A, which changes sign as the real A passes through real roots of that polynomial. Note that if one c_i changes sign there is no guarantee that some other c_j may not change sign to compensate the number of sign changes in the sequence. Therefore

the roots of the various polynomials for the c_j's need not correspond to real parts of actual roots z_i of the original polynomial f(z). However, the roots obtained do provide starting guesses for these parts of the roots z_i. Since one is looking for real roots of real polynomials there may be some simplification in solving the equations for A rather than solving for the complex roots of the original complex polynomial.

A related point is that if z has positive real part then so does 1/z = s and if z satisfies f(z) = 0 then s satisfies

$$s^n + \frac{(a_{n-1}+ib_{n-1})}{(a_n+ib_n)}s^{n-1} + \frac{(a_{n-2}+ib_{n-2})}{(a_n+ib_n)}s^{n-2} + \cdots + \frac{a_0}{(a_n+ib_n)} = 0 \qquad (3.9)$$

The use of the techniques given later in this chapter and in Appendix 1 on this polynomial may lead to some simplification, or at least gives a different starting point so that with a_0', a_1', c_2' one can see some of the effects of changing an a_{n-1}, b_n and b_{n-1} without involving large determinants.

3.3 Application to Field's Instability

As an application of the methods suggested in section 2, Field's equation may be treated. This equation may be derived as done in Chapter 2, keeping $\delta x = 0 = \delta I$, and may be written

$$\beta^3 + (\gamma-1)\frac{\mu L_T}{k_B}\beta^2 + \frac{k^2 P\gamma}{\mu n}\beta + k^2(\gamma-1)(TL_T - nL_n) = 0 \qquad (3.10)$$

$P = nk_BT$, the subscript has been dropped from γ.

$$a_o = 1$$

$$a_1 = L_T \frac{(\gamma-1)\mu}{k_B}$$

$$c_2 = -\begin{vmatrix} a_1 & a_o & 0 \\ -b_2 & -b_1 & a_1 \\ a_3 & a_2 & b_2 \end{vmatrix} = \begin{vmatrix} L_T \frac{\mu}{k_B}(\gamma-1) & 1 & 0 \\ 0 & 0 & L_T \frac{\mu}{k_B}(\gamma-1) \\ K^2(\gamma-1)(TL_T - nL_n) & \frac{K^2 PY}{\mu n} & 0 \end{vmatrix}$$

$$= K^2 L_T \frac{\mu}{k_B}(\gamma-1)^2 \left[(\gamma-1)TL_T + nL_n\right] = a_1(\gamma-1)K^2\left[(\gamma-1)TL_T + nL_n\right]$$

$$c_3 = -\begin{vmatrix} a_1 & a_o & 0 & 0 & 0 \\ -b_2 & -b_1 & a_1 & a_o & 0 \\ a_3 & a_2 & b_2 & b_1 & a_1 \\ -b_4 & -b_3 & a_3 & a_2 & -b_2 \\ a_5 & a_4 & b_4 & b_3 & a_3 \end{vmatrix} = K^2(\gamma-1)(c_2)^2\left[TL_T - nL_n\right]$$

(3.11)

If any of

$$L_T,\ ((\gamma-1)\ TL_T + nL_n),\ \ (TL_T - nL_n)$$

are negative then roots corresponding to instabilities exist.

If one changes from β to z where

$$z = \beta - A \qquad (3.12)$$

and $A = \frac{1}{\tau_c}$, τ_c being some critical timescale, for example the free fall timescale, or A may be treated as an investigatory tool, the cubic equation becomes, upon substituting,

$$z^3 + z^2\left\{L_T(\gamma-1)\frac{\mu}{k_B} + 3A\right\} + z\left\{\frac{K^2P\gamma}{\mu m} + 2AL_T(\gamma-1)\frac{\mu}{k_B} + 3A^2\right\}$$

$$+\left\{K^2(\gamma-1)(TL_T + mL_n) + \frac{AK^2P\gamma}{\mu m} + A^2L_T(\gamma-1) + A^3\right\} = 0 \qquad (3.13)$$

Use of the methods detailed above shows that if any of

$$a_1' = L_T(\gamma-1)\frac{\mu}{k_B} + 3A \quad ;$$

$$c_2' = K^2(\gamma-1)\left[(\gamma-1)TL_T + mL_n\right] + 2A\left[\frac{K^2P\gamma}{\mu m} + (\gamma-1)^2\left(\frac{L_T\mu}{k_B}\right)^2\right]$$

$$+ A^2\left[8L_T(\gamma-1)\frac{\mu}{k_B}\right] + 8A^3 \quad ;$$

$$c_3' = K^2(\gamma-1)\left[TL_T - mL_n\right] + A\frac{K^2P\gamma}{\mu m} + A^2L_T(\gamma-1)\frac{\mu}{k_B} + A^3 \qquad (3.14)$$

are negative then there are roots with real part greater than A.

c_3' is just the original expression and so changes sign as A passes through real roots of the original equation.

With given positive A there may be an instability with growth timescale less than τ_c if

$$L_T(\gamma-1)\frac{\mu}{k_B} + 3A < 0 \qquad (3.15)$$

$$\text{i.e.} \quad -L_T > \frac{3Ak_B}{(\gamma-1)\mu} \quad \text{from } a_1'$$

Alternatively if A is to be determined, there may be a root with positive real part given by

$$A = - L_T \frac{\mu}{3k_B} (\gamma-1) \text{ (if } L_T < 0) \qquad (*)$$

The sign variation of the other terms must be examined as A passes through the above value in order to determine whether there is such a root.

Similarly, if A is given then c_2' may be examined. For example if $a_1 > 0$, $c_2 < 0$ then $c_2' < 0$ for k greater than k_{c_2} where

$$k_{c_2}^2 = \frac{-2A(a_1+2A)^2}{\left(\frac{2AP\gamma}{\mu n}\right) + (\gamma-1)\left[T L_T(\gamma-1) + n L_n\right]} \qquad (3.16)$$

It will be seen that k_{c_2} is real only if

$$\left[(\gamma-1)T L_T + n L_n\right](\gamma-1) + \frac{2AP\gamma}{\mu n} < 0$$

$$\text{i.e.} \quad \frac{2AP\gamma}{\mu n(\gamma-1)} < -\left[(\gamma-1)T L_T + n L_n\right] \qquad (3.17)$$

If $-c_2$ is less than the above value, there may not be an instability of sufficiently short timescale, but again the sign of the other test functions must be considered.

As before, if A is regarded as a variable then there may be a root of (3.13) with positive real part given by

$$A = -(\gamma-1)\left[(\gamma-1)T L_T + n L_n\right]\frac{\mu n}{2P\gamma} \qquad (**)$$

$$\text{if} \quad k > k_{c_2}$$

For fixed $A > 0$, if $L_T > 0$ and $c_3 < 0$ one can see that c_3' will be negative for $k < k_c$ where k_c is given by

$$k_{c_3}^2 = \frac{-A^2\left[(\gamma-1)\frac{\mu}{k_B}L_T + A\right]}{\left[\frac{AP\gamma}{\mu n} + (\gamma-1)(TL_T - nL_n)\right]} \tag{3.18}$$

k_{c_3} will not be real unless

$$-(TL_T - nL_n) > \frac{AP\gamma}{\mu n(\gamma-1)} \tag{3.19}$$

If $a_1 > 0$, $c_2 > 0$, $c_3 < 0$ then there is only one root with positive real part and this real part can only have real part greater than A if

$$-(TL_T - nL_n) > \frac{AP\gamma}{\mu n(\gamma-1)} \tag{3.20}$$

and then only for $k > k_{c_3}$.

Conversely A given by

$$A = -(TL_T - nL_n)\,\frac{\mu n(\gamma-1)}{P\gamma} \qquad ***$$

will definitely be the value of the real part of the root.

Field (1965) derived approximate solutions to this equation which had real parts equivalent to *, ** and ***. The latter equations may be tabulated

a_0	b_1	b_2	b_3	No. of instabilities	A
+	+	+	+	0	
+	+	+	−	1	$-b_3 \frac{\mu n}{P\gamma}(\gamma-1)$
+	+	−	+	2	$-b_2 \frac{\mu n(\gamma-1)}{2P\gamma}$
+	+	−	−	1	$-b_3 \frac{\mu n(\gamma-1)}{P\gamma}$
+	−	+	+	2	
+	−	+	−	3	
+	−	−	+	2	
+	−	−	−	1	

where

$$b_1 = L_T$$

$$b_2 = (\gamma-1)\, TL_T + nL_n$$

$$b_3 = TL_T - nL_n$$

and $(\gamma-1)$ has been taken as positive. If $L_T > 0$ then Field's approximate expressions can be seen to be exact, with the restriction that k must be sufficiently large, so that whenever the coefficient of k^2 in the expression for c_2' or c_3' is negative then the whole expression becomes negative.

On the other hand if one is interested in instabilities occuring with a finite wavenumber then the above analysis gives sufficient conditions on k and b_1, b_2, b_3 in order that the timescale of the instability is less than some chosen value.

3.4 Applications to more general systems

It may help to understand what is going on to consider some very similified conditions, allowing only two out of δx, δn, δT and δI to be non-zero, the condition for the other two to be zero is that their coefficients in certain of the equations are effectively infinite. The first special case will make this clear.

The full set of linearised equations is

$$\text{a)}\quad \left(m\beta^2+\frac{k^2P}{n}\right)\delta n + \frac{k^2P}{T}\delta T + \frac{k^2P}{n}\delta x = 0 \qquad (3.21)$$

$$\text{b)}\quad \chi_n \delta n + \chi_T \delta T + (\chi_x+\beta)\delta x + \chi_I \delta I = 0 \qquad (3.22)$$

$$\text{c)}\quad \left(-\frac{P\beta}{n^2 m}+L_n\right)\delta n + \left(AR\beta+L_T\right)\delta T + \left(BRT\beta+L_x\right)\delta x + L_I \delta I = 0 \qquad (3.23)$$

$$\text{d)}\quad \rho_n I \delta n + \rho_T I \delta T + \rho_x I \delta x + \left(\frac{\beta}{c}+ik+\rho\right)\delta I = 0 \qquad (3.24)$$

where ρ is the opacity per unit volume

(i) $\delta x \simeq 0 \simeq \delta I$.

If $(\beta + \chi_x)$ is sufficiently large then no matter what the size of the other terms, for δx small with respect to δn, δT, δI equation (b) will be satisfied. Similarly, since from (d)

$$\delta I = \frac{-\left[\rho_x I \delta x + \rho_T I \delta T + \rho_n I \delta n\right]}{\left(\frac{\beta}{c} + ik + \rho\right)} \qquad (3.25)$$

it follows that if $\rho x I$, $\rho T I$, $\rho n I$ are all very small compared to $(\beta/c + \rho + ik)$; then δI will be very small compared to δx, δT and δn.

When both these conditions hold one sees that

$$\left| \frac{K^2 P}{T} \delta T \right| \gg \left| \frac{K^2 P}{M} \delta x \right|$$

and

$$\left| \left(m\beta^2 + \frac{K^2 P}{n} \right) \delta n \right| \gg \left| \frac{K^2 P}{M} \delta x \right|$$

so that (a) may be written

$$\left(m\beta^2 + \frac{K^2 P}{n} \right) \delta n + \frac{K^2 P}{T} \delta T = 0 \qquad (3.26)$$

to a very good approximation. The approximation becomes better the larger is $(\beta + X_x)$. In the same way the δx and δI terms may be missed out of (c), unless of course β is such as to make the coefficients of δn and δT sufficiently small and all four terms become of the same magnitude. This last possiblity will be neglected below.

Having now set δx and δI effectively equal to zero (a) and (c) reduce to Field's equations which have been considered at the beginning of this section. The use of this limited set does imply certain constraints on the sizes of the various terms. For example, setting $\delta I = 0$ for simplicity, (b) may be written

$$\delta x = \frac{-1}{(X_x + \beta)} \left[X_n \delta n + X_T \delta T \right]$$

Substituting this into (a)

$$\delta n \left[m\beta^2 + \frac{K^2 P}{n} - \frac{K^2 P \, X_n}{M (X_x + \beta)} \right] + \delta T \left[\frac{K^2 P}{T} - \frac{K^2 P \, X_T}{M (X_x + \beta)} \right] = 0$$

(3.27)

The constraints become

$$\left| \frac{m\beta^2}{k^2 P} + \frac{1}{n} \right| \gg \left| \frac{\chi_n}{M(\chi_x + \beta)} \right|$$

$$\left| \frac{1}{T} \right| \gg \left| \frac{\chi_T}{\chi_x + \beta} \right|$$

Using the equation for δI another set of constraints may be obtained and in order to see whether or not it is valid to use Field's equations one should test whether or not these constraints are satisfied. An equivalent method, which is used here, is to solve the full set of equations to see if any new features are shown.

There are other alternatives within this special case. It may be that for example L_I and ρ_x are very large compared to the other coefficients in which case (c) and (d) are satisfied with δx and δI very small compared to δn and δT and, assuming the implied constraints are satisfied (a) and (b) reduce to

$$\left(m\beta^2 + \frac{k^2 P}{n} \right) \delta n + \frac{k^2 P}{T} \delta T = 0$$

$$\chi_n \delta n + \chi_T \delta T = 0 \qquad (3.28)$$

The dispersion relation is

$$\beta^2 + \frac{k^2 P}{m \chi_T} \left\{ \frac{\chi_T}{n} - \frac{\chi_n}{T} \right\} = 0 \qquad (3.29)$$

and there will be a growing instability if $\frac{k^2 P}{m\chi_T} \left(\frac{\chi_T}{n} - \frac{\chi_n}{T}\right)$ is negative which will occur if either $\chi_T < 0$ or

$$\left(\frac{\partial X}{\partial T}\right)_{p,x} < 0 \qquad (3.30)$$

corresponding to the isochoric and isobaric instabilities of Field.

(ii) $\delta T = 0 = \delta x$

(a) immediately implies that either $\delta n = 0$ or $\beta = \pm i\sqrt{(\frac{k^2 P}{nm})}$. If any two of (b), (c) and (d) had been chosen so that the coefficient of δx and that of δT were sufficiently large that these two equations were automatically satisfied, the third equation would then show that if δn equals 0, δI must also be zero, or else $\frac{\delta I}{\delta n}$ must be determined.

Although formally possible to assume that either $k^2 P/T$ or $k^2 P/M$ were effectively infinite, so allowing a different dispersion relation to be found for β, it seems an unreasonable thing to do because neither M nor T should be zero and the factor $k^2 P$ is common to all the terms.

(iii) $\delta n = 0 = \delta T$

Unless (a) is chosen as one of the two equations which are satisfied for very small δn or δT, it follows that δx is zero. If β is sufficiently large, however, it does seem reasonable to say that the coefficient of δn is effectively infinite. Assuming also that (c) is satisfied because $(AR\beta + L_T)$ is also sufficiently large, (b) and (d) remain as

$$\begin{aligned} &(X_x + \beta)\delta x + X_I\, \delta I \\ &\rho_x I\, \delta x \qquad + (\beta/c + \rho + ik)\, \delta I = 0 \end{aligned} \qquad (3.31)$$

The dispersion relation

$$\frac{\beta^2}{c} + \beta\left\{\frac{X_x}{c} + \rho + ik\right\} + \left\{(\rho X_x - IX_I\rho_x) + ikX_x\right\} = 0 \qquad (3.32)$$

has test functions

$$\begin{aligned} c_o &= \frac{1}{c} \\ c_1 &= \frac{X_x}{c} + \rho \\ c_2 &= k^2 \rho X_x + \left(\frac{X_x}{c} + \rho\right)^2 [\rho X_x - IX_I \rho_x] \end{aligned} \qquad (3.33)$$

c_2 shows that if $\rho X_x - IX_I \rho_x$ is negative there is an instability if k is sufficiently small when X_x is positive.

$\rho X_x - IX_I \rho_x$ may be written

$$X_x\left(\frac{\partial \rho}{\partial I}\right)_x$$

since if x is constant $X_I\ \delta I + X_x\ \delta x = 0$.

(iv) $\delta T = \delta I = 0$

If (c) and (d) are assumed to be the ones with large coefficients of δT and δI respectively, (a) and (b) reduce to

$$\left(m\beta^2 + \frac{k^2 P}{n}\right)\delta n + \frac{k^2 P}{M}\delta x = 0 \qquad (3.34)$$

$$X_n\,\delta n + (X_x + \beta)\,\delta x = 0$$

$$\therefore\quad m\beta^3 + m X_x \beta^2 + \frac{k^2 P}{n}\beta + \frac{k^2 P}{nM}(MX_x - nX_n) = 0 \qquad (3.35)$$

$$c_0 = m$$

$$c_1 = m X_x$$

$$c_2 = -\begin{vmatrix} m X_x & m & 0 \\ 0 & 0 & m X_{x} \\ \frac{K^2 P}{nM}\left(M X_x - n X_n\right) & \frac{K^2 P}{n} & 0 \end{vmatrix} = m X_x \frac{K^2 P}{n}\left\{ X_{x} - X_x + \frac{n}{M} X_n \right\}$$

$$= m^2 \frac{K^2 P}{M} X_x X_n$$

$$c_3 = \left[m \frac{K^2 P}{M} X_n \right]^2 \frac{K^2 P}{nM}\left(M X_x - n X_n \right)$$

(3.36)

If any of the conditions

$$\text{(i)} \quad X_x < 0$$

$$\text{(ii)} \quad X_n < 0 \qquad (3.37)$$

$$\text{(iii)} \quad X_x - \frac{n}{M} X_n < 0$$

there will be at least one root corresponding to an instability.

(v) $\delta n = 0 = \delta I$

Taking (c) and (d) as the ones with large coefficients of δn and δI the remaining equations read

$$\frac{\delta T}{T} + \frac{\delta x}{M} = 0$$

$$X_T \, \delta T + (X_x + \beta) \, \delta x = 0 \qquad (3.38)$$

$$\therefore \qquad \beta + (X_x - \frac{T}{M} X_T) = 0 \tag{3.39}$$

and $\beta > 0$ if $X_x - \frac{TX_T}{M} < 0$

i.e. $(\frac{\partial X}{\partial x})_{P,n} < 0.$

Alternatively one may assume that X_n is very large so that the equations to be used are

$$\frac{\delta T}{T} + \frac{\delta x}{M} = 0 \tag{3.40}$$

$$(AR\beta + L_T)\delta T + (BRT\beta + L_x)\delta x = 0$$

$$\beta\left[B - \frac{A}{M}\right]R + \left(\frac{L_x}{T} - \frac{L_T}{M}\right) = 0 \tag{3.41}$$

$$\beta > 0 \quad \text{if} \quad \frac{1}{R\left(B - \frac{A}{M}\right)}\left\{\frac{L_x}{T} - \frac{L_T}{M}\right\} < 0 \tag{3.42}$$

When $B - A/M > 0$ the condition is

$$(\frac{\partial L}{\partial T})_{P,n} > 0$$

(iv) $\delta x = 0$

$$\left(m\beta^2 + \frac{K^2 P}{n}\right)\delta n + \frac{K^2 P}{T}\delta T = 0$$

$$\left(-\frac{P\beta}{n^2 m} + L_n\right)\delta n + (AR\beta + L_T)\delta T + L_I \delta I = 0 \tag{3.43}$$

$$qI\,\delta n + \left(\beta^2 + mq + ik\right)\delta I = 0$$

which lead to

$$\left[AR\frac{nm}{c}\right]\beta^4 + \left[nm\left(\frac{L_T}{c} + nqAR\right) + ikmAR\right]\beta^3$$

$$+ \left[\frac{k^2P}{nc}\left(AR + \frac{P}{nmT}\right) + n^2qmL_T + iknmL_T\right]\beta$$

$$+ \frac{k^2P}{nT}\left[\frac{1}{c}\left(TL_T - nL_n\right) + nq\left(AR + \frac{P}{nmT}\right) + ik\left(AR + \frac{P}{nmT}\right)\right]$$

$$+ \frac{k^2P}{nT}\left[nq\{TL_T - nL_n\} + nqIL_I + ik\{TL_T - nL_n\}\right] = 0$$

(3.44)

The corresponding test functions are

$$c_o = \frac{ARnm}{c}$$

$$c_1 = nm\left(\frac{L_T}{c} + nqAr\right)$$

$$c_2 = k^2\left[n^4m^3q\left(AR\right)^2L_T + \frac{mP^2}{c^2T}\left(\frac{L_T}{c} + nqAR\right)\left(L_T + \frac{ARn^2m}{P}L_n\right)\right] \quad (3.45)$$

$$+ \left[nm\left(\frac{L_T}{c} + nqAR\right)\right]^2 n^2mqL_T$$

If L_T is positive it follows that if $(L_T + \frac{ARn^2m}{P} L_n)$ is sufficiently negative that the coefficient of k^2 in c_2 is also negative then for sufficiently large k, c_2 will be negative. A necessary condition for c_2 to be negative is that

$$(L_T + \frac{ARn^2m}{P} L_n) < 0. \quad (3.46)$$

CHAPTER 4

MODELS FOR THE INTERSTELLAR GAS

4.1 Description of Heating and Cooling Mechanisms in L

This Chapter contains detailed models of the thermal properties of a gas composed of hydrogen, carbon ions, electrons and grains as elements. The species included are hydrogen atoms and molecules, carbon ions and atoms; electrons and grains; the parameters are labelled by H_1, H_2, C, C^+, e, and g respectively. Simple reactions are also included and the self-shielding thermo-reactive instabilities mentioned in Chapter 1 are investigated in the situations considered.

The heat input in these models may come from cosmic ray destruction and photodestruction of hydrogen atoms and molecules, photodissociation of carbon atoms, hydrogen molecule formation and the photoelectric effect on the grains. The gas is cooled by infra-red radiation from collisionally excited ions, atoms and molecules, and this radiation is assumed to escape freely from the gas. Unless otherwise stated the details below are taken from the paper by Dalgarno and McCray (1972).

Cosmic rays dissociate and destroy hydrogen atoms and molecules giving an average heat input to the gas of 6.6 eV and 19 eV respectively per interaction (Glassgold and Langer 1973a).

$$\Gamma_{CR} = n\zeta\left[x_H\, 6.6 + x_{H_2}\, 1.65.19\right] \text{ eV s}^{-1}\text{cm}^{-3} \qquad (4.1)$$

where Γ_{CR} is the heat input per unit volume per second from cosmic rays.

ζ is the primary ionisation rate for hydrogen atoms (H)

n is the number density of hydrogen nucleii.

Hydrogen molecules (H_2) are assumed to have a cross section 1.65 times as great.

Photodissociation of H_2 and carbon atoms (C) contribute 0.5 eV (Milgrom et al. 1973) and 2 eV (Spitzer 1968) respectively, providing a heating rate of Γ_{ph} per unit volume

$$\Gamma_{ph} = n\left[\sigma^{(2)}_{H_2} x_{H_2} \bar{I}_{H_2} * 0.5 + x_c \sigma^{(2)}_c \bar{I}_c * 2\right] eV\,s^{-1}cm^{-3} \qquad (4.2)$$

where $\sigma^{(2)}_j$ is the photodestruction cross section for species j, defined similarly to $\sigma^{(1)}_j$ in (2.10).

Watson (1973) and Jura (1976) have suggested that the photoelectric effect on grains can provide a significant heating source for the interstellar gas. Jura estimates a heating rate Γ_{pe} for photoemission from a population of very small grains, radius about 50 Å, for which the photoelectric yield is high, as is the energy of the photoelectrons.

$$\Gamma_{pe} = 1.69\,10^{-13} n f e^{-\tau_d}\left[1 - 4\,10^{-4}\frac{\sqrt{T}}{n_e} e^{-\tau_d}\right] eV s^{-1}cm^{-3} \qquad (4.3)$$

where n_e is the number density of electrons,

f the fraction of the extinction at 1000 Å that is produced by the small (50 Å) grains. Jura suggests f may be as high as 0.5

τ is the optical depth at 1000 Å assuming that where $\tau_d = 0$ the incident U-V flux is the observed ambient field.

Spitzer and Cochran (1973) had suggested that the H_2 molecules formed on grains were ejected from those grains with a kinetic energy of about 3 eV per molecule; they suggested this in order to explain certain observed line profiles of ultra-violet H_2 absorption. However, more recently Spitzer and Jenkins (1975) state that higher resolution observations show that the previous observations can be explained by the overlap of two sets of absorption profiles. Not-with-standing this latter paper Barlow and Silk (1976) have estimated the average kinetic energy with which a newly formed H_2 leaves a graphite grain's surface as 2.3 eV, which would provide a heating rate Γ_{Hr} of

$$\Gamma_{Hr} = 2.3\, n^2 \frac{x_H}{2} x_g \gamma v_H \sigma_{gr} \quad eVs^{-1}cm^{-3} \qquad (4.4)$$

where v_H is the average thermal velocity of H atoms, σ_{gr} the average grain cross section and γ is an efficiency factor of about unity (Hollenbach et al. 1971).

Excitation of C^+ by collisions with electrons gives a cooling rate Λ_1:-

$$\Lambda_1 = \frac{9.9\,10^{-20}}{1.6\,10^{-12}\sqrt{T}} n^2 x_e x_{c^+} \left[e^{-92/T} + 10\, e^{-413/T} + 2.6\, e^{-554/T} + 3.4\, e^{-921/T} \right] \qquad (4.5)$$

$$eVs^{-1}cm^{-3}$$

using the collisional factors given by Kaplan and Pikelner (1968).

Collisions between C^+ and H or H_2 gives a cooling rate Λ_2 given by

$$\Lambda_2 = n^2 x_{C^+}\left[x_H + \frac{x_{H_2}}{4}\right] e^{-92/T} \frac{1.8}{1.6} 10^{-11} \quad \text{eV s}^{-1}\text{cm}^{-3} \tag{4.6}$$

where, as suggested by Dalgarno and McCray, H_2 is taken to be 0.25 times as efficient as H for collisional excitation. Collisional rather than radiative de-excitation predominates for $n \geq 2 \times 10^4$ cm^{-3} for C^+ whereas neutral carbon (C), which is also excited by collisions, collisional de-excitation becomes important for $n > 100$ cm^{-3} so that one must consider the details of the changes in populations of the fine structure energy levels for carbon atoms, as has been done by Penston (1970). Penston included only collisions with hydrogen atoms, neglecting hydrogen molecules and electrons. The rate constant for collisional de-excitation by H atoms is about 2×10^{-10} cm^3 s^{-1}, that for electrons one may estimate as 10^{-9} cm^3 s^{-1} by analogy with oxygen and nitrogen atom de-excitation (Kaplan and Pikelner 1968) so that their relative importance in determining the level populations is

$$\frac{n x_e 10^{-9}}{n x_H 2 10^{-10}} = 5\frac{x_e}{x_H}$$

which is always much less than unity in the models considered. Therfore Penston's results will be used except that we shall include collisions with hydrogen molecules, assuming them again to be $\frac{1}{4}$ as efficient as hydrogen atoms in collisions, which means that

$$n_1 = n(x_H + \tfrac{1}{4} x_{H_2})$$

will be used instead of nx_H.

One may state Penston's results in the form of the following cooling rates Λ_3 and Λ_4:

$$\Lambda_3 = 9.3\,10^{-10} f_1 \quad eVs^{-1}cm^{-3} \tag{4.7}$$

$$\Lambda_4 = 1.2\,10^{-10} f_2 \quad eVs^{-1}cm^{-3} \tag{4.8}$$

where f_1, f_2 are the number densities of carbon atoms in the levels $2p^2\ ^2P_1$ and $2p^2\ ^3P_2$ respectively. By considering the various processes by which the level populations change, both collisional and radiative, one may write

$$f_1 = (FC - HD) / (AF - ED)$$

$$f_2 = (HA - CE) / (AF - ED)$$

$$A = 0.83 + \frac{580}{n_1 T^{1/3}} + 3e^{-39/T} + 2.5\,e^{-23/T}$$

$$D = -1.8 - \frac{27000}{n_1 T^{1/3}} + 2.5\,e^{-23/T}$$

$$C = 2.5\,e^{-23/T}\, n x_c$$

$$E = 4e^{-62/T} - 3e^{-39/T} \tag{4.9}$$

$$F = 2.6 + \frac{27000}{n_1 T^{1/3}} + 4\,e^{-92/T}$$

$$H = n x_c e^{-62/T}$$

Λ_5 represents the cooling by H_2 and HD. The long lifetimes of the excited states of HD and H_2 mean that the populations of the levels are approximately those given by the Boltzmann distribution

$$\Lambda_5 = \frac{n x_{H_2} k_B}{1.6\,10^{-12}} \left[512 * 1.5\,10^{-10} e^{-512/T} + \left(\frac{x_{HD}}{x_{H_2}}\right) 1.6\,10^{-5} e^{-128/T} \right] \qquad (4.10)$$

$\frac{x_{HD}}{x_{H_2}}$ was taken as 10^{-6} in all cases below unless otherwise stated.

The function L is made up of contribution of some or all of the above Λ's and Γ's. If all are included then

$$L = \frac{1}{\mu n}\left(\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4+\Lambda_5-\Gamma_{CR}-\Gamma_{ph}-\Gamma_{pe}-\Gamma_{Hr}\right) eVs^{-1}g^{-1} \qquad (4.11)$$

The factor $1/\mu n$ is included because L is the net cooling rate per unit mass.

4.2 Limitations on L

The gas is assumed to be optically thin to the infra-red cooling radiation. The condition for this is that the total column density of hydrogen is less than $2.5\ 10^{20}\ (\Delta v)\ cm^{-2}$ for C^+ radiation, (Δv) is the line width in km s^{-1} for the cooling lines.

Cosmic ray attenuation is ignored above. The range of 2 MeV protons in the gas corresponds to a column density of $210^{21}\ cm^{-2}$ of hydrogen. Attenuation of the ultraviolet radiation becomes appreciable for a column density of $210^{+21}\ cm^{-2}$, assuming that grains provide the major contribution to the extinction and

$\sigma_g x_g = 510^{-22}$ cm^2. However if

$$x_{H_2} > \frac{\sigma_g}{\sigma_{H_2}} x_g \qquad (4.12)$$

then H_2 molecules will themselves be the main attenuating agent. Taking σ_{H_2} as about 10^{-15} cm^2 one sees that if

$$x_{H_2} > 5.10^{-7}$$

then the self shielding of Hydrogen molecules must be taken to govern the attenuation of the radiation.

In a situation where sources of radiation are not smoothly distributed throughout the gas but the radiation is incident on one side of a slab of gas from sources outside the gas one may expect that the condition of the gas is fairly constant over lengths of about $\frac{10^{21}}{n}$ cm $\simeq \frac{300}{n}$ pc. and so they will be constant over about one wavelength of a perturbation if the wavevector k is greater than $610^{-21} n$ cm^{-1}. If, however, x_{H_2} is greater than about 510^{-7} then the limiting k will be σ_{H_2} n x_{H_2}.

It has been assumed that the energy given to the gas which is destined to be thermalised is in fact immediately thermalised. This is a good approximation since the thermalising time is of order the time between collisions of atoms, $\frac{10^{10}}{n}$ sec. and we deal with timescales much longer than this.

Dalgarno and McCray (1972) mention many coolants which are not used here, the reason is that for temperatures of 100 k or less C^+ cooling is the most important, unless carbon is depleted by a factor 100 more than are Silicon and Iron. Therefore for many situations

in interstellar space the model above may be a good approximation. However, the other species may affect the ionisation balance which may in turn affect, for example, L_T. A more detailed model is certainly desirable, including for example the conversion of C^+ to CO which certainly occurs at the higher densities considered in these models.

4.3 Formation and Destruction Mechanisms

The H_2 molecules in these models are formed on grain surfaces and destroyed by ultra-violet and cosmic radiation (Hollenbach et al. 1971, Soloman et al. 1971). Carbon atoms are photoionised to produce carbon ions (C^+) and reformed by gas phase radiative recombination of C^+ and electrons (Werner 1970). These mechanisms give

$$\chi_{H_2} = \frac{1}{n}\left[\frac{n x_H}{2} x_g (\sigma_g v_H \gamma) - n x_{H_2} \sigma_{H_2}^{(2)} \bar{I}_{H_2} - 1.65\, n x_{H_2} \zeta\right] \quad s^{-1} \qquad (4.13)$$

where σ_g is the geometric cross section of the grains, v_H the thermal velocity of hydrogen atoms and γ is an efficiency factor of about unity. The product $x_g \sigma_g v_H \gamma$ is taken here as $2 10^{-18} \sqrt{T}$, with the temperature dependence coming from the temperature dependence of v_H. ζ is the cosmic ray destruction rate constant for hydrogen atoms; the factor 1.65 in the last term arises because the cross section for interaction between H_2 and cosmic rays is 1.65 greater than that between hydrogen atoms and cosmic rays (Solomon et al. 1971)

$$X_e = \frac{1}{n}\left[n^2 \alpha(T)\, x_e x_{c+} - n x_c \sigma_c^{(2)} I_c \right] s^{-1} \qquad (4.14)$$

where

$$\alpha(T) = \frac{10^{-11}}{\sqrt{T}} \left[12.6 - 2.4 \log_{10} T \right] cm^3 s^{-1} \qquad (4.15)$$

$\alpha(T)$ is the radiative recombination rate coefficient (Werner 1971).

$\underline{R_k}$

R_{H_2} and R_c are taken to be the same, the contribution to the opacity of the dust. As mentioned in Chapter 2 R_k will be taken to be proportional to n, and indeed for dust opacity

$$R_{H_2} = R_c = R_o = \sigma_g x_g n. \qquad (4.16)$$

In other words the shielding of C by H_2 and visa versa will be ignored. Detailed calculations by Viala and Walmsley (1976) confirm that this is justified in some cases at least.

Limitations of X

H_2 is dissociated via line absorption. Bulk velocities could therefore greatly alter the situation because of the doppler effect on the frequencies which are absorbed. This effect is ignored here but is treated in a simple-minded way in Chapter 7.

4.4 Values of the parameters

γ_i

At low temperature the internal degrees of freedom of hydrogen molecules are inactive so that γ_{H_2} will be taken as that of an ideal monatomic gas, as will γ_i for all the other species, that is

$$\gamma_j = 5/3 \text{ for all } j. \qquad (4.17)$$

χ_i and B

χ_k is that part of the heat of the reaction which forms species k which is converted into thermal energy of the gas and which is not otherwise included in L. In this model, however, both formation and destruction are accompanied by a heating of the gas rather than the two processes being reciprocal with the energy released of one being balanced by an energy absorption by the other in the steady-state.

The real part of each of the equation in Chapter 2 is the physically interesting part, but the corresponding imaginary part has been added so that the exponential notation may be used. This reminder is given here for the following reason. When $\frac{d\delta x}{dt}$ is positive the details given in the discussion of the function L will assign to a certain value since $\chi^+ \frac{d\delta x}{dt}$ corresponds to the heating obtained when H_2 is dissociated and x is increasing. When $\frac{d\delta x}{dt}$ is negative a similar analysis will show that B has a different value because even though $\frac{d\delta x}{dt}$ is now negative, part of the term BRT $\frac{d\delta x}{dt}$ namely $\chi^- \frac{d\delta x}{dt}$ still corresponds to a heating process, and so χ^- must have the opposite sign to χ^+, and may also have different

magnitude. This point is discussed again in Chapter 7. The equations have been solved for two values of B corresponding to χ^+ and χ^- and any differences are pointed out in the results.

$\sigma_i^{(1)}$, $\sigma_i^{(2)}$, $\bar{I}_i$, $\Delta\nu_i$, $\int$, $\frac{n_c}{n_H}$, $\frac{n_e}{n_H}$

In these models we take the following values for the above parameters:-

$0.5\ \sigma_c^{(1)} = \sigma_c^{(2)} = 1.1\ 10^{-17}$ cm^{-2} as suggested by Werner (1970) and assuming about half the absorptions lead to dissociation.

$I_c = 1.3\ 10^5$ photons cm^{-2} s^{-1} in some models. I_c is reduced by factors of $\frac{1}{10}$, $\frac{1}{100}$, etc. in others. This values gives $\sigma_c^{(2)}\ I_c = 1.4\ 10^{-12}$ s^{-1} which agrees with the estimates of the photo dissociation rate of carbon in unshielded space (Werner 1970).

$\sigma_{H_2}^{(2)}$, and I_{H_2} are rather tricky to define. If there are ℓ lines of approximately equal strength $\bar{\sigma}$ each of width Δ, the Doppler width of a single line, then the dissociation rate of H_2 in unshielded space will be $3 \times 10^{-8}\ \ell\bar{\sigma}\Delta$ since the photon flux is 310^{-8} photons cm^{-2} s^{-1} Hz^{-1} (Hollenbach et al. 1971). This rate has been calculated as about 10^{-10} s^{-1}

$$\therefore \quad \ell\Delta\bar{\sigma} = \frac{310^{-11}}{310^{-8}} = 310^{-3}$$

$$\Delta \simeq 10^9\sqrt{T} \quad \text{Hz}$$

$$\therefore \quad \ell\bar{\sigma} \simeq \frac{310^{-12}}{\sqrt{T}}$$

Taking a geometrical cross section for $\bar{\sigma}$, $\bar{\sigma} = 10^{-15}$ cm^2

$$\ell = \frac{310^3}{\sqrt{T}} = 300 \text{ if } T = 100 \text{ k}$$

and $\ell\Delta = 310^{12}$ which is a rather large fraction of the frequencies of the photons. Making $\bar{\sigma}$ smaller would require making ℓ and $\ell\Delta$ uncomfortably large, and would imply that a large fraction of the spectrum corresponded to lines of photodissociation of H_2 and even more would correspond to other H_2 lines. We do not want to make $\bar{\sigma}$ greater than the geometrical cross section and so we set $\sigma_{H_2}^{(2)} = \bar{\sigma} = 10^{-15}$ cm^{-2}.

I_{H_2} was taken to be 310^4 and fractions of this, corresponding to a maximum dissociation rate of 310^{-11} s^{-1}.

$\sigma_{H_2}^{(1)} = 2.5\ \sigma_{H_2}^{(2)}$ because Hollenbach et al. estimated that about 40% of the absorptions lead to dissociation.

ζ was taken as 310^{-16} s^{-1} and the attenuation of the cosmic rays was ignored.

$\frac{n_C}{n_H}$ was taken to be $410^{-4}\ \delta$ where δ is the depletion factor, taken in these models to be 0.1.

k_T was taken to be that of the neutral gas, namely $= \frac{AR\sqrt{3RT}}{3\pi n \sigma_T^2}$ where $\sigma_T = 8.94\ 10^{-9}$ cm.

$x_e = \frac{n_e}{n_i}$ was set to (1) x_{C^+} (2) $x_{C^+} + 410^{-4}$ and (3) $x_{C^+} + 10^{-3}$ in different models.

4.5 Photon Mean free path

With the reaction scheme suggested the dependence of the mean free path (m.f.p.) of the molecule destroying photons may be derived

1) H_2 destroying photons

$$\text{m f p for } \underset{\text{photons}}{H_2} = \left[n_1 x_{H_2} \sigma_{H_2}^{(2)} + n_1 x_g \sigma_j\right]^{-1} = \frac{1}{n_1}\left[\sigma_{H_2}^{(2)}\left(\frac{1-x_H}{2}\right) + 10^{-21}\right]^{-1}$$

$$= \frac{1}{n}\left[\frac{n g x_H}{2 I_{H_2}} + 10^{-21}\right]^{-1} = \frac{1}{n}\left[\frac{n}{I_{H_2}} 10^{-17} + 10^{-21}\right]^{-1} \text{ cm} \quad (4.18)$$

since $x_H \approx 1$

and $g \ll \sigma_{H_2}^{(2)} I_{H_2}$ in these models

If $I_{H_2} = 10^5$ photons $cm^{-2} s^{-1}$, m.f.p. $= \frac{10^{21}}{n}\left(\frac{n}{10} + 1\right)^{-1}$ cm

corresponding to $k_c = 6n\left[\frac{n}{10} + 1\right] 10^{-21}$ cm^{-1}

n	30	300	3000	30,000	cm^{-3}
k_c	610^{-20}	610^{-18}	610^{-16}	610^{-14}	cm^{-1}

If $I_{H_2} = 10^3$ photons cm^{-2} s^{-1}

n	30	300	3000	30,000	cm^{-3}
k_c	610^{-18}	610^{-16}	610^{-14}	610^{-12}	cm^{-1}

In order to apply the linear perturbation method detailed in Chapter 2 a necessary condition is that k the wave number of the perturbation is restricted by

$k_c \ll k$ for the conditions used.

Carbon photoionising photons

$$mfp = \left[n_1 x_c \sigma_c^{(2)} + n_1 \sigma_g x_g \right]^{-1}$$

$$= \left[n_1^2 \alpha \frac{x_e x_{c+}}{I_c} + n\, 10^{-21} \right]^{-1} \qquad (4.19)$$

$$\alpha \approx 10^{-11}, \quad x_{c+} = 4\, 10^{-4} \delta$$

$$\delta = 0.1, \quad I_c \approx 10^7 \text{ photons cm}^{-2} \text{s}^{-1}$$

$$= \frac{1}{n} \left[n \frac{10^{-20}}{I_c} + 10^{-21} \right]^{-1} = \frac{10^{21}}{n_1} \left[\frac{10 n_1}{I_c} + 1 \right]^{-1}$$

$$= \frac{10^{21}}{n \left[10^{-6} n + 1 \right]} = \frac{2\pi}{k_c}$$

n	30	300	3000	30,000	cm^{-3}
k_c	210^{-19}	210^{-18}	210^{-17}	210^{-16}	cm^{-1}

while of $I_c = 10^5$ photons cm^{-2} s^{-1} one obtains

n	30	300	3000	30,000	cm^{-3}
k_c	210^{-19}	210^{-18}	210^{-17}	710^{-16}	cm^{-1}

As before one requires that

$$k_c << k$$

for any perturbation considered.

4.6. Method of Solution

The dispersion relation was solved numerically for a large range of k, the wavenumber of the perturbation, and n, the number density of hydrogen. The various derivatives, L_T, etc. were calculated numerically and the subroutine CØ2ABF of the Nottingham Algorithm Group, made available by the Oxford Computing Centre, was used. CØ2ABF uses a version of Newton's method to find the roots of complex polynomials, it however suffered from the defect of occasionally producing numbers which were not roots of the equation presumably because of overflow or underflow when multiplying very large or very small numbers. It was therefore necessary to check the roots of substitution into the equation and the algebraic sum of the terms was compared to the magnitude of the largest terms. Furthermore it was necessary to check against error introduced by the fact that numbers smaller than a certain non-zero value do not register in the computer, so that multiplication of two non-zero numbers together can give zero in the machine. The check consisted of using the calculated value of β in all but one of the original equations to obtain $\frac{\delta n}{\delta x}$, $\frac{\delta T}{\delta x}$, $\frac{\delta I}{\delta x}$ etc. and these values were substituted into the last equation. Again the algebraic sum was compared to the magnitude of the largest terms, the ratio of these two quantities should much less than unity if the value of β is a good approximation to a root of the equation. The ratio was 10^{-7} or less in most cases.

4.7 Results and Conclusions

Tables 1 and 2 give the values of T, L_T, etc. for various combinations of n and I, and $J \equiv I_c$ for two models. The first is heated by cosmic rays, H_2 dissociation and carbon photoionisation, the second by these and also the heating mechanism associated with the formation of H_2 on grains. None of the situations considered was unstable in the Field sense.

Tables 3 and 4 gives the value of $1/Re(\beta)$, whenever it is positive, for a variety of wavevectors k, density n and radiation flux I, in the two models in which we look at perturbations in the hydrogen molecule fraction $x_{H_2} = (\frac{1-x}{2})$. J is taken to be constant throughout these tables.

The only point to come out of these results is that, although there do seem to be instabilities under certain conditions, these instabilities occur on timescales greater than the free-fall time-scale. It will be remembered that the parameters used are rather optimistic in that they tend to overestimate the effect of radiative transfer.

The calculations were repeated using neutral carbon as the photodissociated species, with values of I_C between 13 and $1.3\ 10^7$ photons $cm^{-2}\ s^{-1}$ in multiples of 100, and δ equal to 0.1 and 1.0, in model 1. Very few instabilities were found and none had timescales less than the free-fall time.

CHAPTER 5

A MODEL FOR PURE HYDROGEN GAS

5.1 The Heating and Cooling mechanisms

Instabilities in pure hydrogen gas are of interest when the formation of the first generation of stars is discussed. Saslaw (1967) gives an approximate expression for the rate of cooling by collisional excitation of the rotational levels of H_2 followed by radiation of the energy. This may be written

$$\Lambda_{H_2} = 1.5\ 10^{-7}\ T^{2.4}\ n^{2/3}\ x\ \frac{(1-x)}{2}\ \text{ergs gm}^{-1}\ \text{s}^{-1} \qquad (5.1)$$

and this is valid for

$$500 < T < 5000 \qquad (5.2)$$

and $10 < n < 10^4$.

We assume the heating here is some unspecified density dependent effect giving the heating required to maintain T at 1000^{o}k, after including that part of Γ_{Hr} which comes from H_2 dissociation. γ was taken to be 7/5 for H_2 and 5/3 for the atoms.

5.2 Formation and Destruction mechanisms

Even if there are no grains present, H_2 can still form via the reactions

$$H + e \rightarrow H^- + \text{photon} \qquad (5.3)$$

$$H^- + H \rightarrow H_2 + e \qquad (5.4)$$

where the rate coefficient for (5.3) is $6.06\ 10^{-19}\ T\ cm^3\ s^{-1}$ (Peebles 1969) and the rate of (5.4) is much larger than that of (5.3). Peebles suggests an initial fractional ionisation of 310^{-5} and so we get an effective value of g (see the previous chapter) of $1.8\ 10^{-20}$. In this calculation we use $g = 10^{-20}$.

From Yoneyama's paper we can take the rate of collisional dissociation to be

$$X^-_{H_2,\,collisional} = n x(1-x) \frac{2\,10^{-6}}{\sqrt{T}} \exp\left(-5{\cdot}2\,10^4/T\right) \qquad (5.5)$$

With our usual expression of $X^-_{photo} = \sigma^{(2)}(1 - x)\ I$, we find

$$\frac{X^-_{photo}}{X^-_{H_2,\,coll}} = \frac{\sigma^{(2)}(1-x) I \sqrt{T}\ e^{5{\cdot}2\,10^4/T}}{2\,n(1-x)x\,10^{-6}} \qquad (5.6)$$

$$= \frac{\sigma^{(2)} I}{nx}\ 5\,10^5 \sqrt{T}\ e^{5{\cdot}2\,10^4/T}$$

$$= \frac{\sigma^{(2)} I}{nx}\ 6\,10^{29} \qquad \text{for } T = 10^3\ K$$

$$= \frac{\sigma^{(2)} I}{nx}\ 9\,10^{9} \qquad \text{for } T = 10^4\ K$$

For the values of I and n taken here, photodissociation is easily the dominant process if $T = 10^3$ k, therefore since we fix the temperature at 10^3 k we take only this process of destruction.

Yoneyama states that in this case the system is stable. We now investigate the effects of the radiative transfer. Table 5 gives the values of the various derivatives for different I and n. Table 6 shows the resulting timescales of the instabilities.

5.3 Conclusions

There is quite a variety of conditions which give rise to instabilities, although long wavelengths and high densities seem most favoured. Table 5 gives the values of L_x etc. The timescales shown in Table 6 are quite short, especially in regions with large values of I. However, none satisfy the condition that k should be much greater than $n\sigma(1-x)$ and so we conclude that, unless there are exceptional circumstances giving rise to uniformity over longer scale lengths, these instabilities will not be very helpful in fragmenting gas clouds of pure hydrogen. Calculations for the three dimensional problem may change this conclusion.

CHAPTER 6

CRITICISM OF SCHATZMAN'S PAPER

6.1 Description of Schatzman's model

This chapter is devoted to a model studied by Schatzman (1958) of a slab of hydrogen gas ionised and heated by a flux of photons coming from both sides. Schatzman reasoned that motion of the gas would change the density of the gas and so affect the radiation field. Changes in the radiation field alter the ionisation structure producing a change in the number density of particles in the gas which in turn affects the motion of the gas.

Schatzman's notation is used in the outline of his paper in this section. We give next the equations used.

Ionisation

$$\frac{\partial n_H}{\partial t} = q\, n_e n_p - \frac{\sigma}{h\nu}\, S\, n_H \qquad (6.1)$$

where n_H = number density of hydrogen atoms

$n_e = n_p$ = number of ionised hydrogen atoms

q = recombination coefficient

S = mean total intensity of ionising radiation

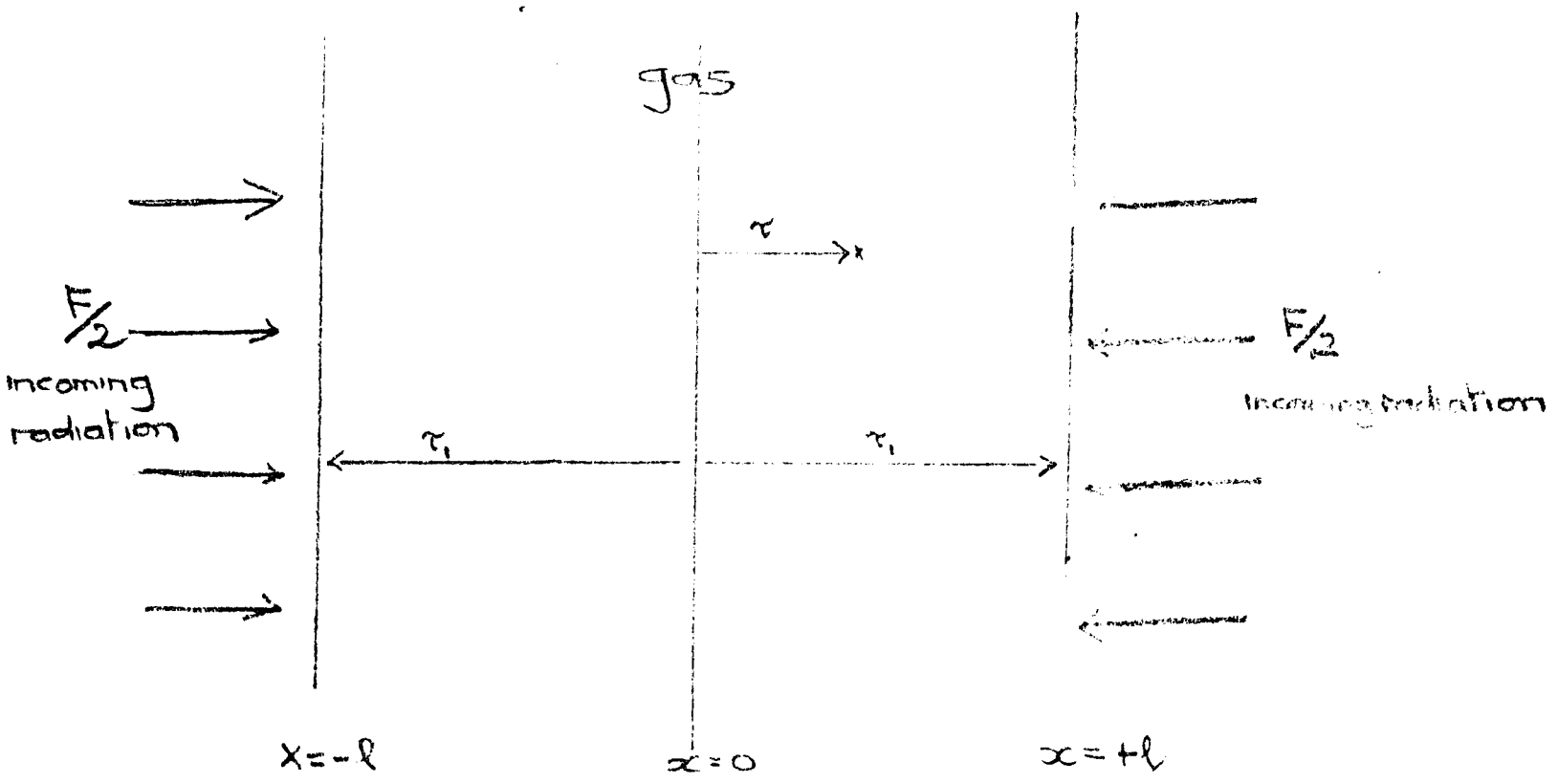
σ = cross section for photoionisation at the Lyman limit

h_ν = photon energy at Lyman limit.

Radiation field

$$\frac{\cos\theta}{\sigma n_H}\left[\frac{1}{c}\frac{\partial I}{\partial t} + \frac{\partial I}{\partial x}\right] = -I + \omega\bar{I} + \omega F e^{-\tau_1}\cosh\tau \qquad (6.2)$$

$$S = \bar{I} + Fe^{-\tau_1} \cosh\tau \qquad (6.3)$$



$2\tau_1$ = optical thickness of slab at Lyman limit

F/2 is the radiation flux incident on either side of normal to the surface

ω = $\frac{\text{no. of re-emissions in the Lyman continuum}}{\text{total no. of re-emissions}}$

τ = optical depth from centre to point.

Equations of motion

Pressure $P = (2N - n_H)\, k_B\, T \qquad (6.4)$

where N = total no. of hydrogen nucleii

T = gas kinetic temperature

k_B = Boltzmann's constant.

Small disturbances propagate according to the equation

$$\frac{\partial^2 N}{\partial t^2} = \frac{\partial^2}{\partial x^2} \left\{ \frac{k_B T}{m_o} (2N - n_H) \right\} \qquad (6.5)$$

m_o = mass of proton.

Schatzman next argued that the timescale for thermal relaxation in an HII region at T = 5000^{o}k is $\frac{1.4\ 10^{4}}{n_p}$ years (for example see Spitzer 1968) and furthermore T varies very little with density so that the temperature could be supposed to be constant. The timescale for ionisational relaxation is also very short, about 1 yr. in cases where $n_H << N$, taking $\sigma = 6\ 10^{-18}$ cm^2 and a radiation temperature $2\ 10^4$ K with dilution factor 10^{-14}, therefore he assumed ionisation equilibrium. Since perturbations in the radiation field propagate with about the speed of light and taking typical dimensions of interstellar gas clouds as 1 pc then the timescale for radiative relaxation will be about 1 yr. so Schatzman assumed radiative equilbrium. These considerations together meant that he could neglect the time derivatives in (6.1) and (6.2) and also regard T as constant in all considerations of perturbations.

The steady state equations are from (6.1) and (6.2) with $\frac{\partial}{\partial t} \equiv 0$, together with

$$P = \text{constant}$$

for hydrostatic equilibrium, assuming the gas is somehow contained within the slab, for example by external gas pressure. These steady state equations are soluble, exactly if $\omega = 0$ and approximately if $\omega \neq 0$, and S, n_H and N may be given in terms of τ. Given this steady, but non-uniform solution the equations may be linearised around them.

(6.1) gives

$$\delta N = \frac{N + n_H}{2n_H}\ \delta n_H + \frac{N - n_H}{2}\ \frac{\delta S}{S} \qquad (6.6)$$

where δN, δn_H, δS are the infinitesimal perturbations to N, n_H and S respectively.

Assuming that all perturbations vary in time as $e^{\lambda t}$ (6.5) becomes

$$\lambda^2 \, \delta N = \frac{k_B T}{m_o} \, \frac{\partial^2}{\partial x^2} \, (2 \, \delta N - \delta n_H). \qquad (6.7)$$

If δS can be found using (6.2) in terms of δN and δn_H then (6.6) may be used to find δn_H and δS in terms of δN so that finally (6.7) is an eigenvalue equation for λ if appropriate boundary conditions are imposed on δN. Having found an eigenvalue the function δN may be calculated.

Schatzman assumed that $\lambda = 0$ was an eigenvalue and tried to find the values of τ_1 and ℓ_c to which this corresponded with boundary conditions $\delta N = 0$ at $x = \pm \ell$. He claimed that the appropriate values for the case $\omega = 0$ were

$$\tau_1 = \tfrac{1}{2} \ln\left(\frac{N}{n_o}\right)$$

$$\ell_c = \frac{1}{2\sigma n_o}$$

where n_o is the number density of hydrogen atoms in a radiation field of dilution W

$$n_o = N^2 \, W^{-1} \, 10^{-18.44}$$

He argued that if the slab were larger, that is $\ell > \ell_c$, then it would collapse.

For $W = 10^{-14}$, $\ell_c = \frac{675}{N^2}$ pc, a sphere of this radius would contain a mass $\frac{8.3 \, 10^7}{N^5} \, M_\odot$.

For $\omega \neq 0$ Schatzman found that ℓ_c increased so for example if $\omega = 0.39$, $\ell_c = \frac{1000}{N^2}$ pc. After this spontaneous collapse

Schatzman speculated that the gas would recombine almost entirely and form a cool HI cloud.

Examination of the equations for $\omega = 0$ shows that his analysis is wrong.

6.2 Analysis of the equations for $\omega = 0$, $\lambda = 0$

(6.7) with $\lambda = 0$ is

$$\frac{\partial^2}{\partial x^2} (2\delta N - \delta n_H) = 0. \qquad (6.8)$$

Now $$\tau = \sigma \int_o^x n_H(x')\, dx' \qquad (6.9)$$

so (6.8) may be written, dropping the subscript H

$$\frac{\partial}{\partial \tau} \left(n\sigma \frac{\partial}{\partial \tau} (2\delta N - \delta n)\right) = 0 \qquad (6.10)$$

$$\therefore \quad \frac{\partial}{\partial \tau} (2\delta N - \delta n) = \frac{A}{n(\tau)} \qquad (6.11)$$

$$2\,\delta N - \delta n = \int_o^\tau \frac{A}{n(\tau')}\, d\tau' + B \qquad (6.12)$$

A, B constant

$\delta N = 0$ at $\tau = \pm\tau_1$ are the assumed boundary conditions.

With $\omega = 0$ (6.2) can be solved exactly

$$S = F\, e^{-\tau_1} \cosh\tau. \qquad (6.13)$$

If δn is an even function of τ then δS and δN are also even functions of τ. At $\tau = +\tau_1$

$$-\delta n(\tau_1) = \int_o^{\tau_1} \frac{A}{n(\tau')}\, d\tau' + B$$

at $\tau = -\tau_1$

$$-\delta n(-\tau_1) = \int_0^{-\tau_1} \frac{A}{n(\tau')}\, d\tau' + B$$

$$= -\int_0^{+\tau_1} \frac{A}{n(\tau')}\, d\tau' + B$$

since n is an even function of τ by the symmetry of the situation combined with the fact that $n \geq 0$. The symmetry requires $n(\tau)=-n(-\tau)$ or $n(\tau) = n(-\tau)$. As $n > 0$ one sees immediately that the first is not possible.

$\delta n(\tau_1) = \delta n(-\tau_1)$ gives

$$\int_0^{\tau_1} \frac{A}{n(\tau)}\, d\tau + B = -\int_0^{\tau_1} \frac{A}{n(\tau)}\, d\tau + B$$

$$\therefore \quad A = 0 \qquad (6.14)$$

Since there is no restriction on the sign of δn as long as $n \neq 0$, $\delta n(\tau)$ may be an odd function. This is considered below.

$$S+\delta S = \frac{F}{2}\left[\exp\left\{-\int_{-\ell}^{x}\sigma(n+\delta n)\,dx\right\} + \exp\left\{-\int_{x}^{\ell}\sigma(n+\delta n)\,dx\right\}\right]$$

using equation (6.13) and (6.9) with $(n + \delta n)$ instead of n.

$$S+\delta S = \frac{F}{2}\left[\exp\left\{-\sigma\int_{-\ell}^{x} n\,dx\right\}\exp\left\{-\sigma\int_{-\ell}^{x}\delta n\,dx\right\} + \exp\left\{-\sigma\int_{x}^{\ell} n\,dx\right\}\exp\left\{-\sigma\int_{x}^{\ell}\delta n\,dx\right\}\right]$$

$$= \frac{F}{2}\left[\exp\left\{-\sigma\int_{-\ell}^{0} n\,dx - \sigma\int_{0}^{x} n\,dx\right\}\exp\left\{-\sigma\int_{-\ell}^{0}\delta n\,dx - \sigma\int_{0}^{x}\delta n\,dx\right\}\right.$$
$$\left. + \exp\left\{-\sigma\int_{0}^{\ell} n\,dx + \sigma\int_{0}^{x} n\,dx\right\}\exp\left\{-\sigma\int_{0}^{\ell}\delta n\,dx + \sigma\int_{0}^{x}\delta n\,dx\right\}\right]$$

$$\tau_1 = \sigma \int_o^{\ell} n\, dx = \sigma \int_{-\ell}^{o} n\, dx$$

writing $\omega = \sigma \int_o^x \delta n\, dx$

and remembering that δn is an infinitesimal quantity so that the exponential containing it may be expanded and only the terms up to first order in δn need be retained.

$$S+\delta S = \frac{F}{2} e^{-\tau_1} \left[\begin{array}{l} e^{-\tau} \left(1 - \omega(\tau=\tau_1) - \omega(\tau)\right) \\ + e^{\tau} \left(1 - \omega(\tau=\tau_1) + \omega(\tau)\right) \end{array} \right] \ldots$$

$$\therefore \delta S = \frac{F}{2} e^{-\tau_1} \left[\begin{array}{l} e^{-\tau} - e^{-\tau}\left(\omega(\tau_1) + \omega(\tau)\right) \\ + e^{\tau} - e^{-\tau}\left(\omega(\tau_1) - \omega(\tau)\right) - e^{\tau} - e^{-\tau} \end{array} \right]$$

$$= \frac{F}{2} e^{-\tau_1} \left[-\omega(\tau_1)\left[e^{\tau} + e^{-\tau}\right] + \omega(\tau)\left[e^{\tau} - e^{-\tau}\right] \right]$$

$$\therefore \frac{\delta S}{S} = \omega(\tau)\tanh(\tau) - \omega(\tau_1)$$

(6.15)

Using this in equation (6.6)

(6.15) $2\,\delta N - \delta n = \frac{N}{n}\,\delta n + (N - n)\,[\tanh(\tau)\,\omega(\tau_1) - \omega(\tau)]$

$= B$ from (6.12) with $A = 0$

$$\frac{\delta n}{n} = \frac{\sigma \delta n}{\sigma n} = \frac{d\omega}{d\tau} \quad .$$

In the steady state

$$2N - n = \frac{Pm}{k_B T} = \text{constant} = D \tag{6.16}$$

$$\therefore N = \frac{D+n}{2}, \quad N - n = \frac{D-n}{2}$$

(6.15) may then be written

$$\frac{D+n}{2}\frac{d\omega}{d\tau} + \frac{D-n}{2}(\omega \tanh(\tau) - \omega_1) = B \tag{6.17}$$

where $\omega_1 = \omega(\tau_1)$

$$\therefore \quad \frac{d\omega}{d\tau} + \frac{D-n}{D+n}(\tanh \tau)\omega = \frac{2B}{D+n} + \omega_1 \frac{(D-n)}{(D+n)} \tag{6.18}$$

This equation has an integrating factor $\exp\left\{\int_0^\tau \frac{D-n}{D+n}\tanh\tau\, d\tau\right\}$

$$\omega \exp\left\{\int_0^\tau \frac{D-n}{D+n}\tanh(\tau)d\tau\right\} =$$

$$G + \int_0^\tau \left(\frac{2B}{D+n} + \frac{D-n}{D+n}\omega_1\right)\left[\exp\int_0^{\tau'}\frac{D-n}{D+n}\tanh\tau''d\tau''\right]d\tau' \tag{6.19}$$

Since $\omega = \sigma \int_0^x \delta n\, dx$ then $\omega(\tau = 0) = 0$

$$\therefore G = 0 \tag{6.20}$$

Consider now the boundary conditions $\delta N(\tau = \pm\tau_1) = 0$

(6.15) gives $2\delta N = \frac{D+3n}{2}\frac{d\omega}{d\tau} + \frac{D-n}{2}(\omega \tanh(\tau) - \omega_1)$

so at $\tau = \tau_1$

$$\left.\frac{d\omega}{d\tau}\right|_{\tau=\tau_1} = -\left(\frac{D-n}{D+3n}\right)\omega_1(\tanh \tau_1 - 1).$$

Now using the second part of (6.15) B must be given by

$$B = -\frac{(D+n_o)(D-n_o)}{2(D+3n_o)}\omega_1(\tanh\tau_1 - 1) + \frac{D-n_o}{2}\omega_1(\tanh\tau_1 - 1) = B$$

$$n_o \equiv n(\tau = \pm\tau_1)$$

$$= \omega_1 \frac{(D-n_o)n_o}{(D+3n_o)}(\tanh\tau_1 - 1)$$

$$= \omega_1 D(\tanh\tau_1 - 1)\frac{(1-\frac{n_o}{D})(\frac{n_o}{D})}{(1-3\frac{n_o}{D})} \qquad (6.21)$$

$$= \omega_1 A$$

Note that $B < 0$ since $\tanh\tau_1 < 1$ and $|A| \ll D$ if $(\frac{n_o}{D}) \ll 1$.

An explicit formula for n in terms of τ may be found using (6.1), neglecting the time derivative, and using (6.13) for S

$$q(N-n)^2 = \frac{F\sigma e^{-\tau_1}}{h\nu}\cosh\tau\, n \qquad (6.22)$$

ie $$(D-n)^2 = (\frac{4F\sigma e^{-\tau_1}}{h\nu q})\cosh\tau\, n \qquad (6.23)$$

using (6.16).

Let $$2RD = \frac{4F\sigma e^{-\tau_1}}{h\nu q} \qquad (6.24)$$

$$n^2 - 2nD[1 + R\cosh\tau] + D^2 = 0$$

$$\therefore \quad n = D[(1 + R\cosh\tau) - \sqrt{(1 + R\cosh\tau)^2 - 1}] \qquad (6.25)$$

Where the negative sign for the square root has been taken because $n \leq N$.

Let $$\cosh\theta = 1 + R\cosh\tau \qquad (6.26)$$

$$n = D\,[\cosh\theta - \sinh\theta] = D\,e^{-\theta} \tag{6.27}$$

$$\frac{D-n}{D+n} = \frac{1-e^{-\theta}}{1+e^{-\theta}} = \frac{e^{\theta/2} - e^{-\theta/2}}{e^{\theta/2} + e^{-\theta/2}} = \tanh(\theta/2) \tag{6.28}$$

$$\int_0^{\tau} \frac{D-n}{D+n}\tanh\tau\, d\tau = \int_{\theta_0}^{\theta} \frac{\sinh\theta}{(1+\cosh\theta)} \frac{R\sinh\theta}{R(\cosh\theta - 1)}\, d\theta$$

$$= \int_{\theta_0}^{\theta} \frac{\sinh^2\theta}{\cosh^2\theta - 1}\, d\theta = \int_{\theta_0}^{\theta} d\theta = \theta - \theta_0$$

(6.29)

(6.19) then is

$$we^{\theta} = \int_{\theta_0}^{\theta} \left\{ \frac{2A\omega_1}{D(1+e^{-\theta})} + \omega_1 \tanh\left(\frac{\theta}{2}\right) \right\} e^{\theta}\, d\theta \left(\frac{d\tau}{d\theta}\right) .$$

$$\theta_0 = \cosh^{-1}(1+R)$$

$$\theta = \cosh^{-1}(1+R\cosh\tau)$$

Neglecting the term $\dfrac{A}{D(e^{-\theta}+1)}$ at first, since $A \ll D$ if $n_o \ll D$ we require that at $\tau = \tau_1$

$$\omega_1 e^{\theta_1} = \omega_1 \int_{\theta_0}^{\theta_1} e^{\theta} \tanh\left(\theta/2\right) \left(\frac{d\tau}{d\theta}\right) d\theta$$

for consistency

ie. $$e^{\theta_1} = \int_{\theta_0}^{\theta_1} \tanh\left(\frac{\theta}{2}\right) e^{\theta} \frac{\sinh\theta}{\sqrt{(\cosh\theta - 1)^2 - R^2}}\, d\theta \tag{6.30}$$

Now let $z = e^{\theta}$ (6.31)

$$z_1 = \int_{z_0}^{z_1} \frac{\left(\sqrt{z} - \frac{1}{\sqrt{z}}\right)^2}{\left(\frac{(z-1)^4}{z^2} - 4R^2\right)^{1/2}} dz$$

$$= \int_{z_0}^{z_1} \frac{dz}{\sqrt{1 - \frac{4R^2z^2}{(z-1)^4}}} \qquad (6.32)$$

To summarise, the above equation must be solved for z_1 in order to find τ_1, the critical optical depth to the centre of a slab which is marginally stable against perturbations to N, n and S. If no z_1 can be found this means that the choice of $\lambda = 0$ in (6.7) is not correct. However, this does not rule out the possibility that for some optice depths τ_1 the situation is unstable against such perturbations; the λ's, however, may be complex so that $Im(\lambda) \neq 0$ when $Re(\lambda) = 0$.

Schatzman's result was $\tau_1 = \frac{1}{2} \ln(\frac{N}{n_o})$

so that $\frac{N}{n_o} = e^{2\tau_1} >> 1$ (6.33)

(6.23) and (6.24), with $D >> n_o$ give

$$D = 2Rn_o \cosh\tau_1 \qquad (6.34)$$

at $\tau = \tau_1$

$$\frac{D}{2n} = \frac{N}{n} = R\cosh\tau_1 \approx \frac{Re^{\tau_1}}{2} \quad \text{if } \tau_1 \gg 1$$

$$\therefore \frac{Re^{\tau_1}}{2} = e^{2\tau_1}$$

$$R = 2e^{\tau_1} \gg 1 \qquad (6.35)$$

$$\therefore \quad \theta_o = \cosh^{-1}(1 + R) >> 0$$

$$\text{so} \quad z_o = e^{\theta_0} >> 1 \qquad (6.36)$$

(6.32) may then be approximately written as

$$z_1 = \int_{z_0}^{z_1} \frac{dz}{\sqrt{1 - 4\frac{R^2}{z^2}}}$$

since z is always >> 1

$$= \int_{z_0}^{z_1} \frac{z\,dz}{\sqrt{z^2 - 4R^2}}$$

$$= \left[\sqrt{z^2 - 4R^2}\right]_{z_0}^{z_1}$$

$$= \left[z_1^2 - 4R^2\right]^{1/2} - \left[z_0^2 - 4R^2\right]^{1/2} \qquad (6.37)$$

Now the RHS must be less than z_1 therefore no z_1 can be found to satisfy (6.32), so that one cannot use Schatzman's value for τ_1.

The integral involving $\frac{A}{D(1+e^{-\theta})}$ is negative so that its inclusion makes the inequality worse.

The assumption that $R >> 1$ forces the conclusion that (6.32) has no solution. It remains to be seen whether a solution is possible if $R << 1$, in this case z_o is approximately unity so that one cannot neglect 1 with respect to z throughout the range. Numerical integrations with $R = 5, 1, 10^{-2}, 10^{-3}$ show no solution.

The above calculations assumed that $\delta n(\tau)$ was an even function of τ. If δn were an odd function of τ, then using similar methods

$$\frac{\delta S}{S} = \tanh(\tau)\,[\omega(\tau) - \omega(\tau_1)] \qquad (6.38)$$

(6.12) evaluated at $\tau = \tau_1$ gives

$$\delta n(\tau_1) = \int_o^{\tau_1} \frac{A\,d\tau'}{n(\tau)} + B \text{ since } \delta N(\tau_1) = 0$$

evaluated at $\tau = -\tau_1$

$$\delta n(-\tau_1) = -\delta n(\tau_1) = \int_o^{\tau_1} \frac{A}{n(\tau)} d\tau + B$$

combining these two,

$$B = 0 \qquad (6.39)$$

$$\therefore \quad 2\,\delta N - \delta n = \int_0^{\tau} \frac{A\,d\tau}{n(\tau)}$$

$$= \frac{N}{n}\,\delta n + (N-n)\tanh(\tau)\,(\omega-\omega_1)$$

$$= \frac{D+n}{2}\frac{d\omega}{d\tau} + \frac{D-n}{2}\tanh(\tau)\,(\omega-\omega_1) \qquad (6.40)$$

which one can write

$$\frac{d}{d\tau}(\omega - \omega_1) + \left(\frac{D-n}{D+n}\right)\tanh(\tau)\,(\omega - \omega_1) = \int_0^{\tau} \frac{A}{n(\tau')}\,d\tau' \qquad (*)$$

$$2\delta N = \left(\frac{D+3n}{2}\right)\frac{d\omega}{d\tau} + \left(\frac{D-n}{2}\right)\tanh(\tau)\,(\omega - \omega_1) \qquad (6.41)$$

evaluating at $\tau = \tau_1$

$$0 = 2\delta N(\tau = \tau_1) = \left(\frac{D+3n_0}{2}\right)\left.\frac{d\omega}{d\tau}\right|_{\tau=\tau_1} \qquad (6.42)$$

$$\therefore \quad \left.\frac{d\omega}{d\tau}\right|_{\tau=\tau_1} = 0 \qquad (6.43)$$

Evaluating (*) at $\tau = \tau_1$

$$0 = A\int_0^{\tau_1} \frac{d\tau}{n(\tau)}$$

and since $n > 0$

$$A = 0$$

$$\therefore \quad 2\delta N - \delta n = 0$$

$$\therefore \quad 2\delta N = \delta n \qquad (6.44)$$

Also

$$\frac{d}{d\tau}\left[(\omega-\omega_1)\exp\left\{\int_0^{\tau}\frac{D-n}{D+n}\tanh(\tau')\,d\tau'\right\}\right] = 0$$

$$(\omega-\omega_1)\exp\left\{\int_0^{\tau}\frac{D-n}{D+n}\tanh(\tau')\,d\tau'\right\} = C = \text{constant.}$$

at $\tau = \tau_1$, then $C = 0$ (6.45)

$$\therefore \quad \omega = \omega_1 \quad \text{for all } \tau \qquad (6.45)$$

$$= 0 \quad \text{since at } \tau = 0,\ \omega = 0$$

$$\therefore \text{ since } \frac{d\omega}{d\tau} = \frac{\delta n}{n}$$

$$\text{therefore } \delta n = 0 \text{ for all } \tau \tag{6.46}$$

$$\text{and } \delta N = 0. \tag{6.47}$$

If δn is assumed to be an odd function of τ then the only solution possible is $\delta n = 0$ for all τ.

6.3 Linear Analysis of the Homogeneous case

It is interesting to analyse the uniform analogue of the above model, that is the situation of a uniformly ionised slab of hydrogen, using the methods of Chapters 2 and 3. Making the same assumption about neglecting the time derivative in the equation of radiative transfer but retaining the time derivative in the ionisation equation and taking T constant, the basic equations become

$$\frac{\partial n}{\partial t} = \frac{B}{2}(N-n)^2 - An\,I \tag{6.48}$$

$$+\frac{\partial I}{\partial x} = -n\sigma I \tag{6.49}$$

$$\frac{\partial^2}{\partial t^2} N = \frac{\partial^2}{\partial x^2}\left[\left(\frac{k_B T}{m}\right)(2N - n)\right]. \tag{6.50}$$

Initially N, n and I are constant; changing n to n + δn etc. the linearised equations are

$$\beta\,\delta n = B(N-n)(\delta N - \delta n) - An\,\delta I - AI\,\delta n \tag{6.51}$$

$$\text{i.e. } \delta n\left[\beta + B(D-n) + AI\right] - \delta N\left[B(D-n)\right] + An\delta I = 0 \tag{6.52}$$

$$-\delta n\,K^2\frac{k_B}{m}T + \delta N\left[\beta^2 + 2K^2\frac{k_B}{m}T\right] = 0 \tag{6.53}$$

$$\delta n\,\sigma I \;+\; [n\sigma + ik]\,\delta I = 0 \qquad (6.54)$$

where all perturbations are assumed to have time dependence $e^{\beta t}$.

The condition for consistency is

$$\begin{vmatrix} \beta + B(D-n) + AI & -B(D-n) & An \\ -k^2 RT & \beta^2 + 2RTk^2 & 0 \\ \sigma I & 0 & n\sigma + ik \end{vmatrix} = 0 \qquad (6.55)$$

where $R = \dfrac{k_B}{m}$

$$\beta^3 + \beta^2\left[B(D-n) + AI - \frac{n\sigma IA}{n^2\sigma^2 + k^2}(n\sigma - ik)\right] + 2RTk^2\beta$$

$$+ RTk^2\left[B(D-n) + 2AI\right] - 2RTk^2\frac{\sigma IAn}{n^2\sigma^2 + k^2}(n\sigma - ik) = 0$$

$$\beta^3 + \beta^2\left[B(D-n) + \frac{k^2 AI}{n^2\sigma^2 + k^2} + i\left(\frac{n\sigma IAk}{n^2\sigma^2 + k^2}\right)\right]$$

$$+ \quad 2RTk^2\beta$$

$$+ RTk^2\left[B(D-n) + \frac{2k^2 AI}{n^2\sigma^2 + k^2} + i\,\frac{2n\sigma AIk}{n^2\sigma^2 + k^2}\right] = 0 \qquad (6.56)$$

Using the notation of Chapter 3

$$a_0 = 1 \quad , \quad a_1 = B(D-n) + \frac{k^2 AI}{n^2\sigma^2 + k^2} \tag{6.57}$$

$$c_2 = -\begin{vmatrix} a_1 & a_0 & 0 \\ -b_2 & -b_1 & a_1 \\ a_3 & a_2 & b_2 \end{vmatrix} = \begin{vmatrix} B(D-n)+\frac{k^2AI}{n^2\sigma^2+k^2} & 1 & 0 \\ 0 & \frac{-n\sigma IAk}{n^2\sigma^2+k^2} & B(D-n)+\frac{k^2AI}{n^2\sigma^2+k^2} \\ RTk^2\left\{B(D-n)+\frac{2k^2AI}{n^2\sigma^2+k^2}\right\} & 2RTk^2 & 0 \end{vmatrix}$$

$$= \left[B(D-n)+\frac{k^2AI}{n^2\sigma^2+k^2}\right]\left[2RTk^2\left[B(D-n)+\frac{k^2AI}{n^2\sigma^2+k^2}\right] - RTk^2\left[B(D-n)+\frac{2k^2AI}{n^2\sigma^2+k^2}\right]\right]$$

$$= \left[B(D-n)+\frac{k^2AI}{n^2\sigma^2+k^2}\right]\left[RTk^2B(D-n)\right] \tag{6.58}$$

$$c_3 = -\begin{vmatrix} a_1 & a_0 & 0 & 0 & 0 \\ -b_2 & -b_1 & a_1 & a_0 & 0 \\ a_3 & a_2 & b_2 & b_1 & a_1 \\ -b_4 & -b_3 & a_3 & a_2 & -b_2 \\ a_5 & a_4 & b_4 & b_3 & a_3 \end{vmatrix} =$$

$$
-\begin{vmatrix}
B(D-n)+\frac{K^2AI}{\Delta} & 1 & 0 & 0 & 0 \\
0 & -\frac{n\sigma IAK}{\Delta} & B(D-n)+\frac{K^2AI}{\Delta} & 1 & 0 \\
RTK^2\left[B(D-n)+k^2AI/\Delta\right] & 2RTK^2 & 0 & \frac{n\sigma IAK}{\Delta} & B(D-n)-\frac{K^2AI}{\Delta} \\
0 & -2RTK^3\frac{n\sigma AI}{\Delta} & RTK^2\left[B(D-n)+2K^2AI/\Delta\right] & 2RTK^2 & 0 \\
0 & 0 & 0 & 2RTK^3\frac{n\sigma AI}{\Delta} & RTK^2\left[B(D-n)+2K^2AI/\Delta\right]
\end{vmatrix}
$$

$$
= \left[RTK^2B(D-n)\right]^2\left[\left(B(D-n)+\frac{K^2AI}{\Delta}\right)\left(n\sigma\frac{IAK}{\Delta}\right)^2 + RTK^2\left(B(D-n)+2\frac{K^2AI}{\Delta}\right)\right]
$$

(6.59)

where $\Delta = n^2\sigma^2 + k^2$ (6.60)

Since $D - n > 0$, $B > 0$ etc. it may be seen that a_0, a_1, c_2 c_3 all have the same sign therefore there are no roots with positive real part, which is to say there are no instabilities of the form considered, unless c_3 is negative, that is unless

$$
T < \frac{(AIn\sigma)^2}{R\Delta^2}\,\frac{\left[\Delta B(D-n)+k^2AI\right]}{\left[\Delta B(D-n)+2k^2AI\right]} < \frac{(AIn\sigma)^2}{\Delta^2}
$$

CHAPTER 7

INSTABILITIES INVOLVING VELOCITY DEPENDENT EFFECTS

7.1 The effects of the Doppler shift for line absorption

In Chapter 4 it was mentioned that the Doppler effect may introduce some new features when considering macroscopic motions in a gas. This section investigates the general formulae connected with the effect, section 7.2 deals with an application of line absorptions with the temperature constant, section 7.3 includes variations in temperature and section 7.4 deals with continuum absorptions.

Suppose that species k has many absorption lines which lead to photodissociation of the molecule. Let $\sigma_{ki}(\nu)$ be the cross section for dissociation via absorption in line i. The total contribution to X_K from photodissociation is

$$\dot{X}_{ph,k} = \alpha_K \sum_i \int_0^\infty \sigma_{ki}(\nu)\, I'(\nu)\, d\nu \qquad (7.1)$$

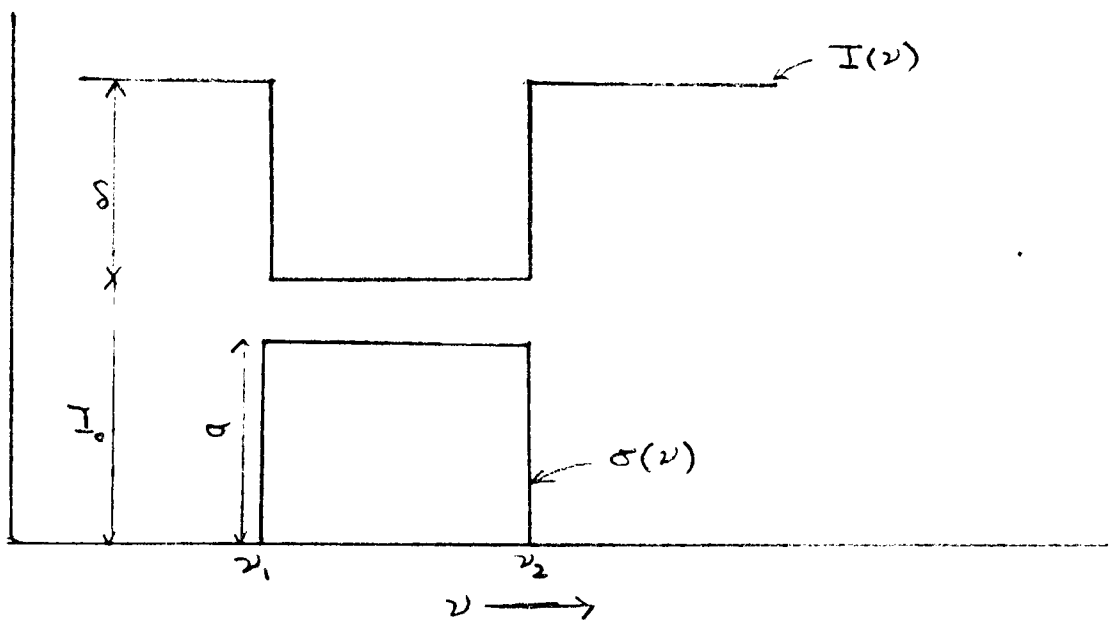
I'(ν) is the photon flux received by the molecule, which will depend on the velocity $\underset{\sim}{v}$ of the gas. If the gas moves with bulk velocity v, much less than c the velocity of light, each photon is seen to be shifted in frequency by a factor $(1 - \frac{v\mu}{c})$ where μ is the cosine of the angle between the photon's and the gas's respective paths. The number of photons incident per second per unit area will be changed by the same factor

Consider the case of a rectangular cross section

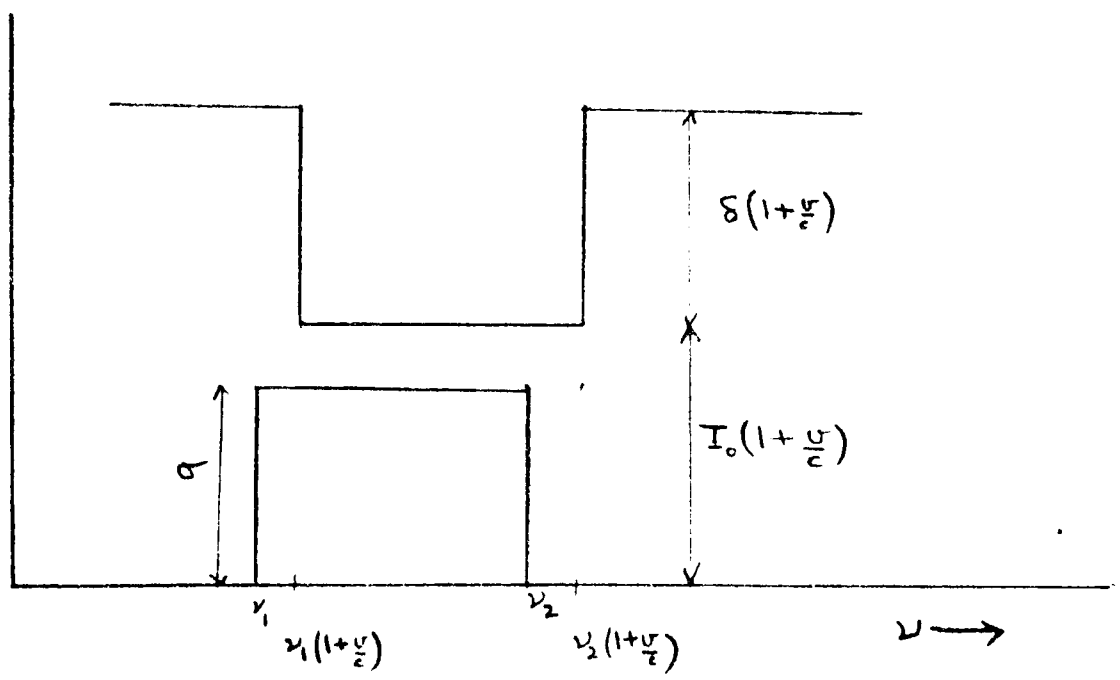
$$\sigma(\nu) = \begin{cases} \sigma & \nu_2 > \nu > \nu_1 \\ 0 & \nu > \nu_2 \,, \quad \nu < \nu_1 \end{cases} \qquad (7.2)$$

where the subscript k has been dropped.

In a uniform initial state it seems reasonable to assume that $I(\nu)$ has been selectively reduced in the region $\nu_2 > \nu > \nu_1$ by molecular absorptions. The situation may be represented as follows



A moving molecule sees the radiation changed in the manner described and this situation may be shown as



in the case where the gas is moving in the opposite direction to the photons, which themselves are all moving in a single direction.

The integral for $\bar{X}_{ph}$ may be seen to be

$$\bar{X}_{ph}(v=0) = \varkappa\sigma I_0(\nu_2-\nu_1) \tag{7.3}$$

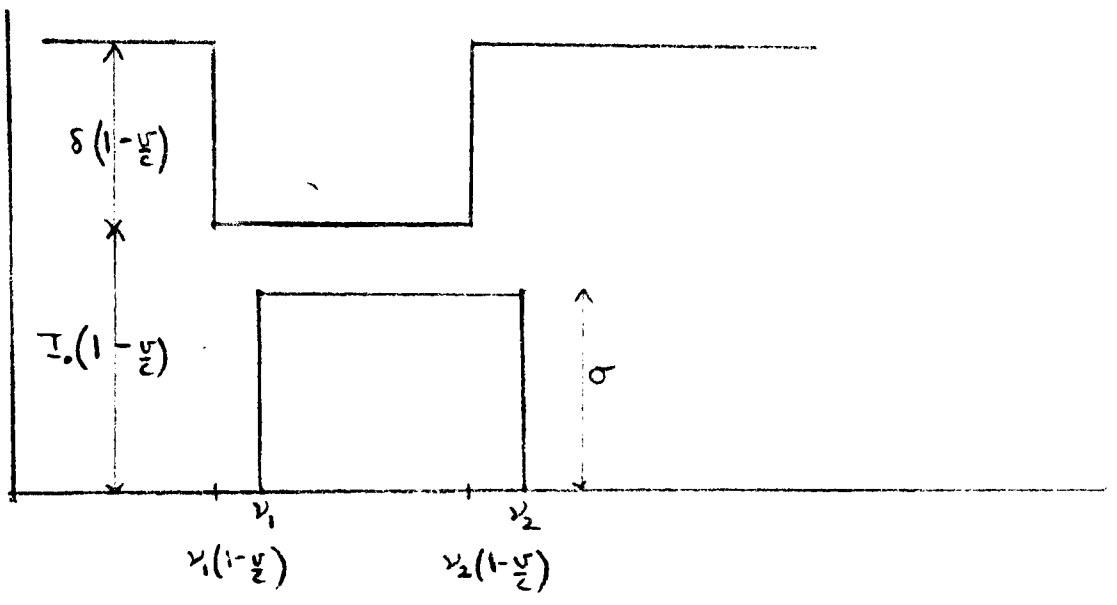
$$\bar{X}_{ph}(v) = \varkappa\sigma I_0\left(1+\tfrac{v}{c}\right)\left[\nu_2-\nu_1\left(1+\tfrac{v}{c}\right)\right] + \sigma(I_0+\delta)\left(1+\tfrac{v}{c}\right)\left[\nu_1\left(1+\tfrac{v}{c}\right)-\nu_1\right] \tag{7.4}$$

$$= \varkappa\sigma I_0(\nu_2-\nu_1)+\tfrac{v}{c}\left[\varkappa\sigma I_0(\nu_2-\nu_1)+\varkappa\sigma\delta\nu_1\right] \tag{7.5}$$

$$= \bar{X}_{ph}(v=0) + \tfrac{v}{c}\left[\bar{X}_{ph}(v=0)+\varkappa\sigma\delta\nu_1\right] \tag{7.6}$$

to first order in (v/c).

When v is reversed the picture becomes



$$\bar{X}_{ph}(v) = \varkappa\sigma\left[\nu_2\left(1-\tfrac{v}{c}\right)-\nu_1\right]I_0\left(1-\tfrac{v}{c}\right) + \varkappa\sigma\left[\nu_2-\nu_2\left(1-\tfrac{v}{c}\right)\right](I_0+\delta)\left(1-\tfrac{v}{c}\right) \tag{7.7}$$

$$= \varkappa\sigma I_0(\nu_2-\nu_1)+\tfrac{v}{c}\left[\varkappa\sigma I_0(\nu_2-\nu_1)+\varkappa\sigma\nu_2\delta\right] \tag{7.8}$$

$$= \bar{X}_{ph}(v=0) + \frac{v}{c}\left[\bar{X}_{ph}(v=0) + x\sigma\nu_2\delta\right] \qquad (7.9)$$

to first order in (v/c).

$\bar{X}_{ph,k}(v)$ is not therefore a simple linear function of v, it is, rather, linear in the mutually exclusive regimes of v in the same and in the opposite direction of the photons, with different slopes in the two regimes. It will be noted that the photons have been assumed to be all moving in the same direction, as may be a reasonable approximation to the situation at the edge of a large gas cloude, where all but those photons incident normally to the near surface have been absorbed in gas.

Since ν_1 is very close to ν_2 for line absorption we can write

$$\bar{X}_{ph}(v) = \bar{X}_{ph}(v=0) + \frac{|v|}{c}\left[\bar{X}_{ph}(v=0) + x\sigma\nu_2\delta\right] \qquad (7.10)$$

When there are many lines, the effects simply add if the lines are not too close together. One can see that this non-linear behaviour is true for a more general line shape, at least when v is sufficiently large, by considering a molecule with such a profile in an undirectional radiation field, which has a deep absorption dip corresponding to the profile. Macroscopic motion in either direction exposes the gas to destructive photons and the number of such photons will be proportional to the magnitude of the frequency shift, which is to say proportional to the magnitude of the velocity.

When we have several rectangular absorption lines with different values for σ an interesting effect occurs, as shown by the case with two such lines

$$\sigma_1(\nu) = \begin{cases} \sigma_1 & \nu_{12} > \nu > \nu_{11} \\ 0 & \nu < \nu_{11}\ ,\ \nu > \nu_{12} \end{cases} \qquad (7.11)$$

$$\sigma_2(\nu) = \begin{cases} \sigma_2 & \nu_{22} > \nu > \nu_{21} \\ 0 & \nu < \nu_{21}\ ,\ \nu > \nu_{22} \end{cases} \qquad (7.12)$$

There will be correspondingly I_1, I_2 and δ_1, δ_2 and bearing in mind the model of a slab of gas illuminated normally on one side we would expect

$$I_1 = J\, e^{-N\sigma_1}, \quad I_2 = J\, e^{-N\sigma_2} \qquad (7.13)$$

N = column density of the molecular species. (7.14)

If $\sigma_1 >> \sigma_2$, there will come a point where $\sigma_2\, I_2(\nu_{22} - \nu_{21}) >> \sigma_1\, I_1(\nu_{12} - \nu_{11})$ and line 2 will dominate the photodissociation rate. However,

$$\delta_1 = J - I_1, \quad \delta_2 = J - I_2 \text{ in the simplest case} \qquad (7.15)$$

and so in the expression for $\bar{X}_{ph}(v)$

$$\bar{X}_{ph}(v) = \propto \left[\begin{array}{l} \sigma_1 \bar{I}_1(\nu_{12}-\nu_{11}) + \sigma_2 \bar{I}_2(\nu_{22}-\nu_{21}) \\ + \frac{|v|}{c} \left\{ \begin{array}{l} \sigma_1 \bar{I}_1(\nu_{12}-\nu_{11}) + \sigma_2 \bar{I}_2(\nu_{22}-\nu_{21}) \\ + \sigma_1 \delta_1 \nu_{11} + \sigma_2 \delta_2 \nu_{22} \end{array} \right\} \end{array} \right] \qquad (7.16)$$

the term x σ_1 δ_1 ν_{11} can completely dominate the coefficient of v, even though line 2 dominates the static photodestruction rate.

A final point to note is that $\sigma(\nu)$ above is that obtained after averaging over thermal velocities of the individual molecules. The non-linearity is dealt with next.

In an analysis of the stability of the system one would like to expand X in first order in the various perturbations and write

$$\beta\delta x_j + X_{j,n}\,\delta n + X_{j,T}\,\delta T + \sum_K X_{j,I_K}\,\delta I_K + \sum_K X_{j,x_K}\,\delta x_K + X_{j,v}\,\delta v = 0 \qquad (7.17)$$

$$X_{j,v} = \left(\frac{\partial}{\partial v} X_j\right)_{n,T,I_K,x_K\ldots} \qquad (7.18)$$

δ will also change as the moving gas absorbs the photons at frequencies around the absorption lines but this will be a second order effect and so is not included in the linearised equation.

The term describing the doppler shift in absorption lines is not linear in the velocity for rectangular profiles because as seen from

$$X_v|\delta v| = \begin{cases} X_v\,\delta v & \text{when } \delta v > 0 \\ -X_v\,\delta v & \text{when } \delta v < 0 \end{cases} \qquad (7.19)$$

the coefficient of δv is not independent of δv. It may therefore seem that a linear stability analysis cannot be made, since one of the terms is essentially non-linear. However, in the case that δv is positive, its coefficient is constant the equations can be linearised and as long as δv remains positive these equations must

describe the motion of the system. The dispersion relation may be found and for a given k, the roots for β may be calculated. For each β the corresponding values of $\delta n/\delta x$, $\frac{\delta T}{\delta x}$ etc. may be calculated. Given any initial values of δn, δT, δx, etc., the linear time development may be followed up to the point where δv changes sign. When δv is negative a different set of equations must be used in that X_v is changed to $-X_v$ from the first set, and then all the previous procedure may be followed. In particular the values of δn, δT, δx, etc. obtained from the previous analysis may be used as initial conditions, and as before these values may be analysed into the modes corresponding to different β's. If there are growing modes in one or both regimes of δv the only way that the system could remain stable is that if, given <u>any</u> initial conditions, the values of δn etc when δv changes sign are such that an analysis of these values into modes of the other regime of δv yields zero contribution from any growing modes of this other regime. This situation corresponds to one in which the decaying modes in one regime cancel the effects of the growing modes in the other.

In the examples considered here, that is, without including the equation of radiative transfer, the complex coefficients in the dispersion relation depend upon X_v so that changing from X_v to $-X_v$ is just the same as taking the complex conjugate of each coefficient. Therefore the roots of the two equations with X_v and $-X_v$ are complex conjugates of one another, so that if there are roots with positive real parts for one case there will be roots with the same positive real parts in the other case and so the difficulty mentioned above will probably not arise.

7.2 Applications with T constant

Case I

The simplest case to investigate which includes the velocity dependent term in the reaction rate is that with constant temperature and radiation field. The linearised equations are, if we use the notation of Chapter 2,

$$\beta\,\delta n + ikn\,\delta v = 0 \tag{7.20}$$

$$nm\beta\,\delta v + ikP\left(\frac{\delta n}{n} + \frac{\delta x}{M}\right) = 0 \tag{7.21}$$

$$X_n\,\delta n + X_v\,\delta v + (\beta + X_x)\,\delta x = 0 \tag{7.22}$$

$$X_x = \left(\frac{\partial X}{\partial x}\right)_{n,v} \text{ etc.}, \qquad M^{-1} = \frac{1}{P}\left(\frac{\partial P}{\partial x}\right)_n \tag{7.23}$$

These equations give a dispersion relation for β in the form

$$\begin{vmatrix} \beta & ikn & 0 \\ \frac{ikP}{n} & nm\beta & \frac{ikP}{M} \\ X_n & X_v & \beta + X_x \end{vmatrix} = 0 \tag{7.24}$$

i.e.
$$\beta^3 + \beta^2 X_x + \beta\,\frac{k^2P}{nm}\left[1 - \frac{i\,X_v}{kM}\right] + \frac{k^2P}{nm}\left[X_x - \frac{n}{M}X_n\right] = 0 \tag{7.25}$$

In order to determine whether any of the roots of this equation has positive real part the methods of Chapter 3 may be used. The sequence of test functions is:

$$c_0 = 1 \tag{7.26}$$

$$c_1 = X_x \tag{7.27}$$

$$c_2 = -\begin{vmatrix} X_x & 1 & 0 \\ \frac{K^2 P X_v}{nmkM} & 0 & X_x \\ \frac{K^2 P}{nm}\left[X_x - \frac{n X_n}{M}\right] & \frac{KP}{nm} & -\frac{K^2 P X_v}{nmkM} \end{vmatrix}$$

$$= \frac{K^2 P}{[nmM]^2}\left[mn^2 M X_x X_n - P X_v^2\right] \tag{7.28}$$

$$c_3 = \left[X_x - \frac{n}{M} X_n\right]\left[n^3 m X_n^2 - P X_v^2\right] \tag{7.29}$$

apart from some essentially positive factors.

There will therefore be an instability in the system if any of the following conditions hold:

(i) $$X_x < 0 \tag{7.30i}$$

(ii) $$\frac{mn^2 M}{P} X_x X_n < X_v^2 \tag{7.30ii}$$

(iii) (a) $$\frac{n^3 m}{p} X_n^2 < X_v^2 \qquad (7.30iii)$$

OR

(b) $$\left[X_x - \frac{n}{M} X_n\right] < 0 \qquad (7.30iv)$$

If both (iii)(a) and (iii)(b) are satisfied simultaneously c_3 is positive but (ii) is automatically satisfied.

The stability of a stationary system of hydrogen atoms and molecules can now be investigated, assuming that the temperature and radiation field can be treated as unperturbed. As noted in Chapter 4 the H_2 molecules are dissociated by line absorptions in interstellar space so that the methods in Section 1 can be used. Later on in this section the effect of the true line shape is investigated and is found to alter X_v, but the change can easily be included by multiplying δ by a correcting factor in many cases. Therefore before going on in detail we stop here and investigate in a general way the possible occurrence of the instability.

Using the notation of Chapter 4

$$X_n = gx$$

$$X_x = ng + \sigma I_0 \Delta$$

$$|X_v| = \frac{\sigma}{c} \nu_1 \delta \frac{(1-x)}{2} \qquad \text{if } \frac{\nu_1}{\Delta}\frac{\delta}{I_0} \gg 1, \quad \Delta = \nu_2 - \nu_1$$

$$\text{and} \qquad \frac{1-x}{2} = \frac{ng\,x}{2\sigma I_0 \Delta}$$

$$\therefore \quad |X_v| = \frac{1}{c}\left(\frac{\nu_1}{\Delta}\right)\left(\frac{\delta}{I_0}\right)\frac{ngx}{2} \approx \frac{ng}{2c}\left(\frac{\nu}{\Delta}\right)\left(\frac{\delta}{I_0}\right)$$

since x is of order 1 in most circumstances. Condition (i) then may be written

(i) $$n g + \sigma I_0 \Delta < 0$$

which is never true.

(ii) requires

$$\frac{2m}{k_B T}(n g + \sigma I_0 \Delta) < \frac{n g x}{(2c)^2}\left(\frac{\nu}{\Delta}\right)^2\left(\frac{\delta}{I_0}\right)^2$$

ie $$\frac{mc^2}{k_B T} < \frac{n g x}{8(n g + \sigma I_0 \Delta)}\left(\frac{\nu}{\Delta}\right)^2\left(\frac{\delta}{I_0}\right)^2$$

and if $ng \approx \sigma I$, that is the fraction of hydrogen in molecular form is near unity we arrive at

$$\frac{mc^2}{k_B T} < \frac{1}{8}\left(\frac{\nu}{\Delta}\right)^2\left(\frac{\delta}{I_0}\right)^2$$

or

$$8\left(\frac{mc^2}{k_B T}\right)\left(\frac{\Delta}{\nu_1}\right)^2 < \left(\frac{\delta}{I_0}\right) \approx e^{2\tau}$$

where τ is the optical depth from the surface.

If the lines are thermally broadened $\left(\frac{\Delta}{\nu}\right) \approx b\sqrt{\left(\frac{k_B T}{mc^2}\right)}$

$\therefore$ we need

$8b^2 < e^{2\tau}$ where b is a constant depending upon the averaging over the lines.

i.e. $3b < e^{\tau}$

In fact the assumption that $(1 - x) \approx 1$ is a stronger restraint on τ because in unshielded space $\sigma I_o \Delta >> ng$. Assuming now that $ng << \sigma I_o \Delta$ we arrive at

$$\frac{mc^2}{k_B T} < \frac{nqx}{8\sigma I_o \Delta}\left(\frac{\nu_\perp}{\Delta}\right)^2\left(\frac{\delta}{I_o}\right)^2$$

i.e.

$$\frac{8\sigma I_o \Delta}{ng} b^2 < e^{2\tau} \quad ..$$

$$I_o = J e^{-\tau}$$

so

$$\frac{8\sigma \Delta J}{ng} b^2 < e^{3\tau}$$

Using $g = 10^{-17}$, $\sigma\Delta J = 310^{-11}$ we require

$$2.4 10^7 \frac{b^2}{n} < e^{3\tau}$$

i.e. approximately

$$300\left(\frac{b^2}{n}\right)^{1/3} < e^{\tau}$$

Now $\tau = \sigma \int_o^L n\frac{(1-x)}{2}dz$ for a uniform gas of density n where L is the path length to the surface

so

$$\frac{5.7 + \frac{1}{3}\ln\left(b^2/n\right)}{\sigma} < \int_o^L n\frac{(1-x)}{2}\,dz$$

$$\frac{\left[5.7 + \frac{1}{3}\ln\left(b^2/n\right)\right] 10^{19}\,cm^{-2}}{\left(\frac{\sigma}{10^{-19}\,cm^2}\right)} < \int_o^L \frac{n}{2}(1-x)\,dz =$$

column density of H_2 to the surface.

Condition (iii) reads

a)

$$X_x - \frac{n X_n}{1+x} = \Delta\sigma I_0 + ng - \frac{ngx}{1+x} = \sigma I_0 \Delta + \frac{ng}{1+x} < 0$$

which is never true

or b)

$$\frac{2m}{(1+x)k_B T} < \left[\frac{1}{2c}\left(\frac{\nu_1}{\Delta}\right)\left(\frac{\delta}{I_0}\right)\right]^2$$

i.e.

$$\frac{8mc^2}{k_B T} < \left(\frac{\nu_1}{\Delta}\right)^2\left(\frac{\delta}{I_0}\right)^2$$

i.e.

$$8b^2 < e^{2\tau}$$

approx

$$3b < e^{\tau}$$

i.e.

$$\frac{\ln(3b)}{\sigma} < \int_0^L \frac{(1-x)}{2} n \, dz$$

These considerations show that one would expect that these doppler instablities may occur at column densities of H_2 of order $\frac{10^{15}}{(\sigma/10-15)}$ cm^{-2} into a cloud. In the next section we calculate the timescales of the instability.

It remains to find the timescales of the instabilities. If

$$|X_x| > \left|\frac{n}{M} X_n\right| \qquad \text{and} \qquad (7.31)$$

$$1 > \left|\frac{X_v}{kM}\right| \qquad (7.32)$$

the dispersion relation may be written as

$$(\beta + X_x)\left(\beta^2 + \frac{k^2 P}{nm}\right) = 0 \qquad (7.33)$$

to a good approximation.

The roots are

$$\beta_1 = -X_x \quad , \quad \beta_{2,3} = \pm i \left(\frac{k^2 P}{nm}\right)^{1/2} \qquad (7.34)$$

To find the roots of the full equation write the change in β_i to be 'A_i', which will be taken to be negligible in second and higher order. The equation for 'A' is

$$3A_i\beta_i^2 + 2A_i\beta_i X_x - i\beta_i \frac{k^2 P}{nm}\frac{X_v}{kM} + A_i \frac{k^2 P}{nm}\left[1 - i\frac{X_v}{kM}\right] - \frac{k^2 P}{nm}\frac{n}{M}X_n = 0 \qquad (7.35)$$

$$A_i = \frac{\left[i\beta_i \frac{X_v}{kM} + \frac{n}{M}X_n\right]\frac{k^2 P}{nm}}{3\beta_i^2 + 2\beta_i X_x - i\frac{k^2 P}{nm}\frac{X_v}{kM} + \frac{k^2 P}{nm}} \qquad (7.36)$$

$$A_1 = \left[-i\frac{X_x X_v}{kM} + \frac{n}{M}X_n\right]\frac{k^2 P}{nm}\left[X_x^2 - i\frac{k^2 P}{nm}\frac{X_v}{kM} + \frac{k^2 P}{nm}\right]^{-1} \qquad (7.37)$$

if X_x is large and positive the real part of $\beta_i + A_i$ will be negative.

$$A_{2,3} = \frac{\left[\mp \frac{X_v}{kM} \left(\frac{k^2 P}{nm} \right)^{1/2} + \frac{n}{M} X_n \right] \frac{k^2 P}{nm}}{\left[-3 \frac{k^2 P}{nm} \pm 2i X_x \left(\frac{k^2 P}{nm} \right)^{1/2} - i \frac{k^2 P}{nm} + \frac{k^2 P}{nm} \right]}$$

$$= \frac{\left[\mp \frac{X_v}{kM} \left(\frac{k^2 P}{nm} \right)^{1/2} + \frac{n}{M} X_n \right]}{\left[-2 \pm 2i X_x \left(\frac{k^2 P}{nm} \right)^{-1/2} - i \right]} \tag{7.38}$$

$$Re(A_{2,3}) = \frac{\left[\pm \frac{2 X_v}{kM} \left(\frac{k^2 P}{nm} \right)^{1/2} - 2 \frac{n}{M} X_n \right]}{\left[4 + \left(\pm 2 X_x \left(\frac{k^2 P}{nm} \right)^{-1/2} + 1 \right)^2 \right]} \tag{7.39}$$

which shows that if

$$X_n^2 \frac{n^3 m}{P} < X_v^2$$

there will be a root with positive real part; in agreement with a previous result. When

$$X_x \left(\frac{k^2 P}{nm} \right)^{-1/2} \gg 1 \tag{7.40}$$

$$Re(A_{2,3}) = \pm \frac{X_v}{X_x^2 kM} \left(\frac{k^2 P}{nm} \right)^{3/2} - \frac{n}{M} X_n \left(\frac{k^2 P}{nm} \right) \tag{7.41}$$

When

$$X_x \left(\frac{k^2 P}{nm} \right)^{-1/2} \ll 1 \tag{7.42}$$

$$Re(A_{2,3}) = \pm \frac{2 X_v}{5 kM} \left(\frac{k^2 P}{nm} \right)^{1/2} - \frac{2}{5} \frac{n}{M} X_n \tag{7.43}$$

Using the previously suggested values for n, I, g, σ and s,

if $k \ll 10^{-15}$ cm^{-1}

and if $X_n^2 \frac{n^3 m}{p} < X_v^2$

$$\left|Re(A_{2,3})\right| = 5\,|X_v|\,k^2 10^{34} \ll 5 10^4 |X_v| \qquad (7.44)$$

while if $k \ll 10^{-15}$ cm^{-1}

$$\left|Re(A_{2,3})\right| = 2 10^4 |X_v| \qquad (7.45)$$

$$|X_v| = \sigma n_1 \delta \frac{(1-x)}{2c} = \frac{\sigma}{c} J\Delta \left(\frac{n}{\Delta}\right)\left(\frac{\delta}{J}\right)\frac{(1-x)}{2} \qquad (7.46)$$

if $\frac{n}{\Delta}\frac{\delta}{I_o} \gg 1$

$J = 10^{-8}$ ph cm^{-2} s^{-1} str^{-1} Hz^{-1} (Hollenbach et al. 1971).

If the width of a single line is $\Delta_1 = 10^{10}\, a$ Hz ($a = \frac{v_{thermal}}{1 kms^{-1}}$) and $\ell\sigma J\Delta = 3 10^{-11}$, it follows that $\sigma\ell = \frac{3 10^{-13}}{a}$ cm^2 and therefore, summing the effects of ℓ lines

$$|X_v| = \frac{\sigma \ell n \delta (1-x)}{2c} = J\sigma \frac{\ell n}{c}\left(\frac{\delta}{J}\right)\frac{(1-x)}{2}$$

$$= \left(\frac{3}{a}\right)\left(\frac{\delta}{J}\right)\frac{(1-x)}{2} 10^{-16} \text{ cm}^{-1} \qquad (7.47)$$

and the fastest timescale τ associated with these instabilities will be, if $a \approx 1$,

$$\tau = \frac{10^{-4}}{2|X_v|} \text{ sec} \approx \frac{10^4}{(\delta/J)(1-x)} \text{ years} \qquad (7.48)$$

If $(\frac{\delta}{J})\ (1-x) \approx \frac{1}{50}$ the timescale will be 510^5 years while if $(\frac{\delta}{J})\ (1-x) \approx \frac{1}{10}$ it is 10^5 years. For lengthscales $k << k_c = X_x \sqrt{\frac{nm}{P}}$ the timescales are $\tau(\frac{k_c}{k})^2$. The auxiliary condition $X_v^2 > \frac{n^3 m}{P} X_n^2$ implies that

$$\tau < \frac{2.5}{n} 10^{22} \text{ sec.} \qquad (4.49)$$

In the regime $k > k_c$, τ varies as $\sqrt{n}$ if P is constant, and if $k < k_c$, $\tau \propto n^{3/2}$. (7.47) and (7.48) assume a value of $\sigma J \Delta$ of 310^{-11} s^{-1}, the average interstellar value. It seems reasonable to suppose that clouds nearer to hot stars than the average will experience a greater radiation field and τ will vary inversely as the flux density, because it is argued below that $\delta(1-x)$ can be of order J. For example Spitzer (1976) has suggested a flux ten times the one used here in order to explain observations of two particular clouds. The timescale for these instabilities will then be 10^5 years.

The quantity $\delta(1-x)$ will change as one proceeds into the slab of gas. At each point the following equations determine (1-x) and the changes in this and δ with distance into the gas.

$$\delta = I_1 - I_o \qquad (7.50)$$

$$ngx = \sigma\Delta(I_o)\ (1-x) = \sigma\Delta(I_1 - \delta)\ (1-x)$$

$$\therefore\ \delta(1-x) = \frac{ngx\ \delta}{\sigma\Delta(I_1 - \delta)} \qquad (7.51)$$

If the initial spectrum at the surface was constant and given by J then

$$I_1(z) = J\, e^{-nsz} \tag{7.52}$$

$$\delta = Je^{-nsz}\left[1 - \exp\left\{-\int_0^z n\sigma\left(\frac{1-x}{2}\right)dz\right\}\right] \tag{7.53}$$

where ns is the opacity per unit volume due to the dust, and the number density is taken to be constant.

$$\therefore\ \delta(1-x) = \frac{ngx}{\sigma\Delta}\,\frac{Je^{-nsz}}{Je^{-nsz}}\,\frac{\left[1 - e^{-\int_0^z \frac{n\sigma}{2}(1-x)\,dz}\right]}{e^{-\int_0^z \frac{n\sigma}{2}(1-x)dz}}$$

$$= \frac{ngx}{\sigma\Delta}\left[e^{\int_0^z n\sigma\left(\frac{1-x}{2}\right)dz} - 1\right] \tag{7.54}$$

In the steady state

$$ngx = \sigma\Delta I_o(1 - x)$$

$$\therefore\ n\sigma\left(\frac{1-x}{2}\right) = \frac{n^2gx}{2\Delta I_o}$$

The equation of radiative transfer may be written

$$\frac{dI_o}{dz} + nsI_o = -n\sigma\left(\frac{1-x}{2}\right)I_o \tag{7.55}$$

$$= -\frac{n^2gx}{2\Delta}$$

$$\therefore \quad I_o = -\frac{n^2 g x}{2\Delta n s} + \left(J + \frac{n^2 g x}{2\Delta n s}\right) e^{-nsz} \qquad (7.56)$$

if n is constant.

$$\frac{n^2 g x}{2\Delta I_o} = \frac{\frac{n^2 g x}{2\Delta}}{e^{-nsz}\left(J + \frac{n^2 g x}{2\Delta n s}\right) - \frac{n^2 g x}{2\Delta n s}}$$

$$= \frac{n s e^{nsz}}{\left[\frac{2 n s \Delta J}{n^2 g x} + 1 - e^{nsz}\right]} \qquad (7.57)$$

$$\therefore \int_0^z n\sigma\frac{(1-x)}{2} dz = \int_0^z \frac{n^2 g x}{2\Delta I_o}$$

$$= \ln\left[\frac{2\Delta J s / n g x}{\frac{2\Delta J s}{n g x} + 1 - e^{nsz}}\right] \qquad (7.58)$$

This last integral is approximate in that x has been taken as a constant. This is a good approximation if x is very close to unity. Small changes much less than one will leave x still close to unity but can alter (1-x) by a large factor. For $x \simeq 1$ one requires that

$$\frac{ng}{\Delta I_o \sigma} \ll 1 \qquad (7.59)$$

$$\delta(1-x) = \frac{n g x}{\sigma\Delta}\left[\frac{2\Delta J s}{n g x\left\{\frac{2\Delta J s}{n g x} + 1 - e^{nsz}\right\}} - 1\right] \qquad (7.60)$$

It can be seen that

$$1-x = \frac{ngx}{\sigma J \Delta} e^{nsz} e^{\int_0^z n\sigma\frac{(1-x)}{2}dz}$$

$$= \left(\frac{2s}{\sigma}\right) \frac{e^{nsz}}{\left[\frac{2Js\Delta}{ngx} + 1 - e^{nsz}\right]} \qquad (7.61)$$

and so (1 - x) will become an appreciable fraction of 1 when

$$e^{nsz} \simeq 1 + \frac{2Js\Delta}{ng} \qquad (7.62)$$

with the values above and $\delta(1-x)$ can be, independent of σ, about J at $z \simeq \frac{1}{ns} \ln 2$. The formulae above fail as this region is approached but they are sufficient to show that the radiation flux outside the line is only attenuated by a factor of $e^{-nsz} = \frac{1}{2}$ when within the line the flux is reduced to about $(\frac{ng}{\Delta})$, that is, $\sigma I \Delta$ is or order ng. In this region $(\frac{\delta}{J})$ (1-x) can be about $\frac{1}{10}$ or more.

Glassgold and Langer (1973c) have calculated the values of n, T, x, etc. as functions of column density of hydrogen from the surface of a stationary cloud, n is about 10, T about 50 - 100 k. At a column density of 10^{21} cm^{-2} the fraction of hydrogen in molecular form is $\frac{1}{10}$, so $(1-x) \simeq \frac{1}{10}$ and the radiation field outside the line will have been reduced by a factor e^{-2} since they take a grain opacity 210^{-21} cm^2 per H atom. $\frac{\delta}{J}$ (1-x) will be about $\frac{1}{70}$ and the instabilities grow on a timescale 510^5 years.

In summary, bearing in mind the approximations made about uniformity, unperturbed temperature and the form of X_v, this section has shown that a stationary cloud of hydrogen atoms and molecules is subject to an instability which grows on a timescale of about 10^5 years. The instability is driven by doppler shifts in the absorption lines when there are bulk motions of the gas, and, when exposed to the radiation field outside the self shielding line, molecules are dissociated producing a pressure imbalance which in turn leads to bulk motions of the gas. This work implies that model calculations of H_2 in interstellar clouds, for example, by Glassgold and Langer, and Hollenbach et al., are unstable on timescales of interest and so there should perhaps be a revision of ideas about the evolution of interstellar clouds. Hollenbach et al. do discuss the effect of turbulence in broadening the self sheilding lines but do not include the dynamical effects and so they conclude, incorrectly in the light of the above analysis, that its effect is not great. The globules of neutral hydrogen atoms and molecules within H II regions described by Stasinska (1973) may also be subject to this instability.

A final point to note is that as the instability grows one may expect the line width to increase and σ to decrease proportionately. Inspection of eq. (7.62) shows that as Δ increases so does e^{nsz}, for the region in which (1-x) tends to unity. δ will fall approximately as e^{-nsz} so that the maximum value of $\delta(1-x)$ may decrease with increasing Δ, although of course these equations are approximate; a full analysis may show a larger, certainly not smaller, value for the maximum of $\delta(1-x)$. Equation (7.48) shows then that the most rapid timescale may be increased in proportion to $(1 + \frac{2Js\Delta}{ngx})$, not more rapidly

It follows that a large increase in Δ, a factor 10 or more in order to make τ greater than 310^6 years, is allowable before τ becomes much longer than the other timescales of interest. This corresponds to a large increase in the motion of the gas.

In the previous part of this section the frequency dependence of $\sigma(\nu)$ was assumed to consist of a set of rectangles with equal sizes. This is unrealistic. A major fact that must be taken into account is that $\sigma(\nu)$ for each line is more smoothly varying than the rectangle we have assumed, so that there is a more gradual exposure to the unshielded radiation than previously found, tending to reduce X_ν.

For a single line with frequency dependence $\sigma(\nu)$ which is everywhere a smoothly varying function of ν the expression for $\bar{X}_{ph}(v)$ is proportional to

$$\left(1+\frac{v}{c}\right)^2\int_0^\infty k\,\sigma(\nu)\,\bar{I}\left(\frac{\nu}{1+\frac{v}{c}}\right)d\nu \;\approx\; \left(1+2\frac{v}{c}\right)\int_0^\infty k\,\sigma(\nu)\,\bar{I}\left(\nu\left(1-\frac{v}{c}\right)\right)d\nu$$

$$= \left(1+2\frac{v}{c}\right)\int_0^\infty k\,\sigma(\nu)\left[\bar{I}(\nu) - \frac{v}{c}\,\nu\,\frac{\partial \bar{I}}{\partial \nu}\right]d\nu \qquad (7.63)$$

$$= \int_0^\infty k\,\sigma(\nu)\bar{I}(\nu)\,d\nu \;+\; \frac{v}{c}\left\{2\int_0^\infty k\,\sigma(\nu)\bar{I}(\nu)\,d\nu - \int_0^\infty k\nu\sigma(\nu)\frac{\partial \bar{I}}{\partial \nu}\,d\nu\right\} \qquad (7.64)$$

to first order in $(\frac{v}{c})$. k is the probability that absorption leads to dissociation. $\bar{I}(\nu)$ will be assumed to be given by

$$\bar{I}(\nu) = \bar{J}'\exp\left\{-N\sigma(\nu)\right\} \qquad (7.65)$$

$J' = e^{-nsL}J$, the radiation field after dust attenuation which is the spectrum expected at a column density N of molecules from the surface of a slab of gas illuminated on one side by a flux J.

If $\sigma(\nu)$ is symmetrical about a frequency ν_o then

$$\int_{-\infty}^{\infty} \sigma(\nu) \frac{\partial I_\nu}{\partial \nu} d\nu = 0 \qquad (7.66)$$

because of cancellation on either side of ν_o. The lower limit of the integral has been replaced by $-\infty$ to make the formulae symmetrical; this does not affect the value of the integral if $\sigma(\nu) \to 0$ as $\nu \to 0$. If on the other hand $I(\nu)$ is not so symmetrical there is not this cancellation and so the second integral may be replaced by, as a rough approximation

$$\int_0^{\infty} \nu\, \sigma(\nu) \frac{\partial I_\nu}{\partial \nu} d\nu \approx \nu_o \int_{\nu_o}^{\infty} \sigma(\nu) \frac{\partial I_\nu}{\partial \nu} d\nu$$

$$= -NJ'\nu_o \int_{\nu_o}^{\infty} \sigma(\nu) \frac{\partial \sigma}{\partial \nu} e^{-N\sigma(\nu)} d\nu \qquad (7.67)$$

$$= -NJ\nu_o \int_{\sigma_o}^{0} \sigma\, e^{-N\sigma} d\sigma = \frac{J'\nu_o}{N}\left\{1 - e^{-N\sigma_o}(1+N\sigma_o)\right\}$$

where $\sigma_o = \sigma(\nu_o)$

$I(\nu)$ is assumed slightly smaller on the side $\nu_o > \nu$ and $N\sigma(\nu_o)$ is assumed to be large so that the major contribution to the integral comes from around $\nu = \nu_o + \Delta$ where Δ is the line width, $\sigma \to 0$ fairly rapidly for $|\nu - \nu_o| >$ few x Δ.

For $e^{-N\sigma_o}$ to be about 10^{-6}, $N\sigma_o$ is about 14.

$$\int_0^\infty \nu\,\sigma(\nu)\,\frac{\partial I_C}{\partial \nu}\,d\nu \approx \frac{J'_{\nu_0}}{N} \approx \frac{J'_{\nu_0}}{\mu}\,\nu_0 \qquad (7.68)$$

so that X_ν is reduced by about an order of magnitude from that given by a rectangular $\sigma(\nu)$. The amount of asymmetry required is very small, for example if $I(\nu_o + \mu)$ is larger than $I(\nu_o - \mu)$ by a factor $(1 + \eta)$ then on either side of ν_o the contributions to the integrand from $\nu_o + \mu$ and $\nu_o - \mu$ will differ in magnitude by a factor $(1 + \eta)$. The main contribution to the integral is from $\nu_o \pm$ few x Δ so that if the dissymmetry is $(1 + \eta)$, an extra factor η appears in front of $J\nu_o/N$. Such dissymmetry may arise from shielding by another cloud of similar velocity or else by bulk subsonic motions within the cloud itself. It seems strange that the integral is independent of σ_o, the explanation is that the main contribution to that integral comes from small values of the variable σ. The cross-sections with large σ_o eventually come down to these small values of σ, those which have small σ_o reach these values of σ more quickly. The value of the integral is seen to be independent of the shape of $\sigma(\nu)$, as long as $\sigma_o N \gtrsim 1$

When we have a very symmetric line the following analysis may be made:

$$\int_0^\infty \sigma\,\frac{\partial I_C}{\partial \nu}\,d\nu = 0 \qquad (7.69)$$

$$\therefore \int_0^\infty \nu\,\sigma\,\frac{\partial I_C}{\partial \nu}\,d\nu = \int (\nu - \nu_o)\,\sigma\,\frac{\partial I_C}{\partial \nu}\,d\nu$$

$$= \left[(\nu-\nu_o)\sigma I\right]_0^\infty - \int_0^\infty \sigma I\,d\nu - \int_0^\infty (\nu-\nu_o)\,I\,\frac{\partial \sigma}{\partial \nu}\,d\nu \qquad (7.70)$$

The last integral is, in the case of a Maxwellian distribution,

$$\sigma(\nu) = \sigma_o \exp\left\{-\left(\frac{\nu-\nu_o}{\Delta}\right)^2\right\},$$

$$-J'\int_{-\infty}^{\infty} 2\left(\frac{\nu-\nu_o}{\Delta}\right)^2 N\sigma_o^2 e^{-2\left(\frac{\nu-\nu_o}{\Delta}\right)^2} \exp\left\{-N\sigma_o e^{-\left(\frac{\nu-\nu_o}{\Delta}\right)^2}\right\} d\nu \quad ,$$

setting $z = \frac{\nu-\nu_o}{\Delta}$ this integral becomes

$$-2J'N\sigma_o^2\Delta\int_{-\infty}^{\infty} z^2 e^{-z^2} e^{-N\sigma_o e^{-z^2}} dz \qquad (7.71)$$

The factor $e^{-N\sigma_o e^{-z^2}}$ in the integrand tends rapidly to unity so that the integral will be of order

$$\int_{-\infty}^{\infty} z^2 e^{-z^2} dz = \frac{\pi}{2}$$

therefore the whole expression is of order

$$-2J'N\sigma_o^2\Delta = -2(\sigma_o N)(\sigma_o J'\Delta)$$

Because Δ will in general be much less than ν_o the integral in the symmetrical case will be much less than that in the non-symmetrical case.

Case A

$$\chi^-_{ph,\sigma} = \frac{x}{c} k\left\{2\int_o^{\infty}\sigma(\nu)I(\nu)\,d\nu - J'\sigma_o\nu_o\frac{1}{\sigma_o N}\eta\right\} \qquad (7.72)$$

for the non-symmetrical line

Case B

$$\bar{X}_{ph,\nu} \approx \frac{\alpha}{c} K \left\{ 2\int_0^\infty \sigma(\nu) I(\nu) d\nu - \left[[\nu - \nu_0] \sigma(\nu) I(\nu) \right]_0^\infty + \int_0^\infty \sigma(\nu) I(\nu) d\nu \right\}$$

$$+ \quad 2(\sigma_0 N)(J' \sigma_0 \Delta)$$

$$= \frac{\alpha}{c} K \left\{ 2\int_0^\infty \sigma(\nu) I(\nu) d\nu + 2(\sigma_0 N)(J' \sigma_0 \Delta) \right\} \quad (7.73)$$

for the symmetrical line.

If $\sigma_o N$ is greater than unity the integral term in each expression will usually be much less than the other term, that is $\int \sigma(\nu)\, I(\nu)\, d\nu$ must be much less than $\Delta \sigma_o J'$ because of the fact that $I(\nu)$ has a deep minimum where $\nu = \nu_o$, and will hence also be less than $J' \sigma_o \nu_o$ by several orders of magnitude.

The other point mentioned above was that different lines have different values of σ_o. We assume here that the effect of the individual lines may be simply summed, although this may not be true for lines whose central frequencies are close together compared to the line width. Examination of the expression for $\bar{X}_{ph,\nu}$ for a single line in case A shows that the dominant term is independent of σ_o - as long as this is not too small. $\bar{X}_{ph,\nu}$ in this case is

$$\bar{X}_{ph,\nu} = \frac{\alpha_{H_2}}{cN} \eta \sum_{\substack{\text{all lines} \\ \text{leading to} \\ \text{dissociation}}} K J' \nu_o = \frac{\alpha_{H_2}}{cN} \eta J' \nu_o \sum K$$

if J' and ν_o are about the same for all the lines.

$$\approx 10\, x_{H_2}\, \eta \cdot \frac{J'_{2}}{<N} \qquad (7.74)$$

for H_2, using the tabulated values of k, given by Hollenbach et al. (1971) and dividing by 2 to approximately take into account the fact that different lines occur for ortho and para H_2. This sum is for the Lyman-band lines only, so that the sum should perhaps by increased by a factor 2 or 3 to take into account the Werner band lines.

For case B

$$\bar{X}_{ph,\upsilon} = x_{H_2} \frac{2N}{c} \sum_{\text{all lines}} J' k \sigma_o^2 \Delta$$

$$= x_{H_2} \frac{2N}{c} J' \sum k \sigma_o^2 \Delta \qquad (7.75)$$

if J' is constant for all lines.

The values of σ_o are small so that only that line with the largest value of σ_o will effectively contribute, unless Δ varies dramatically from one line to another. Taking this σ_o to be 10^{-15} cm^2, Δ to be 10^{10} Hz, k to be 1/3

$$\bar{X}_{ph,\upsilon} = \frac{1}{3} \frac{2}{c} N x_{H_2} J' 10^{-30} 10^{10} \approx x_{H_2} \frac{N}{c} J' 10^{-20} \qquad (7.76)$$

$$\frac{\bar{X}_{ph,\upsilon}\ (\text{case A})}{\bar{X}_{ph,\upsilon}\ (\text{case B})} \approx 3 \frac{10^{36}}{N} \eta \qquad (7.77)$$

N it will be remembered is the column density of H_2 to the surface of the cloud, η is a measure of the assymmetry of the absorption line at the point considered, and α_{H_2} is the ratio of the number density of H_2 to the total number density of hydrogen.

7.3 Applications with T perturbed

The next step is to include the energy balance equation in the linearised set. A term L_v appears which reflects the fact that, when the absorption lines are doppler shifted out of the dip in the radiation spectrum, more molecules are dissociated and there may be a contribution to the heating of the gas. L_v will be proportional to X_v and to the energy given to the gas per dissociation. The linearised set to be considered are:

$$\begin{aligned}
&\beta\,\delta n + ikn\,\delta v = 0 \\
&nm\beta\,\delta v + ikP\left\{\frac{\delta n}{n} + \frac{\delta T}{T} + \frac{\delta x}{M}\right\} = 0 \\
&X_n\,\delta n + X_T\,\delta T + X_v\,\delta v + (X_x+\beta)\,\delta x = 0 \qquad (7.78)\\
&\left(L_n - \frac{P\beta}{mn^2}\right)\delta n + (L_T + AR\beta)\,\delta T + L_v\,\delta v + (L_x + BRT\beta)\,\delta x = 0
\end{aligned}$$

The corresponding dispersion relation is

$$\begin{aligned}
&ARnm\beta^4 + \beta^3 nm\left[ARX_x + L_T - BRTX_T\right] \\
&+ \beta^2\left[nm(L_TX_x - L_xX_T) + k^2P\left(AR + \frac{P}{nmT}\right) + ikP\left\{R\left(B - \frac{A}{M}\right)X_v - \frac{L_v}{T}\right\}\right] \\
&+ \beta\left[\begin{array}{l} k^2P\left\{X_x\left(AR + \frac{P}{nmT}\right) - X_T\left(BRT + \frac{P}{nmM}\right) + nRX_n\left(B - \frac{A}{M}\right) + L_T - \frac{n}{T}L_n\right\} \\ + ikP\left\{\left(\frac{L_x}{T} - \frac{L_T}{M}\right)X_v - L_v\left(\frac{X_x}{T} - \frac{X_T}{M}\right)\right\}\end{array}\right] \qquad (7.79)\\
&+ k^2P\left[(L_TX_x - L_xX_T) - \frac{n}{T}(L_nX_x - L_xX_n) + \frac{n}{M}(L_nX_T - L_TX_n)\right] = 0
\end{aligned}$$

$$c_0 = a_0 = \cdot ARnm \tag{7.80}$$

$$c_1 = a_1 = nm\left[ARX_x + L_T - BRTX_T\right] \tag{7.81}$$

$$c_2 = -\begin{vmatrix} nm\left(ARX_x+L_T-BRTX_T\right) & nmAR & 0 \\ -KP\left[\left(B-\frac{A}{M}\right)RX_v - \frac{L_v}{T}\right] & 0 & nm\left(ARX_x+L_T-BRTX_T\right) \\ K^2P\left\{\begin{matrix} X_x\left(AR+\frac{P}{nmT}\right)-X_T\left(BRT+\frac{P}{nmM}\right) \\ +nRX_n\left(B-\frac{A}{M}\right)+\left(L_T-\frac{n}{T}L_n\right)\end{matrix}\right\} & \left\{nm\left(L_TX_x-L_xX_T\right)+K^2P\left(AR+\frac{P}{nmT}\right)\right\} & KP\left[\begin{matrix} R\left(B-\frac{A}{M}\right)X_v \\ -\frac{L_v}{T}\end{matrix}\right] \end{vmatrix}$$

$$= n^3m^3\left(ARX_x+L_T-BRTX_T\right)^2\left(L_TX_x-L_xX_T\right)$$

$$-nmK^2\left[\begin{matrix} nmP\left(ARX_x+L_T-BRTX_T\right)\left\{\begin{matrix}\left(B-\frac{A}{M}\right)R\left(\frac{PX_T}{nm}+nARX_n\right) \\ -\left(\frac{P}{nmT}L_T+\frac{ABn}{T}L_T\right)\end{matrix}\right\} \\ +P^2AR\left[\left(B-\frac{A}{M}\right)RX_v-\frac{L_v}{T}\right]^2 \end{matrix}\right] \tag{7.82}$$

c_3 and c_4 are very complicated expressions which will not be given. The form of c_2 shows that if $(L_TX_x - L_xX_T)$ is negative then c_2 must certainly become negative as k tends to zero, and if (X_v) is sufficiently large to make the term in large brackets positive c_2 must become negative as k tends to infinity. The X_v term again therefore tends to destabalise the system.

Tables 7 to 10 give the results, tabulated as before; of the effect of including various values of X_v in the equations without radiative transfer. The values for X_v taken are the maximum value derived above, namely 510^{-17} (1-x) $(\frac{\delta}{J})$, also this value reduced by a factor $\frac{1}{100}$ and finally zero. The models used are numbers 1 and 2 for the interstellar gas and also the model of pure hydrogen.

It will be noted that there is not very much difference between the results for models 1 and 2. The high value of X_v produces many more instabilities than do the low values. In fact for low X_v the timescales are longer than the free-fall timescale. Those for the high X_v are rather short but fail to satisfy the condition that k should be much greater than $n\sigma(1-x)$, although a few lie on the border-line of $k \simeq n\sigma(1-x)$. We conclude that, although we have not shown the usefulness of these instabilities conclusively, they may be significant in the evolution of clouds of interstellar and pure hydrogen gas, in cases in which the absorption lines corresponding to the frequencies of photodissociation are asymmetrical. The timescales of growth tend to decrease as the hydrogen becomes more self-shielded, but are about 310^4 years on lengthscales of 310^{-4} to 310^{-2} parsecs. The masses of spheres of these radii are small fractions of the solar mass.

The corresponding results for pure H_2 gas are shown in Table 11. The timescales are rather longer and occur at longer wavelengths. As in the previous case we note that the instabilities occur for values of wavelength greater than the scale of uniformity expected.

7.4 Continuum absorption

The methods developed in the earlier sections can be applied not only to line absorption but also to continuum absorption, as occurs in the photoionisation of carbon atoms, and also in the photodestruction of many molecular species. Hydrogen molecules are exceptional in interstellar space because they are dissociated by line absorption at wavelengths greater than 912 Å. Hydrogen atoms are also photoionised in space, in HII regions around hot stars and perhaps more widely in interstellar space.

Suppose that the frequency dependent cross section per molecule for photodissociation for the species k of interest is

$$\sigma(\nu) = \begin{cases} \dfrac{B}{\nu^r} & \nu \geqslant \nu_0 \\ 0 & \nu_0 > \nu \end{cases} \qquad (7.83)$$

B, r constant.

r = 3 in many cases.

With the same assumptions as before the steady-state radiation field I(ν) will be given by

$$I(\nu, z) = \begin{cases} J e^{-ns z} \exp\left\{ -\dfrac{Bn}{\nu^r} \int_0^z \alpha_\kappa \, dl \right\} & \nu \geqslant \nu_0 \\ J e^{-ns z} & \nu < \nu_0 \end{cases} \qquad (7.84)$$

where ns is the grey opacity from dust per unit length. Scattering has been ignored.

The photodestruction rate for one stationary molecule will be $\omega_o(z)$ where

$$\omega_o(z) = \int_0^\infty I(\nu,z)\,\sigma(\nu)\,d\nu$$

$$= \int_{\nu_0}^{\infty} I(\nu,z)\,\frac{B}{\nu^r}\,d\nu \qquad (7.85)$$

If the molecule is moving at a speed v(v << c) in the same direction as the photons, it will see each photon's frequency changed by a factor (1 - v/c) and the photon flux changed by the same factor so that the photodestruction rate of this molecule will be

$$\omega_v^+(z) = \left(1-\frac{v}{c}\right)\int_{\nu_0(1-\frac{v}{c})}^{\infty} I(\nu,z)\,\sigma\left(\nu\left(1-\frac{v}{c}\right)\right)d\nu$$

$$= \left(1-\frac{v}{c}\right)\int_{\frac{\nu_0}{1-v/c}}^{\infty} I(\nu,z)\,\frac{B}{\nu^r\left(\frac{v}{c}-1\right)^r}\,d\nu =$$

$$= -\left(1-\frac{v}{c}\right)\int_{\nu_0}^{\nu_0/1-v/c} I(\nu,z)\,\frac{B}{\nu^r\left(\frac{v}{c}-1\right)^r}\,d\nu + \frac{\left(1-\frac{v}{c}\right)}{\left(\frac{v}{c}-1\right)^r}\int_{\nu_0}^{\infty} I(\nu,z)\,\frac{B}{\nu^r}\,d\nu =$$

$$\approx -\left(1-\frac{v}{c}\right)\left\{\frac{\nu_0}{1-v/c}-\nu_0\right\} I(\nu_0,z)\,\frac{B}{\nu_0^r\left(\frac{v}{c}-1\right)^r} + \frac{\left(1-\frac{v}{c}\right)^r}{\left(\frac{v}{c}-1\right)^r}\,\omega_o(z)$$

$$\approx -\frac{v}{c}\left(1+r\frac{v}{c}\right)\frac{\nu_0}{\nu_0^r}\,BI(\nu_0,z) + \left(1+(r-1)\frac{v}{c}\right)\omega_o(z)$$

$$\approx -\frac{v}{c}\,\frac{B}{\nu_0^{r-1}}\,I(\nu_0,z) + \left(1+(r-1)\frac{v}{c}\right)\omega_o(z) \qquad (7.86)$$

to first order in $(\frac{v}{c})$

$$\omega_v^+ = \omega_v(z) + \frac{v}{c}\left\{(1-1)\nu_0(z) - \frac{B\,I(\nu_0,z)}{\nu_0^{1-1}}\right\} \qquad (7.87)$$

For a molecule moving in the opposite direction to the photons the frequency shift is by a factor (1 + v/c) the flux is changed by this factor also, and the dissociation rate is

$$\omega_v^-(z) = \left(1+\frac{v}{c}\right)\int_0^\infty I(\nu,z)\,\sigma\left(\nu\left(1+\frac{v}{c}\right)\right)\,d\nu$$

$$= \left(1+\frac{v}{c}\right)\int_{\frac{\nu_0}{1+v/c}}^{\nu_0} I(\nu,z)\frac{B}{\nu^1\left(1+\frac{v}{c}\right)^1}\,d\nu + \left(1+\frac{v}{c}\right)\int_{\nu_0}^{\infty} I(\nu,z)\frac{B}{\nu^1\left(1+\frac{v}{c}\right)^1}\,d\nu$$

$$\approx \frac{1}{\left(1+\frac{v}{c}\right)^{1-1}}\left\{\nu_0 - \frac{\nu_0}{1+v/c}\right\} J e^{-n\sigma z}\frac{B}{\nu_0^1} + \frac{1}{(1+\frac{v}{c})^{1-1}}\,\omega_v(z)$$

$$\approx \frac{v}{c}\frac{B}{\nu_0^{1-1}} J e^{-n\sigma z} + \left(1 - \frac{(1-1)v}{c}\right)\omega_0(z)$$

$$\omega_v^-(z) = \omega_0(z) + \frac{v}{c}\left\{\frac{B}{\nu_0^{1-1}} J e^{-n\sigma z} - (1-1)\,\omega_0(z)\right\}$$

(7.88)

If $\omega_0(z)$ dominates both the other terms in the two brackets the formulae may be combined as

$$\omega_v(z) = \omega_0(z) - \frac{v}{c}(1-1)\,\omega_0(z) \qquad (7.89)$$

where now v is the speed, measured positive if in the same direction as the photons. If on the other hand $\omega_o(z)$ is not sufficiently large then $\omega_v^-(z)$ and $\omega_v^+(z)$ will have different forms.

The contribution to X from photodissociation X^- will be

$$X^- = x_k \omega_o$$

when the bulk velocity is zero (1)

$$X^- = x_k \omega_o + x_k \frac{v}{c} \left\{ (\Gamma-1)\omega_o - \frac{B}{\nu_o^{\Gamma-1}} I(\nu_o, z) \right\}$$

when the motion is along that of the photons (2)

$$= x_k \omega_o + x_k \frac{v}{c} \left\{ \frac{B}{\nu_o^{\Gamma-1}} J e^{-n s z} - (1-1)\omega_o \right\}$$

when the motion is antiparallel to the photon .

motion (3)

(7.90)

and

$$\omega_o = J e^{-n s z} \int_{\nu_o}^{\infty} \frac{B}{\nu^{\Gamma}} d\nu = \frac{J e^{-n s z}}{(\Gamma-1)} \frac{B}{\nu_o^{\Gamma-1}}$$

7.5 Instabilities in HII regions

For the reasons outlined in Chapter 6 Schatzmann has argued that perturbations in temperature can be ignored in HII regions, and so the conditions (7.3) should be examined to see if the system is unstable. We take hydrogen atoms as the molecular species k, which is dissociated to protons and electrons, and the latter radiatively recombine to reform the atoms.

Let x = fractional ionisation = $\frac{n_e}{n} = \frac{n_p}{n}$

$$n = n_e + n_H$$

where n_e = no density of electrons

n_p = no density of protons

n_H = no density of atoms

$$X = X^+ - X^-$$

$$X^+ = n\alpha(T)\,(x)^2$$

α = recombination coefficient excluding direct recombination to ground level

$$\approx \frac{210^{-11}}{T}\ s^{-1}\ cm^{-3} \qquad \text{(Spitzer 1968)}$$

apart from factors of order unity.

$$X^- = (1-x)\int_0^\infty I(\nu,z)\,\sigma(\nu)\,d\nu$$

$$X_x = 2xn\alpha(T) + \int_0^\infty I(\nu,z)\sigma(\nu)\,d\nu$$

$$X_n = \alpha(T)x^2$$

$$X_\nu = -\frac{1}{c}\begin{cases} 2X_o^- - \dfrac{B(1-x)}{\nu_o^2}I(\nu_o,z) & \text{in case 2} \\ -2X_o^- + \dfrac{B(1-x)}{\nu_o^2}Je^{-nsz} & \text{in case 3} \end{cases}$$

(7.92)

In the steady state X = 0 so

$$X_o^+ = X_o^- = (1-x)\int_0^\infty I(\nu,z)\sigma(\nu)\,d\nu = n\alpha(T)x^2$$

$$\therefore \quad X_n = \frac{1}{n}X_o^-$$

and if $(1 - x) \ll 1$

$$X_x = \frac{1}{(1-x)}(X_o^-)$$

Furthermore one expects that $I(\nu_o, z)$ will be much less than Je^{-nsz} because the former must include the effects of photon absorptions by hydrogen. Therefore if

$$2X_o^- \gg \frac{B}{\nu_o^2}Je^{-nsz}$$

then

$$|X_\nu| = \frac{2}{c}X_o^-$$

and (7.30) can be examined.

$$X_x > 0$$

and

$$X_x - \frac{n}{M} X_n > 0 \qquad , \quad M = 1 + x$$

so that (7.30)(i) and (iv) do not hold. (7.30) (ii) may be written

$$\frac{n^2 m}{P} M \frac{X_o^-}{(1-x)} \frac{X_o^-}{n} < \frac{4}{c^2} (X_o^-)^2$$

$$\frac{nm}{4P} M c^2 < 1-x$$

$P = MnkT$, $T \approx 10^4 K$ in HII regions

$$L.H.S. = 2 \frac{n}{kT} 10^{20} \approx 2\,10^{8}$$

and so (7.30) (ii) cannot hold since 1-x is less than unity.

(7.30) (iii) becomes

$$n^3 \frac{m}{P} \frac{(X_o^-)^2}{n} < \frac{4}{c^2} (X_o^-)^2$$

$$\text{ie} \quad \frac{nmc^2}{4P} < 1$$

but the $\frac{nmc^2}{4P}$ is greater than unity and so (7.30) (ii) cannot hold either. Case 2 or Case 3 in (7.92).

The possibility remains that X_o^- is not the dominant term in either or both of the two cases

(7.30) (ii) requires that

$$\frac{nmM}{P} \frac{(X_o^-)^2}{(1-x)} < \frac{B^2}{c^2 \nu_o^4} (1-x)^2 J^2 e^{-2n\sigma_{\nu}}$$

$$\frac{c^2 \nu_0^4 (X_0^-)^2 m}{(1-x)\, kT} < \left[B(1-x) J e^{-n\sigma z} \right]^2$$

$$\therefore \quad \frac{c |X_0^-|}{\nu_0 \sqrt{RT}} < \frac{B}{\nu_0^3} (1-x)^{3/2} J e^{-n\sigma z}$$

$$X_0^- = \alpha n_e x \approx 2 10^{-13} n$$

if $x \approx 1$

$$\sqrt{RT} \approx 10^6 \qquad \text{if } T \approx 10^4$$

$$\frac{c}{\nu_0} = 10^{-5} \text{ cm}$$

$$\frac{B}{\nu_0^3} = 6.6\ 10^{-18} \text{ cm}^2 \text{ s}^{-3}$$

(Spitzer 1968)

$\therefore$ we require

$$1.5\, n\, 10^{-6} < (1-x)^{3/2} J e^{-n\sigma z} \qquad (7.93).$$

(7.30) (iii) requires

$$\frac{nm}{P} (X_0^-)^2 < \frac{B^2}{c^2 \nu_0^4} (1-x)^2 J^2 e^{-2n\sigma z}$$

$$\frac{1}{MkT} \alpha^2 n^2 < \frac{B^2}{\nu_0^6} \frac{\nu_0^2}{c^2} (1-x)^2 J^2 e^{-2n\sigma z}$$

i.e. $$\alpha\, n \sqrt{\frac{m}{kT}}\, \frac{\nu^3}{B}\, \frac{c}{\nu_0} < (1-x) J e^{-n\sigma z}$$

$$2\frac{10^{-13}}{10^{6}}\, n\, 1.5\, 10^{18}\, 10^{-5} < (1-x) J e^{-n\sigma z}$$

$$3n\, 10^{-6} < (1-x) J e^{-n\sigma z}$$

(7.94)

To calculate J one can, as a first approximation, take the star to radiate as a black body of radius r_* and temperature T_*. The photon flux per unit frequency interval per unit area of the star surface will be $2\pi \frac{\nu^2}{c^2} \left(\exp\left(\frac{h\nu}{kT_*}\right) - 1\right)^{-1}$ cm^{-2} s^{-1} Hz^{-1}

A factor $\left(\frac{r_*}{r_*+Z}\right)^2$ will take into account the dilution of the radiation with distance since $(r_* + Z)$ is the distance to the centre of the star.

$$\frac{h\nu_0}{kT_*} = \frac{157000}{T_*}, \quad \left(\frac{\nu_0}{c}\right)^2 \approx 1.2\, 10^{10}$$

$$J \approx \frac{6 \times 1.2\, 10^{10}}{\left(e^{\frac{157000}{T_*}} - 1\right)} \left(\frac{r_*}{r_*+Z}\right)^2 \qquad (7.95)$$

T_* = 56000 K for the O5 star

and

$$J = \left(\frac{1}{1+\frac{Z}{r_*}}\right)^2 4\, 10^{9} \quad \text{cm}^{-2}\,\text{s}^{-1}\,\text{Hz}^{-1}$$

$T_* = 21000$ K for a B0 star

and

$$J = \left(\frac{1}{1+\frac{z}{r}}\right)^2 4 \cdot 10^7 \quad \text{cm}^{-2}\,\text{s}^{-1}\,\text{Hz}^{-1}$$

(1-x) varies through the HII region but it will have its minimum value at the surface of the star where the radiation is unattenuated. The equation for ionisation steady-state is

$$x^2 n^2 \alpha = n(1-x)\int_{\nu_0}^{\infty} \sigma(\nu)\, I(\nu)\, d\nu$$

$$\int_{\nu_0}^{\infty} I(\nu)\, d\nu = \frac{3.1\,10^{49}}{4\pi r_*^2}$$

for an 05 star $r_* = 7$ x solar radius

$$= \frac{4\ 10^{46}}{4\pi r_*^2}$$

$r_* = 5.3$ x solar radius for a B0 star

(Spitzer 1968)

$= 510^{24}$ cm^{-2} s^{-1} for 05

$= 510^{22}$ cm^{-2} s^{-1} for B0

= no. of ionising photons radiated per cm^2 per second

(7.96)

Replacing $\sigma(\nu)$ by its peak value 710^{-18} cm^{-2}

$$1-x = \frac{3n\,10^{-14}}{7\,10^{-18}\,5\,10^{24}} \approx n\,10^{-21}$$

for an 05

$$= \frac{3n\,10^{-14}}{7\,10^{-18}\,5\,10^{22}} \approx n\,10^{-19} \ldots$$

for a BO, since $x \approx 1$

$$J(1-x) = 4n\,10^{-14} \quad \text{for 05}$$

$$= 4n\,10^{-12} \quad \text{for BO}$$

(7.97)

On the other hand at the edge of the Stromgren sphere (1-x) rapidly increases with increasing radius, tending to unity outside the sphere, at the same time J has been reduced by a factor$(r_*/r_s)^2$ where r_s is the Stromgren radius

$$r_s = 310^{20}\, n^{-2/3} \quad \text{for 05}$$

$$= 3.3\,10^{19}\, n^{-2/3} \quad \text{for BO}$$

$$J(r=r_s) = 410^{9}\left(\frac{710^{10}\cdot 7}{310^{20}}\right)^2 n^{4/3}$$

$$= 2.7\,10^{-10}\, n^{4/3} \quad \text{for 05}$$

$$J = 4\,10^{7}\left(\frac{7\,10^{10}\cdot 5\cdot 5}{3\cdot 3\,10^{19}}\right)^{2} n^{4/3}$$

$$= 5\cdot 2\,10^{-9}\,n^{4/3}$$

for B0

$\therefore$ we require

$$3n\,10^{-6} \lesssim 5\,10^{-9}\,n^{4/3}$$

for a B0 star at $r \approx r_s$ when $1 - x \approx 1$

i.e.

$$10^{3} \lesssim n^{1/3}$$

$$10^{9} \lesssim n$$

At intermediate values of r,

$$1-\omega \approx \frac{3\,\nu_0^3}{n\,B\,r_s}\,\frac{\left(r/r_s\right)^2}{\left\{1-\left(\frac{r}{r_s}\right)^3\right\}} \qquad \text{if } 1-x \ll 1$$

(see Field 1974)

and so if s = 0, J(1-x) is

$$3\,J(r=r_*)\,\frac{\nu_0^3}{n\,B}\,\frac{r_*^2}{r_s^3}\,\frac{1}{\left\{1-\left(\frac{r}{r_s}\right)^3\right\}}$$

which remains approximately constant until r is very close to r_s

We can reconsider Schatzman's model, which has been examined in Chapter 6 but in that chapter the doppler effect was not taken into account. Schatzman gives an expression for the neutral concentration in a radiation field corresponding to a radiation temperature $2 10^4$K diluted by a factor W which may be written

$$1-x = n \frac{10^{-13.44}}{W} \qquad (7.99)$$

W is about 10^{-14} in unshielded space. If the radiation field is further reduced by a factor $e^{-\tau}$ in the ionising radiation

$$1-x = \frac{n}{W} e^{-\tau} 10^{-13.44}$$

If s = 0, J for wavelength just greater than 912 A will be given by

$$J = W \frac{7.2 10^{10}}{e^{7.85}-1} = 3 10^7 W$$

Therefore

$$J(1-x) = 3 10^7 W \frac{n}{W} e^{\tau} 10^{-13.44} \simeq n e^{\tau} 10^{-11}$$

In order to satisfy (7.30) (iii) we need

$$3n 10^{-6} < n e^{\tau} 10^{-11}$$

$$\text{ie} \quad 3 10^5 < \tau \qquad (7.100)$$

This in turn implies

$$1-x = \frac{n e^2 10^{-4.44}}{\left(W/10^{-14}\right)} > \frac{n\ 10}{\left(W/10^{-14}\right)}$$

and so, since 1-x is less than unity,

$$n < \left(\frac{W}{10^{-14}}\right)\frac{1}{10} \qquad (7.101)$$

within Schatzman's approximations. Therefore in regions of sufficiently low density which are sufficiently well shielded, the instabilities discussed may arise.

7.6 Instabilities due to photoionisation of carbon in the interstellar medium

Using the maximum allowable value for X_v for carbon ionisation in the interstellar medium the roots of the dispersion relation were found in a similar way to those for H_2 photodissociation in Section 3. A few instablities were found but they had timescales very much longer than the free-fall time and on lengthscales very much longer than the scale of uniformity. We conclude that these instabilities are not of importance in realistic calculations about the interstellar clouds.

CHAPTER 8

A MODIFIED VERSION OF FIELD'S DISCUSSION OF INSTABILITIES DRIVEN BY RADIATION PRESSURE

8.1 Instabilities driven by Radiation pressure

In this chapter the influence of radiation directly on the momentum of the gas is discussed; the effect was ignored in previous chapters and correspondingly in this chapter thermal and reactive effects will be ignored. Following Field (1971), we divide the material components of the system into two parts, namely the dust grains which can interact directly with the radiation and the gas which cannot, but which collides with, and hence exerts a viscous drag upon the dust. Two different forms of interaction have been considered by various authors. Field and several others, for example see Spitzer (1968), had used the transfer of momentum from an incident photon to the grain, the momentum per photon being $\frac{h\nu}{c}$ where ν is the photon's frequency. On the other hand Gerola and Schwartz (1976) considered the effect of photoejection of gas atoms which had struck and then stuck onto the grains. This idea had been put forward originally by Reddish (1971), the point being that the momentum transfer to the grains is about $\sqrt{mh\nu}$, where m is the mass of a gas atom, and this is much larger than the original photon's momentum. The ejected gas atom takes up an equal and opposite amount of momentum if that of the photon is neglected. Provided that the grain had not rotated appreciably between the absorption of the photon and the ejection of the atom, the grain will be given on average a push in the direction in which the photon had been travelling. Although the mechanism can be more efficient in

pushing the grains than is radiation pressure it is limited by the necessity of there being atoms sufficiently loosely bound to the surface to be photoejected by those photons available in interstellar space. In diffuse clouds this means that the rate of momentum transfer to a grain is limited by the atomic flux onto the surface and is independent of the magnitude of the radiation flux, at least so long as thermal evaporation from the grain is ignored. The disadvantage of the process, from the point of view of wanting to form clumps in the gas, is that according to Gerola and Schwartz the net momentum transfer to the gas is zero. This is because the photoejected particle and the grain have equal and opposite momenta, and these momenta are both absorbed into the gas by collisions. Since ordinary dust grains are believed to constitute only about one percent by mass of the interstellar medium it follows that, unless some other effects come into play in the non-linear regime, the mechanism, affecting as it does only the grains, cannot produce appreciable mass density variations in the gas. They point out that despite this, the associated instability can lead to significant segregation of gas and dust during the lifetime of a cloud. We note here that this segregation can affect the rate of molecule formation and destruction because the formation rate depends to some extent upon the surface area of grains per unit volume and the destruction rate depends upon grain shielding from the destructive ultra-violet radiation which is absorbed whether or not there is an associated jet effect. The molecules may then help drive an instability of the sort described in the earlier chapters. However, in the linear regime, ignoring thermal and reactive effects Gerola and Schwartz's mechanism does not appear to be very useful for the purpose of

forming clumps. Field's mechanism, however, may be more useful and is investigated further in this chapter.

In his paper Field derived equations for the separate motion of the gas and dust, coupled by the viscous drag of the gas on the dust. He then stated that those modes in which the gas and dust move relative to one another contained little essentially different from the solutions he gave in detail. These latter solutions corresponded to the case of close coupling between gas and dust, that is, no relative motion of gas and dust. In this chapter we ignore the possibility of relative motion between gas and dust but we have not investigated these modes.

Field also included the possibility that a photon, once absorbed, could be re-emitted isotropically in the rest frame of the gas. The probability for re-emission was taken as frequency independent and about equal to 0.5 for interstellar grains. Setting this probability equal to zero does not alter Field's final results appreciably in this case. In this chapter we neglect this elastic scattering. A connected point is that the energy of the absorbed photon is re-radiated by the grain at frequencies such that there is no further interaction with the grains. This assumption is made here also. A fuller treatment of the problem should certainly include frequency dependent elastic and inelastic scattering and the possibility of radiation trapping, even if we keep the restrictions of constant temperature and composition.

The extension we make to Field's treatment here is to include the possibility that the absorption cross-section of the grains is frequency dependent, rather than frequency independent as Field assumed. This can produce some interesting changes as shown in the next section.

8.2 The Inclusion of Frequency Dependence

The model consists of a uniform gas immersed in a uniform isotropic radiation field of specific intensity $I(\nu)$ergs cm^{-2} s^{-1} Hz^{-1} $ster^{-1}$; the system is initially stationary. We assume that the interaction between the radiation and the gas is through certain absorbers, tightly coupled to the rest of the gas, and the average absorption cross section per unit volume is $n\sigma(\nu)$ where n is the total particle number density of the gas. Thermal effects are neglected here, the temperature is assumed constant throughout.

The equation of motion, continuity and radiative transfer are respectively

$$nm\frac{dv}{dt} + \frac{\partial P}{\partial z} = \frac{2\pi}{c}\int_0^\infty\int_{-1}^{1} \mu n\sigma(\nu,\mu) I(\nu,\mu) d\mu d\nu \qquad (8.1)$$

$$\frac{dn}{dt} + n\frac{\partial v}{\partial z} = 0 \qquad (8.2)$$

$$\frac{1}{c}\frac{\partial I}{\partial t} + \mu\frac{\partial I}{\partial z} + n\,\sigma(\nu,\mu)\,I(\nu,\mu) = j(\nu,\mu) \qquad (8.3)$$

$$P = n\,k_B T \qquad \text{the pressure} \qquad (8.4)$$

k_B = the Boltzmann's constant

$$= n\,A \quad (A = k_B T = \text{constant}) \qquad (8.5)$$

v is the macroscopic motion of the gas which is along the direction of the wavevector k, itself directed along the z axis.

μ is the cosine of the angle between $\underset{\sim}{k}$ and the direction of the beam $I(\nu,\mu)$. The angular dependence has been included in σ and I because relativistic effects introduce anisotropy in the rest frame

even though the cross-section is isotropic in the frame of reference moving with the gas. The velocity time and space dependences are not explicitly indicated.

$j(\nu,\mu)$ comes from the uniformly distributed sources which maintain the initial uniform radiation field.

As seen in the initial rest frame the cross section of gas moving at speed v along the positive z direction will be given by

$$\sigma(\mu,\nu) = (1 - \frac{\mu v}{c})\ \sigma(\nu\ (1 - \frac{\mu v}{c})) \qquad (8.6)$$

(see Pomraning 1972) and to first order in v/c

$$\sigma(\nu,\mu) = (1 - \frac{\mu v}{c})\ \{\sigma(\nu) - \mu\nu\ \frac{v}{c}\ \frac{\partial\sigma(\nu)}{\partial\nu}\} \qquad (8.7)$$

σ_o is the isotropic cross section in the rest frame of the gas. Writing

$$\sigma(\nu,\mu) = \sigma_o(\nu) + \sigma_1(\nu,\mu) \qquad (8.8)$$

$$\sigma_1(\nu,\mu) = -\frac{\mu v}{c}\ \{\sigma_o + \frac{\nu\partial\sigma_0}{\partial\nu}\} \qquad (8.9)$$

Indicating initial values by zero subscripts and perturbations by '1' subscripts, the equations describing these perturbations are, to first order in these perturbations and assuming a space and time dependence $\exp\{\beta t + ikz\}$ for all the quantities

$$\beta n_1 + ikn_o v_1 = 0 \qquad (8.10)$$

$$I_1(\nu,\mu)\left\{\frac{\beta}{c} + ik\mu\right\} + n_1\sigma_o(\nu)I_o(\nu) + n_o I_o(\nu)\sigma_1(\nu,\mu)$$

$$+ n_o\sigma_o(\nu)I_1(\nu,\mu) = 0 \qquad (8.11)$$

$$n_0 m\beta v_1 + ikA n_1 = \frac{2\pi}{c}\int_0^\infty\int_{-1}^{1}\mu\left\{n_0\sigma_1(\nu,\mu)I_0(\nu) + n_1\sigma_0(\nu)I_1(\nu) + n_0\sigma_0(\nu)I_1(\nu,\mu)\right\}d\mu\,d\nu$$

$$= \frac{2\pi}{c}n_0\int_0^\infty I_0(\nu)\int_{-1}^{1}\mu\,\sigma_1(\nu,\mu)\,d\mu\,d\nu + \frac{2\pi}{c}n_0\int_0^\infty\sigma_0(\nu)\int_{-1}^{1}\mu\,I_1(\nu,\mu)\,d\mu\,d\nu \qquad (8.12)$$

since initially $$\int_{-1}^{1}\mu\,\sigma_0(\nu)\,I_0(\nu)\,d\mu = 0. \qquad (8.13)$$

These equations may be rewritten

$$n_1 = \frac{-ikn_0}{\beta}v_1 \qquad (8.14)$$

$$I_1\left\{\frac{\beta}{c} + ik\mu + n_0\sigma_0\right\} - \frac{v_1}{c}n_0 I_0\left\{\frac{ikc}{\beta}\sigma_0 + \mu\left(\sigma_0 + \nu\frac{\partial\sigma_0}{\partial\nu}\right)\right\} = 0$$

$$\therefore\quad I_1 = \frac{v_1 n_0 I_0}{c}\,\frac{\left\{\mu\left(\sigma_0 + \nu\frac{\partial\sigma_0}{\partial\nu}\right) + \frac{ikc\sigma_0}{\beta}\right\}}{\left\{\frac{\beta}{c} + n_0\sigma_0 + ik\mu\right\}} \qquad (8.15)$$

Substituting these values for n_1, I_1 and σ_1 into (8.12) we arrive as

$$n_0\left(m\beta + \frac{k^2A}{\beta}\right)v_1 = -\frac{v_1}{c}\frac{n_0}{c}2\pi\int_0^\infty I_0(\nu)\int_{-1}^{1}\mu^2\left(\sigma_0(\nu) + \nu\frac{\partial\sigma_0}{\partial\nu}\right)d\mu\,d\nu$$

$$+ \frac{2\pi v_1}{c}n_0^2\int_0^\infty\sigma_0(\nu)I_0(\nu)\int_{-1}^{1}\frac{\left\{\mu^2\left(\sigma_0 + \nu\frac{\partial\sigma_0}{\partial\nu}\right) + \frac{ikc\sigma_0\mu}{\beta}\right\}}{\left\{\frac{\beta}{c} + ik\mu + n_0\sigma_0\right\}} \qquad (8.16)$$

The μ integrals may be performed resulting in the dispersion relation, after cancelling v_1:

$$0 = n_o\left(in\beta + \frac{k^2 A}{c}\right) + \frac{4\pi}{3c^2} n_o \int_0^\infty I_o(\nu)\left(\sigma_o(\nu) + \nu\frac{\partial\sigma_o}{\partial\nu}\right) d\nu$$

$$- \frac{2\pi}{k^2c^2} n_o^2 \int_0^\infty \sigma_o I_o \left\{ \left(\frac{\beta}{c} + n_o\sigma_o\right)\left(\sigma_o + \nu\frac{\partial\sigma_o}{\partial\nu}\right) + \frac{k^2 c\sigma}{\beta} \right\} \left\{ 2 - \frac{\left(\frac{\beta}{c} + n_o\sigma_o\right)}{ik} \ln\left(\frac{\frac{\beta}{c} + n_o\sigma_o + ik}{\frac{\beta}{c} + n_o\sigma_o - ik}\right) \right\} d\nu$$

(8.17)

Several simplifying assumptions can be made. If the term (β/c) in the radiative transfer equation is ignored, that is if

$$|\beta| \ll c|n_o\,\sigma_o + ik| \tag{8.18}$$

and since $\sigma_o \to 0$ as $\nu \to \infty$, we require

$$|\beta| \ll c|k|, \tag{8.19}$$

the equation becomes, after dividing by n_o

$$in\beta + \frac{k^2 A}{\beta} + \frac{4\pi}{3c^2}\int_0^\infty I_o\left(\sigma_o + \nu\frac{\partial\sigma_o}{\partial\nu}\right) d\nu$$

$$- \frac{2\pi n_o}{k^2c^2}\int_0^\infty \sigma_o I_o \left\{ n_o\sigma_o\left\{\sigma_o + \nu\frac{\partial\sigma_o}{\partial\nu}\right\} + \frac{k^2 c\sigma_o}{\beta}\right\}\left\{2 - \frac{n_o\sigma_o}{ik}\ln\left(\frac{n_o\sigma_o + ik}{n_o\sigma_o - ik}\right)\right\}$$

$$= 0$$

(8.20)

Furthermore, if $k \gg n_o \sigma_o(\nu)$ for all ν the equation reduces to

$$m\beta + \frac{K^2 A}{\beta} + \frac{4\pi}{3c^2}\int_0^\infty I_o\left(\sigma_o + \nu\frac{\partial\sigma_o}{\partial\nu}\right)d\nu = 0$$

$$-\frac{4\pi n_o}{K^2c^2}\int_0^\infty \sigma_o^2 I_o\left\{n_o\left(\sigma_o + \nu\frac{\partial\sigma_o}{\partial\nu}\right) + \frac{K^2c}{\beta}\right\}d\nu \qquad (8.21)$$

i.e. $$m\beta^2 + \beta\left\{\frac{4\pi}{3c^2}\int_0^\infty I_o\left(\sigma_o + \nu\frac{\partial\sigma_o}{\partial\nu}\right)d\nu - \frac{4\pi n_o^2}{K^2c^2}\int_0^\infty \sigma_o^2 I_o\left(\sigma_o + \nu\frac{\partial\sigma_o}{\partial\nu}\right)d\nu\right\}$$

$$+\left[K^2A - \frac{4\pi}{c}n_o\int_0^\infty \sigma_o^2 I_o\,d\nu\right] = 0 \qquad (8.22)$$

We may specialise even further and assume $I(\nu)$ is independent of frequency, at least over those frequencies where $\sigma(\nu)$ is non-zero. The various integrals now may be considered

$$\int_0^\infty\left(\sigma_o + \nu\frac{\partial\sigma_o}{\partial\nu}\right)d\nu = \int_0^\infty \frac{\partial}{\partial\nu}(\nu\sigma_o)\,d\nu = \left[\nu\sigma_o(\nu)\right]_0^\infty$$

$= 0$ if $\sigma_o(\nu)$ tends to zero as $\nu \to \infty$ faster than ν^{-1}.

(8.23)

$$\int \sigma_o^2 \left(\sigma_o + \nu \frac{\partial \sigma_o}{\partial \nu}\right) d\nu = \int \sigma_o^2 \frac{\partial}{\partial \nu}(\nu \sigma_o)\, d\nu = \left[\sigma_o^3 \nu\right]_0^\infty - \int_0^\infty 2\nu \sigma_o^2 \frac{\partial \sigma_o}{\partial \nu} d\nu$$

$$= \left[\sigma_o^3 \nu\right]_0^\infty - \frac{2}{3}\left[\nu \sigma_o^3\right]_0^\infty + \frac{2}{3}\int_0^\infty \sigma_o^3 \, d\nu$$

$$= \frac{2}{3}\int_0^\infty \sigma_o^3 \, d\nu \qquad (8.24)$$

The dispersion relation is then

$$m\beta^2 - \beta \frac{4\pi n_o^2}{k^2 c^2} I_o \int_o^\infty \sigma_o^3 \, d\nu + \left[k^2 A - \frac{4\pi n}{c} I_o \int_o^\infty \sigma_o^2 \, d\nu\right] = 0 \qquad (8.25)$$

σ_o is always positive and so the coefficient of β is negative showing that there must be a root with positive real part. Hence for sufficiently large k there must be an instability in the system independent of the value of I, as long as I is not zero. This may be contrasted with Field's work in which he found that a necessary condition for instability was that the radiation pressure exceed the gas pressure. The difference can be traced to the fact that included in the coefficient of β the $\nu \frac{\partial \sigma}{\partial \nu}$ integral cancels exactly with the σ integral. But for this cancellation the first integral would completely dominate the second, negative, contribution in the coefficient of β because k is assumed to be very much greater than $n\sigma_o$.

If

$$\left[\frac{4\pi n_o^2}{k^2 c^2} I_o \int_0^\infty \sigma_o^3 \, d\nu\right]^2 - 4m\left[k^2 A - \frac{4\pi}{c} n_o I_o \int_0^\infty \sigma_o^2 \, d\nu\right] < 0 \qquad (8.26)$$

as will be the case if I is sufficiently small or k sufficiently large, then the timescale of the instability will be

$$\tau = \frac{1}{Rc(\beta)} = \frac{k^2 mc^2}{4\pi n_o^2 I_o \int_0^\infty \sigma_o^3 d\nu} \qquad (8.27)$$

τ increases as k^2 but decreases as I increases. If σ_o is approximately constant, equal to q, over a frequency range Δ and then falls off rapidly to zero the integral in the denominator is approximately $q^3\Delta$ and

$$\tau = \frac{nmc^2}{4\pi I\Delta} \quad \frac{k^2}{(nq)^3} = \frac{mc^2}{4\pi(qI\Delta)} \quad \left(\frac{k}{nq}\right)^2 . \qquad (8.28)$$

With the values

$q = 10^{-21}$ cm^{-2}, corresponding to the cross section of grains of radius 310^{-5}cm and with number abundance 10^{-12} that of hydrogen.

m = mass of hydrogen, since most of the mass of the gas is in the form of hydrogen.

$\Delta = 10^{14}$ Hz, corresponding to a width of 100 $\mathring{A}$ around wavelengths 1000 $\mathring{A}$.

$I = 310^{-8}$ photons cm^{-2} s^{-1} Hz^{-1} as before

$$\tau = \frac{k^2}{n^2} 410^{52}\ s = \left(\frac{k}{nq}\right)^2 q^2\ 410^{52} s = \left(\frac{k}{nq}\right)^2 410^{10} s \cong \left(\frac{k}{nq}\right)^2 10^3 \text{yrs}$$

$$\text{and by assumption } \left(\frac{k}{nq}\right) > 1 \qquad (8.29)$$

With the above numerical values (8.26) is satisfied but in other situations even when (8.26) is not satisfied there must be an instability with a shorter timescale than that given in eqn. (8.27).

The assumption that $k > nq$ made through most of this section is in fact quite interesting for the following reason. In a more realistic non-uniform model, a JWKB treatment may well show the growth rates for k values much greater than nq are approximately as given; $\frac{1}{nq}$ being the length scale for uniformity.

It will be remembered that these results depend upon the auxiliary assumptions that $\left| \frac{\beta}{c} + n\sigma_o(\nu) \right| \ll k$ for all ν, and also that $I(\nu)$ was constant. If on the other hand $I(\nu)$ is not constant but instead goes to zero before the $\nu\frac{\partial \nu}{\partial \nu}$ is significant, or alternatively that $\frac{\partial \sigma}{\partial \nu} = 0$ we arrive at the equation

$$m\beta^2 + \beta\left\{\frac{4\pi\sigma_o}{3c^2}\int_o^\infty I_o(\nu)\,d\nu - \frac{4\pi n_o^2\sigma_o^3}{k^2c^2}\int_o^\infty I(\nu)\,d\nu\right\}$$

$$+ k^2A - \frac{4\pi}{c}n_o\sigma_o^2\int_o^\infty I_o(\nu)\,d\nu = 0 \qquad (8.30)$$

Here we have again assumed $\left| \frac{\beta}{c} + n\sigma \right| \ll |k|$. Field's equations come to this same equation in the same short wavelength approximation. As mentioned previously it follows from his equations that a necessary condition for instability is that the radiation pressure $\frac{1}{c}\int_o^\infty I(\nu)\,d\nu$ exceeds the gas pressure. This same fact may be seen from (8.30), which is a rather cruder approximation than is Field's treatment:

> The coefficient of β is positive because $\frac{n\sigma}{k} < 1$ and this is independent of $\int_o^\infty I(\nu)\,d\nu$.

The coefficient independent of β is negative only if

$$\frac{1}{c}\int_0^\infty I_0 \, d\nu > \frac{k^2}{n_0^2\sigma_0^2}\frac{1}{4\pi} nA = \left(\frac{k}{n_0\sigma_0}\right)^2 \frac{1}{4\pi} \cdot \text{gas pressure} \tag{8.31}$$

and by assumption $\frac{k}{n\sigma} > 1$.

Therefore the radiation pressure must exceed the gas pressure for an instability, at least in this short wavelength approximation. Taking the values $n\sigma \simeq 710^{-21}$ cm^{-1}, $\frac{1}{c}\int I(\nu)\, d\nu = 10^{-13}$ erg cm^{-3} and $n = 10$ cm^{-3}, he derived a timescale of growth of about 10^7 years. This timescale is a little longer than that for graviational collapse (τ_{grav}) and varies with n as $n^{-\frac{1}{2}}$, as does τ_{grav} and so it remains longer than τ_{grav} for all densities. The instability discussed in this section varies as n^{-2} and hence decreases faster than does τ_{grav} as n increases. It is therefore possible that if, with a given set of values for q, I, Δ, etc., the timescale τ is greater than τ_{grav} for a certain n, it can rapidly fall below τ_{grav} as n increases. Furthermore in Field's application he found that, just as in the case of the gravitational instability, there was a critical wavelength below which there was no instability. This limiting wavelength suggests that there is a corresponding minimum mass, namely that mass contained in a sphere of radius the critical wavelength, which will collapse. Field found a minimum mass of about 10^6 $M_\odot$ using the above numerical values, which he found was too large for his purposes. On the other hand the instability discussed here shows no minimum wavelength and may therefore be more useful in the discussion of the formation of clumps of small masses.

Presumably a realistic evaluation of the integrals leads to some situation intermediate between the two extremes mentioned above. Neither $\sigma(\nu)$ nor $I(\nu)$ will be constant over sufficiently wide ranges for either extreme to hold. One might also expect some selective absorption of photons, by gas near the sources of radiation, at frequencies near those for which $\sigma(\nu)$ is maximum.

CHAPTER 9

CONCLUSIONS

It seems appropriate now to summarise and discuss the implications of the results obtained in the previous chapters. One result rather separate from the rest of the thesis is a method of finding the number roots of a complex polynomial which have positive real part, and may also give estimates of the values of the real parts. This method has been used in this thesis but clearly has other applications.

In Chapter 6 we saw that, despite Schatzman's conclusions, a slab of HII is not marginally unstable, ~~and~~ but a study of the linearised equations suggests that it ~~is not~~ may be unstable ~~at~~ after all. However, at the beginning of this work it was hoped that a self shielding instability might operate in clouds containing H_2 molecules, similar to the one imagined to occur in HII regions by Schatzman. Such instabilities would have been of great use in the study of framentation of inter-stellar clouds, and in particular it would have aided Reddish's scenario of galactic evolution; without some additional physical effects it seems that this fragmentation mechanism cannot produce fragments of greater than stellar mass and may not be able to form any fragments without extra compression. The hoped for instabilities were not found, as detailed in Chapter 4 and also Chapter 5.

Examination of Reddish's ideas show that there are two rather separate parts. The first part involves the idea that the property which defines a fragment is that its radius should correspond to unit grain opacity, taking an approximately geometric cross-section for the grains. With this definition of a fragment he then goes on to show that, when applied to fragments forming within a gaseous, polytropic sphere, the resulting mass spectrum is very similar to the deduced initial

mass spectrum of stars. It is worth noting that the different fragments have different densities and hence different free-fall times therefore one might perhaps think that a very close matching of the two spectra is not desirable; one might instead expect the stellar mass function to be proportional to the cloud mass function multiplied by the inverse free fall time of the initial fragment. Reddish himself has made this point in another connection (Reddish 1977). On the other hand one should not forget the effects of magnetic fields and rotation as a fragment collapses, either of which may hold up the final collapse for a time independent of density and hence allowing all the fragments to form stars of essentially the same time. Thus a matching of the cloud fragment and the stellar mass spectra is not a sufficient condition, and may not even be a necessary condition, for a satisfactory theory of star formation.

The second part of Reddish's thesis is that molecule formation on grains is the actual mechanism causing the fragmentation. Molecules form most rapidly when the grains are at a certain temperature, and grains within about unit optical depth are at about the same temperature. Now in order for the fragments to maintain their identity one requires that the timescale of molecule formation should be less than the free-fall time. This latter condition gives essentially a lower limit to the density required and hence an upper limit to the mass of the fragment. The Jeans criterion, assuming a temperature of about 5K, gives a lower limit to the mass spectrum. Only the numerical value of the upper limit to the mass is dependent upon the actual fragmentation mechanism, although the mechanism itself must be linked to the unit grain optical depth. The strong point of Reddish's theory is that the first part provides one of the most

plausible, and least ad hoc, explanations for the initial mass function.

The relevance of the above considerations to this thesis is that there are many processes going on inside clouds and one related to the previous investigation is the possibility that Carbon ions, the main coolants in such clouds, are subject to a self-shielding instablity. But once again no such instabilities were found.

Now these investigations have concerned only a one dimensional model, have neglected scattering, and concern only infinitesimal perturbations. The lack of self-shielding instabilities has not therefore been proved, but it is rather discouraging not to have found any, despite having used values of the parameters, such as the photo-dissociation cross-section, which one would think would have emphasised the importance of the radiative transfer.

Having started upon the track of possible instabilities in stationary interstellar clouds, we next looked at the idea that self shielding plays a vital role in allowing the build-up of H_2 in the clouds. Chapter 7 contains the investigation. In that chapter we found that assuming the self absorption line is not exactly symmetrical, a fairly reasonable assumption in interstellar space, then the clouds are subject to an instability which tends to break them up. For densities in the range 30 - 300 cm^{-3} the timescales were about 10^5 years, or sometimes less, in interstellar space, and decreased as the unshielded flux increased. The instabilities did not seem to develop quickly unless the fractional abundance (f) of H_2 was about 10^{-4} or more.

Bearing in mind the limitations of the model and the method of analysis, this instability may well go some way to explain the apparently significant lack of clouds with f in the range 210^{-4} to 510^{-2} (Spitzer and Jenkins 1975). Those clouds with f less than about 10^{-4} are not unstable on rapid timescales; those with f greater than a certain value may be stable because the approximation that the wavelength of the rapidly growing perturbations is less than the scale of uniformity fails badly. The latter point remains to be confirmed by more detailed calculations. In the case of large clouds it may be that the instability tends to fragment the outer layers off in stages, but it is difficult to draw firm conclusions with this level of calculation. This instability will affect the model calculations of interstellar clouds, and perhaps of globules of neutral gas in HII regions.

Applying similar considerations to carbon ionisation we find no instability, despite the most favourable values of the parameters. In HII regions on the other hand this instability may affect the gas but not in the way Schatzman imagined.

Carrying on the investigation into the effect of line-shape on the gas we next re-examined Field's work on radiation pressure driven instabilities. In Chapter 9 the analysis was restricted to showing that at least in one limiting case there are rapidly growing instabilities. While the particular limiting case may not itself be of immediate importance at least it has been shown that further, more detailed, investigations may be of interest in more realistic studies of the fragmentation of interstellar clouds.

APPENDIX 1

A1.1 Derivation of the test functions (a)

In Chapter 3 it was shown that in order to find the number of roots with positive real part of the equation

$$f(z) = a_o z^n + (a_1 + ib_1)z^{n-1} \ldots\ldots + (a_n + ib_n) = 0$$

one method involved evaluation of the leading coefficients of f_o, f_1, f_2, ... f_n where

$$f_o(y) = \{a_o\, y^n + b_1\, y^{n-1} - a_1\, y^{n-2} \ldots\ldots \}D$$

$$f_1(y) = -\{a_1\, y^{n-1} + b_2\, y^{n-2} - a_3\, y^{n-3} \ldots \}D$$

where $D = (-1)^{n/2}$ if n is even

$= (-1)^{(n+1)/2}$ if n is odd

$f_2(y)$ = -Remainder of (f_o/f_1)

$f_3(y)$ = -Remainder of (f_1/f_2) etc.

$f_2(y)$ can be calculated explicitly by dividing f_o by f_1 as follows:

To find f_2:

$$a_1y^{n-1}+b_2y^{n-2}-a_3y^{n-3}-\cdots$$

$$\overline{\left[a_0y^n+b_1y^{n-1}-a_2y^{n-2}-b_3y^{n-3}+a_4y^{n-4}+b_5y^{n-5}-\cdots\right.}\quad \frac{a_0}{a_1}y+\frac{1}{a_1}\left(b_1-b_2\frac{a_0}{a_1}\right)$$

$$\underline{a_0y^n+\frac{b_2}{a_1}a_0y^{n-1}-a_3\frac{a_0}{a_1}y^{n-2}-b_4\frac{a_0}{a_1}y^{n-3}+a_5\frac{a_0}{a_1}y^{n-4}+b_6\frac{a_0}{a_1}y^{n-5}\cdots}$$

$$\left(b_1-b_2\frac{a_0}{a_1}\right)y^{n-1}-\left(a_2-a_3\frac{a_0}{a_1}\right)y^{n-2}-\left(b_3-b_4\frac{a_0}{a_1}\right)y^{n-3}+\left(a_4-a_5\frac{a_0}{a_1}\right)y^{n-4}+\left(b_5-b_6\frac{a_0}{a_1}\right)y^{n-5}\cdots$$

$$\underline{\left(b_1-b_2\frac{a_0}{a_1}\right)y^{n-1}+\frac{b_2}{a_1}\left(b_1-b_2\frac{a_0}{a_1}\right)y^{n-2}-\frac{a_3}{a_1}\left(b_1-b_2\frac{a_0}{a_1}\right)y^{n-3}-\frac{b_4}{a_1}\left(b_1-b_2\frac{a_0}{a_1}\right)y^{n-4}+\frac{a_5}{a_1}\left(b_1-b_2\frac{a_0}{a_1}\right)y^{n-5}\cdots}$$

$$\therefore \quad f_2(y)=-\text{remainder}=\frac{+1}{a_1^2}\left\{\left[a_1(a_2a_1-a_3a_0)+b_2(a_1b_1-b_2a_0)\right]y^{n-2}+\left[a_1(b_3a_1-b_4a_0)-a_3(a_1b_1-b_2a_0)\right]y^{n-3}\right.$$

$$\left.-\left[a_1(a_4a_1-a_5a_0)+b_4(a_1b_1-b_2a_0)\right]y^{n-4}-\left[a_1(b_5a_1-b_6a_0)-a_3(a_1b_1-b_2a_0)\right]y^{n-5}+\cdots\right.$$

$$=\frac{1}{a_1^2}\left\{\begin{vmatrix}a_1 & a_0 & 0\\ -b_2 & -b_1 & a_1\\ -a_3 & -a_2 & -b_2\end{vmatrix}y^{n-2}+\begin{vmatrix}a_1 & a_0 & 0\\ -b_2 & -b_1 & a_1\\ -b_4 & -b_3 & a_3\end{vmatrix}y^{n-3}-\begin{vmatrix}a_1 & a_0 & 0\\ -b_2 & -b_1 & a_1\\ -a_5 & -a_4 & -b_4\end{vmatrix}y^{n-4}-\begin{vmatrix}a_1 & a_0 & 0\\ -b_2 & -b_1 & a_1\\ -b_6 & -b_5 & a_5\end{vmatrix}y^{n-5}\right.$$

$$+\cdots$$

$f_2(y)$ has been written with the sequence of signs in front of the coefficients as ++ -- ++ -- etc. which is the sequence of signs in $f_o(y)$ and $f_1(y)$. Because of this similarity to obtain $f_3(y)$ one need only replace a_o by a_1 etc. in f_o and a_1 by

$$\frac{-1}{a_1^2}\begin{vmatrix} a_1 & a_o & 0 \\ -b_2 & -b_1 & a_1 \\ a_3 & -a_2 & -b_2 \end{vmatrix} = c_2 \text{ etc.}$$

in f_1, in all the manipulations above and $f_3(y)$ is given by following this substitution in $f_2(y)$. Similarly if one has the expression for $f_3(y)$ then $f_4(y)$ may be obtained by replacing a_o by a_1 etc. and a_1 by c_2 etc. in f_3.

The leading coefficients of f_t is, apart from a positive factor

$$c_t = (-1)^{\frac{t(t-1)}{2}}\begin{vmatrix} a_1 & a_o & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ -b_2 & -b_1 & a_1 & a_o & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_3 & a_2 & b_2 & b_1 & a_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ -b_4 & -b_3 & a_3 & a_2 & -b_2 & -b_1 & a_1 & a_o & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & & & & \\ a_{2t-1} & a_{2t-2} & b_{2t-2} & b_{2t-3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

Column 1 consists of the sequence a_1, $-b_2$, a_3, $-b_4$ ..... a_{2t-1}, and each row consists of a continuation in pairs of a_j b_j, with all b_j terms having a negative sign in even rows. To show that the above formula is true one first notes that it is true for $r = 0, 1, 2$. If one now replaces a_o by a_1 and a_1 by c_2 etc. one ends up with c_{r+1}, and it remains to show that this expression can be written in determinantal form

$$c_{t+1} = (-1)^{\frac{(t+1)t}{2}} \begin{vmatrix} a_1 & a_o & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{2t+1} & a_{2t} & b_{2t-1} & \cdots & \end{vmatrix}$$

Making the substitutions in c_t:

$$c_{t+1} = \frac{(-1)^{\frac{t(t-1)}{2}}}{a_1^{2t}} \begin{vmatrix} a_1(a_1a_2-a_3a_o) & a_1 & 0 & 0 & \cdots \\ +b_2(a_1b_1-b_2a_o) & & & & \\ -a_1(a_1b_3-b_4a_o) & -b_2 & a_1(a_1a_2-a_3a_o) & a_1 & \cdots \\ +a_3(a_1b_1-b_2a_o) & & +b_2(a_1b_1-b_2a_o) & & \\ a_1(a_1a_4-a_5a_o) & a_3 & a_1(a_1b_3-b_4a_o) & \cdots & \\ +b_4(a_1b_1-b_2a_o) & & -a_3(a_1b_1-b_2a_o) & & \\ \vdots & & \vdots & & \\ a_1(a_1a_{2t}-a_{2t+1}a_o) & a_{2t-1} & \cdots & \cdots & \\ +b_{2t}(a_1b_1-b_2a_o) & & & & \end{vmatrix}$$

Adding, for m = 1, 2, 3, 4, etc., $(-1)^{t+1}$ $(a_1b_1 - b_2a_o)$ times column 2m to column 2m + 1 one obtains

$$(-1)^{\frac{t(t-1)}{2}} \frac{a_1^{(t-1)}}{a_1^{2t}} \begin{vmatrix} a_1(a_1a_2-a_3a_o) & a_1 & (a_1b_1-b_2a_o) & 0 & 0 & \cdot \\ +b_2(a_1b_1-b_2a_o) & & & & & \\ -a_1(a_1b_3-b_4a_o) & -b_2 & (a_1a_2-a_3a_o) & a_1 & -(a_1b_1-b_2a_o) & \cdot \\ +a_3(a_1b_1-b_2a_o) & & & & & \\ \cdot & a_3 & (a_1b_3-b_4a_o) & b_2 & (a_1a_2-a_3a_o) & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

Moving column 3 to the left hand side this is equivalent to

$$(-1)^{\frac{t(t-1)}{2}} \frac{a_1^{(t-1)}}{a_1^{2t}} \begin{vmatrix} a_1 & a_o & 0 & 0 & 0 & \cdot & \cdot \\ b_2 & b_1 & a_1(a_1a_2-a_3a_o) & a_1 & 0 & \cdot & \cdot \\ & & +b_2(a_1b_1-b_2a_o) & & & & \\ a_3 & a_2 & \cdot & -b_2 & a_1 & -(a_1b_1-b_2a_o) & \cdot \\ b_4 & b_3 & \cdot & a_3 & b_2 & (a_1a_2-a_3a_o) & \cdot \\ \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ b_{2t} & b_{2t-1} & & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

Next subtracting column 1 * $(a_1b_1 - b_2a_o)$ from column 3 and taking out a factor a_1:

$$(-1)^{\frac{t(t-1)}{2}} \frac{a_1^{t}}{a_1^{2t}} \begin{vmatrix} a_1 & a_o & -(a_1b_1-b_2a_o) & 0 & \cdot & \cdot & \cdot \\ b_2 & b_1 & (a_1a_2-a_3a_o) & a_1 & 0 & 0 & \cdot \\ a_3 & a_2 & (a_1b_3-b_4a_o) & -b_2 & a_1 & -(a_1b_1-b_2a_o) & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & & \\ b_{2t} & b_{2t-1} & (a_2a_{2t}-a_{2t+1}a_o) & & & & \cdot \end{vmatrix}$$

Moving column 3 to the left hand side we can obtain the equivalent form

$$(-1)^{t\left(\frac{t-1}{2}\right)} \frac{a_1^t}{a_1^{2t}} \begin{vmatrix} a_1 & a_o & 0 & 0 & 0 & & \cdot & \cdot & \cdot \cdots \\ -b_2 & -b_1 & a_1 & a_o & 0 & & & & \\ a_3 & a_2 & b_2 & b_1 & a_1 & 0 & & & \\ -b_4 & -b_3 & a_3 & & -b_2 & a_1 & -(a_1b_1-b_2a_o) & 0 & \\ \cdot & \cdot & \cdot & & & & (a_1a_2-a_3a_o) & a_1 & +(a_1b_1-b_2a_o \\ \cdot & \cdot & \cdot & & & & -(a_1b_3-b_4a_o) & & \\ \cdot & \cdot & \cdot & & & & & & \\ a_{2t+1} & a_{2t} & b_{2t} & b_{2t-1} & & & \cdot & \cdot & \cdot \cdots \end{vmatrix}$$

Expanding at column 7 we obtain

col 7 col 8

$$(-1)^{t\left(\frac{t-1}{2}\right)} \frac{a_1^t}{a_1^{2t}} \left|\begin{array}{cccc|cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & a_1 & a_o & 0 & 0 & 0 \\ a_1 & a_o & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b_2 & -b_1 & a_1 & a_o & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a_3 & a_2 & b_2 & b_1 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b_4 & -b_3 & a_3 & a_2 & -b_2 & a_1 & -b_2 & -b_2 & 0 & 0 & \\ a_5 & a_4 & b_4 & b_3 & a_3 & b_2 & a_2 & a_2 & a_1 & (a_1b_1-b_2a_o) & \cdot \\ \cdot & \cdot & & & & & & & & & \cdot \\ \cdot & \cdot & & & & & & & & & \cdot \end{array}\right|$$

Now move column 8 two places to the left then add row 1 to row 4 and expand about row 1; the cofactor of a_o is zero, having two columns equal:

$$-(-1)^{\frac{t(t-1)}{2}}\frac{a_1^{t+1}}{a_1^{2t}}\begin{vmatrix} a_1 & a_o & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot \\ -b_2 & -b_1 & a_1 & a_o & 0 & 0 & 0 & 0 & & \cdot \\ a_3 & a_2 & b_2 & b_1 & a_1 & a_o & 0 & 0 & & \\ -b_4 & -b_3 & a_3 & a_2 & -b_2 & -b_1 & a_1 & 0 & & \\ a_5 & \cdot & \cdot & \cdot & \cdot & \cdot & b_2 & a_1 & (a_1b_1-b_2a_o) & \cdot \\ & & & & & & & & & \\ \cdot & \cdot & \cdot & \cdot & & & & & & \\ & & & & & & & & & \\ a_{2t+1} & a_{2t} & b_{2t} & b_{2t-1} & & & & & \cdot & \cdot \end{vmatrix}$$

We can now expand about column 9 and proceed as with column 7 previously. This introduces another factor $(-a_1)$. Repetition of this process leads to

$$C_{t+1} = (-1)^{\frac{t(t-1)}{2}}\frac{(-1)^t}{a_1^{2t}}\begin{vmatrix} a_1 & a_o & 0 & 0 & \cdot & \cdot \\ -b_2 & -b_1 & a_1 & a_o & \cdot & \cdot \\ a_3 & & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & \\ a_{2t+1} & a_{2t} & b_{2t} & & \cdot & \cdot \end{vmatrix}$$

$$= \frac{(-1)^{\frac{t(t+1)}{2}}}{a_1^{2t}}\begin{vmatrix} a_1 & a_o & 0 & 0 & \cdot & \cdot \\ -b_2 & -b_1 & a_1 & a_o & & \\ a_3 & & & & & \\ & & & & & \\ & & & & & \\ a_{2t+1} & a_{2t} & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

as expected, and it may be seen that the positive factor previously missed is $[a_1 c_2 \cdots c_{t-1}]^{-2}$ for c_t. It will be noted that it has been assumed that none of the c_i's are zero.

The cases in which some of the c_j are zero or r is not equal to n are treated next.

A1.2 If f_o and f_1 have a common polynomial factor:

It may happen that f_o and f_1 have a common polynomial factor; real roots of the common factor correspond to entirely imaginary roots of $f(z) = 0$ but complex roots may occur. In these cases $f_{r+1}(y) = 0$ but $f_r(y)$ is not a constant, but is in fact a multiple of the common factor since in the sequence $f_o, f_1, f_2 \ldots f_r, 0$, if f_o and f_1 are $q\, g_o$, $q\, g_1$ where q is the common factor then $-\text{Rem}(\frac{f_o}{f_1}) = -\text{Rem}(\frac{g_o}{g_1}) = f_2$ and $g_o q = f_o = p\, g_1\, q - f_2$ so $f_2 = q(pg_1 - g_o)$ where p is some polynomial. f_2 and f_1 have common factor q and so f_3 has also. The sequence will terminate at f_r = constant $+q$ because then $f_{r-1}/f_r = qp/q\ \text{const} = p/\text{const}$ so that $\text{Rem}(f_{r-1}/f_r) = 0$.

If f_r can be found explicitly one must then try to find the number of its roots with real positive part. Routh, using another theorem of Sturm proved, in Appendix 2, suggests that f_{r+1} be replaced by $\frac{d}{dy} f_r(y)$ and the sequence continued. Repeated imaginary roots of f will lead to another occurrence of $f_s \neq \text{const}$, $f_{s+1} = 0, s \neq n$ although this may occur for other reasons. Repeated imaginary roots are of significance in that they correspond to solutions growing not exponentially but linearly so that if the type of zeros just mentioned appears twice then the roots of q must be checked explicitly,

A1.3 Zeros in the sequence of test functions

It may happen that one or several of the c_j's in the middle of the sequence are zero,(§2) above is a special case of this as $c_i = 0$ for $i = r + 1$ to n. If it is the existence of any roots with positive real part that is of interest then the following procedure may be adopted. The simplest case has some of the non-zero c_j's negative; there must be instabilities. The other possibility is that none of the c_j's are negative. The coefficients a_i, b_i can be varied by ε_i, η_i to remove the zeros, and ε_i, η_i may be chosen sufficiently small so that the non-zero c_j's are unchanged to first order but the zero terms may be written keeping only the lowest order terms. The ε_i and η_i can be varied in sign and magnitude to see if any negative terms may be produced.

Making all the ε_i, η_i non-zero makes a very complicated problem but one can make selected ε_i's, η_i's non-zero, selected after inspection of the zero determinants. A less random procedure is suggested in (§4) below.

If it is possible to find the maximum number of sign changes (Δ_{max}) and the minimum (Δ_{min}) then

$$\Delta_{max} = n_a + n_o$$

where n_a = no. of roots with real part = 0

n_o = no. of roots with real part > 0,

and $$\Delta_{min} = n_o$$

because the roots on the axis have been shifted all to the right half plane and all to the left half plane respectively

$$n_a = \Delta_{max} - \Delta_{min}$$

$$n_o = \Delta_{min}.$$

An example of this is given in Section 5.

A1.4 An algorithm for dealing with zeros

An alternative perhaps more convenient algorithm is one which follows Routh's method rather closely. As shown in (§1) f_3 can be written

$$f_3(y) = y^{n-2}\begin{vmatrix} a_1 & a_0 & 0 \\ -b_2 & -b_1 & a_1 \\ -a_3 & -a_2 & -b_2 \end{vmatrix} + y^{n-3}\begin{vmatrix} a_1 & a_0 & 0 \\ -b_2 & -b_1 & a_1 \\ -b_4 & -b_3 & a_3 \end{vmatrix}$$

$$- y^{n-4}\begin{vmatrix} a_1 & a_0 & 0 \\ -b_2 & -b_1 & a_1 \\ -a_5 & -a_4 & -b_4 \end{vmatrix} - y^{n-5}\begin{vmatrix} a_1 & a_0 & 0 \\ -b_2 & -b_1 & a_1 \\ -b_6 & -b_5 & a_5 \end{vmatrix} + \cdots$$

If $a_1 = 0$ it may be replaced by ε_1, and the determinants above may be calculated explicitly. If the leading coefficient is zero it can be replaced by ε_2 and the coefficients of $f_4(y)$ can be calculated. $f_r(y)$ may be differentiated to provide $f_{r+1}(y)$ if $f_r(y) = 0$ but $r \neq n$ or alternatively some new parameter may be introduced for the leading term. When all the f_i's , $i = 0, 1, \cdots n$ have been calculated the ε_1, ε_2 etc. are allowed to tend to zero and only the lowest order terms in these quantities need be retained. The signs and sizes of the small quantities ε_1, can be varied and Δ_{max} and Δ_{min} found, usually with considerably more ease than using §2. A convenient format for setting out calculations of the type described in §3 is suggested below.

A1.5 Some simple applications of the method

Consider

$$f(z) = (z-1)\,(z+1)\,(z-i)^2 = (z^2-1)\,(z^2-2iz-1)$$

$$= 1z^4 + z^3\{0-2i\} + z^2\{-2+0i\} + z\{0+2i\} + 1$$

$$= a_o z^4 + z^3\{a_1+b_1 i\} + z^2\{a_2+b_2 i\} + z\{a_3+b_3 i\} + \{a_4+b_4 i\}$$

$$a_o = 1$$

$$a_1 = 0$$

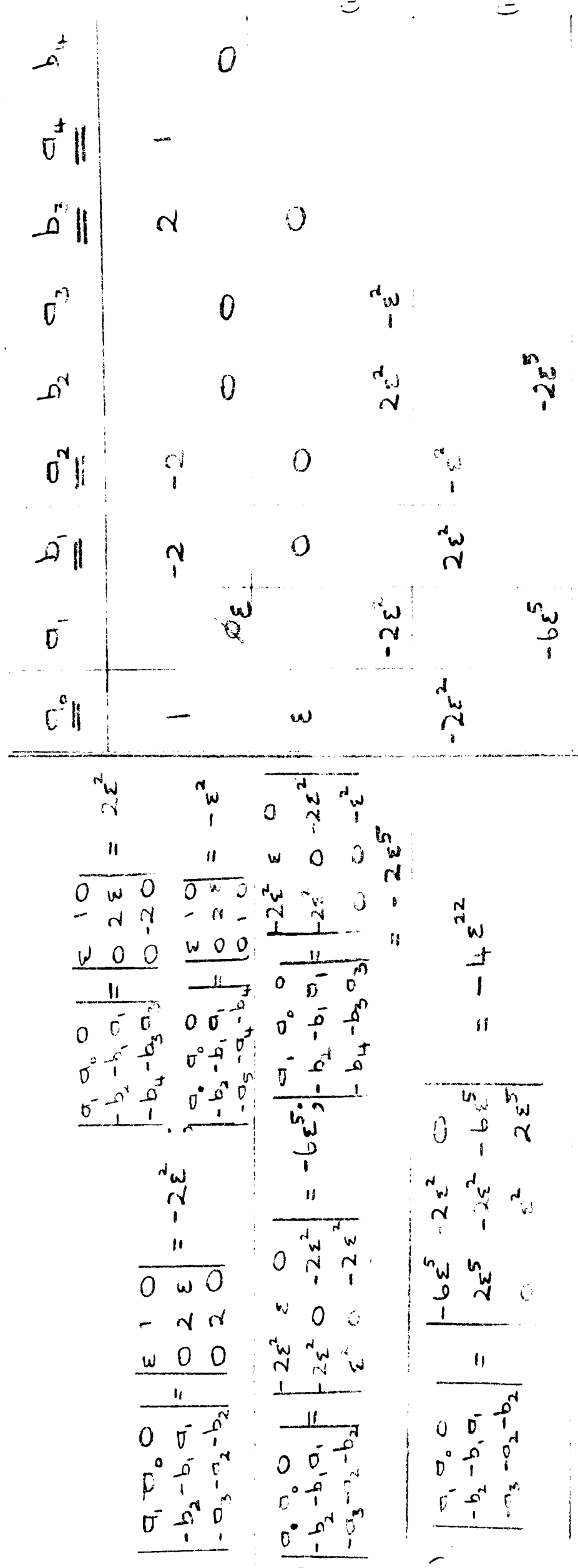
$$c_2 = -\begin{vmatrix} a_1 & a_o & 0 \\ -b_2 & -b_1 & a_1 \\ a_3 & a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & 0 \end{vmatrix} = 0$$

$$c_3 = -\begin{vmatrix} a_1 & a_o & 0 & 0 & 0 \\ -b_2 & -b_1 & a_1 & a_o & 0 \\ a_3 & a_2 & b_2 & b_1 & a_1 \\ -b_4 & -b_3 & a_3 & a_2 & -b_2 \\ a_5 & a_4 & b_4 & b_3 & a_3 \end{vmatrix} = -\begin{vmatrix} 0 & 1. & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & -2 & 0 & -2 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{vmatrix}$$

$$= 0$$

$$c_4 = +\begin{vmatrix} a_3 & a_o & 0 & 0 & 0 & 0 & 0 \\ -b_2 & -b_1 & a_1 & a_o & 0 & 0 & 0 \\ a_3 & a_2 & b_2 & b_1 & a_1 & a_o & 0 \\ -b_4 & -b_3 & a_3 & a_2 & -b_2 & -b_1 & a_1 \\ a_5 & a_4 & b_4 & b_3 & a_3 & a_2 & b_2 \\ -b_6 & -b_5 & a_5 & a_4 & -b_4 & -b_3 & a_3 \\ a_7 & a_6 & b_6 & b_5 & a_5 & a_4 & b_4 \end{vmatrix} = 0$$

It will be noted that putting $a_1 = \varepsilon$ and letting $\varepsilon < 0$ there must be a change in sign in the sequence so there is indeed at least one root with positive real part.


(i)
(ii)
(iii)
(iv)
(v)

The sequence a_o, a_1, c_2, c_3, c_4 is

$$1 \quad , \quad \epsilon \quad , \quad -2\epsilon^2 \quad , \quad -6\epsilon^5 \quad , \quad -4\epsilon^{22}$$

$\epsilon > 0$	+	+	−	−	−	$\Delta = 1$
$\epsilon < 0$	+	−	−	+	−	$\Delta = 3$

$$\Delta_{max} = 3 \quad , \quad \Delta_{min} = 1$$

so the number of roots on the axis is $\Delta_{max} - \Delta_{min} = 3 - 1 = 2$ number of roots with real part > 0 is $\Delta_{min} = 1$. In fact the roots are -1, +1, i, i, i.e. 2 on the axis and one with real part > 0.

Line (i) contains f_o and f_1 or rather the values of a_i, b_i; the upper part of line (ii) is just f_1 repeated but moved along the underlined columns a_o, b_1, a_2 .... etc. The lower part of (ii) is calculated using the determinants given above, it is in fact f_3. f_3 is then moved into the underlined columns in line (iii). For f_4, the lower line of (ii) can be easily calculated since the necessary numbers are under the appropriate column names to use in the general determinant form.

Another alternative way to treat the equation

$$(z - 1)\,(z + 1)\,(z - i)^2 = 0$$

is to calculate $\frac{df_i}{dy}$ if $f_{i+1} = 0$, but $i \neq n$. Because f_i has been explicitly calculated it is easy to differentiate, noting that the highest power of y decreases by unity going from one pair to the next pair below, and decreases by unity going along each line to the right.

$$f_o = y^4 - 2y^3 + 2y^2 - 2y + 1$$

$f_1 = 0$ so that f_1 is replaced by

$$f_1 = 4y^3 - 6y^2 + 4y - 2 = 4y^3 + (-6)\,y^2 - (-4)\,y - (+2).$$

Similarly at (iv) below,

$$f_3 = -y + 1$$

$$f_4 = 0$$

so f_4 is replaced by -1.

	a_0	a_1	b_1	a_2	b_2	a_3	b_3	a_4	b_4	
(i)	1		-2	-2			2	1		
		0			0	0			0	replaced by
		4			-6	-4			2	
(ii)	4		-6	-4			2			divide by 2
	2		-3	-2			1			
		-1			4	3				
(iii)	-1		4	3						
		-1			1					
(iv)	-1		1							
		0								replaced by
		-1								
(v)	-1									

The sequency of leading coefficients is

$$1, 2, -1, -1, -1.$$

The number of changes in sign is $\Delta = 1$. Moreover since two differentiations have been performed there must be either a repeated root of $f(z)$ on the axis, or $f_o(y)$ and $f_1(y)$ have some common factor of the form

$$(y - a - ib)^2, \quad b \neq 0$$

The last replacement involved $f_3(y) = -y + 1$; this must be the square root of the common factor, and since

$$f_3(y) = 0$$

has a real root $y = 1$ the original equation $f(z) = 0$ must have a double root $z = iy = i$.

APPENDIX 2

Proof of a theorem due to Sturm.

The following theorem due to Sturm has been used in Chapter 3:-

Given $$f_o(x) = c_o x^n + c_1 x^{n-1} + c_2 x^{n-2} \ldots + c_n = 0 \qquad (c_n \neq 0)$$

$$f_1(x) = d_o x^m + d_1 x^{m-1} + d_2 x^{m-2} \qquad + d_m = 0 \qquad (m \leq n)$$

$$f_2(x) = -\text{Rem}\ (f_o/f_1)$$

$$f_3(x) = -\text{Rem}\ (f_1/f_2)$$

$$f_r(x) = -\text{Rem}\ (f_{r-2}/f_{r-1})$$

$$f_{r+1}(x) = 0.$$

When x increases in the interval $A < x < B$ the number of real root of $f_o(x) = 0$ for which f_o and f_1 change from unlike to like signs, minus the number of real roots of $f_o(x) = 0$ for which f_o and f_1 change from like to unlike signs, is equal to the number of variations of sign in the sequence

$$f_o(A),\ f_1(A),\ \ldots\ f_r(A) \qquad \text{(A2.1)}$$

minus the number of variations of sign in the sequence

$$f_o(B),\ f_1(B),\ \ldots\ f_r(B) \qquad \text{(A2.2)}$$

Multiple roots are counted only once and any zero terms are omitted from (A2.1) and (A2.2). If, however, all of the terms in (A2.1) or (A2.2) are zero then A or B must be changed by a small amount.

Proof

$$f_{i-1}(x) = q_i(x)\, f_i(x) - f_{i+1}(x) \qquad i = 1, 2, \ldots r \quad \text{(A2.3)}$$

$$f_{r+1}(x) = 0$$

where $q_i(x)$ is a polynomial. The degree of f_i decreases as i increases.

(i) If $f_r(x)$ = constant ($\neq 0$) (A2.4)

As x increases $f_o(x)$, $f_1(x)$,.....$f_r(x)$ can change in the number of variations in sign only when one or more of the $f_i(x)$ change in sign. Now no two neighbouring members can become zero simultaneously because otherwise the next in the sequence will also be zero, from (A2.3), and all the members will be zero contradicting (A2.4). Also when $f_i(x) = 0$ it follows that $f_{i+1}(x)$ and $f_{i-1}(x)$ have opposite sign. Therefore the passage of an internal term $f_i(x)$ through zero does not change the number of variations in sign of the sequences. It follows that the number of sign changes ($\Delta(x)$) only alters when the end terms change sign, and since $f_r(x)$ is constant only changes in sign of $f_o(x)$ contribute. If as $f_o(x)$ passes through zero $f_o(x)$ and $f_1(x)$ change from like to unlike sign then $\Delta(x)$ increases by unity; if they change from unlike to like sign then Δ decreases by unity.

(ii) $f_r(x) \neq$ const.

This implies

$$f_{r-1}(x) = q_r(x)\, f_r(x)$$

where $q_r(x)$ is some polynomial. $f_{r-1}(x)$ and $f_r(x)$ therefore have a common factor and

$$f_{r-2}(x) = q_{r-1}(x)\, f_{r-1}(x) - f_r(x)$$

$$= q_{r-1}(x)\,(q_r(x) - 1)\, f_r(x)$$

so that $f_{r-1}(x)$ and $f_{r-2}(x)$ have common factor $f_r(x)$. In a similar way one can show that all the $f_i(x)$, $i = 0, 1 \ldots\ldots r$ have $f_r(x)$ as a common factor.

Defining

$$h_i(x) = \frac{f_i(x)}{f_r(x)} \qquad i = 0, 1, \ldots\ldots r$$

the $h_i(x)$ will be polynomials and $h_r(x)$ is constant so that (i) above may be applied. One can then multiply the sequence by $f_r(x)$; $\Delta(x)$ cannot be changed by this operation so that the theorem is true in this case also.

One can investigate the properties of $f_r(x)$ further by defining

$$f_o'(x) = f_r(x)$$

$$f_1'(x) = \frac{df_r}{dx}(x)$$

As $f_o'(x)$ passes through zero $f_o'(x)$ and $f_1'(x)$ must change from unlike to like signs so that applying the procedure in (i) to $f_o'(x)$, $f_1'(x)$ one obtains the number of real roots of $f_r(x) = 0$ which have $A < x < B$, multiple roots being counted once and zeros are omitted from the evaluation of Δ. Note that if there are any repeated roots in $f_r(x) = 0$ then $f_o'(x)$ and $f_1'(x)$ will also have a common factor.

To see the application of Sturm's theorem to Routh's work, detailed in Chapter 3, one needs to remember that

$$\text{Arg}(f(z)) = \text{Arg}(P(z)+iQ(z)) = \tan^{-1}\left(\frac{Q}{P}\right)$$

$$= \text{Arg}\left(-\frac{f_1(y)}{f_0(y)}\right)$$

if $z = iy$,

and consider the variation of $\tan\theta$ as θ increases. Passing through a zero of $f_o(y)$, if $\arg(f(z))$ is increasing then $f_1(y)$ and $f_2(y)$ change from like to unlike signs, but if $\arg(f(z))$ is decreasing then they change from unlike to like signs. $\text{Arg}(f(z))$ increases by π going between two zeros of $f_o(z)$, if $f_1(z)$ changes sign once in between. Therefore the number $\Delta(A) - \Delta(B)$ given by application of Sturm's theorem is minus the increase in $\frac{1}{\pi}\text{Arg}(f(iy))$ going from $y = A$ to $y = B$. As A tend to $+\infty$ and B to $-\infty$ the leading term in each polynomial $f_i(y)$ dominates the numerical value of the expression so that in Routh's method one need only consider the initial terms.

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INDEX OF TABLES

Table 1 values of I(photons $cm^{-1}s^{-1}$), n(cm^{-3}), T(K) and the derivatives of L and X.
Units of L ergs $s^{-1}gm^{-1}$
Units of X s^{-1}
I_c =J = 1.3(7)photons $cm^{-1}s^{-1}$
Model 1 of interstellar gas.

Table 2 as Table 1, but for model 2.

Table 3 values of 1/Re( ) (s) for model 1, in cases where this is positive. is a root of equation 2.24, in which the effects of radiative transfer are included.

Table 4 as Table 3, but for model 2.

Table 5 as Table 1,but for pure Hydrogen gas.

Table 6 as Table 3,but for pure Hydrogen gas.

Table 7 as Table 3,but for roots of equation 7.79 describing the Doppler induced instabilities.
X_v = (1-x)(/J).5(-17) cm^{-1}

Table 8a as Table 7, but X_v = (1-x)(/J)5(-17) cm^{-1}

Table 8b as Table 7, but X_v =0

Table 9 as Table 7, but for model 2.

Table 10 as Table 8a, but for model 2.

Table 11 as Table 7, but for pure Hydrogen gas.

Throughout these tables the notation ( ) denotes power of ten, so that for example 5.7(-2)=5.7 10^{-2}.

Most numbers are given to only two significant figures.

I	n	T	L_T	L_n	L_x	L_I	X_T	X_n	X_x	X_I
3.0(4)	30	48.9	7.9(-5)	6.8(-5)	1.4(1)	-4.8(-9)	3.0(-18)	1.0(-17)	3.0(-11)	-1.0(-20)
	300	24.9	5.0(-4)	1.1(-5)	1.4(1)	-4.8(-8)	6.0(-17)	1.0(-17)	3.0(-11)	-1.0(-19)
	3000	20.9	3.4(-3)	5.4(-6)	1.4(1)	-4.8(-7)	7.2(-16)	1.0(-17)	3.0(-11)	-1.0(-18)
	30000	21.0	3.0(-2)	4.8(-6)	1.5(1)	-4.7(-6)	7.1(-15)	1.0(-17)	3.0(-11)	-1.0(-17)
3.0(2)	30	49.0	7.9(-5)	6.8(-5)	1.5(-1)	-4.8(-7)	3.0(-18)	1.0(-17)	3.0(-13)	-1.0(-18)
	300	25.0	5.0(-4)	1.1(-5)	1.5(-1)	-4.8(-6)	6.0(-17)	1.0(-17)	3.0(-13)	-9.9(-18)
	3000	20.9	3.2(-3)	5.1(-6)	1.6(-1)	-4.4(-5)	6.5(-16)	9.1(-18)	3.3(-13)	-9.1(-17)
	30000	20.4	1.6(-2)	2.5(-6)	2.6(-1)	-2.4(-4)	3.7(-15)	5.0(-18)	6.0(-13)	-5.0(-16)
3.0(0)	30	53.0	7.4(-5)	7.3(-5)	5.8(-3)	-3.8(-5)	2.6(-18)	9.2(-18)	3.8(-15)	-7.9(-17)
	300	29.9	3.9(-4)	1.3(-5)	9.3)-3(	-2.2(-4)	2.7(417)	5.4(-18)	6.5(-15)	-4.6(-16)
	3000	22.5	9.8(-4)	1.8(-6)	2.3(-2)	-4.3(-4)	6.9(-17)	1.0(-18)	3.3(-14)	-8.9(-16)
	30000	15.7	2.1(-3)	1.8(-7)	3.2(-2)	-4.8(-4)	1.1(-16)	1.1(-17)	3.0(-13)	-9.9(-16)

TABLE 1

I	n	T	L_T	L_n	L_x	L_I	X_T	X_n	X_x	X_I
3.0(-2)	30	66.0	1.1(-4)	7.7(-5)	4.0(-3)	-1.7(-4)	1.4(-18)	6.4(-18)	8.3(-16)	-3.6(-16)
	300	41.9	2.2(-4)	1.4(-5)	1.5(-2)	-4.1(-4)	5.3(-18)	1.5(-18)	3.5(-15)	-8.5(-16)
	3000	23.4	7.3(-4)	1.5(-6)	2.5(-2)	-4.7(-4)	1.1(-17)	1.7(-19)	3.1(-14)	-9.8(-16)
	30000	15.1	1.7(-3)	1.4(-7)	2.5(-2)	-4.8(-4)	1.7(-17)	1.7(-20)	3.0(-13)	-1.0(-15)

TABLE 1 (continued)

I	n	T	L_T	L_n	L_x	L_I	X_T	X_n	X_x	X_I
3.0(4)	30	57.6	7.7(-5)	6.8(-5)	1.4(1)	-4.8(-9)	2.6(-18)	1.0(-17)	3.0(-11)	-1.0(-20)
	300	35.3	7.3(-4)	1.1(-5)	1.4(1)	-4.8(-8)	4.2(-17)	1.0(-17)	3.0(-11)	-1.0(-19)
	3000	33.2(	6.9(-3)	5.4(-6)	1.4(1)	-4.8(-7)	4.5(-16)	1.0(-17)	3.0(-11)	-1.0(-18)
	30000	34.1	6.3(-2)	4.8(-6)	1.4(1)	-4.8(-6)	4.3(-15)	1.0(-17)	3.0(-11)	-1.0(-17)
3.0(2)	30	57.6	7.5(-5)	6.8(-5)	1.5(-1)	-4.8(-7)	2.6(-18)	1.0(-17)	3.0(-13)	-1.0(-17)
	300	35.5	7.3(-4)	1.1(-5)	1.5(-1)	-4.8(-6)	4.2(-17)	1.0(-17)	3.0(-13)	-9.9(-18)
	3000	33.1	6.4(-3)	5.1(-6)	1.5(-1)	-4.4(-5)	4.1(-16)	9.1(-18)	3.3(-13)	-9.1(-17)
	30000	32.6(	3.5(-2)	2.5(-6)	8.5(-2)	-2.4(-4)	2.3(-15)	5.0(-18)	6.0(-13)	-5.0(-16)
3.0(0)	30	61.1	7.7(-5)	7.2(-5)	5.1(-3)	-3.8(-5)	2.3(-18)	9.2(-18)	3.8(-15)	-7.9(-17)
	300	38.3	4.6(-4)	1.3(-5)	7.7(-3)	-2.2(-4)	2.1(-17)	5.4(-18)	6.5(-15)	-4.6(-16)
	3000	28.3	1.4(-3)	1.8(-6)	-1.9(-2)	-4.3(-4)	5.5(-17)	1.0(-18)	3.3(-14)	-9.0(-16)
	30000	18.4	3.5(-3)	1.8(-7)	-6.0(-1)	-4.8(-4)	9.4(-17)	1.2(-19)	3.0(-13)	-9.9(-16)

TABLE 2

I	n	T	L_T	L_n	L_x	L_I	X_T	X_n	X_x	X_I
3.0(-2)	30	69.9	1.3(-4)	6.8(-5)	2.8(-3)	-1.7(-4)	1.4(-18)	6.4(-18)	8.3(-16)	-3.6(-16)
	300	46.4	2.3(-4)	1.4(-5)	1.2(-2)	-4.1(-4)	4.8(-18)	1.5(-18)	3.5(-15)	-8.5(-16)
	3000	24.9	8.1(-4)	1.5(-6)	-3.6(-2)	-4.7(-4)	1.0(-17)	1.7(-19)	3.1(-14)	-9.8(-16)
	30000	15.7	2.0(-3)	1.4(-7)	-6.4(-1)	-4.8(-4)	1.7(-17)	1.7(-20)	3.0(-13)	-1.0(-15)

TABLE 2 (continued)

			k(cm^{-1})				
I	n	T	1(-13)	1(-15)	1(-17)	1(-19)	1(-21)
3(4)	30	48.9					5.9(20)
	300	24.5				3.8(16)	3.8(19)
	3000	20.9					
	30000	21.0		8.0(13)	8.0(13)	1.2(15)	1.1(17)
3(2)	30	49.0				5.6(15)	9.0(17)
						3.8(15)	7.7(17)
3	30	53.0				5.6(15)	9.0(17)
						5.6(15)	9.0(17)
3(-2)	300	41.9				9.0(15)	4.0(19)
						9.0(15)	4.0(19)

TABLE 3

I	n	T	1(-13)	1(-15)	1(-17)	1(-19)	1(-21)
3(4)	30	57.6					1.8(21)
	300	35.4				2.1(16)	2.2(19)
	3000	33.2					
	30000	34.2		6.3(13)	6.3(13)	6.3(13)	7.1(17)
3(2)	30	57.6					4.8(19)
	300	35.5					
	3000	33.1		1.5(16)	1.5(16)	1.5(16)	3.6(17)

TABLE 4

TABLE 5

I	n	T	L_T	L_n	L_x	L_I	X_T	X_n	X_x	X_I
3.0(4)	30	1000	8.3(-10)	7.7(-9)	-4.3	-7.2(-12)	4.5(-22)	3.0(-20)	3.0(-11)	-3.0(-22)
	300	1000	3.8(-8)	3.6(-8)	-4.6(1)	-7.2(-11)	4.5(-21)	3.0(-20)	3.0(-11)	-3.0(-22)
	3000	1000	1.8(-6)	1.6(-7)	-2.4(+2)	-7.2(-10)	4.5(-20)	3.0(-20)	3.0(-11)	-3.0(-21)
	30000	1000	8.2(-5)	7.7(-7)	-1.1(3)	-7.2(-9)	4.5(-19)	3.0(-20)	3.0(-11)	-3.0(-20)
3.0(2)	30	1000	8.2(-8)	7.6(-7)	-1.1(1)	-7.2(-10)	4.5(-22)	3.0(-20)	3.0(-13)	-3.0(-21)
	300	1000	3.8(-6)	3.5(-6)	-5.3(1)	-7.2(-9)	4.5(-21)	3.0(-20)	3.0(-13)	-3.0(-20)
	3000	1000	1.8(-4)	1.6(-5)	-2.5(+2)	-7.2(-8)	4.5(-20)	3.0(-20)	3.0(-13)	-3.0(-19)
	30000	1000	8.2(-3)	7.6(-5)	-1.1(3)	-7.2(-7)	4.5(-19)	3.0(-20)	3.0(-13)	-3.0(-18)
3.0	30	1000	7.1(-6)	6.6(-5)	-1.1(1)	-6.2(-8)	4.5(-22)	3.0(-20)	3.5(-15)	-2.6(-19)
	300	1000	3.3(-4)	3.0(-4)	-5.3(1)	-6.2(-7)	4.5(-21)	3.0(-20)	3.5(-15)	-2.6(-18)
	3000	1000	1.5(-2)	1.3(-3)	-2.3(2)	-6.0(-6)	4.4(-20)	2.9(-20)	3.6(-15)	-2.5(-17)
	30000	1000	4.5(-1)	4.2(-3)	-6.8(2)	-4.9(-5)	3.6(-19)	2.4(-20)	4.4(-15)	-2.0(-16)
3.0(-2)	30	1000	4.7(-5)	4.4(-4)	-1.1(1)	-4.1(-7)	4.5(-22)	3.0(-20)	5.3(-16)	-1.7(-18)
	300	1000	2.1(-3)	2.0(-3)	-5.1(1)	-4.0(-6)	4.4(-21)	2.9(-20)	5.3(-16)	-1.7(-17)

I	n	T	k (cm^{-1}) 1(-13)	1(-15)	1(-17)	1(-19)	1(-21)
3(4)	30	1000					8.3(15)
	300	1000					2.0(15)
	3000	1000				2.1(13)	2.0(15)
	30000	1000			5.1(11)	2.0(13)	2.0(15)
3(2)	30	1000					6.7(14)
	300	1000				3.4(13)	2.0(15)
	3000	1000		4.8(13)	5.0(12)	2.8(13)	2.0(15)
	30000	1000		3.8(12)	4.0(12)	2.8(13)	2.0(15)
3	30	1000				5.0(14)	3.6(15)
	300	1000			4.5(14)	4.5(14)	3.6(15)
	3000	1000		4.5(14)	4.5(14)	4.5(14)	3.7(15)
	30000	1000	7.1(14)	6.3(14)	6.3(14)	6.3(14)	5.3(15)

TABLE 6

			k (cm^{-1})				
I	n	T	1(-13)	1(-15)	1(-17)	1(-19)	1(-21)
3(2)	3000	20.9				3.1(19)	3.1(23)
	30000	20.4			4.5(15)	3.2(18)	3.1(22)
3	30	53.0		3.0(12)	1.4(12)	1.6(14)	1.0(16)
	300	29.9		9.0(10)	9.0(11) 7.1(14)	3.3(16)	3.2(20)
	3000	22.5		4.3(10)	3.7(12) 6.3(14)	6.7(15)	6.3(19)
	30000	15.7	2.8(10)	5.0(10)	1.7(13)	2.6(17)	2.6(20)
3(-2)	30	66.4		1.9(11)	5.3(11)	1.0(14) 1.0(15)	2.4(15)
	300	41.9		4.8(10)	4.2(11)	7.1(13)	9.0(14)
	3000	23.4		3.8(10)	1.9(12) 1.4(14)	1.5(17)	1.5(21)
	30000	15.1	2.6(10)	4.3(10)	1.7(13)	2.1(16)	2.0(20)

TABLE 7

k (cm^{-1})

I	n	T	1(-13)	1(-15)	1(-17)	1(-19)	1(-21)
3	30	53.0				7.7(15)	2.2(19)
	300	29.9				2.6(17)	1.6(21)
3(-2)	30					7.7(15)	9.0(17)
	300	41.9				1.1(15)	2.0(18)
	3000	23.4				2.2(17)	2.2(21)
	30000	15.1			7.1(14)	3.6(18)	3.6(22)

TABLE 8a

I	n	T	1(-13)	1(-15)	1(-17)	1(-19)	1(-21)
3	30	53.0				5.0(16)	9.0(19)
						5.0(16)	9.0(19)
3(-2)	300	41.9				1.0(16)	4.5(19)
						1.0(16)	4.5(19)

TABLE 8b

			k (cm^{-1})				
I	n	T	1(-13)	1(-15)	1(-17)	1(-19)	1(-21)
3(2)	30	57.6					
	300	35.5		4.8(12)		1.9(17)	1.7(21)
	3000	33.1		2.8(11)	5.3(13)	1.4(17)	1.4(21)
	30000	32.6	5.0(10)	1.0(12)	4.3(13)	2.3(17)	2.3(21)
3	30	61.1		1.9(11)	5.0(11) 1.0(15)	2.2(13) 1.2(16)	9.0(19)
	300	38.3		2.3(10)	2.6(11) 1.8(14)	5.9(14)	1.0(16)
	3000	28.3	1.4(10)	1.5(10)	5.6(11) 3.2(13)	6.7(13)	6.7(16)
	30000	18.4	5.3(9)	1.7(10) 3.4(12)	3.7(12)	2.8(14)	2.6(18)
3(-2)	30	69.9		3.7(10)	2.2(11) 5.9(14)	9.0(12) 8.3(14)	1.0(15)
	300	46.4		1.5(10)	7.7(14) 1.5(11)	1.7(17) 1.8(13)	1.7(21)
	3000	24.9	1.0(10)	1.3(10)	2.8(11) 2.7(13)	5.0(13)	1.6(16)
	30000	15.7	4.8(9)	1.4(10) 2.3(12)	1.0(13) 3.6(12)	9.0(13)	7.3(17)

TABLE 9

k (cm^{-1})

I	n	T	1(-13)	1(-15)	1(-17)	1(-19)	1(-21)
3	30	1000				1.1(16)	2.8(19)
	300	1000				8.3(15)	4.9(18)
	3000	1000				8.3(17)	8.3(21)
3(-2)	30	1000				1.8(15)	1.1(17)
	300	1000			1.9(14)	2.6(15)	9.1(18)
	3000	1000			1.9(14)	9.0(15)	9.0(19)
	30000	1000			6.3(13)	2.9(17)	2.9(21)

TABLE 10

I	n	T	1(-13)	1(-15)	1(-17)	1(-19)	1(-21)
3(2)	30	1000				1.4(15)	4.3(15)
	300	1000				5.3(13)	4.2(16)
	3000	1000				4.8(13)	3.4(17)
	30000	1000			1.9(12)	5.3(13)	4.2(17)
3	30	1000				1.1(13)	9.1(13)
	300	1000				7.7(12)	9.0(13)
	3000	1000			1.6(12)	7.7(12)	9.0(13)
	30000	1000			1.5(12)	8.3(12)	1.0(14)
3(-2)	30	1000				8.3(12)	7.7(13)
	3000	1000			2.2(12)	7.7(12)	7.7(13)

TABLE 11