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ON THE FOURIER SERIES OF THE SQUARE OF A FUNCTION

by

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Abstract

The problem set in this paper was that of finding restrictions on a function which were to be sufficient to insure the convergence of the Fourier series of the square of the function whenever the Fourier series for the function itself converged. It is known that whenever the Fourier series of a function is absolutely convergent[1] the Fourier series of the square of the function also converges absolutely which is itself a positive result of the type desired. The interest of this investigation lies in the fact that the convergence of the Fourier series of the function is of itself not enough to insure the convergence of the Fourier series for the square of the function. In fact, R. Salem[2] has exhibited a continuous periodic function whose Fourier series converges uniformly everywhere but yet the Fourier series for the square of this function diverges at an everywhere dense set of points.

In this paper the author proves two theorems, the first dealing with the situation at a single point, while the second is an extension of the first throughout an interval. In both cases the convergence of the Fourier series of the function together with the fact that the Fourier coefficients of the function are $o(\frac{1}{n})$ insure the convergence of the Fourier series of the square of the function. This hypothesis is obviously less stringent than that of absolute convergence, nevertheless

(1) Wiener, Levy, p.140, Trigonometric Series, Zygmund.

(2) "A Singularity of the Fourier Series of Continuous Functions" Duke Mathematical Journal, 4, 10(1943).

there is still considerable uncharted land between these series and those of Salem's examples which have terms $O(\frac{1}{\log n})^{\frac{1}{2}}$.

The main theorems have been stated and proved for Fourier series but the author points out they are also true for power series on the circle of convergence.

Let f(x) be a Lebesgue integrable function of period 2π, and let

$$S_n(x) = \frac{a_0}{2} + \sum_1^n (a_n \cos nx + b_n \sin nx)$$

where

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos nt \, dt$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin nt \, dt$$

are the Fourier coefficients of f(x).

Theorem I: If the Fourier coefficients of f(x) are $o(\frac{1}{n})$ then at a point x_0 where $s_n(x_0) \to f(x_0)$ as $n \to \infty$, the Fourier series for $f^2(x)$ converges to $f^2(x_0)$.

The proof is divided into two parts. In the first step it is shown that the Cauchy product for $[s_n(x_0)]^2$ approaches $[f(x_0)]^2$ as $n \to \infty$. In the second step it is shown that the difference between the n th partial sum of the Fourier series for $f^2(x)$ and the Cauchy product for $s_n^2(x)$ can be made as small as desired by choosing n large enough. By combining these steps the desired result is obtained.

First Step: For brevity write

$$S_n(x_0) = \frac{a_0}{2} + \sum_1^n (a_n \cos nx_0 + b_n \sin nx_0) = \sum_0^n A_n$$

Then the Cauchy product of order k is

$$P_c(x_0,k) = \sum_{n=0}^{k} \sum_{p=0}^{n} A_p A_{n-p} .$$

An idea of G. Hardy[1] is used to show that $P_c(x_0,k) \to f(x_0)$ as $k \to \infty$.

$$P_c(x_0,k) = \sum_{p=0}^{0} A_p A_{-p} + \sum_{p=0}^{1} A_p A_{1-p} + \sum_{p=0}^{2} A_p A_{2-p} + \cdots + \sum_{p=0}^{K} A_p A_{K-p} .$$

Factor, A_0 out of every summation, A_1 out of every summation but the first, etc.

$$P_c(x_0,k) = A_0 s_K + A_1 s_{K-1} + A_2 s_{K-2} + \cdots + A_K s_0 .$$

Now let N be the integral part of $\frac{k}{2}$.

$$P_c(x_0,k) = A_0 s_K + A_1 s_{K-1} + \cdots + A_N s_{K-N} + A_{N+1} s_{K-N-1} + \cdots + A_K s_0 . \qquad (1)$$

The first N+1 terms are transformed as follows: In the first term A_0 multiplied by the last term of s_k, namely A_k, is extracted and written as $A_k s_0$. This leaves the first term as $A_0 s_{k-1}$.

(1) "Multiplication of Conditionally Convergent Series" Proc. London Math.Soc. (2) 6(1908) pp. 410 - 23.

Now from the first two terms the last term of s_{k-1}, namely A_{k-1}, is extracted with its multipliers A_0 and A_1, and written as $A_{k-1}s_1$. Continue in this fashion until the term $A_{k-N+1}s_{N-1}$ has been extracted and written. At this stage the first N+1 terms look as follows

$$A_0 s_{K-N} + A_1 s_{K-N} + \cdots + A_N s_{K-N}.$$

This may be written shortly as $s_N s_{k-N}$, so that finally (1) is transformed into

$$P_c(x_0, k) = A_K s_0 + A_{K-1} s_1 + \cdots + A_{K-N+1} s_{N-1} + s_N s_{K-N} + \\ + A_{N+1} s_{K-N-1} + \cdots + A_K s_0 . \tag{2}$$

The above transformation is essentially that of Abel's Lemma. Rearranging (2) gives

$$P_c(x_0, k) - s_N s_{K-N} = \sum_{p=0}^{N-1} s_p A_{K-p} + \sum_{p=0}^{K-N-1} s_p A_{K-p} .$$

It is given that $s_n \to f(x_0)$ as $n \to \infty$. From this it can be inferred that

$$|s_n| < M$$

, a positive constant, for all n. Further, since the Fourier coefficients of f(x) are $o(\frac{1}{n})$, it is

possible, given any $\epsilon > 0$, to find an n_o such that

$$|A_n| < \frac{\epsilon}{n}$$

when $n > n_o$.

Thus

$$|P_c(x_o,k) - f^2(x_o)| \leq |P_c(x_o,k) - S_N S_{K-N}| + |S_N S_{K-N} - f^2(x_o)|$$

$$< M\left(\sum_{K-N+1}^{K} |A_p| + \sum_{N+1}^{K} |A_p|\right) + |S_N S_{K-N} - f^2(x_o)|$$

$$< M\epsilon\left(\sum_{K-N+1}^{K} \frac{1}{p} + \sum_{N+1}^{K} \frac{1}{p}\right) + |S_N S_{K-N} - f^2(x_o)|, \quad K > 2n_o.$$

As $k \to \infty$ so does N, and $k - N$. Then the parenthesis above is bounded, and the term on the right approaches zero as $k \to \infty$. Hence finally

$$P_c(x_o,k) \to f^2(x_o) \qquad \text{as } k \to \infty.$$

Second Step: It is to be shown here that the k th partial sum of the Fourier series for $f^2(x)$ minus the Cauchy product of order k approaches zero as $k \to \infty$. The difficulty lies in reducing these two expressions to a similar form so that a convenient expression for their difference can be obtained. It seems to be most advantageous to use the exponential form of the Fourier series. Let

$$c_{-n} = \frac{1}{2}(a_n + i b_n)$$

$$c_n = \frac{1}{2}(a_n - i b_n)$$

$$\mathbb{C}_n = c_n e^{inx}$$

so that

$$A_n = a_n \cos nx + b_n \sin nx = \mathbb{C}_n + \mathbb{C}_{-n}$$

$$A_0 = \frac{a_0}{2} = \mathbb{C}_0$$

$$S_n = \sum_{p=0}^{n} A_p = \sum_{p=-n}^{n} \mathbb{C}_p$$

The Cauchy product can be transformed as follows.

$$P_c(x,K) = \sum_{n=0}^{K} \sum_{p=0}^{n} A_p A_{n-p}$$

$$= \mathbb{C}_0^2 + \sum_{n=1}^{K} \sum_{p=1}^{n} (\mathbb{C}_p + \mathbb{C}_{-p})(\mathbb{C}_{n-p} + \mathbb{C}_{-n+p})$$

$$= \mathbb{C}_0^2 + \sum_{n=1}^{K} \sum_{p=1}^{n} (\mathbb{C}_p \mathbb{C}_{n-p} + \mathbb{C}_{-p} \mathbb{C}_{-n+p} + \mathbb{C}_{-p} \mathbb{C}_{n-p} + \mathbb{C}_p \mathbb{C}_{-n+p})$$

Since A_p is the sum of two $\mathbb{C}$'s except when p = 0, the summation on p extends only down to one. The two terms arising from p = 0 are placed in with the two terms arising from p = n to give the four terms shown when p = n.

$$P_c(x,k) = C_0^2 + \sum_{n=1}^{K} \sum_{p=1}^{n} C_p C_{n-p} + \sum_{n=-1}^{-K} \sum_{p=-1}^{n} C_p C_{n-p}$$

$$+ \sum_{\substack{1 \le s-r \le K \\ r \le -1 \\ s \ge 0}} C_r C_s + \sum_{\substack{-K \le m-q \le -1 \\ q \ge 1 \\ m \le 0}} C_q C_m$$

Expanding the first three terms gives

$$P_c(x,k) = \sum_{p=1}^{K} C_p C_{k-p} + \sum_{p=1}^{K-1} C_p C_{K-1-p} + \sum_{p=1}^{K-2} C_p C_{K-2-p} + \cdots$$

$$\cdots + \sum_{p=1}^{1} C_p C_{1-p} + C_0 C_0 + \sum_{p=-1}^{-1} C_p C_{-1-p} + \cdots$$

$$\cdots + \sum_{-1}^{-K+2} C_p C_{-K+2-p} + \sum_{-1}^{-K+1} C_p C_{-K+1-p} + \sum_{-1}^{-K} C_p C_{-K-p} + \qquad (3)$$

$$+ \sum_{\substack{1 \le s-r \le K \\ r \le -1 \\ s \ge 0}} C_r C_s + \sum_{\substack{-K \le m-q \le -1 \\ q \ge 1 \\ m \le 0}} C_q C_m .$$

It remains to put the last two summations into the form of the first 2k + 1 summations. This is done in the following step by step manner. It is first noted that the index sums r + s and q + m are both bounded by $\pm k$. Hence the last two summations will be completely exhausted if for each index sum r + s, and q + m between +k and -k the corresponding terms are removed and added to that summation in the 2k + 1 summations which has the

same index sum. To begin, all terms with r + s or q + m equal to k are removed and included in the first summation. This is seen to be just the single term $\mathbb{C}_k\mathbb{C}_0$ (q = k, m = 0). To add this to the first summation extend the lower index from 1 to 0. Similarly the index sum k − 1 gives only a single term $\mathbb{C}_{k-1}\mathbb{C}_0$ (q = k − 1, m = 0). To add this to the second summation extend the lower index from 1 to 0. Continue in this fashion until the last two summations have been completely assimilated by the first 2k + 1. In order to get explicit values for the new index limits it is necessary to make an assumption regarding the odd-or-even-ness of k. Let k be even. Then, for example, the terms with index sums equal to zero are those with (r = −1, s = +1) to $(r = -\frac{k}{2}, s = \frac{k}{2})$ and those with (q = 1, m = −1) to $(q = \frac{k}{2}, m = -\frac{k}{2})$. To add these to the term $\mathbb{C}_0\mathbb{C}_0$ write

$$\sum_{-\frac{k}{2}}^{\frac{k}{2}} \mathbb{C}_p \mathbb{C}_{-p}$$

Finally (3) becomes

$$P_c(x,k) = \sum_{p=0}^{k} \mathbb{C}_p \mathbb{C}_{k-p} + \sum_{p=0}^{k-1} \mathbb{C}_p \mathbb{C}_{k-1-p} + \sum_{p=-1}^{k-1} \mathbb{C}_p \mathbb{C}_{k-2-p} + \cdots$$
$$\cdots + \sum_{-\frac{k}{2}+1}^{\frac{k}{2}} \mathbb{C}_p \mathbb{C}_{1-p} + \sum_{-\frac{k}{2}}^{\frac{k}{2}} \mathbb{C}_p \mathbb{C}_p + \sum_{-\frac{k}{2}}^{\frac{k}{2}-1} \mathbb{C}_p \mathbb{C}_{-1-p} + \cdots \qquad (4)$$
$$\cdots + \sum_{p=-k+1}^{\prime} \mathbb{C}_p \mathbb{C}_{-k+2-p} + \sum_{p=-k+1}^{\circ} \mathbb{C}_p \mathbb{C}_{-k+1-p} + \sum_{p=-k}^{\circ} \mathbb{C}_p \mathbb{C}_{-k-p}.$$

The kth partial sum of the Fourier series of $f^2(x)$ is given by the formal product

$$g(x,k) = \sum_{n=-K}^{K} \left(\sum_{p=-\infty}^{\infty} c_p c_{n-p} \right) e^{inx} = \sum_{n=-K}^{K} \sum_{p=-\infty}^{\infty} \mathbb{C}_p \mathbb{C}_{n-p} . \quad (5)$$

Since $C_n = o(\frac{1}{n})$ the summation on p is convergent for every finite n.

It can be seen that $g(x,k) - P_c(x,k)$ will be a double series in which the terms of (4) are omitted from (5). It is useful to note that in every term $\mathbb{C}_p \mathbb{C}_{n-p}$ of the difference one of the indices p or n-p is always equal or larger in absolute value than $\frac{k}{2}$, also neither index is ever zero.

To show $g(x,k) - P_c(x,k) \to 0$ as $k \to \infty$ it is possible to use a dominating series. Since the Fourier coefficients of f(x) are $o(\frac{1}{n})$, given an $\epsilon_o > 0$, it is possible to find an n_o such that

$$|\mathbb{C}_n| < \frac{M}{|n|} \qquad \text{for all } n \neq 0$$

$$|\mathbb{C}_n| < \frac{\epsilon_o}{|n|} \qquad \text{for } |n| > n_o$$

where M is a positive constant. Then for every term $\mathbb{C}_p \mathbb{C}_{n-p}$ in $g(x,k) - P_c(x,k)$ a dominating term is

$$\frac{\epsilon_o M}{|p||n-p|}$$

whenever $k > 2n_0$.

Thus $$|g(x,k) - P_c(x,k)| < \sum \text{(dominating terms)}.$$

Further, due to the symmetry of the dominating terms it is necessary to study only the quarter region where both n and p are positive.

$$|g(x,k) - P_c(x,k)| < 4\left\{\sum_{p=\frac{k}{2}+1}^{\infty} \frac{\epsilon_0 M}{|p||1-p|} + \sum_{p=\frac{k}{2}+2}^{\infty} \frac{\epsilon_0 M}{|p||2-p|} + \cdots \right.$$

$$\left. \cdots + \sum_{p=k}^{\infty} \frac{\epsilon_0 M}{|p||k-1-p|} + \sum_{p=k+1}^{\infty} \frac{\epsilon_0 M}{|p||k-p|} \right\} +$$

$$+ 2 \sum_{p=\frac{k}{2}+1}^{\infty} \frac{\epsilon_0 M}{|p||-p|}$$

Since a monotonic decreasing series may be replaced by an integral with an error less than the magnitude of the first term, it is permissible to write

$$|g(x,k) - P_c(x,k)| < 4\epsilon_0 M \left\{ \log \frac{\frac{k}{2}+1}{\frac{k}{2}} + \frac{1}{2} \log \frac{\frac{k}{2}+2}{\frac{k}{2}} + \cdots \right.$$

$$\left. \cdots + \frac{1}{k-1} \log \frac{k}{1} + \frac{1}{k} \log \frac{k+1}{1} \right\} + \tag{6}$$

$$+ 2 \frac{\epsilon_0 M}{\frac{k}{2}+1} + 8\epsilon_0 M \frac{k+1}{k},$$

where the last term represents the k+1 error terms since in each series the first term is always less then $\frac{2\epsilon_0 M}{k}$.

It remains to show that the series of logarithms in the brackets stays bounded as $k \to \infty$. The alternate logarithmic terms can be summed as follows,

$$\sum_{p=1}^{\frac{k}{2}} \frac{1}{2p} \log \frac{\frac{k}{2}+p+1}{\frac{k}{2}-p+1} + \sum_{p=0}^{\frac{k}{2}-1} \frac{1}{2p+1} \log \frac{\frac{k}{2}+p+1}{\frac{k}{2}-p} =$$

$$= \sum_{\substack{x=\frac{2}{k}, \\ \Delta x=\frac{2}{k}}}^{1} \frac{\Delta x}{2x} \log \frac{1+x+\frac{2}{k}}{1-x+\frac{2}{k}} + \sum_{x=\frac{1}{k}}^{x=1-\frac{1}{k}} \frac{\Delta x}{2x} \log \frac{1+x+\frac{1}{k}}{1-x+\frac{1}{k}}, \tag{7}$$

by making the substitutions $x = \frac{2p}{k}$ in the first sum and $x = \frac{2p+1}{k}$ in the second sum.

When $0 \leq x \leq 1$, and $\epsilon > 0$, a straightforward analysis shows that

$$\log \frac{1+x}{1-x} \geq \log \frac{1+x+\epsilon}{1-x+\epsilon}$$

hence (7) is dominated by

$$\sum_{x=\frac{2}{k}}^{1} \frac{\Delta x}{2x} \log \frac{1+x}{1-x} + \sum_{x=\frac{1}{k}}^{1-\frac{1}{k}} \frac{\Delta x}{2x} \log \frac{1+x}{1-x}, \quad \Delta x = \frac{2}{k}.$$

As $k \to \infty$, both these summations approach the following integral,

$$\int_0^1 \frac{dx}{x} \log \frac{1+x}{1-x},$$

which is an absolute constant, since the integral converges at the logarithmic singularity at x = 1 and the integrand is bounded elsewhere. Hence the series of logarithms in (6) is bounded and so

$$| g(x,k) - P_c(x,k) | < \epsilon_0 B, \qquad k > 2n_0, \tag{8}$$

where B is a constant, independent of k and x. It will be remembered that earlier, for convenience of exposition, k was chosen even. To complete the proof the argument above could be repeated with k odd or it is enough to use (8) for k even together with the fact that

$$| g(x,k+1) - g(x,k) | \longrightarrow 0$$

and $$| P_c(x,k+1) - P_c(x,k) | \longrightarrow 0, \quad \text{as } k \to \infty,$$

This together with the result of Step One proves Theorem I. It should be noted that in Step One the convergence of $s_n(x)$ at a point was required, while the result of Step Two was independent of x. However both steps required that the Fourier coefficients of f(x) were $o(\frac{1}{n})$.

If the hypotheses of Theorem I are enlarged to give the convergence of $s_n(x)$ to f(x) uniformly throughout an interval, $a \leq x \leq b$ then it is seen by a study of Step One that the Cauchy product would converge uniformly to $f^2(x)$ in $a \leq x \leq b$. Finally due to the uniformity of the result of Step Two it is possible to state:

<u>Theorem II</u>. If the Fourier coefficients of f(x) are $o(\frac{1}{n})$ and $s_n(x) \to f(x)$ uniformly in $a \leq x \leq b$ then the Fourier series for

$f^2(x)$ converges uniformly to $f^2(x)$ in $a \leq x \leq b$.

In particular if $a = 0$ and $b = 2\pi$, then $f(x)$ is necessarily a continuous function. The interest of this result lies in the fact that R. Salem[1] has exhibited a continuous function of period 2π whose Fourier series converges uniformly in $0 \leq x \leq 2\pi$ and yet the Fourier series for $f^2(x)$ diverges at an everywhere dense set. The order of the coefficients in this example was roughly $\frac{1}{(\log n)^{\frac{1}{2}}}$ so that there is still a considerable gap between the positive result and the negative result.

The results of Theorems I and II apply equally well to power series on the circle of convergence. Thus if a power series for $f(z)$ converges to $f(z_0)$ where z_0 is a point on the circle of convergence, and the coefficients of the power series are $o(\frac{1}{n})$ then the power series for $f^2(z)$ will converge to $f^2(z_0)$ at z_0. The proof here is much simpler than the corresponding case for Fourier series since the formal product for power series is precisely the Cauchy product, and hence Step One is the complete proof.

The proof of Step One for Fourier series, given above is easily adapted to power series as follows. Let

$$f(z) = \sum_0^\infty d_n z^n$$

have the circle of convergence $|z| = 1$. Then with

[1] "A Singularity of the Fourier Series of Continuous Functions" Duke Mathematical Journal, 4, 10(1943).

$$S_n = \sum_0^n d_n e^{in\theta}$$

$$A_n = d_n e^{in\theta}$$

the proof given is entirely adequate.

In the case where the power series for f(z) converges uniformly on the circle of convergence the adaptation of Theorem II proves that the power series for $f^2(z)$ also converges uniformly on the circle of convergence. This result is then applicable to the series discussed by G. Hardy(1) and E. Landau(2) which converge uniformly on the circle of convergence but have coefficients which are $O(\frac{1}{n \log n})$.

(1) "A Theorem Concerning Taylor Series" Quarterly Journal of Pure and Applied Mathematics, 44, 1913, p.157 et seq.

(2) Ergebnisse der Funktionentheorie 1929 Kap.4.

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Biographical Note

Stephen H. Crandall was born December 2, 1920, Cebu, Philippine Islands. After a scattered elementary school training he attended Eastside High School in Paterson, New Jersey and was graduated in February 1937. He entered Stevens Institute of Technology immediately, but,due to sustained illness, did not graduate until May 1942. After a summer of graduate work in the Department of Mathematics of M.I.T. he joined the staff of the Radiation Laboratory where he worked on the problem of stabilizing shipboard radar antennas. During the spring term of 1944 the author began as an instructor in the Department of Mathematics, where he has remained to date.