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EXTENSIONS OF RINGS AND MODULES

by

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ABSTRACT

The primary objective of this thesis is to present a unified account of the various generalizations of the concept of ring of quotients given by K. Asano (1949), R. E. Johnson (1951), Y. Utumi (1956), G. D. Findlay and J. Lambek (1958). A secondary objective is to investigate how far the commutative localization can be carried over to the noncommutative case.

We begin with a formulation of the notion of D-system of right ideals of a ring R. The investigation of the D-systems was motivated by the fact that each maximal right quotient ring of R consists precisely of semi R-homomorphisms into R with domains in a specific D-system of right ideals of R or of R^1, the ring obtained from R by adjoining identity. A nonempty family X of right ideals of R is called a D-system provided the following three conditions hold:

D1. Every right ideal of R containing some member of X is in X.

D2. For any two right ideals A and B of R belonging to X, $\varphi^{-1}B$ belongs to X for each R-homomorphism φ of A into R.

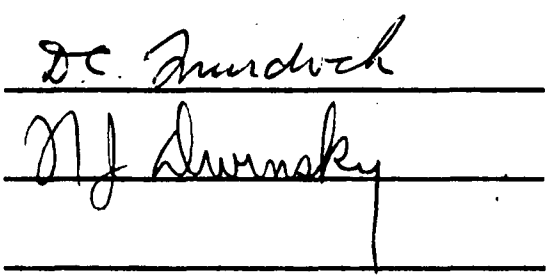
D3. If A belongs to X and if for each a in A there exists B_a in X, then the ideal sum of aB_a (a in A) is in X.

Each D-system X of right ideals of R induces a modular closure operation on the lattice L(M) of all submodules of an R-module M and hence gives rise to a set $L^X(M)$ of closed submodules of M. We are able to set up an isomorphism between the lattice of all modular closure operations on L(R) and the lattice of all D-systems of right ideals of R and characterize the D-systems X

used in Asano's, Johnson's and Utumi's constructions of quotient rings in terms of properties of $L^X(R)$.

In view of the intimate relation between the rings of quotients of a ring R and the extensions of R-modules, we generalize the concepts of injective R-module, rational and essential extensions of an R-module corresponding to a D-system Y of right ideals of R^1. The existence and uniqueness of the maximal Y-essential extension, minimal Y-injective extension and maximal Y-rational extension of an R-module and their mutual relations are established.

Finally, we come to the actual constructions of various extensions of rings and modules. The discussions center around the centralizer of a ring over a module, the maximal essential and rational extensions and the different types of rings of right quotients. We include here also a partial, though not satisfactory, solution of the noncommutative localization problem.



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INTRODUCTION

It is well-known that every commutative ring R with regular elements possesses a full ring of quotients and that for any multiplicatively closed subset S of R, a ring of quotients R_S can be constructed and a closure operation can be defined on the lattice of all ideals of R. If R is Noetherian and S is the (set theoretical) complement in R of a prime ideal P, then R_S is a local ring in which the extension of P is the unique maximal prime ideal.

During the last two decades, interest has been shown in generalizing the concept of ring of quotients and developing the theory of noncommutative localization. For an arbitrary ring R with regular elements, a ring Q containing R is said to be a classical right quotient ring of R provided the following conditions hold:

(1) Q has an identity element.

(2) Each regular element of R has a (two sided) inverse in Q.

(3) Every element q of Q has the form $q = ab^{-1}$ where a and b are elements of R with b regular.

A necessary and sufficient condition for a ring with regular elements to have a classical right quotient ring (which is unique up to isomorphism) is that the set G of all regular elements of R possesses the following common right multiple property: For each r in R and each g in G, there exist r' in R and g' in G such that $rg' = gr'$. (See [34].)

In a paper published in 1949, Asano generalized the notion of classical right quotient ring. An overring Q of a ring R with regular elements is called a right quotient ring of R relative to

a set A of regular elements of R in case

1. Q has an identity element;

2. each element in A has an inverse in Q; and

3. for each element x of Q, there exists an element a of A such that $xa \in R$.

Corresponding to the classical case, Asano established that a right quotient ring of R relative to A exists if and only if A is multiplicatively closed and possesses the common right multiple property. In the proof, he constructed an overring Q of R which is, in essence, the set

$$Q = \bigcup_K \mathrm{Hom}_R(K,R)$$

where K varies through all right ideals of R containing some elements of A, with equality, addition and multiplication defined as follows: For any α and β in Q, $\alpha = \beta$ if and only if $\alpha x = \beta x$ for every x in some right ideal K of R, $K \subset \mathrm{dom}\,\alpha \cap \mathrm{dom}\,\beta$, $K \cap A \neq \emptyset$;

$$(\alpha + \beta)x = \alpha x + \beta x, \quad (x \text{ in } \mathrm{dom}\alpha \cap \mathrm{dom}\beta);$$

$$(\alpha\beta)x = \alpha(\beta x), \quad (x \text{ in } \beta^{-1}\mathrm{dom}\alpha).$$

The ring R can be isomorphically embedded in Q under the correspondence $r \to r_L$, where r_L is the left multiplication of R by r. On identifying r with r_L, Q consists precisely of elements of the form ba^{-1} with b in R and a in A. The main contribution that Asano has made is that instead of considering pairs of elements of R he has chosen semi R-homomorphisms as elements of the overring. This ingenious method is also adopted by later contributors in constructing overrings of a ring.

Since then numerous generalizations of the concept of ring of quotients have been given. Of utmost importance are the ones formulated by R. E. Johnson (1951), Y. Utumi (1956), G. D. Findlay and J. Lambek (1958).

Johnson [15] has called an overring Q of R a right quotient ring of R if for every nonzero element q of Q there exists r in R such that $0 \neq qr \in R$. It has been proved that every ring R with zero right singular ideal possesses a unique maximal right quotient ring Q which is incidently regular. The ring Q consists precisely of all semi R-homomorphisms of R into itself whose domains are essential right ideals of R. Each right ideal A of R has a unique maximal essential extension A^S in R. The mapping $A \to A^S$ is a closure operation on the lattice L(R) of all right ideals of R. The set of all closed right ideals of R, when ordered by inclusion, forms a complete complemented modular lattice.

It is interesting to enquire what conditions should be imposed on the ring R so that the Johnson's maximal right quotient ring of R coincides with the classical right quotient ring. A. W. Goldie [12] came up with an answer in 1960. It has been shown that for a semi-prime ring R with ascending chain conditions on annihilator right ideals and complemented right ideals, the right singular ideal of R is zero and a right ideal is essential if and only if it contains some regular element of R. The Johnson's maximal right quotient ring Q of such a ring coincides with the classical right quotient ring. In fact, Goldie has proved a more significant result: R has a classical right quotient ring which is semi-simple right Artinian if and only if R is semi-prime with ascending chain conditions on annihilator right ideals and complemented right ideals.

Utumi's definition [36] of a right quotient ring is as follows: An overring Q of a ring R is called a right quotient ring of R if for any pair of elements x and y of Q with $x \neq 0$, there exists an element a of R such that $ya \in R$ and $xa \neq 0$. Evidently every Utumi's right quotient ring of R is also a Johnson's right quotient

ring of R. The converse is true if R has zero right singular ideal. In case R is left faithful, the set Q of all semi R-endomorphisms of R with domains the right ideals A of R such that R is a right quotient ring of A, forms a maximal right quotient ring of R.

Findlay and Lambek [10] have called an overring Q of R a right quotient ring of R if for any pair x and y of elements of R^Q with $x \neq 0$, there exist an element a of R and an integer n such that $ya + ny \in R$ and $xa + nx \neq 0$. It should be remarked that each Utumi's right quotient ring of R is also a Findlay and Lambek's right quotient ring. The converse is true if R is left faithful. Any ring R has a maximal right quotient ring Q which is unique up to isomorphism over R. Again, Q consists of all semi R-homomorphisms into R whose domains are those right ideals K of R^1 (ring obtained from R by adjoining identity) satisfying the following condition: Given $0 \neq r \in R$ and $s \in R^1$, there exist k in K and an integer n such that $rk + nr \neq 0$ and $sk + ns \in K$.

The aim of the present thesis is two-folded. On one hand it is to present a unified account of the theory of quotient rings which embodies the various generalizations, each of which is obtained from it by a suitable specializations; while on the other hand it is to investigate how far the theory of commutative localization can be carried over to noncommutative case.

The starting point of the passage towards the unification is the definition of a D-system of right ideals of a ring R. A non-empty family X of right ideals of R is called a D-system provided the following three conditions hold:

D1. Every right ideal of R containing some member of X is in X.

D2. For any two right ideals A and B of R belonging to X, $\varphi^{-1}B \in X$ for every R-homomorphism φ of A into R.

D3. If $A \in X$ and if for each a in A there exists $B_a \in X$, then the ideal sum $\sum_{a\in A} aB_a$ is in X.

The investigation of the D-systems is motivated by the outstandingly important role that has been played by them in the constructions of various quotient rings. In fact, as memtioned above, each maximal right quotient ring of R consists of semi R-homomorphisms into R with domains in a specific D-system. The first part of the thesis is devoted to a systematic study of the D-systems of right ideals of a ring R. Each D-system X of right ideals of R induces a modular closure operation on the lattice L(M) of all submodules of an R-module M and hence gives rise to a set $L^X(M)$ of closed submodules of M. We are able to set up an isomorphism between the lattice of all modular closure operations on L(R) and the lattice of all D-systems of right ideals of R and characterize the D-systems X used in Johnson's, Utumi's and classical constructions of maximal right quotient rings in terms of the properties of $L^X(R)$.

Historically there has been an intimate relation between the rings of quotients and the extensions of modules. An overring Q of R when regarded as a right R-module, is an essential (resp. rational) extension of R if Q is a Johnson's (resp. Utumi's) right quotient ring of R. The converse is true if R is left faithful. Furthermore Q is a rational extension of R if and only if Q is a Findlay and Lambek's right quotient ring of R. In view of the importance of the notions of rational and essential extensions of an R-module, we dedicate the second part of the thesis to an exposition of generalized injective R-modules, generalized rational

and essential extensions of an R-module corresponding to a D-system Y of right ideals of R^1. The main theme of study is the existence and uniqueness of the maximal Y-essential extension, minimal Y-injective extension and maximal Y-rational extension of an R-module and their mutual relations.

The final part of the thesis deals with the constructions of various extensions of rings and modules by a method originated by Asano [1]. The discussions center around the centralizer of a ring over a module, the maximal essential and rational extensions of modules and the various types of rings of right quotients. We include here also a partial, though not satisfactory, solution of the noncommutative localization problem.

PRELIMINARIES

We assume here the basic properties of groups and rings. (See for example [37], Chapter 1.) The rings we refer to are always associative but not necessarily commutative, nor do they have identity element.

If R is a ring and if $(M,+)$ is an additive abelian group, then M is called a <u>right R-module</u> (or simply R-module) in case there exists a mapping $(m,r) \to mr$ of $M \times R$ into M satisfying the following conditions:

(1) $(m + n)r = mr + nr$ for any $m, n \in M$ and $r \in R$.

(2) $m(r + s) = mr + ms$ for any $m \in M$ and $r, s \in R$.

(3) $m(rs) = (mr)s$ for any $m \in M$ and $r, s \in R$.

A <u>left R-module</u> is defined symmetrically. A right (resp. left) R-module M is said to be <u>unital</u> if R contains an identity element 1 and $m1 = m$ (resp. $1m = m$) for each m in M.

If R and S are rings, then an additive abelian group $(M,+)$ is called an <u>(S,R)-module</u> if M is a left S-module as well as a right R-module and $(sm)r = s(mr)$ for each m in M, r in R and s in S. Note that the ring R itself can be regarded either as a right R-module, or a left R-module, or an (R,R)-module.

Let M be a (right) R-module. Then by a <u>submodule</u> of M we mean an additive subgroup N of M such that $nr \in N$ for any $n \in N$ and $r \in R$. We shall denote the set of all submodules of M by $L(M)$. Incidently $L(R)$ coincides with the set of all right ideals of R.

Suppose that N is a submodule of the right (resp. left) R-module M and H is a subset of M. Then the right ideal $\{r \mid r \in R, Hr \subset N\}$ (resp. left ideal $\{r \mid r \in R, rH \subset N\}$) of R will be denoted by $H^{-1}N$ (resp. NH^{-1}). M is said to be <u>faithful</u> if

$m^{-1}0 = R$ (resp. $0m^{-1} = R$) implies $m = 0$ where m is an element of M. The ring R is said to be <u>left</u> (resp. <u>right</u>) <u>faithful</u> if it is a faithful right (resp. left) R-module. In case S is a subset of R, $S^{-1}0$ (resp. $0S^{-1}$) is called <u>annihilator right</u> (resp. <u>left</u>) <u>ideal</u> of R and will be denoted alternatively by S^r (resp. S^l). An element x of R is said to be <u>right</u> (resp. <u>left</u>) <u>regular</u> if $x^r = 0$ (resp. $x^l = 0$). If x is both right and left regular, then x is called a <u>regular element</u> of R.

Let M and M' be two R-modules. An R-homomorphism of M into M' is a mapping φ of M into M' such that

$$\varphi(m + n) = \varphi(m) + \varphi(n)$$

and

$$\varphi(mr) = \varphi(m)r$$

for any m, n in M and r in R. For the sake of brevity, we usually drop the parentheses in $\varphi(m)$ and write simply φm. We shall denote the domain and image of φ by dom φ and im φ. The set $\{m \mid m \in M, \varphi m = 0\}$ is called the kernel of φ and will be designated by ker φ. If im $\varphi = M'$, then φ is said to be an R-homomorphism of M <u>onto</u> M'. If ker $\varphi = 0$, then φ is said to be an <u>R-isomorphism</u> of M into M'. An R-homomorphism (resp. R-isomorphism) of M into itself is called an <u>R-endomorphism</u> (resp. <u>R-automorphism</u>) of M.

If K is a submodule of an R-module M, then the difference group $M - K$ is an R-module under the definition $(m+K)r = mr+K$ for any coset $m+K$ of $M - K$ and any r in R. The natural homomorphism of the group M onto $M - K$ is also an R-homomorphism. On the other hand if φ is an R-homomorphism of M onto M', then the kernel K of φ is a submodule of M and $m+K \to \varphi m$ is an R-isomorphism of K onto M'.

Let $\{M_i\}_{i\in I}$ be any family of R-modules. Then by the <u>complete direct sum</u> of the R-modules M_i's we mean the set $\prod_{i\in I} M_i$ consisting of all mappings x of I into $\bigcup_{i\in I} M_i$ such that $x_i = x(i)\in M_i$ for each i in I, together with the operations defined by $(xr)_i = x_i r$ and $(x+y)_i = x_i + y_i$ where x and y are elements of $\prod M_i$ and r is an element of R. The R-homomorphism $\pi_i : x \to x_i$ of $\prod M_i$ onto M_i is called the projection of $\prod M_i$ onto M_i. For each i, the injection $m \to m^*$ is an R-isomorphism of M_i into $\prod M_i$ where $m \in M_i$ and $m^*\in \prod M_i$ is such that $(m^*)_i = m$ and $(m^*)_j = 0$ if $j \neq i$. We shall identify m with m^* and consider M_i as a submodule of $\prod M_i$. The <u>discrete direct sum</u> $\sum_{i\in I}\oplus M_i$ is the submodule of $\prod_{i\in I} M_i$ consisting of all x such that $x_i = 0$ for all but a finite number of i in I.

Assume now M and N are two R-modules. If K is a submodule of M, then an R-homomorphism of K into N is called a <u>semi R-homomorphism</u> of M into N. Two semi R-homomorphisms α and β of M into N are said to be equal if dom α = dom β and $\alpha x = \beta x$ for every x in dom α = dom β. A semi R-homomorphism β of M into N is called an <u>extension</u> of the semi R-homomorphism α of M into N and α is called a <u>restriction</u> of β, we write $\alpha < \beta$, if dom $\alpha \subset$ dom β and $\alpha x = \beta x$ for each x in dom α. A semi R-homomorphism α which has no proper extensions is said to be <u>irreducible</u>. The relation "<" defined on the set of all semi R-homomorphisms of M into N is clearly reflexive, anti-symmetric and transitive and so is a partial ordering. It is true that every nonempty family G of semi R-homomorphisms of M into N has a common restriction. However G may or may not have a common extension. A necessary and sufficient condition that G has a

common extension is that every finite subfamily of $\mathcal{G}$ has one. Thus we need only consider common extensions of a finite family $\mathcal{G}$. Let $\mathcal{G} = \{\alpha, \beta, .., \omega\}$ and suppose $\mathcal{G}$ has a common extension μ. Then for any $x \in \text{dom } \alpha$, $y \in \text{dom } \beta$, ..., $z \in \text{dom } \omega$,

$$(\cdot) \qquad \mu(x + y + \cdot\cdot + z) = \alpha x + \beta y + \cdot\cdot + \omega z.$$

It follows that

$$(\cdot\cdot) \quad \text{if } x + y + \cdot\cdot + z = 0, \text{ then } \alpha x + \beta y + \cdot\cdot + \omega z = 0.$$

Conversely if $(\cdot\cdot)$ is true, then μ defined by $(\cdot)$ is a common extension of $\mathcal{G}$. In particular

Proposition. Two semi R-homomorphism of M into N have a common extension if and only if they coincide on the intersection of their domains. (See [10], Prop. 0.2.)

Now suppose $\mathcal{G}$ is a nonempty family of semi R-homomorphisms of M into N having a common extension. Consider the set $\mathcal{B}$ of all common extensions of $\mathcal{G}$. Let $\lambda_1 < \lambda_2 < \cdots$ be an ascending chain in $\mathcal{B}$. Then λ defined by $\text{dom } \lambda = \bigcup_{n=1}^{\infty} \text{dom } \lambda_n$ and $\lambda x = \lambda_n x$ if $x \in \text{dom } \lambda_n$, is an extension of each λ_i. By Zorn's lemma, $\mathcal{B}$ has maximal elements. Obviously such maximal common extensions are irreducible.

Proposition. A nonempty family of semi R-homomorphisms of M into N having a common extension possesses irreducible common extensions.

Corollary. Every semi R-homomorphism of M into N can be extended to irreducible one. (See [10], Prop. 0.1.)

The sum $\alpha + \beta$ of two semi R-homomorphisms α and β of M into N is a semi R-homomorphism of M into N defined by

$$\text{dom } (\alpha+\beta) = \text{dom } \alpha \cap \text{dom } \beta,$$

$$(\alpha+\beta)x = \alpha x + \beta x \quad \text{for each } x \text{ in dom } (\alpha+\beta).$$

It is easy to verify that $\alpha + \beta = \beta + \alpha$ and $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ for any semi R-homomorphisms α, β, γ of M into N. If α is a semi R-homomorphism of M into N and β is a semi R-homomorphism of N into another R-module K, then the product $\beta\alpha$ of β and α is a semi R-homomorphism of M into K defined by

$$\text{dom } (\beta\alpha) = \alpha^{-1}\text{dom } \beta,$$

$$(\beta\alpha)x = \beta(\alpha x) \quad \text{for each } x \text{ in dom } (\beta\alpha).$$

As usual we shall denote the set of all R- homomorphisms of M into N by $\text{Hom}_R(M,N)$. Under the operations defined above, $\text{Hom}_R(M,M)$ and $\text{Hom}_R(N,N)$ become rings and $\text{Hom}_R(M,N)$ is an $(\text{Hom}_R(N,N), \text{Hom}_R(M,M))$-module.

Closure operations on lattices are fundamental in our present work. We shall assume the basic properties of lattices, especially complete modular lattices. (See for example, [4].) Let $\mathcal{K}$ be a complete lattice with identity elements 0 and I relative to $\cup$ and $\cap$ respectively. A mapping $A \to A^c$ of $\mathcal{K}$ into itself is called a <u>closure operation</u> on $\mathcal{K}$ provided the following conditions are satisfied:

C1. $A^c \supset A$ for each A in $\mathcal{K}$.

C2. If $A \supset B$, then $A^c \supset B^c$.

C3. $A^{cc} = A^c$ for each A in $\mathcal{K}$.

The set of all closure operations on $\mathcal{K}$ will be designated by $\mathfrak{R}(\mathcal{K})$. We may define two operations $\cup$ and $\cap$ and a relation $<$ on $\mathfrak{R}(\mathcal{K})$ as follows:

$$b < c \text{ if and only if } A^c \supset A^b \text{ for every A in } \mathcal{K};$$

$$A^{\cap c_i} = \cap\, A^{c_i} \quad \text{for each A in } \mathcal{K};$$

$$A^{\cup c_i} = \cap \{B \mid A \subset B \in \mathcal{K},\ B^{c_i} = B \text{ for all } i\} \quad \text{for each A in } \mathcal{K}.$$

It is clear that $\{\mathfrak{R}(\mathcal{K});\ \cup,\ \cap,\ <\}$ is a complete lattice having $o : A \to A$ and $i : A \to I$ as identity elements.

A subset W of the complete lattice $\mathcal{K}$ that contains I and is closed under the intersection operation is called an inset of $\mathcal{K}$. The family of all insets of $\mathcal{K}$ will be designated by $\tau(\mathcal{K})$. $\tau(\mathcal{K})$ when partially ordered by set inclusion becomes a complete lattice with $\vee$ and $\wedge$ defined by

$$\vee W_i = \{ \cap A_i \mid A_i \in W_i \text{ for all } i \};$$

$$\wedge W_i = \cap W_i.$$

Suppose c is a closure operation on $\mathcal{K}$. An element A of $\mathcal{K}$ is said to be c-closed if $A^c = A$. Obviously the family $\mathcal{K}^c$ of all c-closed elements of $\mathcal{K}$ is an inset of $\mathcal{K}$. On the other hand if W is an inset of $\mathcal{K}$, then associated with each A in $\mathcal{K}$ is the least element A^ω of W containing A. The mapping $A \to A^\omega$ is a closure operation on $\mathcal{K}$. Moreover W becomes a complete lattice (not necessarily a sublattice of $\mathcal{K}$) if we define for a subfamily $\{A_i\}$ of W, $\wedge A_i = \cap A_i$ and $\vee A_i = (\cup A_i)^\omega$. The lattice $\mathcal{R}(\mathcal{K})$ is dual isomorphic to $\tau(\mathcal{K})$ under the correspondence $c \to \mathcal{K}^c$.

PART I. MODULAR CLOSURE OPERATIONS AND D-SYSTEMS

The theory of closure operations on lattices has been developed extensively by R. E. Johnson and many others. Of special interest are the modular closure operations on the complete modular lattice L(M) of all submodules of an R-module M. In Section 1, we use a method employed by Johnson in the investigation of prime and semi-prime rings to obtain modular closure operations. The basic results here are that there exists a one-to-one correspondence between modular closure operations on L(M) and modular insets of L(M) and that the totality of all modular closure operations on L(M) forms a sublattice of the lattice of all closure operations on L(M). Section 2 is devoted to the discussions of the relation between modular closure operations on L(M) and D-systems of right ideals of R. Notable existing examples of D-systems by K. Asano, R. E. Johnson, Y. Utumi, G. D. Findlay and J. Lambek are included. In view of later applications, a useful account of some special properties of the lattice of closed submodules of M under the modular closure operation induced by a D-sustem of right ideals of R is given in Sections 3-5.

1. Modular closure operations. Johnson [18] calls a right ideal A of a semi-prime ring R prime in case the following condition holds: If $BC \subset A$, where B and C are right ideals of R with $C^{r\ell} = R$, then $B \subset A$. When R is a prime ring, the condition $C^{r\ell} = R$ is equivalent to the condition $C \neq 0$ [16,17]. This notion of prime right ideal is subjected readily to generalization.

Let R be an arbitrary ring and let Z be a family of right ideals of R. A submodule N of an R-module M is said to be Z-prime

if N contains all submodules K of M such that $KA \subset N$ for some A in Z. A right ideal of R is called Z-prime if it is an Z-prime submodule of R, regarded as an R-module. Note that every submodule is Z-prime if Z is empty and that M is always an Z-prime submodule of itself. As usual we shall call an R-module Z-prime if the submodule 0 is Z-prime. Thus a submodule N of M is Z-prime if and only if the difference module M - N is Z-prime.

The collection of all Z-prime submodules of M will be designated by P(M;Z). Some of the distinctive properties of P(M;Z) are described in the following

1.1. Proposition. Let Z be a nonempty family of right ideals of R. Then for every R-module M, P(M;Z) is an inset of the lattice L(M) of all submodules of M and satisfies the following conditions:

I1. If $N \in P(M;Z)$ and K is a submodule of M such that $KR \subset N$, then $K \subset N$.

I2. For each N in P(M;Z) and each R-endomorphism φ of M,

(1) $\varphi^{-1}N \in P(M;Z)$; and

(2) every maximal element H (if any) of the family $\{H \mid H \in L(M), \varphi^{-1}H \subset N\}$ belongs to P(M;Z).

Proof. The property I1 and the fact that P(M;Z) is an inset are self-evident. To prove I2(1), assume $KA \subset \varphi^{-1}N$ for some submodule K of M and A in Z. Then $(\varphi K)A = \varphi(KA) \subset N$. Since N is Z-prime, $\varphi K \subset N$, or $K \subset \varphi^{-1}N$. Hence $\varphi^{-1}N$ belongs to P(M;Z). As for the proof of I2(2), we suppose $KA \subset H$ for some submodule K of M and A in Z. Then $(\varphi^{-1}(K+H))A \subset \varphi^{-1}((K+H)A) \subset \varphi^{-1}H \subset N$. Again, since N is Z-prime, $\varphi^{-1}(K+H) \subset N$. By maximality of H, $K + H = H$, or $K \subset H$. This completes the proof.

As has been mentioned in the Preliminaries, every inset of a lattice induces a closure operation on the lattice. We shall denote the closure operation on L(M) induced by P(M;Z) by p(M;Z), or by p if no confusion seems possible. The set of all p-closed submodules coincides with P(M;Z) which forms a complete lattice with set-intersection as meet and join defined by $\vee N_i = (\sum N_i)P$, where $\{N_i\}$ is a subfamily of P(M;Z) and $\sum N_i$ denotes the module sum of N_i's. Furthermore P(M;Z) is a modular lattice in the sense that $N\cap(K\vee H) = K\vee(N\cap H)$ holds for any N, K, H in P(M;Z) such that $N \supset K$. This is because of the modularity of L(M). Accordingly, associated with each R-module is a complete modular lattice of all Z-prime submodules.

An inset of L(M) is said to be <u>modular</u> if it satisfies the conditions I1 and I2 enumerated in Proposition 1.1. Modular insets are closely connected with modular closure operations. By a <u>modular closure operation</u> on L(M) we mean a closure operation c satisfying the following two conditions:

C4. $(\varphi^{-1}N)^c = \varphi^{-1}N^c$ for every R-endomorphism φ of M and every N in L(M).

C5. $(NR)^c = N^c$ for each N in L(M).

In case M is the ring R, the left multiplication by an element r of R is an R-endomorphism of R. It follows from C4 that

C41. $(r^{-1}N)^c = r^{-1}N^c$ for every r in R and N in L(R).

On the other hand, it is not difficult to show that every closure operation c on L(R) satisfying C41 and C5 is modular. In fact, if N is a right ideal of R and φ is an R-endomorphism of R, then by C41, for every r in R

$$r^{-1}(\varphi^{-1}N)^c = (r^{-1}(\varphi^{-1}N))c = ((\varphi r)^{-1}N)^c = (\varphi r)^{-1}N^c = r^{-1}(\varphi^{-1}N^c).$$

By choosing r from $\varphi^{-1}N^c$ and $(\varphi^{-1}N)^c$ respectively, we derive from the above equation the following two inclusions, $(\varphi^{-1}N^c)R \subset (\varphi^{-1}N)^c$ and $\varphi(\varphi^{-1}N)^cR \subset N^c$. By virtue of C5, $\varphi^{-1}N^c \subset (\varphi^{-1}N)^c$ and $\varphi(\varphi^{-1}N)^c \subset N^c$. Hence $\varphi^{-1}N^c = (\varphi^{-1}N)^c$. This completes the proof of the following

1.2. Proposition. A closure operation on L(R) is modular if and only if it satisfies C41 and C5.

1.3. Proposition. If c is a modular closure operation on L(M), then

C21. $(N \cap K)^c = N^c \cap K^c$ for any N and K in L(M); and

C42. $\varphi N^c \subset (\varphi N)^c$ for every N in L(M) and every R-endomorphism φ of M.

Proof. The condition C42 follows immediately from C4. To prove C21, we observe that it is enough to show $(N \cap K)^c \supset N \cap K^c$. Let r be any element of $N \cap K^c$. Then

$$r^{-1}(N \cap K)^c = (r^{-1}(N \cap K))^c = ((r^{-1}N) \cap (r^{-1}K))^c = (r^{-1}K)^c = r^{-1}K^c = R.$$

This is true for every r in $N \cap K^c$. It follows then $(N \cap K)^c \supset (N \cap K^c)R$. By C5. $(N \cap K)^c \supset N \cap K^c$.

For a closure operation c on L(M), we shall use $L^c(M)$ to signify the set of all c-closed submodules of M. If c is a modular closure operation, then C21 and C4 reveal that $L^c(M)$ is closed under finite intersection and also under the operation of taking the inverse image, $\varphi^{-1} : N \to \varphi^{-1}N$, where φ is an R-endomorphism of M. Conditions C5 and C1 imply I1 for $L^c(M)$. That I2 is satisfied by $L^c(M)$ is a consequence of C4 and C1. This clarifies the "only if" part of the following theorem.

1.4. Theorem. A closure operation c on L(M) is modular if and only if the corresponding inset $L^c(M)$ is modular.

Proof. Suppose $L^c(M)$ is modular. Let N be a submodule of M and let φ be an R-endomorphism of M. Then by I2(1),

$$\varphi^{-1}N^c = \varphi^{-1}(\cap\{K \mid K\in L^c(M), K\supset N\}) = \cap\{\varphi^{-1}K \mid K\in L^c(M), K\supset N\}$$

$$\supset \cap\{K \mid K\in L^c(M),\ K\supset\varphi^{-1}N\} = (\varphi^{-1}N)^c.$$

On the other hand if K is a member of $L^c(M)$ containing $\varphi^{-1}N$, let H be a maximal element of the set $\{H \mid N\subset H\in L(M), \varphi^{-1}H\subset K\}$ (H exists by Zorn's lemma). By I2(2), H belongs to $L^c(M)$ and so $N^c \subset H^c = H$. Thus $\varphi^{-1}N^c \subset \varphi^{-1}H \subset K$. Since this is true for all c-closed submodules K containing $\varphi^{-1}N$, $\varphi^{-1}N^c \subset (\varphi^{-1}N)^c$. This proves C4. To prove C5, it is enough to show that $(NR)^c \supset N^c$. Let K be any c-closed submodule containing NR. Then by I1, K contains N and hence N^c. Thus $(NR)^c \supset N^c$.

1.5. Corollary. An inset W of L(R) is modular if and only if it satisfies I1 and

I21. For each N in W and each r in R,

(1) $r^{-1}N \in W$; and

(2) every maximal element (if any) of the family $\{K \mid K\in L(R),\ r^{-1}K\subset N\}$ belongs to W.

Proof. Follows from Proposition 1.2 and Theorem 1.4.

1.6. Corollary. For each nonempty family Z of right ideals of R, p(M;Z) is a modular closure operation on L(M).

Proof. Follows immediately from Proposition 1.1 and Theorem 1.4.

We shall denote the set of all closure operations (resp. all modular closure operations) on L(M) by $\mathfrak{R}(M)$ (resp. by $\mathfrak{R}_m(M)$) and the set of all insets (resp. modular insets) of L(M) by $\tau(M)$

(resp. by $\tau_m(M)$). The lattice $\mathfrak{R}(M)$ is dual isomorphic to the lattice $\tau(M)$ (see Preliminaries) under the correspondence $c \to L^c(M)$. It has just been shown in Theorem 1.4 that this correspondence induces a one-to-one mapping of $\mathfrak{R}_m(M)$ onto $\tau_m(M)$. Hence $\mathfrak{R}_m(M)$ is a sublattice of $\mathfrak{R}(M)$ if and only if $\tau_m(M)$ is a sublattice of $\tau(M)$. In such case, $\mathfrak{R}_m(M)$ is dual isomorphic to $\tau_m(M)$.

1.7. Theorem. $\mathfrak{R}_m(M)$ is a sublattice of $\mathfrak{R}(M)$.

Proof. According to the above remark, it suffices to show that $\tau_m(M)$ is a sublattice of $\tau(M)$, or equivalently, $\cap W_i$ and $\vee W_i = \{\cap N_i \mid N_i \in W_i \text{ for all } i\}$ are modular insets for any subfamily $\{W_i\}$ of $\tau_m(M)$. The verification of modularity of $\cap W_i$ and $\vee W_i$ is simple and straight-forward. As an example, we show that $\vee W_i$ satisfies I2(2). For each i, let N_i be a member of W_i. Suppose H is a submodule maximal with respect to $\varphi^{-1}H \subset \cap N_i$, where φ is an R-endomorphism of M. By Zorn's lemma, there exists a submodule H_i containing H maximal with respect to $\varphi^{-1}H_i \subset N_i$. Since each W_i is modular, H_i belongs to W_i. Now $\varphi^{-1}(\cap H_i) = \cap \varphi^{-1}H_i \subset \cap N_i$. By maximality of H, $H = \cap H_i$. Hence H belongs to $\vee W_i$.

For any two R-modules M and N, the lattices $\mathfrak{R}_m(M)$ and $\mathfrak{R}_m(N)$ are, by no means, completely independent. As a simplest example, if N is a submodule of M such that every R-endomorphism of N can be extended to an R-endomorphism of M, then every modular closure operation c on L(M) induces a modular closure operation $K \to K^c \cap N$ on L(N). The following proposition exhibits a more significant correlation between $\mathfrak{R}_m(M)$ and $\mathfrak{R}_m(R)$.

1.8. Theorem. Let c be a modular closure operation on $L(R)$ and let M be an arbitrary R-module. Then c gives rise to a modular closure operation $N \to N^{c(M)}$ on $L(M)$, where

$$N^{c(M)} = \{ m \mid m \in M,\ (m^{-1}N)^c = R \}.$$

Moreover $c = c(R)$. [Whenever there is no fear of confusion, we shall use c instead of the cumbersome notation $c(M)$.]

Proof. We show first that N^c is a submodule of M. Let m and n be any two elements of N^c and let r be in R. Then, since

$$((m+n)^{-1}N)^c \supset ((m^{-1}N) \cap (n^{-1}N))^c = (m^{-1}N)^c \cap (n^{-1}N)^c = R,$$

and $$((mr)^{-1}N)^c = (r^{-1}(m^{-1}N))^c = r^{-1}(m^{-1}N)^c = r^{-1}R = R,$$

the elements $m + n$ and mr are in N^c. Hence N^c is a submodule of M. The conditions C1 and C2 are clearly satisfied by $c(M)$. To prove C3, let m be any element of N^{cc}. For each r in $m^{-1}N^c$, $r^{-1}(m^{-1}N)^c = ((mr)^{-1}N)^c = R$. Thus $(m^{-1}N)^c \supset (m^{-1}N^c)R$. By C5 for c, $(m^{-1}N)^c \supset (m^{-1}N^c)^c = R$. This shows that m belongs to N^c. Hence $N^{cc} \subset N^c$. This together with C1 and C2 proves C3. Since

$$m \in \varphi^{-1}N^c \leftrightarrow \varphi m \in N^c \leftrightarrow ((\varphi m)^{-1}N)^c = R \leftrightarrow (m^{-1}(\varphi^{-1}N))^c = R \leftrightarrow m \in (\varphi^{-1}N)^c,$$

C4 follows readily. To prove C5, it is enough to show that $(NR)^c \supset N^c$. Suppose n is any element of N^c. Then $R = (m^{-1}N)^c = ((m^{-1}N)R)^c \subset (m^{-1}(NR))^c$, i.e., m belongs to $(NR)^c$. This completes the proof that $c(M)$ is a modular closure operation on $L(M)$. Finally, to prove $c = c(R)$, let A be any right ideal of R. If r is an element of A^c, then $(r^{-1}A)^c = r^{-1}A^c = R$ and so r belongs to $A^{c(R)}$. Conversely for every r in $A^{c(R)}$, $R = (r^{-1}A)^c = r^{-1}A^c$, or $A^c \supset A^{c(R)}R$. By C5 for c, $A^c \supset A^{c(R)}$. Hence $A^c = A^{c(R)}$ for all right ideals A of R, i.e., $c = c(R)$.

In the sequel, till the end of this section, we shall assume c is a modular closure operation on $L(R)$. If N is a submodule of

the R-module M, the resulting modular closure operations $c(M)$ and $c(N)$ are related as follows $K^{c(N)} = N \cap K^{c(M)}$ for each submodule K of N.

1.9. Proposition. N is a c-closed submodule of M if and only if $L^{c}(N)$ is a subset of $L^{c}(M)$. In this case, $K^{c(N)} = K^{c(M)}$ for every submodule K of N.

Proof. If N is a c-closed submodule of M, then $K^{c(N)} = N \cap K^{c(M)} = N^{c(M)} \cap K^{c(M)} = (K \cap N)^{c(M)} = K^{c(M)}$ for every submodule K of N. The first statement then follows immediately.

Given a closure operation c on L(M), a submodule N of M is said to be c-dense if $N^{c} = M$. The study of dense submodules will be our undertaking in the next section. For the present, we shall be satisfied by proving the following

1.10. Proposition. A submodule N of M is c-dense if and only if the lattice $L^{c}(N)$ is isomorphic to the lattice $L^{c}(M)$ under the correspondence $\mu : K \to K^{c(M)}$.

Proof. Suppose N is a c-dense submodule of M. Since $(J \cap N)^{c(N)} = (J \cap N)^{c(M)} \cap N = J \cap N^{c(M)} \cap N = J \cap N$ for every c-closed submodule J of M, $\nu : J \to J \cap N$ defines a mapping of $L^{c}(M)$ into $L^{c}(N)$. For any K in $L^{c}(N)$ and any J in $L^{c}(M)$,

$$(\nu\mu)K = \nu(\mu K) = K^{c(M)} \cap N = K^{c(N)} = K,$$

$$(\mu\nu)J = \mu(\nu J) = (J \cap N)^{c(M)} = J^{c(M)} \cap N^{c(M)} = J \cap M = J.$$

Hence $\nu\mu$ and $\mu\nu$ are identity maps. It follows that μ is a one-to-one mapping of $L^{c}(N)$ onto $L^{c}(M)$ with inverse ν. Since μ and ν are inclusion-preserving, μ is a (lattice) isomorphism of $L^{c}(N)$ onto $L^{c}(M)$. Conversely if $L^{c}(N)$ is isomorphic to $L^{c}(M)$ under μ, then $M = \mu N = N^{c(M)}$, i.e., N is c-dense.

Our next task is to find the relation between $L^c(N)$ and $L^c(M)$ where N is a homomorphic image of M. In this connection, we require the following definition. Given two elements A and B of a lattice $\mathfrak{K}$ with $A > B$. By the closed interval [A,B] of $\mathfrak{K}$ we mean the sublattice consisting of all K of $\mathfrak{K}$ such that $A > K > B$. It is evident that every closed interval of a complete modular lattice is complete and modular. By Proposition 1.9, if N is a c-closed submodule of M, then $L^c(N)$ is precisely the closed interval $[N, 0^{c(M)}]$ of $L^c(M)$. By a similar argument given in the proof of Theorem 1.8, we can show the following

1.11. Lemma. Let M and M' be any two R-modules and let N be a submodule of M'. If φ is an R-homomorphism of M into M', then $(\varphi^{-1}N)^c = \varphi^{-1}N^c$.

1.12. Corollary. If c is a modular closure operation on L(R) such that 0 is a c-closed right ideal of R, then every annihilator right ideal of R is c-closed.

Proof. Let $S^{-1}0$ be an annihilator right ideal of R, where S is a subset of R. Then $S^{-1}0 = \bigcap_{s\in S} s^{-1}0$. By Lemma 1.11, each $s^{-1}0$ is c-closed. Hence $S^{-1}0$ is c-closed.

In general if φ is an R-homomorphism of M onto M', then $\varphi^{-1}: N \to \varphi^{-1}N$ is a one-to-one mapping of L(M') onto the set of all submodules of M containing the kernel of φ such that $N \subset K$ if and only if $\varphi^{-1}N \subset \varphi^{-1}K$. The inverse of φ^{-1} is $\varphi : K \to \varphi K$. By Lemma 1.11, φ^{-1} induces a (lattice) isomorphism of $L^c(M')$ into $[M, (\ker\varphi)^{c(M)}]$. If K is a c-closed submodule of M containing $\ker \varphi$, then $(\varphi K)^c = \varphi\varphi^{-1}(\varphi K)^c = \varphi(\varphi^{-1}\varphi K)^c = \varphi K^c = \varphi K$, i.e., φK is c-closed. Hence

1.13. Proposition. Let M and M' be two R-modules. If φ is an R-homomorphism of M onto M', then the lattice $L^c(M')$ is isomorphic to the closed interval $[M,(\ker\varphi)^{c(M)}]$ of $L^c(M)$ under the correspondence $N \to \varphi^{-1}N$.

2. D-systems of right ideals. The concept of prime-ness of submodules for an arbitrary family Z of right ideals of R, being too general, is rather poor in content. A further investigation would not be fruitful without imposing additional conditions on Z. In this section, we shall deal with Z-prime submodules in a more restricted sense where Z is allowed to enjoy various properties. We shall begin with a brief survey of the set of all c-dense submodules of an R-module M, where c is a modular closure operation on L(M). For a closure operation c on L(M), the family of all c-dense submodules will be desgnated by $D^c(M)$. It follows from Theorem 1.8 that for a modular closure operation c on L(R) and a submodule N of an R-module M, m belongs to $N^{c(M)}$ if and only if $mA \subset N$ for some A in $D^c(R)$. We shall use this fact in the proof of the following proposition.

2.1. Proposition. Let c be a modular closure operation on L(R) and let M be an R-module. Then the family $X = D^c(M)$ satisfies the following conditions:

D1. Every element of L(M) that contains a member of X is in X.

D2. If N and K are two members of X and φ is an R-homomorphism of N into M, then $\varphi^{-1}K$ belongs to X.

If M is a ring containing R as a subring, then X possesses the following additional property:

D3. For $N \in X$, if corresponding to each n in N, there is a member K_n of X, then the sum $\sum_{n\in N} nK_n$ is in X.

Proof. D1 is clearly true. To prove D3, let $K = \sum nK_n$. For each x in N, the left multiplication by x is an R-endomorphism of M and so by C4, $x^{-1}K^c = (x^{-1}K)^c \supset K_x{}^c = M \supset R$. This implies that $K^c \supset NR$. Thus by C5, $K^c \supset N^c = M$. This proves D3. Finally in proving D2, we observe that by Lemma 1.11, $\varphi^{-1}K$ is a c-dense submodule of N. We know also N is a c-dense submodule of M. Applying D3 for $D^c(R)$, $\varphi^{-1}K$ is a c-dense submodule of M.

A nonempty family X of right ideals of R is said to be a D-system if X satisfies conditions D1-D3 with M replaced by R throughout. It follows from Proposition 2.1 that for a modular closure operation c on L(R), the set of all c-dense right ideals of R is a D-system.

Let Z be an arbitrary family of right ideals of R. A submodule N of an R-module M is said to be strongly Z-prime if $mA \subset N$ where $m \in M$ and $A \in Z$ implies $m \in N$. An R-module is said to be strongly Z-prime if the submodule 0 is strongly Z-prime. By a remark preceeding Proposition 2.1, for a modular closure operation c on L(R) a submodule N of an R-module M is c-closed if and only if it is strongly $D^c(R)$-prime. In particular if $c = p(R;Z)$ for some Z, then by Theorem 1.8, $c = c(R)$. A right ideal of R is Z-prime if and only if it is strongly $D^c(R)$-prime.

We shall make a few remarks concerning the above-mentioned conditions D1-D3 for a family X of right ideals of R. If X satisfies D1, the condition D3 is equivalent to the following condition:

D31. For $N \in X$, if K is a right ideal of R such that $n^{-1}K \in X$ for every n in N, then K belongs to X.

In any case, D3 implies the following condition:

D32. If $N, K \in X$, then $NK \in X$;

while the condition D2 implies the following two conditions:

D21. For each N in X and each r in R, $r^{-1}N$ belongs to X.

D22. X is closed under finite intersection.

In the presence of conditions D1 and D3 (or its equivalent D31), D2 follows from D21. The verification of the validity of our remarks except the last statement is simple and will be omitted. For the proof of the last statement, we need only observe that for each n in N, $n^{-1}(\varphi^{-1}K) = (\varphi n)^{-1}K$ which is in X by D21 and apply D31.

Now let X be a D-system of right ideals of R. Then by D21 and D22, for each submodule N of M, the set

$$\overline{N} = \{ m \mid m \in M \text{ and } mA \subset N \text{ for some } A \text{ in } X \}$$

is a submodule of M containing N. If N and K are two submodules of M, then $\overline{N \cap K} = \overline{N} \cap \overline{K}$. By virtue of D3, $\overline{\overline{N}} = \overline{N}$. This shows that $N \to \overline{N}$ is a closure operation on L(M). With respect to this closure operation, a submodule of M is closed if and only if it is strongly X-prime. It is clear that this closure operation satisfies C4. The condition C5 is an easy consequence of D32. Hence $N \to \overline{N}$ is a modular closure operation on L(M), called the modular closure operation induced by X. This furnishes a proof for the first half of the following

2.2. Theorem. Let X be a D-system of right ideals of R and let M be any R-module. Then X induces a modular closure operation on L(M) defined by $N \to \overline{N}$. If x denotes the modular closure operation on L(R) induced by X, then the modular closure operation on L(M) induced by X coincides with x(M). [See Theorem 1.8 for the definition of x(M).]

To complete the proof of the above theorem, it suffices to prove the following lemma.

2.3. Lemma. Let X be a D-system of right ideals of R. Then $X = D^X(R)$.

Proof. Suppose A is any member of X. Then by D21, $A^X = R$ and so A belongs to $D^X(R)$. Conversely let A be a member of $D^X(R)$. Then for each r in R, there exists a right ideal B_r belonging to X such that $rB_r \subset A$. Since X is nonempty, $R \in X$ by D1. According to D3, $\sum_{r \in R} rB_r$ is in X. Hence by D1 again, A belongs to X. This completes the proof of the lemma and hence the proof of the theorem.

2.4. Corollary. The set of all strongly X-prime submodules of an R-module M is a modular inset of L(M).

Consider now the collection $\Delta(R)$ of all D-systems of right ideals of R. It is easily seen that $\Delta(R)$ is an inset of the lattice $2^{L(R)}$ of all subsets of the set of all right ideals of R. Associate with each W of $2^{L(R)}$ is the unique minimal member $\overline{W}$ of $\Delta(R)$ containing W. We shall call $\overline{W}$ the D-system generated by W. The mapping $W \to \overline{W}$ is a closure operation on $2^{L(R)}$ with $\Delta(R)$ as the lattice (not necessarily a sublattice of $2^{L(R)}$) of all closed members of $2^{L(R)}$ (see Preliminaries).

2.5. Theorem. The lattice $\mathfrak{R}_m(R)$ of all modular closure operations on L(R) is isomorphic to the lattice $\Delta(R)$ of all D-systems of right ideals of R under the correspondence $\mu: c \to D^c(R)$.

Proof. By Proposition 2.1, $D^c(R)$ is a D-system of right ideals of R. Denote the modular closure operation on L(R) induced by $D^c(R)$ by ω. We shall show that $c = \omega$. Let J be any right ideal of R. If r is an element of J^c, then $r^{-1}J$ belongs to $D^c(R)$ by C41. By the fact that $r(r^{-1}J) \subset J$, we have r belongs to J^ω. Thus $J^c \subset J^\omega$. Conversely, for each element r of J^ω, there exists a right ideal

A of R belonging to $D^c(R)$ such that $rA \subset J$. By C42, $rR = rA^c \subset (rA)^c \subset Jc$. Thus $J^\omega R \subset J^c$ and by C5, $J^\omega \subset J^c$. This shows that $J^c = J^\omega$ for every right ideal J of R. Hence $c = \omega$ and μ is one-to-one. By Lemma 2.3, μ is "onto".

If Z is any nonempty family of right ideals of R, then Z induces a modular closure operation $p = p(R;Z)$ on L(R) which, by Theorem 2.5, is the same as the modular closure operation induced by the D-system DP(R) of all p-dense right ideals of R.

We collect here some very important examples of D-systems which we shall often have occasion to refer to in the sequel.

Findlay and Lambek's D-system. There has been defined in [10] by Findlay and Lambek a relation among three R-modules N, N' and M as follows: $N' < N$ (M) in case (i) N' is a submodule of N; and (ii) if φ is a semi R-homomorphism of N into M which annihilates N', then im φ = 0.

Observe that $R < R^1$ (M) if and only if M is a faithful R-module. Indeed if $R < R^1$ (M) and if mR = 0 for some m in M, then since m induces an R-homomorphism of R^1 into M which annihilates R, we have $mR^1 = 0$ and so $m = m1 = 0$. This shows that M is faithful. Conversely, suppose M is faithful and let φ be a semi R-homomorphism of R^1 into M which annihilates R. Then $(\varphi \mathrm{dom}\varphi)R = \varphi(\mathrm{dom}\varphi \cdot R) \subset \varphi R = 0$ since R is an ideal of R^1. But since M is faithful, im φ = φ dom φ = 0.

We may suppose then M is a faithful R-module and consider the family X of all right ideals A of R such that $A < R^1$ (M). By the above remark, X is nonempty. It is obvious that X satisfies D1. To prove D21, let A be a member of X and let r be any element of R. Suppose φ is a semi R-homomorphism of R^1 into M which annihilates

$r^{-1}A$. Define a mapping λ of $A + r\cdot\mathrm{dom}\,\varphi$ into M as follows:

$$\lambda(x + ry) = \varphi y, \qquad (x \text{ in } A \text{ and } y \text{ in } \mathrm{dom}\,\varphi).$$

If $x+ry = x'+ry'$ for some x, x' in A and y, y' in dom φ, then $r(y-y') = x'-x \in A$ and so $y-y' \in r^{-1}A$ and $\varphi(y-y') = 0$ or $\varphi y = \varphi y'$. Thus λ is well defined. It is easy to verify that λ is an R-homomorphism, $A \subset \ker \lambda$ and $\mathrm{im}\,\varphi = \mathrm{im}\,\lambda$. Since A belongs to X, $\mathrm{im}\,\varphi = \mathrm{im}\,\lambda = 0$. Hence $r^{-1}A$ is in X. This proves D21. Next, suppose A is a member of X and B is a right ideal of R such that $a^{-1}B$ belongs to X for every a in A. Let φ be a semi R-homomorphism of R^1 into M which annihilates B. For each a in $A \cap \mathrm{dom}\,\varphi$, φa induces an R-homomorphism of R^1 into M which annihilates $a^{-1}B$. Since $a^{-1}B$ belongs to X, $\varphi a = 0$. Thus φ annihilates $A \cap \mathrm{dom}\,\varphi$. Define μ, a mapping of $A + \mathrm{dom}\,\varphi$ into M by

$$\mu(x + y) = \varphi y, \qquad (x \text{ in } A \text{ and } y \text{ in } \mathrm{dom}\,\varphi).$$

If $x+y = x'+y'$ for some x, x' in A and y, y' in dom φ, then $y-y' = x'-x \in A \cap \mathrm{dom}\,\varphi$. Thus $\varphi(y-y') = 0$ or $\varphi y = \varphi y'$. Hence μ is well-defined. It is a semi R-homomorphism of R^1 into M annihilating $A \in X$. Hence $\mathrm{im}\,\varphi = \mathrm{im}\,\mu = 0$. This shows that B belongs to X and hence proves D31. Consequently X is a D-system of right ideals of R, called the Findlay and Lambek's D-system of right ideals of R relative to the faithful R-module M.

In an exactly similar manner, we can show that for any R-module M, the set of all right ideals A of R^1 such that $A < R^1$ (M) is a D-system containing X.

Utumi's D-system. Following Utumi [36], a ring R is said to be a right quotient ring of its subring S and we write $S < R$ provided for any pair of elements x and y in R with $x \neq 0$, there exists an element s in S such that $ys \in S$ and $xs \neq 0$. Note that

if R is not left faithful, then R can never be a right quotient ring of any of its subring. We may then suppose R is left faithful. The Findlay and Lambek's D-system X of R relative to R coincides with the set of all right ideals A of R such that $A < R$. In fact, if $A < R^1$ (R) and if x and y are any two elements of R with $x \neq 0$, then since X is an D-system, $A \cap y^{-1}A$ belongs to X. Since R is left faithful, the left multiplication of R by x is not 0 and so $x(A \cap y^{-1}A) \neq 0$. This means that there exists an element a of A such that $xa \neq 0$ and $ya \in A$. Hence $A < R$. Conversely suppose A is a right ideal of R such that R is a right quotient ring of A. Let φ be a semi R-homomorphism of R^1 into R which annihilates A. If im $\varphi \neq 0$, then $\varphi x \neq 0$ for some x in dom φ. Since R is left faithful, $\varphi(xR) = (\varphi x)R \neq 0$, i.e., $\varphi xr \neq 0$ for some r in R. There exists then an element a in A such that $\varphi xra \neq 0$ and $xra \in A$. This contradicts the assumption that $\varphi A = 0$. Hence im $\varphi = 0$ and so $A \in X$.

We shall call X the Utumi's D-system of right ideals of the left faithful ring R.

Johnson's D-system. Let M be an R-module. Then a submodule N of M is said to be essential (or large) if $N \cap K \neq 0$ for every nonzero submodule K of M, or equivalently, for every nonzero element m of M, there exists r in R^1 such that $0 \neq mr \in M$. The collection Y_M of all essential submodules of M satisfies conditions D1 and D2. We may suspect that Y_R is a D-system. This is far from being true, in general. For example, let R be the set of all integers with addition defined as usual and multiplication defined trivially. Then Y_R which comprises all nonzero ideals of R does not satisfy D3.

An element m of M is called a singular element if $mA = 0$ for some essential right ideal A of R. By D2 (which implies D21 and D22) for Y_R, the set of all singular elements of M constitutes a submodule of M, called the singular submodule of M and denoted by Z(M). In case M = R, Z(R) is an ideal of R, called the right singular ideal of R.

Johnson [15] calls a ring R a right quotient ring of its subring S if for every nonzero element r of R, there exists s in S such that $0 \neq rs \in S$. The notations $S <' R$ will signify that R is a right quotient ring of S. Clearly a right ideal A of R such that $A <' R$ is an essential right ideal of S. The converse is true if and only if the right singular ideal of R is zero.

For a ring R with zero right singular ideal, the Utumi's D-system X of right ideals of R coincides with the set Y_R of all essential right ideals of R (and hence consists of all right ideals A of R such that $A <' R$). To prove this, suppose A is any element of X and b is a nonzero element of R. Then there exists a in A such that $0 \neq ba \in A$. Thus $A <' R$ and by the above remark A is essential. Conversely, let A be any essential right ideal of R and let x and y be two elements of R with $x \neq 0$. Since Y_R satisfies the condition D2, $A \cap y^{-1}A$ is essential. By the hypothesis that R has no nonzero singular elements, $x(A \cap y^{-1}A) \neq 0$. That is, there exists an element a of A such that $ya \in A$ and $xa \neq 0$. Thus A belongs to X.

We shall call X the <u>Johnson's D-system</u> of right ideals of the ring R with zero right singular ideal.

That the right singular ideal of R is zero is a sufficient but not necessary condition for Y_R to be a D-system will be

illustrated by the following example. We observe that every essential right ideal of R contains the socle of R (which is the sum of all minimal right ideals of R if R has minimal right ideals and is 0 otherwise). If R satisfies descending chain condition on right ideals, then a right ideal is essential if and only if it contains the socle of R. Hence for a ring R that satisfies the descending chain condition on right ideals Y_R is a D-system. However not all rings satisfying the descending chain condition on right ideals have zero right singular ideal.

Asano's D-system. Let G be a nonempty multiplicatively closed subset of the ring R. Then the family W_G of all right ideals A of R satisfying the following condition: [For each r in R, there exists g in G such that $rg \in A$.] is a D-system. It is obvious that W_G is contained in the set V of all right ideals A of R such that $A \cap G \neq \emptyset$. The reverse inclusion holds if and only if G satisfies the following condition:

(0) For each g in G and each r in R there exist g' in G and r' in R such that $gr' = rg'$.

In fact if $V = W_G$ and if $g \in G$ and $r \in R$, then $gR \in W_G$. By D21, $r^{-1}(gR) \in W_G$. Thus there exist g' in G and r' in R such that $rg' = gr'$. Conversely, suppose the condition (0) is satisfied by G. If A is a right ideal of R such that $A \cap G \neq \emptyset$, say $g \in A \cap G$, then for each r in R, there exist g' in G and r' in R such that $rg' = gr'$. But since $g \in A$, $rg' \in A$. Hence A belongs to W_G.

A nonempty multiplicative closed subset of R is called an f-set if it satisfies the condition (0). The set of all f-sets of R will be designated by $\mathfrak{F}(R)$. For each D-system X of right ideals of R we shall denote the set of all elements r of R such

that $rR \in X$ by $G_X(R)$. Since $rR = rA^X \subset (rA)^X$ for every A in X, r belongs to $G_X(R)$ if and only if $rA \in X$ for each A in X. The set $G_X(R)$ is multiplicatively closed (though it may be empty). For if r and s are two elemens of $G_X(R)$, then $(rt)^{-1}(rsR) = t^{-1}r^{-1}(rsR) \supset t^{-1}(sR) \in X$ for every rt in rR. By D1 and D31, $rsR \in X$, i.e., rs belongs to $G_X(R)$.

A D-system X is called a D*-system if $G_X(R) \cap A \neq \emptyset$ for every A in X. We shall denote the set of all D*-systems of right ideals of R by $\Delta^*(R)$. If X is an D*-system of right ideals of R, then for each g in $G_X(R)$ and r in R, $r^{-1}(gR) \in X$ and so there exist g' in $G_X(R)$ and r' in R such that $rg' = gr'$. This shows that $G_X(R)$ is an f-set of R. We summarize the above discussion and obtain the following

2.6. Proposition. (1) If G is an f-set of R, then W_G is a D*-system of right ideals of R and $G \subset G_{W_G}(R)$.

(2) If X is a D*-system of right ideals of R, then $G_X(R)$ is an f-set of R and $X = W_{G_X}(R)$.

(3) $X \to G_X(R)$ is a one-to-one inclusion preserving mapping of $\Delta^*(R)$ into $\mathfrak{F}(R)$ and $G \to G_{W_G}(R)$ is a closure operation on $\mathfrak{F}(R)$, partially ordered by inclusion.

In [1], Asano dealt with D*-systems W_G corresponding to f-sets G of R consisting exclusively of regular elements of R. Such D*-systems W_G will be called Asano's D-systems. It is natural to inquire when is a D*-system an Asano's D-system. This question will be answered later in Section 5. (See Corollary 5.14.) For the present, we shall consider f-sets of R composed of one-sided "regular elements" of R.

Before doing this, let us divert our attention for a moment

to a special type of D-systems. A D-system X of right ideals of R is called a D'-system if the ideal 0 is strongly X-prime. A D'-system which is also a D*-system will be called simply a D'D*-system. The D-systems of Findlay and Lambek, of Utumi and of Johnson are examples of D'-systems while the Asano's D-systems are examples of D'D*-systems. The collection of all D'-systems of right ideals of R will be designated by $\Delta'(R)$. Then clearly $\Delta'(R)$ is empty unless R is left faithful.

2.7. Proposition. If R is left faithful, then the Utumi's D-system U of right ideals of R is the unique maximal member of $\Delta'(R)$.

Proof. Let X be an arbitrary D'-system of right ideals of R and let A be any element of X. Given x and y in R with $x \neq 0$, $A \cap y^{-1}A$ is in X by D2. Since X is a D'-system $x(A \cap y^{-1}A) \neq 0$. That is, there exists a in A such that ya A and $xa \neq 0$. Thus A belongs to U.

As an immediately corollary of the above proposition, $\Delta'(R)$ is a sublattice of the lattice $\Delta(R)$ dual isomorphic to the sublattice $\mathfrak{R}'_m(R)$ of $\mathfrak{R}_m(R)$ consisting of all modular closure operations c on L(R) such that $0^c = 0$.

An element r of R is said to be a right (resp. left) X-regular element of R if $r^{-1}0^X = 0^X$ (resp. $0^X r^{-1} = 0^X$). We shall denote the set of all right (resp. left) X-regular elements of R by $G_X^r(R)$ (resp. $G_X^\ell(R)$). Clearly $G_X^r(R)$ and $G_X^\ell(R)$ are multiplicatively closed subsets of R and $G_X(R) \subset G_X^\ell(R)$. In the difference ring $R/0^X$, the images of elements of $G_X^r(R)$ (resp. $G_X^\ell(R)$) are precisely the right regular (resp. left regular) elements of $R/0^X$. Hence if X is a D'-system, then $G_X^r(R)$ (resp. $G_X^\ell(R)$) consists precisely

of all right (resp. left) regular elements of R.

Let G be an f-set of R such that $0 \notin G$. Then $0^{\omega G}$ is an ideal of R consisting of all elements r of R such that $rg = 0$ for some g in G. Clearly $0^{\omega G} \cap G = \emptyset$ and $G \subset G_{W_G}(R)$. Thus if W_G is a D'-system, then G consists exclusively of left regular elements of R. Conversely if G consists exclusively of left regular elements of R, then $0^{\omega G} = 0$ and so W_G is a D'-system. We have thus completed the proof of the following

2.6'. Proposition. (1) Let G be an f-set of R. Then G consists exclusively of left regular elements of R if and only if W_G is a D'D*-system of right ideals of R.

(2) Let X be a D*-system of right ideals of R. Then X is a D'-system if and only if $G_X(R)$ is an f-set of R consisting exclusively of left regular elements of R.

Up to now, we have considered three operators Δ, Δ' and Δ^* each of which associates a ring with a class of families of right ideals of the ring. In case $\Delta'(R)$ is nonempty, it has as its unique maximal element the Utumi's D-system of right ideals of R. For a commutative ring R (or a ring R in which every left regular element is also right regular), $\Delta'(R) \cap \Delta^*(R)$ is precisely the set of all Asano's D-systems of right ideals of R. We postpone the characterization of Johnson's D-system in terms of lattice properties until Section 4. (See Corollary 4.3.)

We proceed to investigate the mutual relation between $\Delta(R_i)$'s where R_i's are rings related in some natural manner.

2.8. Proposition. Let R and S be two rings and let σ be a (ring) homomorphism of R onto S. Then for a D-system X of right ideals of R, the family X^σ of all right ideals of S of the form σA, where A ranges through all elements of X is a D-system. The modular closure operation x on L(R) induced by X and the modular closure operation x^σ on the lattice $L_r(S)$ of all right ideals of S induced by X^σ are related as follows: $x^\sigma = \sigma x \sigma^{-1}$, i.e., $J^{x^\sigma} = \sigma(\sigma^{-1}J)^x$ for every right ideal J of S.

Proof. The condition D1 is clearly satisfied by X^σ. To prove D21, let J be a member of X^σ and let s be any element of S. Then $\sigma^{-1}J \in X$ and $s = \sigma r$ for some r in R. Since $s^{-1}J = (\sigma r)^{-1}J = \sigma(r^{-1}(\sigma^{-1}J))$ and since $r^{-1}(\sigma^{-1}J)$ belongs to X, $s^{-1}J$ belongs to X^σ. This proves D21. Next suppose $J \in X$ and for each j in J there exists I_j in X^σ. Let $A = \sigma^{-1}J$ and $B_a = \sigma^{-1}I_j$ for a in $\sigma^{-1}j$. Then A and B_a are in X and $\sum_{j\in J} jI_j = \sum_{a\in A} (\sigma a)(\sigma B_a) = \sigma(\sum_{a\in A} aB_a)$. Thus the condition D3 for X^σ follows from that for X. This completes the proof that X^σ is a D-system. A simple computation will show the second part of the proposition.

Let X be a D-system of right ideals of R. Then by Theorem 2.2, X induces a modular closure operation $x^\# = x(R^1)$ on the lattice $L(R^1)$ of submodules of the R-module R^1. But since every submodule of R^1 is a right ideal of R^1 and vice versa, $x^\#$ is a closure operation on the lattice $L(R^1)$ of right ideals of R^1 satisfying C4. The condition C5, i.e., $(AR^1)^{x^\#} = A^{x^\#}$ follows from the fact that R^1 possesses an identity. Thus $x^\#$ is a modular closure operation on $L(R^1)$, where R^1 is being regarded as an R^1-module. The set of all $x^\#$-dense right ideals of R^1 will be designated by $X^\#$. Then clearly $X^\# = D^x(R^1)$. By Proposition 2.1, $X^\#$ is a D-system of right ideals of R^1.

2.9. Lemma. A right ideal of R^1 is $x^\#$-dense if and only if it contains some member of X. Hence $X \subset X^\#$.

Proof. By the definition of the closure operation $x(R^1)$, it is clear that if J is an $x^\#$-dense right ideal of R^1, then $J \cap R$ is x-dense in R and hence belongs to X by Lemma 2.3. To prove the converse it suffices to show that every A in X is $x^\#$-dense. For each r in R^1, the left multiplication by r induces an R-endomorphism of R since R is an ideal of R^1. By D2, $r^{-1}A \in X$ and $r(r^{-1}A) \subset A$. Hence A is $x^\#$-dense.

On the other hand if Y is a D-system of right ideals of R^1 such that $R \in Y$, then Y induces a modular closure operation y(R) on the lattice $L_1(R)$ of submodules of the R^1-module R. Since $L_1(R) = L(R)$, y(R) is a closure operation on L(R) satisfying C4. For the proof of C5, we observe that for A in L(R), if $r \in A^{y(R)}$, then $rJ \subset A$ for some J in Y. Since $R \in Y$, $JR \in Y$ by D32 and $rJR \subset AR$. Thus $r \in (AR)^{y(R)}$. This proves C5. Hence y(R) is a modular closure operation on L(R). We shall write $y^\flat$ instead of y(R) and denote the corresponding D-system of right ideals of R by $Y^\flat$. Clearly $Y^\flat = D^Y(R)$.

2.10. Lemma. $A \in Y^\flat$ if and only if $A = J \cap R$ for some J in Y. Hence $Y^\flat \subset Y$.

The proof of the lemma is similar to that of Lemma 2.9 and will be omitted. The following proposition follows easily from Lemmas 2.9 and 2.10.

2.11. Proposition. Let X be a D-system of right ideals of R and let Y be a D-system of right ideals of R^1 such that $R \in Y$. Then $(X^\#)^\flat = X$ and $(Y^\flat)^\# = Y$. Hence there exists a one-to-one correspondence between $\Delta(R)$ and a subset of $\Delta(R^1)$.

Our next target is the connection between the D-systems of a ring which is a discrete direct sum of rings and D-systems of its direct summands. Assume $\{R_i\}$ is a nonempty collection of rings and let R be the discrete direct sum of R_i's. For any right ideal J of R, we shall denote $J \cap R_i$ by J_i and the set of all elements of R_i that appear as i-components of some elements of J by $\bar{J}_i$. Then clearly J_i and $\bar{J}_i$ are right ideals of R_i and $\sum_i J_i \subset J \subset \sum_i \bar{J}_i$.

For each i, let X_i be a D-system of right ideals of R_i and let X be the family of right ideals J of R such that $J_i \in X_i$ for all i. We shall show that X is a D-system of right ideals of R. The condition D1 is clearly satisfied by X. To prove D21, let J be a member of X and let r be any element of R. Then

$$(\cdot) \qquad (r^{-1}J) \cap R_i = r_i^{-1}J_i.$$

By D21 for X_i, $r_i^{-1}J_i$ belongs to X_i. Thus $r^{-1}J$ belongs to X, proving D21. Next, suppose J is a member of X and I is a right ideal of R such that $j^{-1}I$ belongs to X for every j in J. Then by $(\cdot)$, $j_i^{-1}I_i$ belongs to X_i for all j_i in $\bar{J}_i$. Thus I_i is in X_i and I belongs to X. This proves D31. Consequently, X is a D-system of right ideals of R. It is appropriate to write $\oplus X_i$ for X.

2.12. Proposition. Let $\{R_i\}$ be a nonempty collection of rings and let X_i be a D-system of right ideals of R_i. Then $\oplus X_i$ is a D-system of right ideals of the discrete direct sum $\sum \oplus R_i$. Moreover $\oplus X_i$ is a D'-system if and only each X_i is a D'-system.

The second statement of the above proposition can be proved without difficulty. It is worth noting that $\sum \oplus R_i$ is left faithful if and only if each R_i is left faithful.

2.13. Lemma. Let R be the discrete direct sum of a nonempty family $\{R_i\}$ of left faithful rings and let J be a right ideal of R. Then (1) $J < R$ if and only if $J_i = J \cap R_i < R_i$ for all i; and

(2) J is an essential right ideal of R if and only if J_i is an essential right ideal of R_i for all i. Hence the right singular ideal of R is zero if and only if the right singular ideal of each of the rings R_i's is zero. (See p. 27 for the definition of the relation "<".)

Proof. (1) Suppose $J < R$. Let a and b be any two elements of R_i with $a \neq 0$. Since R_i is left faithful $ac \neq 0$ for some c in R_i. Now there exists j in J such that $cj \in J$, $acj \neq 0$ and $bcj \in J$. But cj and bcj belong to R_i too. Thus $J_i < R_i$. Conversely suppose $J_i < R_i$ for each i. Let r and s be two elements of R such that $r \neq 0$. Assume that $r_i \neq 0$. Since $J_i < R_i$, there exists a in J_i such that $r_i a \neq 0$ and $s_i a \in J_i$. That is there exists a in J such that $ra \neq 0$ and $sa \in J$. Hence $J < R$.

(2) Suppose J is an essential right ideal of R. If for some i, J_i is not essential, then $J_i \cap A = 0$ for some nonzero right ideal A of R_i. Since every right ideal of R_i is also a right ideal of R_i and since $J \cap A = J_i \cap A = 0$, $A = 0$. This contradicts the choice of A. Hence each J_i is an essential right ideal of R_i. Conversely suppose J is a right ideal of R such that each J_i is an essential right ideal of R_i. If $J \cap I = 0$ for some right ideal I of R, then $J_i \cap I_i = 0$ for all i. By assumption, J_i is essential $I_i = 0$. Now $\overline{I}_i R_i = I R_i \subset I \cap R_i = I_i = 0$. Since R_i is left faithful, $\overline{I}_i = 0$. This is true for all i. Hence $I = 0$ and J is essential.

2.14. Proposition. Let $\{R_i\}$ be a nonempty family of left faithful rings and let X_i be a D-system of right ideals of R_i. Then $\oplus X_i$ is the Utumi's D-system of right ideals of $R = \sum \oplus R_i$ if and only if each X_i is the Utumi's D-system of right ideals of R_i. If, in addition, for each i, the right singular ideal of R_i is zero, then $\oplus X_i$ is the Johnson's D-system of right ideals of R if and only if each X_i is the Johnson's D-system of right ideals of R_i.

Consider now the ring of all $n \times n$ matrices over the ring R. We shall denote this ring by $[R]_n$. For each right ideal K of $[R]_n$, let K_{ij} be the set of all elements r in R such that the matrix re_{ij} with r at the (i,j)-position and zero elsewhere is in K. Then K_{ij} is a right ideal of R. Assume that X is a D-system of right ideals of R and let $[X]_n$ denote the family of all right ideals of $[R]_n$ such that $K_{ij} \in X$ for all i and j. Note that a right ideal K of $[R]_n$ belongs to $[X]_n$ if and only if there exists a member A of X such that $K \supset [A]_n$.

2.15. Proposition. Let X be a D-system of right ideals of R. Then $[X]_n$ is a D-system of right ideals of $[R]_n$. Moreover X is a D'-system if and only if $[X]_n$ is a D'-system.

Proof. Evidently, $[X]_n$ satisfies D1. Let K be a member of $[X]_n$ and let $a = \sum_{i,j} a_{ij}e_{ij}$ be an element of $[R]_n$. There is no loss of generality to assume that $K = [A]_n$ for some A in X. Then $B = \bigcap_{i,j} a_{ij}^{-1}A$ is in X and for each r in B, $a_{ij}re_{pq} \in K$ for any i, j, p and q. Now $are_{ij} = \sum_p a_{pi}re_{pj}$ is an element of K, i.e., $r \in (a^{-1}K)_{ij}$. Hence $(a^{-1}K)_{ij}$ contains B and so must be in X. It follows that $a^{-1}K$ belongs to $[X]_n$. This proves D21. In proving D31, let K be a member of $[X]_n$ and let H be a right ideal of $[R]_n$

such that $a^{-1}H \in [X]_n$ for all a in K. We may suppose $K = [A]_n$ for some A in X. If $r \in A$, then $r^{-1}H_{ip} = ((re_{ij})^{-1}H)_{jp}$ for any i, j and p. Since $(re_{ij})^{-1}H \in [X]_n$, $r^{-1}H_{ip} \in X$. Thus by D31 for X, $H_{ip} \in X$ for all i and p. Hence H belongs to $[X]_n$. This proves that $[X]_n$ is a D-system of right ideals of $[R]_n$.

If X is a D'-system and if $aK = 0$ for some $a = \sum a_{ij}e_{ij}$ in $[R]_n$ and K in $[X]_n$, then $0 = (\sum_{i,j} a_{ij}e_{ij})(Ae_{pq}) = \sum_i a_{ip}Ae_{iq}$ where $A = \bigcap_{i,j} K_{ij}$. Thus $a_{ip}A = 0$ for all i and p. Since X is a D'-system, $a_{ip} = 0$ and so $a = 0$. Hence $[X]_n$ is a D'-system. The converse is clear.

Let R be a ring with an identity 1. We shall write e_{ij} instead of $1e_{ij}$. If K is a right ideal of $[R]_n$, then K_{ij} is independent of j. For if $r \in K_{ij}$, then $re_{ij} \in K$. Since K is a right ideal of $[R]_n$, $re_{ik} = re_{ij}e_{jk}$ belongs to K and so $r \in K_{ik}$. Thus $K_{i1} = K_{i2} = \cdots = K_{in}$ for i = 1, .., n.

2.16. Lemma. Let R be a ring with an identity and let K be a right ideal of $[R]_n$. Then

(1) $K < [R]_n$ if and only if $K_{ij} < R$ for all i and j; and

(2) K_{ij} (i,j = 1, .., n) are essential right ideals of R if K is an essential right ideal of $[R]_n$. Hence the right singular ideal of $[R]_n$ is zero if that of R is zero.

Proof. (1) Suppose $K < [R]_n$ and let r and s be two elements of R with $r \neq 0$. Then $re_{ii} \neq 0$, e_{ii} and se_{ii} are in $[R]_n$. By assumption, there exists an element $a = \sum_{p,q} a_{pq}e_{pq}$ of K such that $re_{ii}a \neq 0$, $e_{ii}a \in K$ and $se_{ii}a \in K$. Now $0 \neq re_{ii}a = \sum_q ra_{iq}e_{iq}$. Suppose $ra_{it} \neq 0$. Then $a_{it}e_{it} = e_{ii}ae_{tt} \in K$ and $sa_{it}e_{it} = se_{ii}ae_{tt} \in K$. Thus $a_{it} \in K_{it} = K_{ij}$ and $ra_{it} \neq 0$, $sa_{it} \in K_{it} = K_{ij}$. This shows that $K_{ij} < R$. Conversely suppose that $K_{ij} < R$ for all i and j.

We may take X to be the Utumi's D-system of right ideals of R. Then $K_{ij} \in X$, so $K \in [X]_n$. By Proposition 2.15, $[X]_n$ is a D'-system of right ideals of $[R]_n$. Hence $K < [R]_n$.

(2) If $K_{ij} \cap A = 0$ for some right ideal A of R, then $K \cap A^* = 0$, where A^* is the right ideal of $[R]_n$ consisting of all matrices whose ith rows are n-tuples of elements of A and zero elsewhere. Since K is essential, $A^* = 0$. This means that $A = 0$. It follows then that K_{ij} is essential for every pair i and j.

2.17. Proposition. Let R be a ring with an identity and let X be a D-system of right ideals of R. Then X is the Utumi's D-system if and only if $[X]_n$ is the Utumi's D-system. If, in addition, the right singular ideal of R is zero, then $[X]_n$ is the Johnson's D-system provided that X is the Johnson's D-system.

3. Atoms and socles. Let X be a D-system of right ideals of the ring R. As we have shown previously, for each R-module M, X induces a modular closure operation on L(M). The set $L^X(M)$ of all x-closed submodules of M (which are precisely strongly X-prime submodules of M) becomes a complete modular lattice with unique maximal element M and unique minimal element 0^X if we define the join and meet of a subfamily $\{N_i\}$ of $L^X(M)$ by $\vee N_i = (\sum N_i)^X$ and $\wedge N_i = \cap N_i$. A subfamily $\{N_i\}_{i \in I}$ of $L^X(M)$ is said to be join-independent if for each i in I, $N_i \cap (\vee_{j \neq i} N_j) = 0^X$. N is said to be a direct join of N_i $(i \in I)$ if $\{N_i\}_{i \in I}$ is a join-independent family and N is a join of N_i's. In this case, we write $N = \vee_{i \in I} \oplus N_i$ or $N = N_1 \textcircled{\vee} N_2 \textcircled{\vee} \cdots \textcircled{\vee} N_k$ if $I = \{1,2,..,k\}$.

3.1. Lemma. Let $\{N_i\}_{i\in I}$ be a join-independent family of x-closed submodules of M and let N be any x-closed submodule of M such that $N \cap (\underset{i\in I}{\vee} N_i) = 0^x$. Then the family consisting of N and N_i's is join-independent.

Proof. Let m be an element of N_j contained in $N \cap (\underset{i\neq j}{\vee} N_i)$. Then $mA \subset N + \sum_{i\neq j} N_i$ for some A in X. For each a in A, $ma = n+n'$ where n belongs to N and n' to $\sum_{i\neq j} N_i$. Or $n = ma-n' \in N\cap(\underset{i\in I}{\vee} N_i)=0^x$. Thus $nB_a = 0$ for some B_a in X and $maB_a = n'B_a \subset (\underset{i\neq j}{\vee} N_i) \cap N_j = 0^x$. Let $B = \sum_{a\in A} aB_a$. Then B belongs to X and $mB \subset 0^x$ and so $m\in 0^{xx}=0^x$.

As usual we shall call a minimal element of $L^x(M)$ different from 0^x an atom of $L^x(M)$. $L^x(M)$ may or may not have atoms. The X-socle of M is defined to be the join of all atoms of $L^x(M)$ and will be denoted by $S_x(M)$. It is agreed here that the join of an empty collection is 0^x. In case $M = S_x(M)$, M is said to be X-atomic. In particular, the X-socle of any R-module is X-atomic.

3.2. Proposition. Let $\{T_i\}_{i\in I}$ be the family of all atoms of $L^x(M)$ where M is an X-atomic R-module. If N is any x-closed submodule of M, then there exists a join-independent subfamily $\{T_j\}_{j\in J}$ such that $M = N \oplus (\underset{j\in J}{\vee} \oplus T_j)$. In particular the X-socle of an R-module is a direct join of atoms of the lattice of x-closed submodules.

Proof. Consider the collection of all join-independent subfamilies of $\{T_i\}_{i\in I} \cup \{N\}$ that contain N. We partially order this collection by inclusion. Let $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \cdots$ be an increasing chain in the collection and let $\mathcal{G}$ be the union of all $\mathcal{G}_k$. Suppose that $H\in\mathcal{G}$ and m is an element of $H \cap (\underset{H\neq K\in\mathcal{G}}{\vee} K)$. Then $m\in H$ and $mA \subset \sum K$ for some A in X. For each a in A, ma is a finite sum of elements

from $K_1, \ldots, K_n$ say, where $K_k \in \mathcal{G}$. Each of the K_k's is in some $\mathcal{G}_s$ As there are only a finite number of K_k's, H, $K_1, \ldots, K_n$ are contained in some $\mathcal{G}_t$. Since $\mathcal{G}_t$ is join-independent, $maB_a \subset 0^x$ for some B_a in X. Let $B = \sum_{a \in A} aB_a$. Then B belongs to X and $mB \subset 0^x$. Thus $m \in 0^{xx} = 0^x$. Hence $\mathcal{G}$ is join-independent. We can then apply Zorn's lemma to obtain a maximal element $\{N\} \cup \{T_j\}_{j \in J}$ say. For each T_i, either $T_i \cap (N \vee (\vee T_j)) = 0^x$ or $T_i \subset N \vee (\vee T_j)$ since T_i is an atom. By Lemma 3.1 and the maximality of $\{N\} \cup \{T_j\}$, the first case is impossible. Hence the second alternative holds for all i in I and so $M = N \textcircled{\vee} (\underset{j \in J}{\vee} \oplus T_j)$.

Our lattice $L^x(M)$ clearly possesses the following property: For any directed set I, if $N_i \uparrow N$, $i \in I$, then $K \cap N_i \uparrow K \cap N$. By Theorem 13 on p.129 in [4], if M is X-atomic, then every x-closed submodule of M is also X-atomic. Thus every submodule of M is X-atomic by Proposition 1.10. Furthermore Proposition 3.2 tells us that for a submodule N of an X-atomic R-module M, there exists an x-closed submodule K of M such that $M = N^x \textcircled{\vee} K$, i.e., $N \cap K \subset 0^x$ and $M' = N + K$ is an x-dense submodule of M. Thus M'-N is an x-dense submodule of M-N and $M'-N \cong K-(N \cap K)$. By Propositions 1.10 and 1.13, $L^x(M-N) \cong L^x(M'-N) \cong [K, (N \cap K)^x] \cong L^x(K)$. Since K is X-atomic, M-N is X-atomic.

3.3. Theorem. Every submodule and every homomorphic image of an X-atomic R-module is X-atomic.

3.4. Theorem. If $M = \underset{j \in J}{\vee} \oplus T_j = \underset{i \in I}{\vee} \oplus U_i$ where T_j and U_i are atoms of $L^x(M)$, then I and J have the same cardinal number.

Proof. If both I and J are finite, then the theorem of Ore ([4], p. 94, Theorem 12) applies. We may then suppose that I is

infinite. For each j in J, since $T_j \neq 0^x$ and since M is a direct join of U_i's, $0 \neq tA \subset \sum_{i\in I} U_i$ for some t in T_j, $t \notin 0^x$ and A in X. If $ta\in 0^x$ for all a in A, then by D3, t belongs to 0^x. Thus there exists a in A such that $ta \notin 0^x$ and so $T_j = |ta)^x$, where $|ta)$ denotes the submodule of M generated by ta. But since ta is a finite sum of elements from U_i's, T_j is contained in a finite join of atoms U_i. Hence for each j in J there exists a finite subset $\delta(j)$ of I such that $T_j \subset \bigvee_{i\in\delta(j)} U_i$. By a similar argument, we can show that for each i in I, there exists a finite subset $\varepsilon(i)$ of J such that $U_i \subset \bigvee_{j\in\varepsilon(i)} T_j$. Now $U_i \subset \vee\{U_k | k\in \bigcup_{j\in\varepsilon(i)} \delta(j)\}$. Since U_i is join-independent, $i \in \bigcup_{j\in\varepsilon(i)} \delta(j)$. This shows that $I = \bigcup_{j\in J} \delta(j)$ and so I and J have the same cardinal number. The uniquely determined cardinal will be called the <u>X-dimension</u> of M and will be denoted by X-dim M.

Following Bourkabi ([5], p. 158), we shall call an R-module M X-negligible if $0^{x(M)} = M$. The lattice $L^x(M)$ associated with an X-negligible R-module M consists of only one element. Hence every X-negligible R-module is X-atomic with X-dimension 0. For a non X-negligible X-atomic R-module the X-dimension is always greater than 0. We recall that an R-module M is strongly X-prime if $0^{x(M)} = 0$. Accordingly the only R-module which is both X-negligible and strongly X-prime is 0.

3.5. <u>Lemma</u>. Let φ be an R-homomorphism of an R-module M into another R-module N. If T is an atom of $L^x(M)$, then either φT is X-negligible or $(\varphi T)^x$ is an atom of $L^x(M)$.

<u>Proof</u>. Suppose φT is not X-negligible. Then $(\varphi T)^x \neq 0^x$. Let $K \neq 0^x$ be an x-closed submodule of N contained in $(\varphi T)^x$. Then $T \cap \varphi^{-1}K$ is not X-negligible and by Lemma 1.11, $\varphi^{-1}K$ is x-closed.

Since T is an atom, $T \cap \varphi^{-1}K = T$ and so $\varphi T \subset K$. Thus $(\varphi T)^x \subset K$ and $(\varphi T)^x$ is an atom of $L^x(M)$

3.6. Corollary. Let A be an atom of $L^x(R)$ and let m be any element of M. Then either mA is X-negligible or $(mA)^x$ is an atom of $L^x(M)$.

A submodule N of an R-module M which is mapped into itself by every R-endomorphism of M can be regarded as an (E,R)-module, where E is the ring of all R-endomorphisms of M. We shall denote the set of all (E,R)-modules which are x-closed submodules of M by $C^x(M)$. In general, $C^x(M) \neq L^x(M)$. We note that every right ideal of R belonging to $C^x(R)$ is a (two-sided) ideal of R. If R has an identity, then an x-closed right ideal of R is in $C^x(M)$ if it is an ideal of R.

For a subfamily $\{N_i\}$ of $C^x(M)$ and an R-endomorphism φ of M,

$$\varphi(\cap N_i) \subset \cap \varphi N_i \subset \cap N_i,$$

and $\varphi(\sum N_i) = \varphi(\sum N_i)^x \subset (\varphi(\sum N_i))^x = (\sum \varphi N_i)^x \subset (\sum N_i)^x = \sum N_i$.

It follows that $C^x(M)$ is a complete sublattice of $L^x(M)$. In particular $C^x(M)$ is an inset of $L^x(M)$ containing M and induces a closure operation $N \to \overline{N}$ on $L^x(M)$, where $\overline{N}$ is the least member of $C^x(M)$ containing N, or explicitly $\overline{N} = \vee_{\varphi \in E} (\varphi N)^x$.

3.7. Proposition. The X-socle of an R-module M is in $C^x(M)$.

Proof. If $L^x(M)$ contains no atoms, then $S_x(M) = 0^x$ which is clearly in $C^x(M)$. Suppose then $\{T_i\}$ is the set of all atoms of $L^x(M)$. If φ is an R-endomorphism of M, then either $\varphi T_i \subset 0^x$ or $(\varphi T_i)^x$ is an atom by Lemma 3.5. Thus

$$\varphi S_x(M) = \varphi(\sum T_i)^x \subset (\sum \varphi T_i)^x \subset (\sum (\varphi T_i)^x)^x \subset (\sum T_i)^x = S_x(M).$$

3.8. Theorem. If R is X-atomic, then every R-module M has the following property:

($\diamond$) For every x-closed proper submodule N of M, there exists an atom T of $L^x(M)$ such that $N \cap T = 0^x$.

Proof. If R is X-negligible, then every right ideal of R is x-dense and so is in X by Lemma 2.3. Every R-module is X-negligible and there is nothing to prove. We may suppose then R is not X-negligible and let $\{A_i\}$ be the set of all atoms of R. Since R is X-atomic, $\sum A_i$ is x-dense and hence in X. Choose m to be an element of M not in N. Then $m(\sum A_i) \not\subset N$. There exists an atom A_j such that $mA_j \not\subset N$. By Corollary 3.6, $(mA_j)^x$ is an atom of $L^x(M)$. Since $(mA_j)^x \not\subset N$, $N \cap (mA_j)^x = 0^x$.

3.9. Corollary. If R is X-atomic, then every R-module M has the following property:

($\diamond\diamond$) For every pair of x-closed submodules N and K with $N \subsetneq K$, there exists an atom T of $L^x(M)$ such that $T \subset K$ and $T \cap N = 0^x$.

An R-module M satisfying condition ($\diamond\diamond$) automatically satisfies the condition ($\diamond$). On the other hand M satisfies the condition ($\diamond\diamond$) if every x-closed submodule of M satisfies the condition ($\diamond$). The conditions ($\diamond$) and ($\diamond\diamond$) are of interest in themselves, we shall devote the last portion of this section to the study of their implications.

3.10. Proposition. An R-module M satisfies the condition ($\diamond$) relative to X if and only if M is X-atomic.

Proof. Assume M satisfies the condition ($\diamond$) relative to X. If M is X-negligible, then there is nothing to prove. Suppose then M is not X-negligible. The X-socle $S_x(M)$ of M must be M for

otherwise there would exist an atom T of $L^X(M)$ such that $T \cap S_X(M) = 0^X$ which is impossible since $S_X(M)$ is the join of all atoms. Hence M is X-atomic. The converse follows from Proposition 3.2.

In view of Theorem 3.8 and Proposition 3.10, if R is X-atomic, then so is every R-module.

A submodule N of an R-module M is said to be <u>$\cap$-irreducible</u> if $N = N_1 \cap N_2$ where N_1 and N_2 are submodules of M, implies $N = N_1$ or N_2. An x-closed submodule N of M is called <u>meet-irreducible</u> relative to X if $N = N_1 \cap N_2$ where N_1 and N_2 are x-closed submodules of M implies $N = N_1$ or N_2. Clearly every $\cap$-irreducible x-closed submodule of M is meet-irreducible relative to X. Conversely if N is meet-irreducible and x-closed and if $N = N_1 \cap N_2$ for two submodules N_1 and N_2 of M, then $N = N^X = N_1^X \cap N_2^X$. Thus $N = N_1^X$ or N_2^X, i.e., $N = N_1$ or N_2. Hence with respect to x-closed submodules, the concept of $\cap$-irreducibility and that of meet-irreducibility coincide.

3.11. <u>Proposition</u>. If M satisfies the condition ($\diamond\diamond$) relative to X, then (1) M satisfies the condition ($\diamond$) relative to X;

(2) every non X-negligible x-closed submodule of M contains an atom of $L^X(M)$; and

(3) every x-closed submodule is a meet of $\cap$-irreducible x-closed submodules of M.

<u>Proof</u>. (1) and (2) are easy consequences of ($\diamond\diamond$). To prove (3), let N be an x-closed submodule of M. Then N is clearly contained in the meet K of all $\cap$-irreducible x-closed submodules of M that contain N. If $N \neq K$, then by ($\diamond\diamond$), there exists an atom T of $L^X(M)$ such that $T \subset K$ and $T \cap N = 0^X$. This is impossible by the following lemma. Hence $N = K$, a meet of $\cap$-irreducible x-closed submodules of M.

3.12. Lemma. Let M be any R-module and let N be an x-closed submodule of M such that $N \cap T = 0^x$ for some atom T in $L^x(M)$. Then there exists an $\cap$-irreducible x-closed submodule P of M such that $P \supset N$ and $P \cap T = 0^x$.

Proof. Consider the set $\mathfrak{h}$ of all x-closed submodules K of M such that $K \supset N$ and $K \cap T = 0^x$. Since N is in $\mathfrak{h}$, $\mathfrak{h}$ is nonempty. Partially order $\mathfrak{h}$ by inclusion. Let $K_1 \subset K_2 \subset \cdots$ be an ascending chain in $\mathfrak{h}$ and let K be the join of the K_n's. Clearly $K \supset N$. If $k \in K \cap T$, then $k \in T$ and $kA \subset \cup K_n$ for some A in X. For each a in A, $ka \in K_i$ say. Then $ka \in K_i \cap T = 0^x$ and so $kaB_a = 0$ for some B_a in X. Let B be the sum of aB_a. Then B belongs to X and $kB = 0$. Hence $k \in 0^x$ and $K \cap T = 0^x$. Apply Zorn's lemma to obtain a maximal element P of $\mathfrak{h}$. If $P = P_1 \cap P_2$ where P_1 and P_2 are x-closed submodules of M that contain P properly, then $P_1, P_2 \notin \mathfrak{h}$. It follows that $P_1 \cap T \neq 0^x$ and $P_2 \cap T \neq 0^x$ and so $P_1 \supset T$, $P_2 \supset T$ and $P_1 \cap P_2 \supset T$. This is impossible since $P \cap T = 0^x$. Hence P is $\cap$-irreducible.

The following corollary is an immediate consequence of the results 3.8-3.11.

3.13. Corollary. If R is X-atomic, then every R-module M has the following properties:

(1) M is X-atomic.

(2) Each non X-negligible x-closed submodule of M contains an atom of $L^x(M)$.

(3) Every x-closed submodule is a meet of $\cap$-irreducible x-closed submodules of M.

4. Complementation. Assume that X is a D-system of right ideals of R and M is an R-module. By a complement of an element N of $L^X(M)$ is meant an element N' of $L^X(M)$ such that $N \cap N' = 0^X$ and $N \vee N' = M$. The lattice $L^X(M)$ is said to be complemented if all its elements have complements. We shall correlate the concept of complemented lattice with that of essential and complement submodules.

Let N and K be two submodules of M such that $N \cap K = 0$. By Zorn's lemma, there exists a submodule N' of M maximal with respect to $N' \supset K$ and $N' \cap N = 0$. Such a submodule is called a relative complement of N containing K, or simply relative complement of N if K can be neglected or $K = 0$. It is worthwhile noting that N' is a relative complement of N containing K if and only if (i) N' contains K; and (ii) if N" is a submodule of M such that $N'' \supset N'$ and $N'' \cap N = 0$, then $N'' = N'$. Evidently if N' is a relative complement of N, then $N + N'$ is an essential submodule of M. A submodule of M is called a complement submodule if it is a relative complement of some submodule of M.

4.1. Lemma. Let M be any R-module. Then

(1) every complement submodule of $M-0^X$ is x-closed; and

(2) each x-dense submodule of $M-0^X$ is essential.

Proof. Since $M-0^X$ is strongly X-prime, we may suppose M is strongly X-prime, i.e., $0^X = 0$. To prove (1), let N be a complement submodule of M which is a relative complement of K say. Since $N \cap K = 0$, we have $0 = 0^X = (N \cap K)^X = N^X \cap K^X \supset N^X \cap K$. Thus by the above remark, $N^X = N$ and N is x-closed. Next, suppose H is an x-dense submodule of M and m is any element of M. Then $mA \subset H$ for some A in X. Since M is strongly X-prime, $mA \neq 0$. Hence H is essential.

4.2. Theorem. Let M be an R-module. The the following statements are equivalent:

(1) $L^X(M)$ is a complemented lattice.

(2) Every x-closed submodule of M-0^X is a complement submodule.

(3) Each essential submodule of M-0^X is x-dense.

Proof. Since $L^X(M) \cong L^X(M\text{-}0^X)$ and since $M\text{-}0^X$ is strongly X-prime, we may suppose that M is strongly X-prime.

(1) implies (2): Let N be an x-closed submodule of M and let N' be a complement of N in $L^X(M)$. If N" is a relative complement of N' containing N, then N" is x-closed by Lemma 4.1 and $N" = N"\cap M = N" \cap (N \vee N') = N \vee (N" \cap N') = N \vee 0 = N$ by the modularity of $L^X(M)$. Hence N is a complement submodule of M.

(2) implies (3): Let H be an essential submodule of M. Then H^X is x-closed. By (2), H^X is a relative complement of some submodule K of M. Since $H^X \cap K = 0$ and since H is essential, $K = 0$. Hence $H^X = M$ by a remark preceeding Lemma 4.1. Hence H is x-dense.

(3) implies (1): Let N be an x-closed submodule of M and let N' be a relative complement of N. Then by Lemma 4.1, N' is x-closed. Moreover $N + N'$ is essential and hence x-dense by (3). Thus $N\cap N'=0$ and $N \vee N' = M$, i.e., N' is a complement of N in $L^X(M)$. Hence $L^X(M)$ is complement.

4.3. Corollary. There exists a D'-system X of right ideals of R such that $L^X(R)$ is complemented if and only if the right singular ideal of R is zero and X is the Johnson's D-system of right ideals of R.

4.4. Corollary. If X is the Johnson's D-system of right ideals of R, then $L^X(M)$ is complemented for every R-module M. Hence $D^X(M)$ consists of all essential submodule of M while $L^X(M)$ comprises all complement submodules of M if, in addition, M is strongly

X-prime (i.e., the singular submodule of M is zero).

Proof. We may suppose that M is strongly X-prime. Let N be an essential submodule of M and let m be any element of M. Then $m^{-1}N$ is an essential right ideal of R. For otherwise $m^{-1}N \cap T = 0$ for some nonzero right ideal T of R. Then $mT \cap N = 0$. Since N is essential, $mT = 0$ and so $T \subset m^{-1}0 \subset m^{-1}N$. This contradicts the choice of T. Hence $m^{-1}N \in X$ and so m belongs to N^x. This shows that N is x-dense. By Theorem 4.2, $L^x(M)$ is complemented.

We may generalize the concept of relative complement of a submodule of an R-module as follows:

Let N and K be two submodules of M such that $N \cap K$ is X-negligible. Consider the collection $\mathfrak{n}$ of all submodules H of M such that $H \supset K$ and $H \cap N$ is X-negligible. Let $H_1 \subset H_2 \subset \cdots$ be an ascending chain in $\mathfrak{n}$ and let H be the union of all the H_i's. If n is an element of $H \cap N$, then n belongs to $H_i \cap N$ for some i. Hence $H \cap N$ is X-negligible and H belongs to $\mathfrak{n}$. Apply Zorn's lemma to obtain maximal elements of $\mathfrak{n}$. Such submodules are called <u>X-relative complements</u> of N containing K or simply X-relative complements of N. Note that N' is an X-relative complement of N containing K if and only if (i) N' contains K; and (ii) if N" is a submodule of M containing K such that $N" \cap N$ is X-negligible, then N" = N'. A submodule N of M is said to be an <u>X-complement submodule</u> if it is an X-relative complement of some submodule of M.

4.5. <u>Proposition</u>. (1) Every X-complement submodule N of M is x-closed. (2) If $L^x(M)$ is compelemented, then (i) every x-closed submodule of M is an X-complement submodule; and (ii) for an element N in $L^x(M)$, a submodule N' of M is an X-relative complement of N if and only if it is a complement of N in $L^x(M)$.

Proof. Suppose N is an X-relative complement of K. Since $N^x \supset N$ and $N^x \cap K \subset N^x \cap K^x = (N \cap K)x = 0^x$, $N^x = N$ and so N is x-closed. This proves (1). To prove (2)(i) let N be an x-closed submodule of M and let N' be a complement of N in $L^x(M)$. Assume that K is an X-relative complement of N' containing N. Then $K = K \cap (N \vee N') = N \vee (K \cap N') = N \vee 0^x = N$. Thus N is an X-complement submodule of M. Finally in proving (2)(ii), let N' be a complement of N in $L^x(M)$. By a similar argument as given in (i), N' is an X-relative complement of N. Conversely suppose N' is an X-relative complement of N. If $N \vee N' \neq M$, then let N" be a complement of $N \vee N'$ in $L^x(M)$. Clearly N" is not contained in N'. Let n be any element of $N \cap (N'+N")$. Then $n = n'+n"$ for some n' in N' and n" in N". Now $n" = n-n' \in (N \vee N') \cap N" = 0^x$. There exists A in X such that $n"a = 0$ for each a in A. It follows that $na = n'a \in N \cap N' = 0^x$ and so there exists B_a in X such that $naB_a = 0$. Let B be the sum of aB_a $(a \in A)$. Then $B \in X$ and $nB = 0$. Thus $n \in 0^x$ and $N \cap (N'+N") = 0^x$. Since N' is an X-relative complement of N, $N'+N" = N'$. This contradicts the choice of N". Hence $N \vee N' = M$ and N' is a complement of N in $L^x(M)$.

If $L^x(M)$ is complemented, then so is every closed interval [A,B] of $L^x(M)$. For if N is any element of [A,B] and N' is a complement of N in $L^x(M)$, then $N" = A \cap (B \vee N') = B \vee (A \cap N')$ and

$$N \cap N" = N \cap A \cap (B \vee N') = N \cap (B \vee N') = B \vee (N \cap N') = B \vee 0^x = B,$$

$$N \vee N" = N \vee B \vee (A \cap N') = N \vee (A \cap N') = A \cap (N \vee N') = A \cap M = A.$$

Hence N" is a complement of N in [A,B]. This fact together with Propositions 1.9, 1.10 and 1.13 prove the following

4.6. Theorem. If $L^x(M)$ is complemented, then $L^x(N)$ and $L^x(M-N)$ are complemented for every submodule N of M.

By a maximal x-closed submodule of M, we mean a maximal element of $L^x(M) \setminus \{M\}$. An R-module may or may not have maximal x-closed submodules. Since $L^x(M-N)$ is isomorphic to the closed interval $[M,N^x]$ of $L^x(M)$, N^x is a maximal x-closed submodule of M if and only if M-N is an atom of $L^x(M-N)$, i.e., if and only if every non X-negligible submodule of M-N is x-dense.

4.7. Lemma. (1) Every maximal x-closed submodule of M is an $\cap$-irreducible x-closed submodule of M, the converse is true if $L^x(M)$ is complemented.

(2) The complements (if any) of a maximal x-closed submodule of M are atoms of $L^x(M)$ and the complements (if any) of an atom of $L^x(M)$ are $\cap$-irreducible x-closed submodules of M.

Proof. Suppose N is a maximal x-closed submodule of M and let $N = N_1 \cap N_2$ where, we may suppose N_1 and N_2 are x-closed submodules of M. If $N_1 \neq N$, then since N is maximal, $N_1 = M$ and so $N = N_2$. Hence N is $\cap$-irreducible. Conversely suppose $L^x(M)$ is complemented and let N be an $\cap$-irreducible x-closed submodule of M. Then the closed interval $[M,N]$ is complemented. For every K in $[M,N]$ different from M, let K' be the complement of K in $[M,N]$, i.e., $K \cap K' = N$ and $K \vee K' = M$. Since N is $\cap$-irreducible, $K = N$ and so N is a maximal x-closed submodule of M. This proves (1). To prove (2), let T be a complement of N, a maximal x-closed submodule of M. If $K \in L^x(M)$ such that $0^x \neq K \subset T$, then since N is maximal and $K \not\subset N$, $N \vee K = M$. Now $T = T \cap (N \vee K) = K \vee (T \cap N) = K \vee 0^x = K$. Hence T is an atom of $L^x(M)$. Next, let N be a complement of T, an atom of $L^x(M)$. By Lemma 3.12, there exists an $\cap$-irreducible x-closed submodule P of M such that $P \supset N$ and $P \cap T = 0^x$. Now $P = P \cap M = P \cap (N \vee T) = N \vee (P \cap T) = N \vee 0^x = N$. Hence N is $\cap$-irreducible.

We prove in the previous section that if an R-module M satisfies the condition ($\diamond\diamond$) relative to X, then M is X-atomic (and hence by Proposition 3.2, $L^X(M)$ is complemented), every x-closed submodule is a meet of $\cap$-irreducible x-closed submodules (and so maximal x-closed submodules by Lemma 4.7) of M and each non X-negligible x-closed submodule of M contains an atom of $L^X(M)$. The following two propositions show the effect of complementedness of the lattice $L^X(M)$ on the conditions ($\diamond$) and ($\diamond\diamond$). Note that the condition ($\diamond$) for M is equivalent to X-atomicity of M.

4.8. Proposition. If M is an R-module such that $L^X(M)$ is complemented and every x-closed submodule is a meet of $\cap$-irreducible x-closed submodules, then M satisfies the condition ($\diamond$) relative to X.

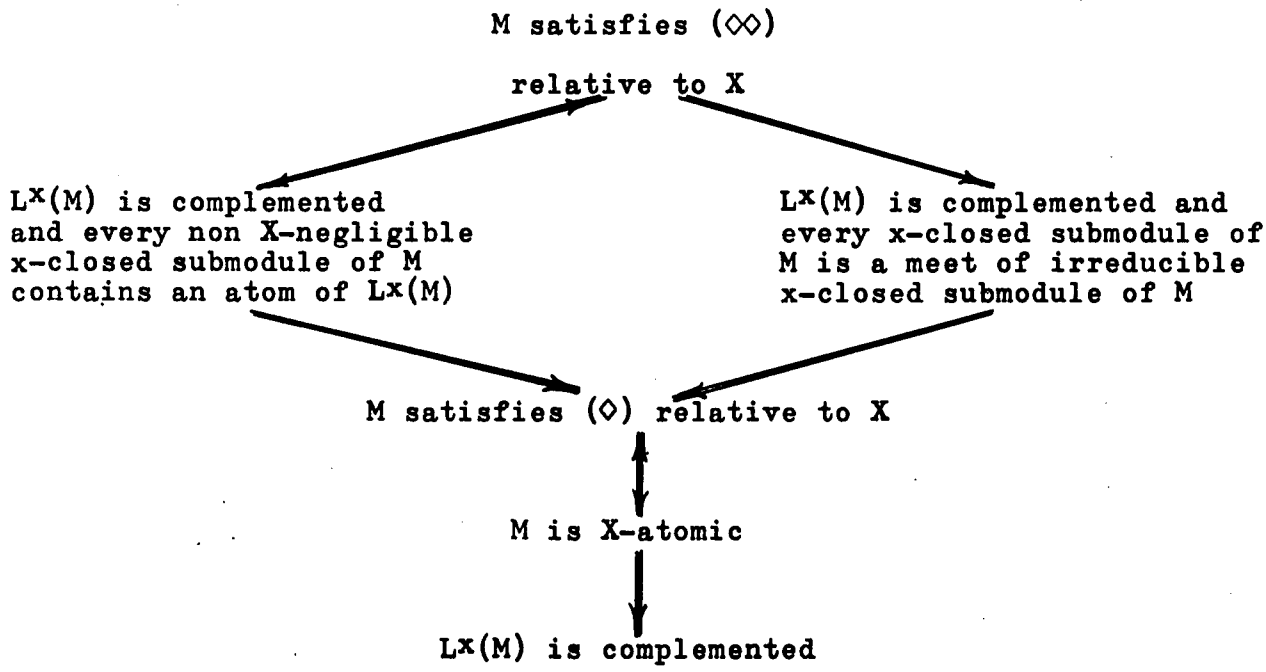
Proof. Let N be a proper x-closed submodule of M. Then N is a meet of $\cap$-irreducible x-closed submodules say P_i ($i \in I$). Since $N \neq M$, there exists P in $\{P_i\}$ such that $P \neq M$. Let T be a complement of P in $L^X(M)$. By Lemma 4.7, P is a maximal x-closed submodule of M and so T is an atom of $L^X(M)$ such that $T \cap N = 0^X$. Hence M satisfies the condition ($\diamond$) relative to X.

4.9. Proposition. If M is an R-module such that $L^X(M)$ is complemented and every non X-negligible x-closed submodule contains an atom of $L^X(M)$, then M satisfies the condition ($\diamond\diamond$) relative to X.

Proof. Let N and K be two x-closed submodules of M such that $N \subsetneq K$. Since $L^X(M)$ is complemented, so is $L^X(K)$. Let N' be a complement of N in $L^X(K)$. Then N' is not X-negligible since $K \vee N' = N$ and $K \neq N$. Because $L^X(K) \subset L^X(M)$, N' is an

x-closed submodule of M. By hypothesis, N' contains an atom T say. Clearly $T \subset K$ and $T \cap N \subset N' \cap N = 0^x$. Hence M satisfies the condition ($\diamond\diamond$) relative to X.

The following diagram, intended primarily for reference, exhibits the interrelations of various conditions studied in the present and previous sections. It should be remarked that for the ring R itself, the condition ($\diamond$) is equivalent to the condition ($\diamond\diamond$). (See Corollary 3.9.)



5. Finiteness assumptions. As usual, the lattice $L^x(M)$ is said to satisfy the ascending (resp. descending) chain condition if every ascending chain $N_1 \subset N_2 \subset \cdots$ (resp. descending chain $N_1 \supset N_2 \supset \cdots$) of elements in $L^x(M)$ terminates at a finite number of steps, i.e., there exists a positive integer n such that $N_n = N_{n+1} = \cdots$. A necessary and sufficient condition for $L^x(M)$ to satisfy ascending (resp. descending) chain condition is that every nonempty collection of elements in $L^x(M)$ has a maximal (resp. minimal) element.

5.1. Theorem. If $L^x(M)$ satisfies the ascending (resp. descending) chain condition, then $L^x(N)$ and $L^x(M-N)$ satisfy the ascending (resp. descending) chain condition for every submodule N of M. Conversely, if $L^x(N)$ and $L^x(M-N)$ satisfy the ascending (resp. descending) chain condition for some submodule N of M, then $L^x(M)$ satisfies the ascending (resp. descending) chain condition.

Proof. We shall prove our proposition for the case where the lattices satisfy ascending chain condition. Suppose $L^x(M)$ satisfies the ascending chain condition. By Propositions 1.9, 1.10 and 1.13, for any submodule N of M, $L^x(N)$ and $L^x(M-N)$ satisfy the ascending chain condition. Conversely, assume that $L^x(N)$ and $L^x(M-N)$ satisfy the ascending chain condition for some submodule N of M. Let $\mathfrak{F}$ be a nonempty collection of elements in $L^x(M)$. Then the set $\mathfrak{G}$ of all submodules of the form $K \cap N$, where K runs through all the elements in $\mathfrak{F}$ is a nonempty collection of elements in $L^x(N)$. Since $L^x(N)$ satisfies ascending chain condition, $\mathfrak{G}$ contains a maximal element say $H \cap N$. Consider the collection $\mathfrak{H}$ consisting of all elements of the form $\overline{K}^x=((K+N)-N)^{x(M-N)}$

where $K \in \mathfrak{F}$ and $K \cap N = H \cap N$. $\mathfrak{B}$ is a nonempty collection of elements in $L^x(M-N)$ and hence contains a maximal element say $\overline{J}^x$. We shall show that J is a maximal element of $\mathfrak{F}$. For if K is a member of $\mathfrak{F}$ containing J, then $K \cap N \supset J \cap N = H \cap N$. Since $H \cap N$ is a maximal element of G, $K \cap N = H \cap N$ and so $\overline{K}^x \in \mathfrak{B}$. Since $\overline{K}^x \supset \overline{J}^x$ and $\overline{J}^x$ is a maximal element of $\mathfrak{B}$, $\overline{K}^x = \overline{J}^x$. If k is any element of K, then $k+N \in \overline{K}^x = \overline{J}^x$. There exists an A in X such that $kA + N = (k+N)A \in \overline{J}$. For each a in A, $ka+n \in J$ for some n in N. Now $n \in K \cap N = J \cap N \subset J$. Thus $ka \in J$ for all a in A. Since J is x-closed, k belongs to J. Hence $K \subset J$ and so J is a maximal element of $\mathfrak{F}$.

5.2. Corollary. Assume $M_1, \ldots, M_n$ are submodules of M such that $M_1 + \cdots + M_n$ is x-dense in M and for each i, $L^x(M_i)$ satisfies the ascending (resp. descending) chain condition, then $L^x(M)$ satisfies the ascending (resp. descending) chain condition.

Proof. Since $L^x(M)$ is isomorphic to $L^x(\sum M_i)$, we may suppose $M = \sum M_i$. It is clearly true for $n = 1$. Assume that it is true for $n = k-1$. If $M = M_1 + \cdots + M_n$ and each $L^x(M_i)$ satisfies the ascending (resp. descending) chain condition, then $L^x(M')$ satisfies the ascending (resp. descending) chain condition, where $M' = M_2 + \cdots + M_k$. Furthermore $M - M_1 = (M'+M_1) - M_1 \cong M' - (M_1 \cap M')$. By Theorem 5.1, $L^x(M)$ satisfies the ascending (resp. descending) chain condition.

5.3. Corollary. If $L^x(R)$ satisfies the ascending (resp. descending) chain condition and if $m_1, \ldots, m_n$ are elements in an R-module M such that $m_1R + \cdots + m_nR$ is an x-dense submodule of M, then $L^x(M)$ satisfies the ascending (resp. descending) chain condition.

Proof. Since each m_iR is an R-homomorphic image of R, $L^x(m_iR)$

satisfies ascending (resp. descending) chain condition. Apply Corollary 5.2.

We shall associate the ascending (or descending) chain condition with some of the notions introduced previously. First of all, let us consider the case where $L^X(M)$ is complemented.

5.4. Lemma. Let M be an R-module and suppose that $L^X(M)$ is complemented. If N and N' are two x-closed submodules of M such that $N \subsetneq N'$, then there exist complements K and K' of N and N' respectively such that $K \supsetneq K'$.

Proof. By Proposition 1.9 and Theorem 4.6, $L^X(N') \subset L^X(M)$ and is complemented. Let H be a complement of N in $L^X(N')$ and let K' be a complement of N' in $L^X(M)$. Then $N \cap (K'+H)$ is X-negligible. Let K be an X-relative complement of N containing K'+H. By Proposition 4.5, K is a complement of N in $L^X(M)$. Since $H \not\subset K'$, $K \supsetneq K'$.

5.5. Theorem. Let M be an R-module such that $L^X(M)$ is complemented. Then $L^X(M)$ satisfies the ascending chain condition if and only if it satisfies the descending chain condition.

The proof of the above theorem is an easy application of Lemma 5.4.

An R-module M is said to be finitely X-atomic if it is a join of a finite number of atoms of $L^X(M)$.

5.6. Proposition. An R-module M is finitely X-atomic if and only if M is X-atomic and $L^X(M)$ satisfies the ascending chain condition.

Proof. Suppose M is finitely X-atomic and is a join of atoms

$T_1, .., T_n$ of $L^X(M)$. Then M is automatically X-atomic. Since each $L^X(T_i)$ contains only two elements, it satisfies the ascending chain condition. By Corollary 5.2, $L^X(M)$ satisfies the ascending chain condition. The converse is clear.

It follows from Theorem 5.5 that every x-closed submodule of M different from 0^X contains an atom of $L^X(M)$ if $L^X(M)$ is complemented and satisfies the ascending chain condition. By Proposition 4.9, M satisfies the condition ($\diamond\diamond$) relative to X. In view of the diagram presented at the end of the previous section, we have the following

5.7. Theorem. Let M be an R-module such that $L^X(M)$ satisfies the ascending chain condition. Then the following statements are equivalent:

(1) M satisfies the condition ($\diamond\diamond$) relative to X.

(2) M satisfies the condition ($\diamond$) relative to X.

(3) M is finitely X-atomic.

(4) $L^X(M)$ is complemented.

Each of the equivalent conditions (1)-(4) implies (i) every non X-negligible x-closed submodule of M contain an atom of $L^X(M)$; and (ii) each x-closed submodule is a meet of a finite number of maximal x-closed submodules (and hence $\cap$-irreducible submodules) of M.

An implication of Proposition 5.6 and Theorem 5.7 is that an R-module M is finitely X-atomic if and only if $L^X(M)$ is complemented and satisfies the ascending chain condition. By virtue of Theorems 4.6 and 5.1 we have

5.8. Corollary. Every submodule and every homomorphic image of a finitely X-atomic R-module is finitely X-atomic.

As shown in Section 3, an X-atomic R-module M is a direct join of atoms T_i ($i \in I$) of $L^X(M)$. The cardinal number of $\{T_i\}$ is an invariant and we called it the X-dimension of M. Clearly two X-atomic R-modules M and M' with isomorphic lattices $L^X(M)$ and $L^X(M')$ have the same X-dimension. In particular, X-dim N = X-dim M if N is an x-dense submodule of M. In case M is finitely X-atomic, X-dimension of M is finite.

5.9. Proposition. Let M be a finitely X-atomic R-module. If N and K are submodules of M such that $N \cap K$ is X-negligible and N+K is x-dense in M, then (X-dimN) + (X-dim K) = X-dim M.

Proof. By the above remark, we may suppose N and K are x-closed submodules of M. If $N = T_1 \textcircled{\vee} \cdots \textcircled{\vee} T_n$ and $K = U_1 \textcircled{\vee} \cdots \textcircled{\vee} U_m$, where T_i's and U_j's are respectively atoms of $L^X(N)$ and $L^X(K)$ (and hence of $L^X(M)$). By repeated applications of Lemma 3.1, we can prove that the family consisting of T_i's and U_j's is join-independent. Now

$$M = N \vee K = T_1 \vee \cdots \vee T_n \vee U_1 \vee \cdots \vee U_m$$

which is a direct join. Hence X-dim M = n + m = (X-dim N)+(X-dim K).

5.10. Corollary. If N is a submodule of the finitely X-atomic R-module M, then X-dim (M-N) = (X-dim M) - (X-dim N).

Proof. Let N' be a complement of N^X in $L^X(M)$. Then $N \cap N'$ is X-negligible and N+N' is x-dense in M. By the above proposition X-dim N' = (Xdim M) - (X-dim N). Since $(N+N')-N \cong N'-(N \cap N')$ and since (N+N')-N is x-dense in M-N, $L^X(M-N) \cong L^X((N+N')-N) \cong L^X(N'-(N \cap N')) \cong L^X(N')$. Hence X-dim (M-N) = X-dim N' = (X-dim M) - (X-dim N).

Consider now a strictly ascending chain

$$0^X = N_0 \subsetneq N_1 \subsetneq N_2 \subsetneq \cdots \subsetneq N_n = M$$

in $L^X(M)$, where M is finitely X-atomic. Since each $L^X(N_i)$ is complemented, we may suppose H_i is a complement of N_i in $L^X(N_{i+1})$. By Proposition 5.9, X-dim N_{i+1} = (X-dim N_i) + (X-dim H_i). Thus

$$\text{X-dim } M = (\text{X-dim } H_0) + (\text{X-dim } H_1) + \cdots + (\text{X-dim } H_{n-1}).$$

Since every H_i is not X-negligible, X-dim $H_i > 1$. So $n \leqslant$ X-dim M. This show that every strictly ascending (or descending) chain in $L^X(M)$ has at most X-dim M terms.

Assume again M is finitely X-atomic. Let n = X-dim M and let M be a direct join of atoms $T_1, \ldots, T_n$ of $L^X(M)$. Denote $\sum_{j\neq i} T_j$ by H_i and $\bigvee_{j\neq i} T_j$ by P_i. Then $P_i = H_i{}^X$ and by Lemma 4.7, P_i is a maximal x-closed submodule of M. Since $\bigcap_i H_i$ is X-negligible, $\bigcap_i P_i = 0^X$. Clearly the representation of 0^X as the intersection of P_i's is irredundant in the sense that none of the P_i's can be deleted. Conversely, suppose $0^X = P_1 \cap \cdots \cap P_m$ is an irredundant representation of 0^X as an intersection of maximal x-closed submodules of M. Let $T_i = \bigcap_{j\neq i} P_j$. Then by Lemma 4.7, T_i being a complement of P_i is an atom of $L^X(M)$. Now the family consisting of T_i's is clearly join-independent and

$$\begin{aligned} T_1 \vee \cdots \vee T_m &= (P_1 \cap \cdots \cap P_m) \vee (T_1 \vee \cdots \vee T_m) \\ &= (P_1 \vee T_1) \cap \cdots \cap (P_m \vee T_m) = M. \end{aligned}$$

Hence m = X-dim M = n and M is a direct join of T_i's. This proves

5.11. Lemma. (1) Let M be a finitely X-atomic R-module and let $M = T_1 \textcircled{\vee} \cdots \textcircled{\vee} T_n$ where T_i's are atoms of $L^X(M)$. Then

$$0^X = \bigcap_{i=1}^{n} (T_1 \vee \cdots \vee T_{i-1} \vee T_{i+1} \vee \cdots \vee T_n)$$

is an irredundant representation of 0^X as an intersection of maximal x-closed submodules of M.

(2) Let M be an R-module such that 0^X is an

irredundant intersection of maximal x-closed submodules P_1, ..., P_n. The M is finitely X-atomic of X-dimension n and

$$M = \bigvee_{i=1}^{n} \oplus (P_1 \cap \cdots \cap P_{i-1} \cap P_{i+1} \cap \cdots \cap P_n)$$

is a direct join of atoms of $L^x(M)$.

5.12. Proposition. Let M be a finitely X-atomic R-module. Then every x-closed submodule K of M can be represented as an intersection of a finite number of maximal x-closed submodules of M. If the representation is irredundant, then the number of maximal x-closed submodules in it is (X-dim M) - (X-dim K).

Proof. We observe that $N \to N-K$ is an isomorphism between the closed interval $[M,K]$ and $L^x(M-K)$. By Corollaries 5.8 and 5.10, it suffices to prove that 0^x is an intersection of a finite number of maximal x-closed submodules of M and if this representation is irredundant, the number of components is X-dim M. But this follows immediately from Lemma 5.11.

In Section 2, we associated with each D-system X of right ideals of R three multiplicatively closed subsets of R defined as follows:

$$G_X(R) = \{ r \mid r\in R,\ rR\in X\} = \{r \mid r\in R,\ rA\in X \text{ for every } A\in X\};$$

$$G_X^r(R) = \{ r \mid r\in R,\ r^{-1}0^x = 0^x\};$$

$$G_X^l(R) = \{r \mid r\in R,\ 0^x r^{-1} = 0^x\}.$$

We remarked also $G_X(R) \subset G_X^l(R)$.

5.13. Proposition. If X is an D-system of right ideals of R such that $L^x(R)$ satisfies the ascending chain condition, then $G_X(R) \subset G_X^r(R)$. The equality holds if, in addition, $L^x(M)$ is complemented.

Proof. Suppose $L^x(R)$ satisfies the ascending chain condition and let a be any element of $G_X(R)$. By I2(1), for each n, $(a^n)^{-1}0^x$

belongs to $L^X(R)$ and $a^{-1}0^X \subset (a^2)^{-1}0^X \subset \cdots$. There exists a positive integer n such that $(a^n)^{-1}0^X = (a^{n+1})^{-1}0^X = \cdots$. Let $c = a^n$. Then c belongs to $G_X(R)$. If r is an element of $cR \cap c^{-1}0^X$, then $r = cr' \in c^{-1}0^X$ for some r' in R and so $c^2r' \in 0^X$. Since $c^{-1}0^X = (c^2)^{-1}0^X$, $cr' \in 0^X$, i.e., $r \in 0^X$. Hence $cR \cap c^{-1}0^X$ is X-negligible and $0^X = (cR \cap c^{-1}0^X)^X = (cR)^X \cap c^{-1}0^X = R \cap c^{-1}0^X = c^{-1}0^X$. Hence $a^{-1}0^X = 0^X$, i.e., a belongs to $G_X^r(R)$. To prove the second statement, let $L^X(R)$ be complemented and satisfy the ascending chain condition. Then R is finitely X-atomic. In case R is X-negligible, there is nothing to prove. We may suppose R is not X-negligible and let $R = T_1 \vee \cdots \vee T_n$, where $\{T_i\}$ is a join-independent family of atoms of $L^X(R)$. If a is an element of $G_X^r(R)$, then $a^{-1}0^X = 0^X$ and so aT_i is not X-negligible. By Corollary 3.6, $(aT_i)^X$ (i = 1, .., n) are atoms of $L^X(R)$ and form a join-independent family. It follows then $R = \overset{n}{\underset{i=1}{\vee}} \oplus (aT_i)^X$. Since each $aT_i \subset aR$, $R \subset (aR)^X$, i.e., $aR \in X$ and a belongs to $G_X(R)$.

5.14. Corollary. If X is an D'D*-system of right ideals of R such that $L^X(R)$ satisfies the ascending chain condition, then X is an Asano's D-system corresponding to the set $G_X(R)$ of regular elements of R.

In [12], A. W. Goldie considered a semi-prime ring R satisfying the following right quotient conditions:

(1r) Every direct sum of nonzero right ideals of R has a finite number of terms.

(2r) The ascending chain condition holds for the annihilator right ideals of R.

Note that the condition (1r) is equivalent to the following condition:

(1r)' The ascending chain condition holds for complement right ideals of R.

Goldie succeeded in proving that the right singular ideal of R is zero, a right ideal of R is essential if and only if it contains some regular elements of R. If we let X be the Johnson's D-system of right ideals of R, then by Corollary 4.3, $L^x(R)$ is complemented. Since a right ideal of R is x-closed if and only if it is a complement right ideal, $L^x(R)$ satisfies the ascending chain condition. By Proposition 5.13, $G_X^r = G_X(R) \subset G_X^\ell(R)$. Hence X is a D'D*-system of right ideals of R such that R is finitely X-atomic. This proves the "if" part of the following

5.15. Proposition. A ring R possesses a D'D*-system X such that R is finitely X-atomic if and only if R is semi-prime and satisfies the Goldie's right quotient conditions (1r) and (2r). In such case, X is the Johnson's D-system of right ideals of R.

Proof. Suppose R has a D'D*-system X such that R is finitely X-atomic. Then $L^x(R)$ is complemented and X is the Johnson's D-system of right ideals of R by Corollary 4.3. A right ideal of R is x-closed if and only if it is a complement right ideal. Moreover by Corollary 1.12, every annihilator right ideal of R is x-closed. Since $L^x(R)$ satisfies the ascending chain condition, R satisfies the conditions (1r)' and (2r). By Proposition 5.13, $G_X^r(R) = G_X(R) \subset G_X^\ell(R)$. If R were not semi-prime, then $N^2 = 0$ for some nonzero ideal N of R. Let N' be a relative complement of N. Then N+N' is essential and so belongs to X. Thus N+N' contains a regular element g say. Now $N(N+N') \subset N^2+NN' \subset N \cap N' = 0$. This is impossible. Hence R is a semi-prime ring satisfying conditions (1r) and (2r).

PART II

INJECTIVE MODULES, ESSENTIAL AND RATIONAL EXTENSIONS OF MODULES

The present part is devoted to an investigation of the generalized injective R-modules, generalized essential and rational extensions of an R-module. Section 6 begins with a reformulation of the concepts of essential submodules and rational extensions of an R-module in view of further generalizations. For each D-system Y of right ideals of R^1, the notions of Y-essential and Y-rational submodules of an R-module are defined. The theory of essential extensions and that of rational extensions are developed simultaneously. In Section 7, we introduce the concept of Y-injective R-module which is then linked with maximal Y-essential and maximal Y-rational extensions of R-modules.

6. Essential and rational extensions. Recall that a submodule N of an R-module M is essential if for every nonzero element m of M there exists an element r of R^1 such that $0 \neq mr \in N$; or equivalently, if for every nonzero element m of M there exists a right ideal J of R^1 such that $0 \neq mJ \subset N$. M is a rational extension of N if for each pair of elements m and n of M with $m \neq 0$ there exists an element r of R^1 such that $nr \in N$ and $mr \neq 0$; or equivalently, if for each pair of elements m and n of M with $m \neq 0$ there exists a right ideal J of R^1 such that $nJ \subset N$ and $mJ \neq 0$. We may generalize these concept as follows:

Let Y be a D-system of right ideals of R^1. A submodule N of an R-module M is said to be Y-essential if for every nonzero element m of M, there exists a right ideal J of R^1 belonging to Y such that $0 \neq mJ \subset N$. If for every pair m and n of elements in M

with $m \neq 0$, there exists a right ideal J of R^1 belonging to Y such that $mJ \neq 0$ and $nJ \subset N$, then N is called a Y-rational submodule of M. Clearly every Y-rational submodule is Y-essential. As proved in Section 2, for a D-system X of right ideals of R, the family $X^\#$ of all right ideals of R^1 that contain some member of X is a D-system of right ideals of R^1. We agree to call $X^\#$-essential (resp. X -rational) submodules simply X-essential (resp. X-rational) submodules.

The concept of Y-essential (resp. Y-rational) submodule depends vitally on the D-system Y. If Y is the smallest D-system of right ideals of R^1, i.e., $Y = \{R^1\}$, then the only Y-essential (resp. Y-rational) submodule of an R-module is the module itself. On the other hand, if Y is the largest D-system of right ideals of R^1, i.e., Y is the family of all right ideals of R^1, then a submodule N of M is Y-essential if and only if it is an essential submodule of M and a submodule K of M is Y-rational if and only if M is a rational extension of K. In general, if Y and Y' are two D-systems of right ideals of R^1 such that $Y \subset Y'$, then every Y-essential (resp. Y-rational) submodule is also a Y'-essential (resp. Y'-rational) submodule.

6.1. Proposition. A submodule of M is Y-essential if and only if it is y-dense as well as essential in M. If M is strongly Y-prime, then Y-essential submodules are precisely y-dense submodules.

6.2. Proposition. Let N be a submodule of the R-module M. Then the following statements are equivalent:

(1) N is a Y-rational submodule of M.

(2) N is a y-dense submodule of M and every R-homomorphism of N into M has a unique irreducible extension.

(3) N is a y-dense submodule of M and every semi R-endomorphism of M which annihilates N has zero image.

Proof. Suppose the condition (1) holds. Let φ be an R-homomorphism of N into M. Assume φ' and φ'' are two irreducible semi R-endomorphisms of M extending φ. If $\varphi'x \neq \varphi''x$ for some x in dom $\varphi' \cap$ dom φ'', then there exists J in Y such that $(\varphi'x-\varphi''x)J \neq 0$ and $xJ \subset N$. But for each j in J, $(\varphi'x-\varphi''x)j = \varphi'(xj)-\varphi''(xj) = \varphi(xj)-\varphi(xj) = 0$. This is a contradiction. Hence $\varphi'x = \varphi''x$ for every x in dom $\varphi' \cap$ dom φ'' and so $\varphi' = \varphi''$ (see Preliminaries). This shows that (2) follows from (1). The equivalence of (2) and (3) is clear. Suppose (3) holds. Let m and n be two elements of M such that $m \neq 0$. Then $J = \{r \mid r \in R^1, nr \in N\}$ belongs to Y and $nJ \subset N$. If $mJ = 0$, then

$$k + nx \to mx, \qquad (k \text{ in } N \text{ and } x \text{ in } R^1)$$

is a well-defined R-homomorphism of $N+nR^1$ into M annihilating N. By (3), $mR^1 = 0$ or $m = 0$. This contradicts the choice of m. Hence $mJ \neq 0$ and so N is a Y-rational submodule of M.

Let M be an R-module. Then a pair (K,μ) is called an extension of M if K is an R-module and μ is an R-isomorphism of M into K. For any two extensions (K,μ) and (L,λ) of M, if there exists an R-isomorphism Θ of K into L such that $\Theta\mu = \lambda$, then we write $(K,\mu) < (L,\lambda)$, or sometimes $(K,\mu) <_\Theta (L,\lambda)$ to emphasize the existence of Θ. In case Θ is an R-isomorphism of K onto L, we write also $(K,\mu) =_\Theta (L,\lambda)$. Note that for any extension (K,μ) of M, $(M,i) <_\mu (K,\mu)$ where i is the identity map of M. The relation "$<$" defined on the set of all extensions of M is reflexive

and transitive. Two extensions (K,μ) and (L,λ) are said to be equivalent if there exists an R-isomorphism ϱ of K onto L such that $(K,\mu) =_\varrho (L,\lambda)$.

Let Y be a D-system of right ideals of R^1. An extension (K,μ) of M is called a Y-essential (resp. Y-rational) extension if μM is a Y-essential (resp. Y-rational) submodule of K. A submodule N of M is a Y-essential (resp. Y-rational) submodule if and only if (M,i) is a Y-essential (resp. Y-rational) extension of N. In such case we shall simply say that M is a Y-essential (resp. Y-rational) extension of N.

6.3. Proposition. Let (K,μ) and (L,λ) be two extensions of M such that $(K,\mu) <_\varrho (L,\lambda)$. Then (L,λ) is a Y-essential (resp. Y-rational) extension of M if and only (L,ϱ) and (K,μ) are Y-essential (resp. Y-rational extensions of K and M respectively.

Proof. We shall prove the proposition for Y-essential extensions. By a simple modification, we can furnish a proof for Y-rational extensions.

Suppose (L,λ) is a Y-essential extension of M. Let x be a nonzero element of L. Then there exists J in Y such that $0 \neq xJ \subset \lambda M = \varrho\mu M \subset \varrho K$, i.e., (L,ϱ) is a Y-essential extension of K. Next, let x be a nonzero element of K. Then ϱx is a nonzero element of L and so there exists J in Y such that $0 \neq \varrho(xJ) = (\varrho x)J \subset \lambda M = \varrho\mu M$. Thus $0 \neq xJ \subset \mu M$. This shows that (K,μ) is a Y-essential extension of M. Conversely, suppose (L,ϱ) and (K,μ) are Y-essential extensions of K and M respectively. Let x be a nonzero element of L. Then there exists J in Y such that $0 \neq xJ \subset \varrho K$. For each j in J, there exists I_j in Y such that $xjI_j \subset \varrho\mu M = \lambda M$. If $xj \neq 0$, we may choose I_j so that $xjI_j \neq 0$.

Let I be the sum of jI_j (j in J). Then I belongs to Y and $0 \neq xJ \subset \lambda M$. Hence (L,λ) is a Y-essential extension of M.

6.4. Corollary. Let (K,μ) be a Y-essential (resp. Y-rational) extension of M. Then every extension of M equivalent to (K,μ) is also a Y-essential (resp. Y-rational) extension of M.

6.5. Corollary. If K is a Y-essential (resp. Y-rational) submodule of N and N, in turn, is a Y-essential (resp. Y-rational) submodule of M, then K is a Y-essential (resp. Y-rational) submodule of M.

It follows immediately from Corollary 6.5 that the relation of "being a Y-essential (resp. Y-rational) submodule of" is finitely transitive. For each submodule N of M, the set η of all submodules K of M such that K is a Y-essential (resp. Y-rational) extension of N has maximal elements. In case the sum of two elements in η is again in η (this is the case if, for example, M is strongly Y-prime), then η has a unique maximal element.

6.6. Proposition. Let (K,μ) be a Y-essential extension of M. If N is an R-module such that there is an R-homomorphism φ of N into K, then N is a Y-essential extension of $\varphi^{-1}\mu M$.

Proof. Let n be a nonzero element of N. If $\varphi n = 0$, then there is nothing to prove, we may suppose $\varphi n \neq 0$. There exists then J in Y such that $0 \neq \varphi(nJ) = (\varphi n)J \subset \mu M$, i.e., $0 \neq nJ \subset \varphi^{-1}\mu M$. Hence $\varphi^{-1}\mu M$ is a Y-essential submodule of N.

For a given D-system Y of right ideals of R^1, the set of all Y-essential submodules of an R-module M will be designated by $\xi^Y(M)$.

6.7. Corollary. $\xi^{Y}(M)$ satisfies the conditions D1 and D2.

Now let Y* be the family of all Y-essential right ideals of R^1, i.e., $Y^* = \xi^{Y}(R^1)$. Then Y* satisfies D1, D2 and hence D21 and D22. Clearly $Y^* \subset Y$. For each R-module M, the set $Z_y(M)$ of all elements of M which annihilate some member of Y* is a submodule of M. We shall call this submodule the Y-singular submodule of M. Observe that $Z_y(R)$ is an ideal of R and will be called Y-singular ideal of R. Note that if Y and W are two D-systems of right ideals of R^1 such that $Y \subset W$ and if N is a submodule of an R-module M, then

(1) $Z_y(M) \subset Z_w(M)$;

(2) $Z_y(N) = Z_y(M) \cap N$; and

(3) if N is a Y-essential submodule of M, then $Z_y(N)$ is a Y-essential submodule of $Z_y(M)$, hence $Z_y(M) = 0$ if and only if $Z_y(N) = 0$.

6.8. Proposition. (1) Let M be an R-module with zero Y-singular submodule and let K be an R-module. Then every semi R-homomorphism of K into M whose domain is a Y-essential submodule of K has a unique irreducible extension.

(2) Let M be an R-module. If every semi R-homomorphism of R^1 into M whose domain is in Y* has a unique irreducible extension, then the Y-singular submodule of M is zero.

Proof. Let φ be an R-homomorphism of L into M, where L is a Y-essential submodule of K. Suppose φ' and φ'' are two irreducible extensions of φ. Let x be any element of dom $\varphi' \cap$ dom φ''. Then since L is Y-essential, by Proposition 6.6, there is a Y-essential right ideal A of R^1 such that $xA \subset L$. Now $(\varphi'x - \varphi''x)A = (\varphi' - \varphi'')xA \subset (\varphi' - \varphi'')L = 0$. Since the Y-singular submodule of M is zero, $\varphi'x = \varphi''x$. This shows that φ' and φ'' coincide on the intersection

of their domains. Hence $\varphi' = \varphi''$ since φ' and φ'' are irreducible. This proves (1). To prove (2), suppose $mA = 0$ for some m in M and A in Y^*. Then the R-homomorphism $t \to 0$ (t in A) of A into M has the following two R-homomorphisms as irreducible extensions:

$$t \to 0, \quad (t \text{ in } R^1)$$

and

$$t \to mt, \quad (t \text{ in } R^1).$$

Since the irreducible extension is unique, $mt = 0$ for all t in R^1. Hence $m = 0$. This shows that the Y-singular submodule of M is zero.

6.9. Corollary. If the Y-singular submodule of M is zero, then every Y-essential submodule (resp. extension) of M is also a Y-rational submodule (resp. extension).

Proof. By Propositions 6.2 and 6.8 and the fact that $Z_Y(K) = 0$ for every Y-essential extension (K, μ) of M.

Suppose N is a submodule of the R-module M. Let us consider the set

$$N^* = \{ m \mid m \in M, mA \subset N \text{ for some } A \in Y^* \}.$$

Since Y^* satisfies conditions D1 and D2, N^* is a submodule of M containing N and is contained in N^Y. If K is a Y-essential extension of N in M, then by Proposition 6.6, $A = \{ r \mid r \in R^1, kr \in N \}$ belongs to Y^* and $kA \subset N$ for every k in K. Thus $K \subset N^*$. In case the Y-singular submodule of M is zero, N^* is a Y-essential extension of N. This proves the following

6.10. Lemma. If M is an R-module with zero Y-singular submodule, then every submodule N of M has a unique maximal Y-essential (and hence Y-rational) extension in M, namely N^*. Moreover $N \to N^*$ is a modular closure operation on $L(M)$ when M is considered as an R^1-module.

Let X be a D-system of right ideals of R. Then $X^{\#}$ is a D-system of right ideals of R^1 containing X. We may call the $X^{\#}$-singular submodule of an R-module M the X-singular submodule of M and denote the set of all X-essential right ideal of R by X*. Clearly $X^* = \mathcal{E}^{X^{\#}}(R)$.

6.11. Lemma. Let R be a left faithful ring and let X be a D-system of right ideals of R. If A is an X-essential right ideal of R, then so is AR.

Proof. For each nonzero element r of R, $0 \neq rJ \subset A$ for some J in $X^{\#}$. Since R is left faithful, $0 \neq (rJ)R = r(JR) \subset AR$. Now J and R are in $X^{\#}$ and so by D32 for X , JR is in $X^{\#}$. Thus AR is X-essential.

When N is a submodule of M and $n \in N^*$, $nA \subset N$ for some A in X*. By the above lemma, $AR \in X^*$ and $n(AR) = (nA)R \subset NR$, i.e., $n \in (NR)^*$. Hence $N^* \subset (NR)^*$. This completes the proof of the following

6.12. Theorem. Let X be a D-system of right ideals of the left faithful ring R and let M be an R-module with zero X-singular submodule, then $N \to N^*$, where N* is the unique maximal X-essential (hence X-rational) extension of N in M, is a modular closure operation on L(M).

6.13. Corollary. If X is a D-system of right ideals of R such that the X-singular ideal of R is zero, then the set X* of all X-essential right ideals of R is a D-system of right ideals of R contained in X.

Following Findlay and Lambek [10], we define a relation between three R-modules K, N and M as follows: $K < N\ (M;Y)$ if K is a sub-

module of N and for each $0 \neq m \in M$ and each $n \in N$, there exists J in Y such that $mJ \neq 0$ and $nJ \subset K$. Analogously, we can show that $K < N\ (M;Y)$ is equivalent to either one of the following conditions:

(1) K is a y-dense submodule of N and every R-homomorphism of K into M has a unique irreducible extension.

(2) K is a y-dense submodule of N and every semi R-homomorphism of N into M which annihilates K has zero image.

6.14. Lemma. If $K < N\ (M;Y)$ and if φ is an R-homomorphism of an R-module L into N, then $\varphi^{-1}K < L\ (M;Y)$.

Proof. Let $0 \neq m \in M$ and let $x \in L$. Then $mJ \neq 0$ and $\varphi(xJ) = (\varphi x)J \subset K$ for some J in Y. So $xJ \subset \varphi^{-1}K$ and $\varphi^{-1}K < L\ (M;Y)$.

6.15. Lemma. If $L < K\ (M;Y)$ and $K < N\ (M;Y)$, then $L < N\ (M;Y)$.

Proof. Let m be a nonzero element of M and n an element of N. Then there exists J in Y such that $mJ \neq 0$ and $nJ \subset K$. For each j in J such that $mj \neq 0$ there exists I_j in Y such that $mjI_j \neq 0$ and $njI_j \subset L$. If j in J such that $mj = 0$, we may choose I_j in Y so that $njI_j \subset L$. Let I be the sum of jI_j (j in J). Then I belongs to Y and $mI \neq 0$ and $nI \subset L$. Hence $L < N\ (M;Y)$.

For any two R-modules N and M, we shall denote the set of all submodules K of N such that $K < N\ (M;Y)$ by $\mathcal{R}^Y_M(N)$, or simply $\mathcal{R}^Y(N)$ if $N = M$.

6.16. Corollary. $\mathcal{R}^Y_M(N)$ satisfies the conditions D1 and D2.

6.17. Proposition. For each D-system Y of right ideals of R^1 and any R-module M, $\mathcal{R}^Y_M(R^1)$ is a D'-system of right ideals of R^1.

Proof. It remains to show D3. Suppose A is a right ideal of R^1 such that $A < R^1\ (M;Y)$ and for each a in A, there exists a right

ideal B_a of R^1 such that $B_a < R^1$ $(M;Y)$. Given r in R^1 and $0 \neq m \in M$, we have $rJ \subset A$ and $mJ \neq 0$ for some J in Y. If j in J such that $mj \neq 0$, then there exists I_j in Y such that $I_j \subset B_{rj}$ and $mjI_j \neq 0$. If j in J such that $mj = 0$, we may choose I_j so that $I_j \in Y$ and $I_j \subset B_{rj}$. Let I be the sum of jI_j (j in J). Then I belongs to Y, $rI = \sum rjI_j \subset \sum_{a \in A} aB_a$ and $mI \neq 0$. Hence $\sum aB_a < R^1$ $(M;Y)$.

6.18. Proposition. Let X be a D-system of right ideals of R and let M be a faithful R-module. Then $R^X_M(R)$ is a D'-system of right ideals of R.

The proof is similar to that of Proposition 6.17 and will be omitted.

Consider now a discrete direct sum M of R-modules M_i (i in I). If for a fixed j in I, (K_j, μ_j) is a Y-essential (resp. Y-rational) extension of M_j, then (K, μ) is clearly a Y-essential (resp. Y-rational) extension of M, where $K = K_j \oplus (\sum_{i \neq j} \oplus M_i)$ and μ is defined by $\mu(\sum m_i) = \mu_j m_j + \sum_{i \neq j} m_i$, $m_i \in M_i$. We shall use this fact in the proof of the following proposition.

6.19. Proposition. Let $\{M_i\}_{i \in I}$ be a nonempty family of R-modules and let (K_i, μ_i) be a Y-essential (resp. Y-rational) extension of M_i. Then $(\sum \oplus K_i, \mu)$ is a Y-essential (resp. Y-rational) extension of $\sum \oplus M_i$, where μ is the R-isomorphism of $\sum \oplus M_i$ into $\sum \oplus K_i$ defined by $\mu(\sum m_i) = \sum \mu_i m_i$, $m_i \in M_i$.

Proof. We shall give a proof for Y-essential extension.

Consider first the case where I is finite. We prove by induction on the number o(I) of elements in I. It is clearly true for o(I) = 1. Assume that the proposition is true for all discrete direct sums of n-1 modules. Let I be a set consisting of n elements and let j be a fixed element of I. Then by the

above remark (K,λ) is a Y-essential extension of $K' = M_j \oplus \sum_{i\neq j} \oplus K_i$, where $K = \sum \oplus K_i$ and λ is an R-isomorphism of K' into K defined by $\lambda(m_j + \sum_{i\neq j} k_i) = \mu_j m_j + \sum_{i\neq j} k_i$, $m_j \in M_j$, $k_i \in K_i$. By induction assumption and the above remark again, (K',ω) is a Y-essential extension of $M = \sum M_i$, where ω is an R-isomorphism of M into K' defined by $\omega(\sum m_i) = m_j + \sum_{i\neq j} \mu_i m_i$, $m_i \in M_i$. Now (K,μ) is an extension of M. Since $\mu = \lambda\omega$, by Proposition 6.3, (K,μ) is a Y-essential extension of M. This completes the proof of the proposition for the case where I is finite.

Let I be any index set. If k is an element of $\sum \oplus K_i$, then there exists a finite subset J of I such that $k = \sum_{j\in J} k_j$, where $k_j \in K_i$. Since $(\sum_{j\in J} \oplus K_j, \mu')$, where μ' is the restriction of μ to $\sum_{j\in J} \oplus M_j$, is a Y-essential extension of $\sum_{j\in J} \oplus M_j$, there exists A in Y such that $0 \neq kA \subset \mu'(\sum_{j\in J} M_j) \subset \mu(\sum_{i\in I} M_i)$. Hence $(\sum \oplus K_i, \mu)$ is a Y-essential extension of $\sum \oplus M_i$.

A family $\{H_i\}_{i\in I}$ of submodules of M is said to be independent if $H_j \cap (\sum_{i\neq j} H_i) = 0$ for every j in I.

6.20. Corollary. Let $\{N_i\}_{i\in I}$ be an independent family of submodules of the R-module M and let H_i be a Y-essential (resp. Y-rational) extension of N_i in M. Then $\{H_i\}$ is an independent family and $\sum \oplus H_i$ is a Y-essential (resp. Y-rational) extension of $\sum \oplus N_i$ in M.

Proof. We need only show that $\{H_i\}$ is an independent family. If $\{H_i\}$ were not independent, then there would exist a finite subset J of I and nonzero elements $n_j \in H_j$ $(j\in J)$ such that $\sum_{j\in J} n_j = 0$. For simplicity, we denote the elements in J by 1, 2, .., k. Since $n_1 \neq 0$ and H_1 is a Y-essential extension of N_1, there exists r_1 in R^1 such that $0 \neq n_1 r_1 \in N_1$. If not all of the $n_j r_1$ $(j = 2, .., k)$

are zero, say $n_2 r_1 \neq 0$, by a similar argument, we find r_2 in R^1 such that $0 \neq n_2 r_1 r_2 \in N_2$. Continue in this way, we obtain finally $r_1, .., r_s$ in R^1 $(s<k)$ such that $n_s r_1 \cdot\cdot r_s \neq 0$, $n_i r_1 \cdot\cdot r_s \in N_i$ for all i, $n_j r_1 \cdot\cdot r_s = 0$ for $j>s$. But $\sum_{j\in J} n_j r_1 r_s = 0$. This contradicts the hypothesis that the family $\{N_i\}$ is independent.

Let I be a directed set, i.e., a set with a reflexive and transitive relative "<" such that for each pair i, j in I there exists k in I for which $i<k$ and $j<k$. Assume that $\{(M_i, \mu_i)\}_{i\in I}$ is a family of extensions of the R-module M such that for every pair i and j in I with $i<j$ there exists an R-isomorphism λ_{ji} of M_i into M_j for which $(M_i, \mu_i) <_{\lambda_{ji}} (M_j, \mu_j)$. We assume also λ_{ii} is the identity map and $\lambda_{kj}\lambda_{ji} = \lambda_{ki}$ for $i<j<k$ in I. We call $\{(M_i, \mu_i), \lambda_{kj}\}_{i,j,k\in I}$ a directed system of extensions of M over I. Let S denote the discrete direct sum of M_i's and let K be the submodule of S generated by elements of the form $\lambda_{ji} m_i - m_i$ $(m_i \in M_i,\ i<j \text{ in } I)$. Denote the difference module S-K by M_∞. M_∞ is called the direct limit of the system. Define the projection λ_i of M_i into M_∞ as follows: $\lambda_i m_i = m_i + K$. If $i<j$ in I, then $\lambda_j \lambda_{ji} = \lambda_i$.

6.21. Lemma. λ_i is an R-isomorphism of M_i into M_∞.

Proof. Clearly λ_i is an R-homomorphism of M_i into M_∞. If m is any element in the kernel of λ_i, then m belongs to K. There exist a finite subset J of I and j' $(<j)$ in I corresponding to each j in J such that $m = \sum_{j\in J} (\lambda_{jj'} m_{j'}, -m_{j'})$. Let k be an element of I greater than i and every j in J. Then since

$$\lambda_{jj'} m_{j'}, -m_{j'} = (\lambda_{kj'} m_{j'}, -m_{j'}) + (\lambda_{kj}(-\lambda_{jj'} m_{j'}) - (-\lambda_{jj'} m_{j'})),$$

m can be written as a sum of elements of the rorm $\lambda_{kt} m_t - m_t$ $(t<k,\ t\neq k)$. For every $t \neq i$, $m_t = 0$. Thus $m = \lambda_{ki} m_i - m_i$. Since

$\lambda_{ki}m_i \in M_k$, $\lambda_{ki}m_i = 0$ and so $m_i = 0$ and $m = 0$. This shows that the kernel of λ_i is zero, i.e., λ_i is an R-isomorphism.

Given any i and j in I, since I is a directed set, $i < k$ and $j < k$ for some k in I. Now $\lambda_k\mu_k = \lambda_k\lambda_{ki}\mu_i = \lambda_i\mu_i$ and similarly $\lambda_k\mu_k = \lambda_j\mu_j$. Thus $\lambda_i\mu_i = \lambda_j\mu_j$ for every pair i and j in I.

6.22. Proposition. If each (M_i,μ_i) is a Y-essential (resp. Y-rational) extension of M, then $(M_\infty,\lambda_i\mu_i)$ is a Y-essential (resp. Y-rational) extension of M.

Proof. We prove this proposition for the case of Y-essential extensions. By Proposition 6.3, it is enough to show that (M_∞,λ_i) is a Y-essential extension of M_i. Let m be an element of D not in K. Then there exists a finite subset J of I such that $m = \sum_{j\in J} m_j$, $m_j \in M_j$. Choose an element k in I such that $k > j$ for every j in J. Since

$$m = (\sum_{j\in J}\lambda_{kj}m_j) - (\sum_{j\in J}\lambda_{kj}m_j - m_j)$$

and since each $\lambda_{kj}m_j - m_j$ is in K, we may suppose $m = \sum_{j\in J}\lambda_{kj}m_j$, i.e., m belongs to M_k. The condition $m \notin K$ is then equivalent to the condition $m \neq 0$ because λ_k is an R-isomorphism by Lemma 6.21. Since (M_k,μ_k) is a Y-essential extension of M, by Proposition 6.3, (M_k,λ_{ki}) is a Y-essential extension of M_i. There exists then a member A of Y such that $0 \neq mA \subset \lambda_{ki}M_i$. Now

$$0 \neq \lambda_k(mA) \subset \lambda_k(\lambda_{ki}M_i) = \lambda_i M_i,$$

i.e., $0 \neq (m+K)A \subset \lambda_i M_i$. Hence (M_∞,λ_i) is a Y-essential extension of M_i.

By a maximal Y-essential (resp. maximal Y-rational) extension of an R-module M we mean a Y-essential (resp. Y-rational) extension (K,μ) of M satisfying the following condition: If (L,λ) is a

Y-essential (resp. Y-rational) extension of M such that $(K,\mu) <_\theta (L,\lambda)$, then $(K,\mu) =_\theta (L,\lambda)$. As an application of Proposition 6.22, we prove the existence of a maximal Y-rational extension of an R-module.

Denote the totality of Y-rational extensions of M by G. Then "$<$" defines a reflexive and transitive relation on G.

6.23. Lemma. Two Y-rational extensions (K,μ) and (L,λ) of an R-module M are equivalent if and only if $(K,\mu) < (L,\lambda)$ and $(L,\lambda) < (K,\mu)$.

Proof. Suppose $(K,\mu) <_\theta (L,\lambda)$ and $(L,\lambda) <_\sigma (K,\mu)$. Observe that if $m \in M$, then $\sigma\theta\mu(m) = \sigma\lambda(m) = \mu m$ and $\theta\sigma\lambda(m) = \theta\mu(m) = \lambda m$ i.e., $\theta\sigma$ and $\sigma\theta$ induce identity maps on λM and μM respectively. Since K and L are Y-rational extensions of μM and λM respectively, by Proposition 6.2, $\sigma\theta$ and $\theta\sigma$ are the identity maps on K and L respectively. Hence θ is an R-isomorphism of K onto L and (K,μ) and (L,λ) are equivalent under θ. The converse is clear.

6.24. Corollary. Let M be an R-module with zero Y-singular submodule. Then two Y-essential extensions (K,μ) and (L,λ) of M are equivalent if and only if $(K,\mu) < (L,\lambda)$ and $(L,\lambda) < (K,\mu)$.

If we identify equivalent extensions in G, then G together with the relation "$<$" is a partially ordered set. An extension (K,μ) is a maximal Y-rational extension if and only if it is a maximal element of G.

Now if (K_i,μ_i) is a sequence in G such that

$$(K_i,\mu_i) <_{\lambda_{i+1,i}} (K_{i+1},\mu_{i+1})$$

and if we define for $j > i$, $\lambda_{ji} = \lambda_{j,j-1}\lambda_{j-1,j-2}\cdots\lambda_{i+1,i}$ and λ_{ii} = identity map, then $\{(K_i,\mu_i), \lambda_{kj}\}$ is a directed system

of extensions of M over the directed set of positive integers with usual order. As shown in Proposition 6.22, $(M_\infty, \lambda_i \mu_i)$ is a Y-rational extension of M such that $(K_i, \mu_i) \leq_{\lambda_i} (M_\infty, \lambda_i \mu_i)$. This shows that any ascending chain in G has an upper bound. By Zorn's lemma, there exist maximal elements in G.

6.25. Theorem. Let Y be a D-system of right ideals of R^1. Then every R-module M has a maximal Y-rational extension.

7. Injective modules. Let Y be a D-system of right ideals of R^1 and let $\mathcal{K}$ be a nonempty family of R-modules. Then an R-module M is said to be $(\mathcal{K},Y)$-injective if for each K in $\mathcal{K}$, every semi R-homomorphism φ of K into M, whose domain is a Y-essential submodule of K, can be extended to an R-homomorphism φ' of K into M. Observe that if φ is an R-isomorphism, then since $\operatorname{dom} \varphi \cap \ker \varphi' = 0$ and $\operatorname{dom} \varphi$ is essential in K, $\ker \varphi' = 0$, i.e., φ' is an R-isomorphism. We agree to call a $(\mathcal{K},Y)$-injective module M simply

(1) (K,Y)-injective if $\mathcal{K} = \{K\}$;

(2) $(\mathcal{K},X)$-injective if $Y = X^{\#}$ for some D-system X of right ideals of R;

(3) Y-injective if $\mathcal{K}$ is the class of all R-modules;

(4) injective if Y is the family of all right ideals of R^1 and $\mathcal{K}$ is the class of all R-modules;

(5) Y-quasi injective if $\mathcal{K} = \{M\}$; and

(6) quasi injective if Y is the family of all right ideals of R^1 and $\mathcal{K} = \{M\}$.

It should be mentioned that our definitions of injective and quasi injective modules are exactly the same as those defined in the

literature (see for example [6] and [24]). If $\mathcal{K} \supset \mathcal{K}'$ and $\mathcal{Y} \supset \mathcal{Y}'$, then every $(\mathcal{K},\mathcal{Y})$-injective R-module is $(\mathcal{K}',\mathcal{Y}')$-injective.

7.1. Proposition. Let M be a $(R,\mathcal{Y})$-injective R-module. Then for every $\mathcal{Y}$-essential submodule L of an R-module K, every R-homomorphism φ of L into M can be extended to an R-homomorphism of L + KR into M.

Proof. Since for every submodule L' of L+KR containing L, L'+KR = L+KR, we may suppose that φ is an irreducible semi R-homomorphism of L+KR into M. If $KR \subset L$, then there is nothing to prove. Suppose then $KR \not\subset L$ and let k be an element of K such that $kR \not\subset L$. By Proposition 6.6, $k^{-1}L$ is a $\mathcal{Y}$-essential right ideal of R and the mapping $r \to \varphi(kr)$ is an R-homomorphism of $\varphi^{-1}L$ into M. Since M is $(R,\mathcal{Y})$-injective, it can be extended to an R-homomorphism μ of R into M. Define a mapping φ' of L+kR into M by

$$\varphi'(l+kr) = \varphi(l) + \mu(r), \qquad (l \text{ in } L \text{ and } r \text{ in } R).$$

If $l+kr = l'+kr'$ for some l, l' in L and r, r' in R, then $k(r-r') = l'-l$. It follows that r-r' belongs to $k^{-1}L$ and so $\mu(r)-\mu(r') = \mu(r-r') = \varphi(k(r-r')) = \varphi(l'-l) = \varphi(l')-\varphi(l)$ or $\varphi(l)+\mu(r) = \varphi(l')+\mu(r')$. Thus φ' is well defined. Clearly φ' is a semi R-homomorphism of L+KR into M extending φ. Since $kR \not\subset L$, the extension is proper. This contradicts the irreducibility of φ. Hence φ can be extended to an R-homomorphism of L+KR into M.

7.2. Corollary. If M is $(R,\mathcal{Y})$-injective and if (M',φ) is a $\mathcal{Y}$-essential extension of M, then $M'R \subset \varphi M$.

Proof. By Proposition 7.1, there exists an R-homomorphism μ of $\varphi M+M'R$ into M such that $\mu(\varphi m) = m$ for every m in M. Now $\varphi M+M'R = \varphi M + \ker \mu$ and $\varphi M \cap \ker \mu = 0$. Since φM is a $\mathcal{Y}$-essential submodule of M', $\ker \mu = 0$ and so $\varphi M = \varphi M+M'R$, i.e., $M'R \subset \varphi M$.

7.3. Proposition. Let Y be a D-system of right ideals of R^1 and let M be an R-module. Then the following statements (1)-(3) are equivalent:

(1) M is Y-injective.

(2) M is (R^1,Y)-injective.

(3) For each R-module N and each Y-essential extension (K,μ) of N, corresponding to every R-homomorphism φ of N into M is an R-homomorphism φ' of K into M such that $\varphi'\mu = \varphi$.

If, in addition, R has an identity e and M is a unital R-module, then each of the above conditions is equivalent to either one of the following statements:

(4) M is (Δ,Y)-injective, where Δ is the class of all unital R-modules.

(5) M is (R,Y)-injective.

Proof. The statements (2) and (1) follow readily from (1) and (3) respectively. The proof of the implication (2)$\Rightarrow$(3) is similar to that of Proposition 7.1.

Clearly (1) implies (4) and which, in turn, implies (5). It remains to show that (5)$\Rightarrow$(2). Let J be a Y-essential submodule of the R-module R^1. Assume that φ is an R-homomorphism of J into M. We may suppose φ is an irreducible semi R-homomorphism of R^1 into M. If $J \neq R^1$, take a in R^1 which is not in J. By Proposition 6.6, $a^{-1}J$ is a Y-essential right ideal of R and since M is (R,Y)-injective, the R-homomorphism $r \to \varphi(ar)$ of $a^{-1}J$ into M can be extended to an R-homomorphism μ of R into M. Let $m = \mu e$. Define a mapping φ' on $J+aR^1$ into M as follows:

$$\varphi'(j+ar) = \varphi j + mr, \quad (j \text{ in } J \text{ and } r \text{ in } R^1).$$

If $j+ar = j'+ar'$ for some j, j' in J and r, r' in R^1, then $j'-j = a(r-r')$ and $r-r'$ belongs to $a^{-1}J$. It follows that $mr-mr'$

$= m(r-r') = (\mu e)(r-r') = \mu(r-r') = \varphi(a(r-r')) = \varphi(j'-j) = \varphi j' - \varphi j$ or $\varphi j + mr = \varphi j' + mr'$. Thus φ' is well-defined, it is clearly an R-homomorphism extending φ properly. This contradicts the irreducibility of φ. Thus $J = R^1$ and φ is an R-homomorphism of R^1 into M.

By a $(\mathcal{X},\mathcal{Y})$-injective extension of an R-module M, we mean an extension (K,μ) of M such that K is $(\mathcal{X},\mathcal{Y})$-injective. An R-module P is said to have no proper $\mathcal{Y}$-essential extensions if every $\mathcal{Y}$-essential extension of P is equivalent ot (P,i), where i is the identity map of P.

7.4. Corollary. If (K,μ) is a $\mathcal{Y}$-essential extension of M and (L,λ) is a $\mathcal{Y}$-injective extension of M, then $(K,\mu) < (L,\lambda)$.

7.5. Corollary. If M is $\mathcal{Y}$-injective, then M has no proper $\mathcal{Y}$-essential extensions.

7.6. Lemma. Every maximal $\mathcal{Y}$-essential extension of an R-module is $\mathcal{Y}$-injective.

Proof. Let (K,μ) be a maximal $\mathcal{Y}$-essential extension of M and let P be a $\mathcal{Y}$-essential submodule of an R-module Q. Assume that φ is an R-homomorphism of P into K. By Zorn's lemma, we may suppose that φ is an irreducible semi R-homomorphism of Q into K. Consider the R-module $L = (Q \oplus K)-H$, where H is the submodule of $Q \oplus K$ generated by $(p,\varphi p)$, p in P. It is not difficult to verify that

$$\omega : k \to (0,k) + H, \qquad (k \text{ in } K)$$

is an R-isomorphism of K into L and

$$\mu' : m \to (0,\mu m) + H, \qquad (m \text{ in } M)$$

is an R-isomorphism of M into L such that $\omega\mu = \mu'$. We shall show

that (L,μ') is a Y-essential extension of M or equivalently, (L,ω) is a Y-essential extension of K.

Let (q,k) be an element of $Q \oplus K$ not in H and let $J = \{ r \mid r \in R^1,\ qr \in P \}$. Then J belongs to Y and for each j in J, $((q,k)+H)j = (qj,kj)+H = (0,kj-\varphi qj)+H$. If $kj-\varphi qj = 0$ for each j in J, then the mapping

$$p + qr \to \varphi p + kr, \quad (p \text{ in } P,\ q \text{ in } Q,\ r \text{ in } R^1)$$

is well-defined and is an R-homomorphism of $P+qR^1$ into K extending φ. But since φ is irreducible, q belongs to P and so $J = R^1$, $k=\varphi q$ and (q,k) belongs to H. This contradicts the choice of (q,k). Hence $(k-\varphi q)J \neq 0$. Since ω is an R-isomorphism, $(0,k-\varphi q)J \not\subset H$ and so $(q,k)J \not\subset H$. Thus for every nonzero element x of L there exists J in Y such that $0 \neq xJ \subset \omega K$. This shows that (L,ω) is a Y-essential extension of K. By Proposition 6.3, (L,μ') is a Y-essential extension of M. Since (K,μ) is maximal, ω is an R-isomorphism of K onto L. For each $(q,k) + H$ in L, there exists k' in K such that $(q,k) + H = (0,k') + H$, or $(q,k-k') \in H$. Hence $q \in P$ and $k-k' = \varphi q$. Thus φ is an R-homomorphism of Q onto M.

7.7. Corollary. If (K,μ) is a maximal Y-essential extension of M and (L,λ) is a Y-essential extension, then $(L,\lambda) < (K,\mu)$. Hence any two maximal Y-essential extensions of M are equivalent.

By a minimal $(\mathcal{X},Y)$-injective extension of M we mean a $(\mathcal{X},Y)$-injective extension (K,μ) of M satisfying the following property: If (L,λ) is a $(\mathcal{X},Y)$-injective extension of M such that $(L,\lambda) <_0 (K,\mu)$ then $(L,\lambda) =_0 (K,\mu)$.

7.8. Lemma. Let Y and W be two D-systems of right ideals of R^1 such that $Y \subset W$ and let (K,μ) be a maximal W-essential extension of the R-module M. Then $L = \{k \mid k\in K,\ kJ \subset \mu M \text{ for some } J \text{ in } Y\}$ is a submodule of K containing μM and (L,μ) is a maximal Y-essential extension of M.

Proof. We shall show first that (L,μ) is a Y-essential extension of M. Let k be a nonzero element of L. Then there exists J in Y such that $kJ \subset \mu M$. Since Y satisfies D1, we may suppose that $J = \{r \mid r\in R^1,\ kr\in \mu M\}$. By the hypothesis, (K,μ) is a W-essential extension, $kJ \neq 0$. Hence (L,μ) is a Y-essential extension of M.

Suppose (H,λ) is a Y-essential extension of M such that $(L,\mu) <_\Theta (H,\lambda)$. Then (H,λ) is a W-essential extension of M and so by Proposition 6.6, (H,Θ) is a W-essential extension of L. Since (K,μ) is a maximal W-essential extension, K is W-injective by Lemma 7.6. Thus there exists an R-isomorphism ω of H into K such that $\omega\Theta = i$, where i is the injection of L into K. Since (H,λ) is a Y-essential extension of M, for each h in H, there exists J in Y such that $hJ \subset \lambda M = \Theta\mu M$, i.e., $(\omega h)J = \omega(hJ) \subset \omega\Theta\mu M = \mu M$. Thus $\omega h\in L$ and so ω is an R-isomorphism of H into L. So $\Theta\omega$ is the identity map on H. Thus Θ is "onto". Hence (L,μ) is a maximal Y-essential extension of M.

The existence of a maximal essential extension of an R-module M has been shown by R. Baer [2], Eckmann and Schopf [6]. If $\hat{M}$ is a maximal essential extension of M, then for each D-system Y of right ideals of R^1,

$$\hat{M}^Y = \{m \mid m\in\hat{M},\ mJ \subset M \text{ for some } J \text{ in } Y\}$$

is a maximal Y-essential extension of M.

7.9. Corollary. Let Y be a D-system of right ideals of R^1. Then every R-module M possesses a maximal Y-essential extension.

7.10. Theorem. Let Y be a D-system of right ideals of R^1. Then every R-module M possesses an extension (K,μ), unique up to equivalence, which satisfies the following three equivalent conditions:

(1) (K,μ) is a maximal Y-essential extension of M.

(2) (K,μ) is a Y-essential extension of M and K is Y-injective.

(3) (K,μ) is a minimal Y-injective extension of M.

Proof. Lemma 7.8 and Corollary 7.7 show the existence and uniqueness of a maximal Y-essential extension. It remains to demonstrate the equivalence of the conditions (1)-(3). By Lemma 7.6, (2) follows from (1). Suppose (K,μ) is a Y-essential as well as a Y-injective extension of M and let (L,λ) be a Y-injective extension of M such that $(L,\lambda) <_Q (K,\mu)$. Since L is Y-injective and K is a Y-essential extension of μM, K is a Y-essential extension of QL and there exists an R-isomorphism ω of K into L such that $Q^{-1} = \omega i$, where i is the injection of QL into K. Thus ωQ is the identity map on L and so ω is an R-isomorphism of K onto L with inverse Q. Hence (K,μ) is equivalent to (L,λ) under ω and (K,μ) is a minimal Y-injective extension of M. This proves the implication (2)$\rightarrow$(3). To prove the implication (3)$\rightarrow$(1), let (L,λ) be a maximal Y-essential extension of M. Since (K,μ) is Y-injective, $(L,\lambda) <_Q (K,\mu)$. But (L,λ) is Y-injective and (K,μ) is a minimal Y-injective extension of M, $(L,\lambda) =_Q (K,\mu)$. So (K,μ) is a maximal Y-essential extension of M.

The unique maximal Y-essential extension of M will be called the Y-injective hull of M and will be denoted by $\hat{M}^Y$. M will be

regarded as a submodule of $\hat{M}^Y$. If N is a submodule of M, we may take $\hat{N}^Y$ to be a submodule of $\hat{M}^Y$. In case Y and W are two D-systems of right ideals of R^1 such that $Y \subset W$, by virtue of Lemma 7.8, we may take $\hat{M}^Y$ to be the unique maximal Y-essential extension of M in $\hat{M}^W$.

7.11. Proposition. An R-module M is $(\mathcal{K},Y)$-injective if and only if for each K in $\mathcal{K}$ and each φ in $\mathrm{Hom}_R(\hat{K}^Y,\hat{M}^Y)$, $\varphi K \subset M$.

Proof. Suppose M is $(\mathcal{K},Y)$-injective. For each K in $\mathcal{K}$ and φ in $\mathrm{Hom}_R(\hat{K}^Y,\hat{M}^Y)$, let $L = K \cap \varphi^{-1}M$. Then L is a Y-essential submodule of K by Proposition 6.6. Since M is (K,Y)-injective, there exists an R-homomorphism φ' of K into M such that $\varphi x = \varphi' x$ for each x in L. Since $\hat{M}^Y$ is Y-injective and $\hat{K}^Y$ is a Y-essential extension of K, there exists φ'' in $\mathrm{Hom}_R(\hat{K}^Y,\hat{M}^Y)$ such that $\varphi' x = \varphi'' x$ for every x in K. Now if $(\varphi''-\varphi)K \neq 0$, then since M is an essential submodule of $\hat{M}^Y$, $(\varphi''-\varphi)K \cap M \neq 0$. That is $0 \neq m = \varphi''k - \varphi k$ for some m in M and k in K. But $\varphi k = \varphi''k - m$ belongs to M and so k is in L. Thus $\varphi k = \varphi' k = \varphi'' k$ and $m = \varphi''k - \varphi k = 0$. This contradicts the choice of m. Hence $(\varphi''-\varphi)K = 0$, or $\varphi K = \varphi''K = \varphi' K \subset M$.

Conversely suppose for each K in $\mathcal{K}$ and each R-homomorphism φ of $\hat{K}^Y$ into $\hat{M}^Y$, $\varphi K \subset M$. Let L be a Y-essential submodule of K. Then by Corollary 6.5, L is a Y-essential submodule of $\hat{K}^Y$. If Θ is an R-homomorphism of L into M, then since $\hat{M}^Y$ is Y-injective, Θ can be extended to an R-homomorphism Θ' of $\hat{K}^Y$ into $\hat{M}^Y$. By hypotheses, $\Theta K \subset M$. Hence the restriction of Θ' to K is an extension of Θ. This shows that M is $(\mathcal{K},Y)$-injective.

7.12. Corollary. Let $\mathcal{K} = \{K_i\}$ and let $\Delta_i = \mathrm{Hom}_R(\hat{K}_i^Y,\hat{M}^Y)$. Then $\overline{M} = M + \sum_i \Delta_i K_i$ is a minimal $(\mathcal{K},Y)$-injective extension of M.

Proof. Let λ be an element of Δ_i. Then $\lambda K_i \subset \Delta_i K_i \subset \overline{M}$. Since

$\overset{\Delta}{M}^Y = \hat{M}^Y$, $\overline{M}$ is $(\mathcal{K},Y)$-injective by Proposition 7.11. Suppose (N,μ) is a $(\mathcal{K},Y)$-injective extension of M such that $(N,\mu) <_Q (\overline{M},i)$. Let $Q = Q N$. Then Q is a $(\mathcal{K},Y)$-injective extension of M. Now $\hat{M}^Y = \hat{Q}^Y = \overset{\Delta}{M}^Y$. It follows that $\sum_i \Delta_i K_i \subset Q$ because Q is $(\mathcal{K},Y)$-injective. So $\overline{M} \subset Q$ and $\overline{M} = Q = QN$. Thus $(N,\mu) =_Q (\overline{M},i)$ and $\overline{M}$ is a minimal $(\mathcal{K},Y)$-injective extension of M.

7.13. Corollary. Any two Y-essential, minimal $(\mathcal{K},Y)$-injective extension of M are equivalent.

Let M be a $(\mathcal{K},Y)$-injective R-module. Given K in $\mathcal{K}$, by Proposition 7.11, the restriction $f|K$ of an R-homomorphism f of $\hat{K}^Y$ into $\hat{M}^Y$ to K is an R-homomorphism of K into M. The mapping $f \to f|K$ is a (group) homomorphism of $\mathrm{Hom}_R(\hat{K}^Y, \hat{M}^Y)$ into $\mathrm{Hom}_R(K,M)$.

7.14. Proposition. Let M be an R-module with zero Y-singular submodule. Then M is $(\mathcal{K},Y)$-injective if and only if for each K in $\mathcal{K}$, $\mathrm{Hom}_R(\hat{K}^Y, \hat{M}^Y)$ is isomorphic to $\mathrm{Hom}_R(K,M)$ under the correspondence $f \to f|K$.

Proof. Suppose M is $(\mathcal{K},Y)$-injective. Then by the above remark, $f \to f|K$ is a homomorphism and by Proposition 6.8, it is an isomorphism. The converse is a consequence of Proposition 7.11.

The existence of a maximal Y-rational extension of an R-module M has been established at the end of Section 6. We are in the position to prove the uniqueness of the maximal Y-rational extension of M by making use of the Y-injective hull of M.

7.15. Theorem. Let M be an R-module and let

$$\overline{M} = \cap \left\{ \ker \varphi \mid \varphi \in \mathrm{Hom}_R(\hat{M}^Y, \hat{M}^Y),\ \ker \varphi \supset M \right\}.$$

Then $\overline{M}$ is a maximal Y-rational extension of M containing every

Y-rational extension of M which is included in $\overset{\Delta}{M}^{Y}$. If (K,μ) is any maximal Y-rational extension of M, then there exists an R-isomorphism θ of K onto $\overline{M}$ such that $(K,\mu) =_{\theta} (\overline{M},i)$.

Proof. We prove first that $\overline{M}$ is a Y-rational extension of M. Let m and n be two elements of $\overline{M}$ with $m \neq 0$. Then $J=\{r \mid r\in R^1, nr\in M\}$ is in Y and $nJ \subset M$. If $mJ = 0$, then

$$\varphi : k + nx \to mx, \quad (k \text{ in } M,\ x \text{ in } R^1)$$

is a well-defined R-homomorphism of $M+nR^1$ into $\overline{M}$ which annihilates M. Since $\overset{\Delta}{M}^{Y}$ is Y-injective, φ can be extended to an R-endomorphism φ' of $\overset{\Delta}{M}^{Y}$. Now $\varphi' n = m \neq 0$. This contradicts the choice of n. Hence $\overline{M}$ is a Y-rational extension of M.

We prove next that $\overline{M}$ contains every Y-rational extension of M included in $\overset{\Delta}{M}^{Y}$. Let K be such an extension of M. Assume that φ is an R-endomorphism of $\overset{\Delta}{M}^{Y}$ which annihilates M. If $\varphi K \neq 0$, then $\varphi k \neq 0$ for some k in K. Now since K is a Y-rational extension of M, $\varphi kJ \neq 0$, $kJ \subset M$ for some J in Y. This contradicts the assumption that $\varphi M = 0$. Hence $\varphi K = 0$ for every R-endomorphism φ of $\overset{\Delta}{M}^{Y}$ which annihilates M. Thus $K \subset \overline{M}$.

Now let (K,μ) be a Y-rational extension of M for which $(\overline{M},i) <_{\theta} (K,\mu)$. Then since $\overset{\Delta}{M}^{Y}$ is Y-injective, there exists R-isomorphism ω of K into $\overset{\Delta}{M}^{Y}$ such that $i = \omega\mu$. It is clear that ωK is a Y-rational extension of M contained in $\overset{\Delta}{M}^{Y}$. Thus $\omega K \subset \overline{M}$ and $(K,\mu) <_{\omega} (\overline{M},i)$. By Lemma 6.23, $(\overline{M},i)$ is equivalent to (K,μ). This shows that $\overline{M}$ is a maximal Y-rational extension of M.

By a similar argument, we can show that for each maximal Y-rational extension (L,λ) of M, there exists an R-isomorphism ω of L into $\overline{M}$ such that $(L,\lambda) <_{\omega} (\overline{M},i)$. Since (L,λ) is a maximal Y-rational extension, ω is an R-isomorphism of L onto $\overline{M}$. Hence $(L,\lambda) =_{\omega} (\overline{M},i)$.

7.16. Proposition. The complete direct sum of a family $\{M_i\}$ of R-modules is $(\mathcal{K},Y)$-injective if and only if each summand M_i is $(\mathcal{K},Y)$-injective.

Proof. Suppose $M = \sum M_i$ is $(\mathcal{K},Y)$-injective. If K is a member of $\mathcal{K}$ and if φ is an R-homomorphism of a Y-essential submodule L of K into M_i, then since M is (K,Y)-injective, φ can be extended to an R-homomorphism φ' of K into M. Thus $\pi_i\varphi'$ is an R-homomorphism of K into M_i extending φ, where π_i is the projective of M on M_i. Hence M_i is $(\mathcal{K},Y)$-injective.

Conversely, suppose each M_i is $(\mathcal{K},Y)$-injective. Let K be a member of $\mathcal{K}$ and let L be a Y-essential submodule of K. If φ is an R-homomorphism of L into M, then $\pi_i\varphi$ is an R-homomorphism of L into M_i. Since M_i is (K,Y)-injective, $\pi_i\varphi$ can be extended to an R-homomorphism φ_i' of K into M_i. Define a mapping φ' of K into M as follows:

$$\varphi'k = \sum_i \varphi_i'k, \qquad (k \text{ in } K).$$

Then φ' is an R-homomorphism of K into M extending φ. Hence M is $(\mathcal{K},Y)$-injective.

7.17. Corollary. Let $\{M_i\}$ be a finite family of R-modules and let M be the direct sum of M_i's. Then $\hat{M}^Y = \sum \oplus \hat{M}_i{}^Y$.

PART III SEMI HOMOMORPHISMS AND QUOTIENT RINGS

The present part is devoted to the constructions of extensions of rings and modules by a method orginated by Asano. Section 8 deals with the totality $F(\Delta,N)$ of semi R-homomorphisms of an R-module M into another R-module N with domains in a specific family Δ of submodules of M. Imposing suitable conditions on the family Δ, we can make $F(\Delta,N)$ into a bimodule. Subsequent sections give an account of various special cases of $F(\Delta,N)$ including the maximal essential and rational extensions of R-modules and the rings of right quotients of R. The last section deals with the ideal structure of R and its rings of right quotients with the object of developing noncommutative localization theory.

8. Semi homomorphisms. Through this section, we shall assume that M and N are two R-modules and denote the ring of all R-endomorphisms of M (resp. N) by S (resp. T).

If K is a submodule of M, then an R-homomorphism of K into N is called a semi R-homomorphism of M into N. Every semi R-homomorphism of M into N has irreducible extensions and any two semi R-homomorphisms of M into N have a common extension if and only if they coincide on the intersection of their domains. (See Preliminaries.)

The sum $\alpha + \beta$ of two semi R-homomorphisms α and β of M into N is a semi R-homomorphism of M into N defined by

$$\text{dom } (\alpha+\beta) = \text{dom } \alpha \cap \text{dom } \beta,$$

$$(\alpha+\beta)x = \alpha x + \beta x \quad \text{for each } x \text{ in dom } (\alpha+\beta).$$

It is easy to verify that $\alpha+\beta = \beta+\alpha$ and $\alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma$ for any three semi R-homomorphisms α, β and γ of M into N. Given s

in S, t in T and α a semi R-homomorphism of M into N, the "scalar products" αs and $t\alpha$ are defined by

$$\text{dom } \alpha s = s^{-1}\text{dom } \alpha, \qquad (\alpha s)x = \alpha(sx) \text{ for each } x \text{ in dom } \alpha s;$$

$$\text{dom } t\alpha = \text{dom } \alpha, \qquad (t\alpha)x = t(\alpha x) \text{ for each } x \text{ in dom } t\alpha.$$

The system of all semi R-homomorphisms of M into N together with addition and scalar multiplications defined above possesses most of the module-like properties, for instance

$$(t+t')\alpha = t\alpha + t'\alpha, \qquad \alpha(s+s') = \alpha s + \alpha s',$$

$$t(\alpha+\alpha') = t\alpha + t\alpha', \qquad (\alpha+\alpha')s = \alpha s + \alpha' s,$$

$$(tt')\alpha = t(t'\alpha), \qquad \alpha(ss') = (\alpha s)s',$$

$$t(\alpha s) = (t\alpha)s,$$

hold for any t, t' in T, s, s' in S and α, α' semi R-homomorphisms of M into N. Nonetheless it is not, in general, a module.

We shall now investigate the set of all semi R-homomorphisms of M into N whose domains are from a certain family Δ of submodules of M. The properties that Δ may possess are

D1. Every submodule of M that contains a member of Δ is in Δ. (See Proposition 2.1.)

D2. If K and L are two members of Δ and φ is an R-homomorphism of K into M, then $\varphi^{-1}L$ belongs to Δ. (See Proposition 2.1.)

D22. Δ is closed under finite intersection.

D23. For each K in Δ and each R-endomorphism φ of M, $\varphi^{-1}K$ is in Δ.

We suppose first that Δ is a nonempty family of submodules of M satisfying the conditions D22 and D23 listed above. Denote by $\mathfrak{Z}(\Delta,N)$ the totality of semi R-homomorphisms of M into N with domains in Δ, i.e.,

$$\mathfrak{Z}(\Delta,N) = \bigcup_{K\in\Delta} \text{Hom}_R(K,N).$$

Given any α and β in $\mathfrak{F}(\Delta,N)$, s in S and t in T, condition D22 ensures that the sum $\alpha+\beta$ of α and β is again in $\mathfrak{F}(\Delta,N)$ while the condition D23 guarantees that the scalar product αs is in $\mathfrak{F}(\Delta,N)$. It is clear that $t\alpha$ belongs to $\mathfrak{F}(\Delta,N)$.

Two elements α and β in $\mathfrak{F}(\Delta,N)$ are said to be congruent, we write $\alpha \equiv \beta$, if there exists a member K of Δ contained in both dom α and dom β such that $\alpha x = \beta x$ for each x in K. It is not diffifult to verify that "$\equiv$" defines a congruence relation on $\mathfrak{F}(\Delta,N)$ in the sense that it is an equivalence relation having the following property: If $\alpha' \equiv \alpha$ and $\beta' \equiv \beta$, then $\alpha'+\beta' \equiv \alpha+\beta$, $\alpha's \equiv \alpha s$ and $t\alpha' \equiv t\alpha$ for any s in S and t in T. Let $\overline{\alpha}$ be the congruence class containing α and let $F(\Delta,N)$ denote the set of all congruence classes. Then $F(\Delta,N)$ becomes a bimodule with S and T act as rings of right and left operators respectively if we define

$$\overline{\alpha} + \overline{\beta} = \overline{\alpha+\beta}, \qquad \overline{\alpha}s = \overline{\alpha s} \quad \text{and} \quad t\overline{\alpha} = \overline{t\alpha}.$$

Note that the operations are well-defined because "$\equiv$" is a congruence relation.

If, in addition to conditions D22 and D23, Δ satisfies also the condition D1, then $M \in \Delta$ and $\mathrm{Hom}_R(M,N) \subset \mathfrak{F}(\Delta,N)$. The mapping $\alpha \to \overline{\alpha}$ is a (T,S)-homomorphism of $\mathrm{Hom}_R(M,N)$ into $F(\Delta,N)$.

8.1. Proposition. Let Δ be a nonempty family of submodules of M satisfying the conditions D1, D22 and D23. Then $F(\Delta,N)$ is a (T,S)-module containing a homomorphic image of $\mathrm{Hom}_R(M,N)$.

As an illustration, we mention a special type of semi R-homomorphisms. Let X be a D-system of right ideals of R. A semi R-homomorphism α of M into N is said to be an X-full R-homomorphism of M into N if the domain of α is in $D^X(M)$. Recall

that $D^X(M)$ is the family of all x-dense submodules of M which by Proposition 2.1 satisfies conditions D1, D2 and hence also D22 and D23. The set of all X-full R-homomorphisms of M into N will be designated by $\mathfrak{F}^X(M,N)$. If we define a congruence relation on $\mathfrak{F}^X(M,N)$ as above, then the resulting congruence classes form a (T,S)-module $F^X(M,N)$ containing a homomorphism image of $\mathrm{Hom}_R(M,N)$.

In case every semi R-homomorphism of M into N with domain in Δ has a unique irreducible extension, there is another way of looking at the bimodule $F(\Delta,N)$. Let $\bar{\alpha}$ be an element of $F(\Delta,N)$. Then α has a unique irreducible extension α^* which is evidently in $\bar{\alpha}$. If β is another element of $\bar{\alpha}$, then there exists K in Δ, $K \subset \mathrm{dom}\,\alpha \cap \mathrm{dom}\,\beta$ such that the restrictions of α and β to K are identical. Denote the common restriction by γ. Then α^* and β^* are irreducible extensions of γ. By the uniqueness of irreducible extensions, $\alpha^* = \beta^*$. This shows that in each class $\bar{\alpha}$ there exists a unique irreducible semi R-homomorphism of M into N which is an extension of every element in the class. If $\bar{\alpha}$ and $\bar{\beta}$ are distinct classes, then $\alpha^* \neq \beta^*$. For otherwise, since $\alpha = \alpha^*$ and $\beta = \beta^*$, we would have $\alpha = \beta$. Thus $\bar{\alpha} \to \alpha^*$ is one-to-one and $F(\Delta,N)$ can be regarded as a bimodule consisting of all irreducible semi R-homomorphisms of M into N with domains in Δ. Operations are defined by

$$\alpha + \beta = (\alpha+\beta)^*, \quad \alpha s = (\alpha s)^* \quad \text{and} \quad t\alpha = (t\alpha)^*.$$

8.1! <u>Proposition</u>. Let Δ be a nonempty family of submodules of M satisfying the conditions D1, D22 and D23. Assume that every semi R-homomorphism of M into N with domain in Δ has a unique irreducible extension. Then the set $F(\Delta,N)$ of all irreducible semi R-homomorphisms of M into N with domains in Δ forms a (T,S)-module which contains $\mathrm{Hom}_R(M,N)$ as a submodule.

8.2. Corollary. Let Δ and Δ' be two families of submodules of M satisfying the conditions D1, D22 and D23. If $\Delta \subset \Delta'$ and every semi R-homomorphism of M into N with domain in Δ' has a unique irreducible extension, then $F(\Delta,N) \subset F(\Delta',N)$.

8.3. Lemma. Let X be a D-system of right ideals of R and let K be a strongly X-prime submodule of N. If α' is a semi R-homomorphism of M into N which is an extension of the X-full R-homomorphism α of M into K, then α' is a semi R-homomorphism of M into K.

Proof. Let m be in the domain of α'. Then since dom α is x-dense M, there exists A in X such that $mA \subset$ dom α. Now $(\alpha'm)A = \alpha'(mA) = \alpha(mA) \subset K$. Since K is strongly X-prime, $\alpha'm \in K$. Hence α' is a semi R-homomorphism of M into K.

8.4. Corollary. If N is strongly X-prime, then every semi R-homomorphism of M into N has a unique irreducible extension.

8.5. Corollary. If K is a strongly X-prime submodule of N, then $F^X(M,K)$ is a submodule of $F^X(M,N)$.

It follows immediately from Corollary 8.4 and Proposition 8.1' that if N is strongly X-prime, then the bimodule $F^X(M,N)$ can be regarded as the totality of irreducible semi R-homomorphisms of M into N whose domains are x-dense submodules of M.

8.6. Proposition. Let Δ be a nonempty family of submodules of M satisfying the conditions D1 and D2. If M' is a member of Δ, then the family Δ' of all submodules of M' that are contained in Δ satisfies the conditions D1 and D2. Moreover the left T-modules $F(\Delta,N)$ and $F(\Delta',N)$ are isomorphic.

Proof. We shall denote the elements in $F(\Delta',N)$ by $\tilde{\alpha}$, $\tilde{\beta}$, .. . Then clearly $\tilde{\alpha} \to \bar{\alpha}$ is a T-isomorphism of $F(\Delta',N)$ into $F(\Delta,N)$. If $\bar{\alpha}$ is any element of $F(\Delta,N)$, then $\tilde{\alpha}_1 \to \bar{\alpha}$, where α_1 is the restriction of α to $M' \cap \text{dom } \alpha$.

If M' is an x-dense submodule of M and $\Delta = D^x(M)$, then obviously $\Delta' = D^x(M')$. By Proposition 8.6, the left T-modules $F^x(M,N)$ and $F^x(M',N)$ are isomorphic.

8.7. Proposition. Let Δ be a nonempty family of submodules of M satisfying conditions D22 and D23 and let N' be a submodule of N such that $\varphi^{-1}N' \in \Delta$ for each φ in $\mathfrak{F}(\Delta,N)$. Then the right S-modules $F(\Delta,N)$ and $F(\Delta,N')$ are isomorphic.

Proof. Denote the elements in $F(\Delta,N')$ by $\tilde{\alpha}$, $\tilde{\beta}$, .. . Then $\tilde{\alpha} \to \bar{\alpha}$ is a S-isomorphism of $F(\Delta,N')$ into $F(\Delta,N)$. If $\bar{\alpha} \in F(\Delta,N)$, then $\alpha^{-1}N'$ belongs to Δ and so $\tilde{\alpha}_1 \to \bar{\alpha}$, where α_1 is the restriction of α to $\alpha^{-1}N'$.

If N' is an x-dense submodule of N, then by Lemma 1.11 and the fact that every x-dense submodule of an x-dense submodule is again x-dense, N' satisfies the hypotheses of Proposition 8.7. Hence the right S-modules $F^x(M,N)$ and $F^x(M,N')$ are isomorphic.

9. Extended centralizer of a ring over a module. In this section we shall consider the (T,S)-module $F(\Delta,N)$, where Δ is a nonempty family of submodules of M satisfying conditions D1 and D2 and N is a submodule of M. For any two elements α and β in $\mathfrak{F}(\Delta,N)$, since $\beta^{-1}\text{dom } \alpha$ belongs to Δ, the semi R-homomorphism $\alpha\beta$ defined by

$$\text{dom } (\alpha\beta) = \beta^{-1}\text{dom } \alpha,$$

$$(\alpha\beta)x = \alpha(\beta x) \quad \text{for each } x \text{ in dom } (\alpha\beta)$$

is in $\mathfrak{F}(\Delta,N)$. Moreover if $\alpha' \equiv \alpha$ and $\beta' \equiv \beta$, then $\alpha'\beta' \equiv \alpha\beta$, i.e., the mapping $(\alpha,\beta) \to \overline{\alpha\beta}$ is well-defined. Thus the (T,S)-module $F(\Delta,N)$ becomes a ring if we define $\bar{\alpha}\cdot\bar{\beta} = \overline{\alpha\beta}$. It follows then that the mapping $\alpha \to \bar{\alpha}$ of $\mathrm{Hom}_R(M,N)$ into $F(\Delta,N)$ is not only a (T,S)-homomorphism, but also a ring homomorphism.

The family Y_M of all essential submodules of M satisfies conditions D1 and D2. The ring $F(Y_M,M)$ is called by Johnson [15] the extended centralizer of R over M and is proved to be a regular ring in the sense that for every nonzero element $\bar{\alpha}$ of $F(Y_M,M)$, there exists $\bar{\beta}$ in $F(Y_M,M)$ such that $\bar{\alpha}\,\bar{\beta}\,\bar{\alpha} = \bar{\alpha}$. The main feature of Y_M is that for every submodule K of M there exists a submodule K' of M such that $K \cap K' = 0$ and $K+K'$ is in Y_M. This property of Y_M plays a vital role in the proof of regularity of the extended centralizer. We are led to the consideration of the condition:

D4. For each submodule K of M there exists a submodule K' of M such that $K \cap K' = 0$ and $K+K' \in \Delta$.

9.1. Lemma. A family Δ of submodules of M satisfies the condition D4 if and only if Δ contains all essential submodules of M.

The proof of the above lemma is easy and will be omitted.

9.2. Proposition. Let Δ be a family of submodules of M satisfying conditions D1, D2 and D4. Then the ring $F(\Delta,M)$ is a homomorphic image of the extended centralizer $F(Y_M,M)$ of R over M.

Proof. Denote the elements in the extended centralizer by $\tilde{\alpha}$, $\tilde{\beta}$, .. . For each $\tilde{\alpha}$ in the extended centralizer, $\alpha \in \mathfrak{F}(\Delta,M)$ since dom α belongs to Δ by Lemma 9.1. The mapping $\tilde{\alpha} \to \bar{\alpha}$ is

clearly a homomorphism of $F(Y_M,M)$ into $F(\Delta,M)$. If $\bar{\alpha}$ is any element of $F(\Delta,M)$, then $K \cap \text{dom } \alpha = 0$ and $K + \text{dom } \alpha$ is essential for some submodule K of M. If we define $\alpha_1(k+x) = \alpha x$, where $k \in K$ and $x \in \text{dom } \alpha$, then $\tilde{\alpha}_1$ is in the extended centralizer and $\tilde{\alpha}_1 \to \bar{\alpha}$. Hence the homomorphism is "onto".

9.3. Corollary. Let Δ be as in Proposition 9.2. Then $F(\Delta,M)$ is a regular ring.

For the sake of reference, we include two properties of regular rings here. An element e is said to be idempotent if $e^2 = e$. We shall denote by $|a)$ the right ideal generated by a.

9.4. Proposition. A ring A is regular if and only if for each a in A there exists an idempotent element e of A such that $|a)=eA$.

Proof. Suppose A is regular and let a be any element of A. If $a = 0$, then there is nothing to prove. Assume $a \neq 0$. Then $axa = a$ for some x in A. It follows that $e = ax$ satisfies $e^2=e$ and $|a) = eA$. Conversely, suppose for each a in A there exists an idempotent element e of A such that $|a) = eA$. Now $e = e^2 \in eA = |a)$ and so $e = ay$ for some y in A^1. Let $x = yay$. Then $axa = ayaya = e^2a = ea = a$. Hence A is regular.

9.5. Corollary. Every regular ring A is semi-simple in Jacobson's sense [14].

Proof. Let J(A) be the Jacobson's radical of A. If $a \in J(A)$, then $|a) = eA$ for some idempotent element e of A. Now since $e = e^2 \in eA = |a) \subset J(A)$, there exists e' in A such that $e+e'-ee'=0$. But $0 = e0 = e^2 + ee' - e^2e' = e + ee' - ee' = e$. So $a = 0$. This shows that $J(A) = 0$ and A is semi-simple.

By Corollary 8.5, if K is a strongly X-prime submodule of M, then $F^X(M,K)$ is a subgroup of $F^X(M,M)$. It is not difficult to verify that $F^X(M,K)$ is a right ideal of $F^X(M,M)$. In fact it is a principal right ideal if $D^X(M)$ satisfies the condition D4. To prove this we note first that $K \cap K' = 0$ and $K+K'$ is x-dense for some submodule K' of M. The mapping

$$\alpha : k + k' \to k \qquad (k \text{ in } K \text{ and } k' \text{ in } K')$$

is an X-full R-homomorphism of M into itself. Clearly $\overline{\alpha} \cdot F^X(M,M) \subset F^X(M,K)$. On the other hand, if β is an X-full R-homomorphism of M into K, then $\beta \text{ dom } \beta \subset K$ and for every m in dom β, $\alpha\beta m = \beta m$. Thus $\overline{\beta} = \overline{\alpha\beta} = \overline{\alpha}\ \overline{\beta} \in \overline{\alpha} \cdot F^X(M,M)$. Hence $F^X(M,K) = \overline{\alpha} \cdot F^X(M,M)$. This proves the following

9.6. Proposition. Let X be a D-system of right ideals of R such that every essential submodule of M is x-dense. If K is an x-closed submodule of M, then $F^X(M,K)$ is a principal right ideal of the ring $F^X(M,M)$.

Consider now the mapping $\Theta : K \to F^X(M,K)$ of $L^X(M)$ into the set of all principal right ideals of $F^X(M,M)$. Obviously, if $K \subset L$, then $F^X(M,K) \subset F^X(M,L)$. Conversely suppose $F^X(M,K) \subset F^X(M,L)$ for some K and L in $L^X(M)$. By the proof of Proposition 9.6, $F^X(M,K) = \overline{\alpha} \cdot F^X(M,M)$, where α is the projection of $K+K'$ onto K, K' is a relative complement of K. Now $\overline{\alpha} \in F^X(M,K) \subset F^X(M,L)$ and so $K = \alpha(K+K') \subset L$.

9.7. Proposition. Let X be a D-system of right ideals of R such that every essential submodule of M is x-dense. Then $K \to F^X(M,K)$ is a one-to-one mapping of $L^X(M)$ into the set of all principal right ideals of the regular ring such that $K \subset L$ if and only if $F^X(M,K) \subset F^X(M,L)$.

As we have remarked previously, if N is strongly X-prime, then $F^X(M,N)$ can be considered as a (T,S)-module consisting of all irreducible X-full R-homomorphisms of M into N which contains $\mathrm{Hom}_R(M,N)$ as a submodule. Moreover a submodule K of N is X-essential if and only if it is x-dense. If further N is (M,X)-injective, then $F^X(M,N) = \mathrm{Hom}_R(M,N)$.

Now suppose that M is strongly X-prime and (M,X)-injective. Then every x-closed submodule K is strongly X-prime and (M,X)-injective. Thus $F^X(M,K) = \mathrm{Hom}_R(M,K)$.

9.8. Proposition. Let M be a strongly X-prime and (M,X)-injective R-module such that $F^X(M,M)$ is a regular ring. Then every principal right ideal of $F^X(M,M)$ is of the form $F^X(M,K)$ where K is an x-closed submodule of M.

Proof. Suppose $\alpha\cdot\mathrm{Hom}_R(M,M)$ is a principal right ideal of $\mathrm{Hom}_R(M,M)$. By Proposition 9.4, we may choose α to be idempotent. Let $K = \alpha M$. We shall prove that K is an x-closed submodule of M. For if $mA \subset K$ for some m in M and A in X, then $mx = \alpha(mx) = (\alpha m)x$, or $(m-\alpha m)x = 0$ for each x in A. Since M is strongly X-prime, $m = \alpha m \in \alpha M = K$. Hence K is an x-closed submodule of M. It is easy to verify that $\alpha\cdot\mathrm{Hom}_R(M,M) = \mathrm{Hom}_R(M,K)$.

9.9. Corollary. If X is a D-system of right ideals of R such that M is strongly X-prime and $L^X(M)$ is complemented, then $K \to F^X(\hat{M}^X, \hat{K}^X)$ is a (lattice) isomorphism of the lattice $L^X(M)$ of all x-closed submodules of M onto the lattice (ordered by inclusion) of all principal right ideals of the regular ring $F^X(\hat{M}^X, \hat{M}^X)$.

We consider next the case where M is strongly X-prime, X-atomic and (M,X)-injective. By Proposition 3.2, $L^X(M)$ is

complemented. In Section 3 (see the remark prior to Proposition 3.7) we studied briefly the set $C^X(M)$ of all x-closed submodules of M which are mapped into themselves by every s in $S = \mathrm{Hom}_R(M,M)$. $C^X(M)$ is a sublattice of $L^X(M)$ containing the socle of $L^X(M)$ and hence induces a closure operation $N \to \overline{N}$ on $L^X(M)$, where

$$\overline{N} = \cap\{K \mid K \in C^X(M),\ K \supset N\} = \bigvee_{s\in S} (sN)^X.$$

9.10. Lemma. If T is an atom of $L^X(M)$, then $\overline{T}$ is an atom of $C^X(M)$.

Proof. Let N be a nonzero element of $C^X(M)$ such that $N \subset \overline{T}$. Suppose $N \cap T = 0$. Then the R-module $N \vee T$ is strongly X-prime and X-atomic by Theorem 3.3. Moreover, since $L^X(M)$ is complemented and M is (M,X)-injective, $N \vee T$ is $(N \vee T, X)$-injective and $N \in C^X(N \vee T)$. We may suppose $N \vee T = M$. We shall first show that if K is a submodule of M such that $K \cap T = 0$, then $K \subset N$. To do this let K' be a relative complement of T containing K. Then K' belongs to $L^X(M)$ and is a complement of T in $L^X(M)$. Since M is (M,X)-injective,

$$\varphi : k' + t \to t \qquad (k' \in K',\ t \in T)$$

defines an R-endomorphism of M. If $n \in N$, then $nA \subset K' + T$ for some A in X. Given $a \in A$, there exist k' in K' and t in T such that $na = k' + t$. Since $\varphi N \subset N$, $\varphi(na) = t \in T \cap N = 0$ and so $na = k' \in K'$. Thus $nA \subset K'$ and $n \in K'$, i.e., $N \subset K'$. Now $K \subset K' = K' \cap (N \vee T)$ $= N \vee (K' \cap T) = N \vee 0 = N$. Next we show that $\overline{T} = T$. Suppose the contrary, $sT \not\subset T$ for some s in S. Let K be a complement of $T \cap sT$ in $L^X(sT)$. Then $K \neq 0$ and $K \cap T = 0$. By the first part of the proof, $K \subset N$. Consider the submodule L consisting of all t-st (t in T and st in K). It is easy to see that $L \neq 0$, $L \cap N = 0$ and $L \not\subset T$. Let J be a complement of $T \cap L$ in $L^X(L)$. Then $J \neq 0$ and $J \cap T = 0$. Thus $J \subset N$. But $0 \neq J = J \cap N \subset L \cap N = 0$.

This is a contradiction. Hence $\overline{T} = T$ and $N = \overline{T} \cap N = T \cap N = 0$. This contradicts the choice of N. We conclude then $N \cap T \neq 0$. Since T is an atom, $T \subset N$ and so $\overline{T} \subset \overline{N} = N$. Thus $\overline{T} = N$ and $\overline{T}$ is an atom of $C^X(M)$.

Now, since M is X-atomic, M is a join of atoms in $L^X(M)$. By the above lemma, M is a join of atoms in $C^X(M)$. We can show by an argument analogous to that given in the proof of Proposition 3.2 that M is a direct join of atoms in $C^X(M)$, say $M = \underset{i\in I}{\vee} \oplus T_i$.

9.11. Theorem. If M is strongly X-prime, X-atomic and (M,X)-injective, then $M = \underset{i\in I}{\vee} \oplus T_i$ is a direct join of atoms T_i $(i\in I)$ in $C^X(M)$ and

$$F^X(M,M) = \prod_{i\in I} F^X(T_i,T_i).$$

Proof. Since M is (M,X)-injective, we may suppose $M = \sum_{i\in I} T_i$. For each R-endomorphism φ of M, the restriction φ_i of φ to T_i is an R-endomorphism of T_i and $\varphi = \sum_{i\in I} \varphi_i \in \prod_{i\in I} \mathrm{Hom}_R(T_i,T_i)$. Conversely for each R-endomorphism φ_i of T_i $(i\in I)$,

$$(\sum_{i\in I} \varphi_i)(\sum_{i\in I} m_i) = \sum_{i\in I} \varphi_i m_i, \quad (m_i \in T_i)$$

defines an R-endomorphism of M. Thus $F^X(M,M) = \mathrm{Hom}_R(M,M) = \prod_{i\in I} \mathrm{Hom}_R(T_i,T_i)$. Notice that since M is X-atomic and (M,X)-injective, each T_i is (T_i,X)-injective and so $F^X(T_i,T_i) = \mathrm{Hom}_R(T_i,T_i)$ and we have the result.

10. Maximal essential and rational extensions. In this section we shall investigate the (T,S)-module $F(\Delta,M)$, where Δ is a non-empty family of right ideals of R^1 satisfying conditions D1 and D2, M is an R-module, $T = \mathrm{Hom}_R(M,M)$ and $S = \mathrm{Hom}_R(R^1,R^1)$. Since $S \cong R^1$ and $\mathrm{Hom}_R(R^1,M) \cong M$, $F(\Delta,M)$ can be regarded as a right R-module containing a homomorphic image of M. The R-homomorphism of M into

$F(\Delta,M)$ is given by

$$m \to \bar{m}_L \qquad (m \text{ in } M)$$

where $m_L : x \to mx \qquad (x \text{ in } R^1)$.

In case M is strongly Δ-prime, this R-homomorphism becomes an R-isomorphism. On identifying m with $\bar{m}_L$, M may be regarded as a submodule of $F(\Delta,M)$. Note that for α in $\mathfrak{I}(\Delta,M)$ and x in dom α, $\alpha x = \overline{(\alpha x)}_L = \overline{\alpha \cdot x} = \bar{\alpha} \cdot x$, so $\bar{\alpha} \cdot \text{dom } \alpha \subset M$.

Let Y be a D-system of right ideals of R^1. Then by Corollary 6.16, the family $\Delta = \mathfrak{R}^Y_M(R^1)$ of all right ideals A of R^1 such that $A < R^1$ (M;Y) satisfies the conditions D1 and D2 and is such that M is strongly Δ-prime. It follows that $F(\Delta,M)$ is an extension of M. Given $\bar{\alpha}$ and $\bar{\beta}$ in $F(\Delta,M)$ with $\bar{\alpha} \neq 0$, $J = \text{dom } \alpha \cap \text{dom } \beta$ belongs to Δ and $\alpha J \neq 0$. Now $\bar{\alpha}J = \alpha J \neq 0$ and $\bar{\beta}J = \beta J \subset M$. This shows that $F(\Delta,M)$ is a Y-rational extension of M. Suppose K is any Y-rational extension of M and let $\Delta' = \mathfrak{R}^Y_K(R^1)$. It can be verified that $\Delta = \Delta'$. Now for each α in $\mathfrak{I}(\Delta,K)$, $\alpha^{-1}M < \text{dom } \alpha$ (K;Y) by Lemma 6.14. Since dom $\alpha < R^1$ (K;Y), $\alpha^{-1}M < R^1$ (K;Y) by Lemma 6.15. That is $\alpha^{-1}M \in \Delta' = \Delta$. Apply Proposition 8.7, the R-module $F(\Delta,M)$ and $F(\Delta,K)$ are isomorphic. Hence

10.1. Proposition. Let Y be a D-system of right ideals of R^1. Then $F(\mathfrak{R}^Y_M(R^1),M)$ is the unique maximal Y-rational extension of M.

We proved in Corollary 6.7 that the family $\Delta = \mathfrak{E}^Y(R^1)$ of all Y-essential right ideals of R^1 satisfies conditions D1 and D2. If M is strongly Δ-prime, i.e., if the Y-singular submodule of M is zero, then $F(\Delta,M)$ is an extension of M. The Y-injective envelope $\hat{M}^Y$ of M is a maximal Y-essential extension of M and so by Lemma 6.14, $\alpha^{-1}M \in \Delta$ for each α in $\mathfrak{I}(\Delta,\hat{M}^Y)$. By Proposition 8.7, the R-module $F(\Delta,M)$ and $F(\Delta,\hat{M}^Y)$ are isomorphic. Since $\hat{M}^Y$

is strongly Δ-prime, $F(\Delta, \hat{M}^Y)$ can be considered alternatively as the R-module of all irreducible semi R-homomorphisms of R^1 into $\hat{M}^Y$ with domains in Δ. However since $\hat{M}^Y$ is Y-injective, the only irreducible semi R-homomorphisms of R^1 into $\hat{M}^Y$ with domains in Δ are full R-homomorphisms of R^1 into $\hat{M}^Y$. Thus $F(\Delta, \hat{M}^Y)$ $= \mathrm{Hom}_R(R^1, \hat{M}^Y) \cong \hat{M}^Y$. This proves the following

10.2. Proposition. Let Y be a D-system of right ideals of R and let M be an R-module with zero Y-singular submodule. Then $F(\xi^Y(R^1), M)$ is the Y-injective envelope of M.

11. Quotient rings. Let Y be a D-system of right ideals of R^1. An overring S of R is said to be a ring of Y-quotients of R if S is a Y-essential extension of the R-module R, or equivalently, if for every nonzero element x of S there exists J in Y such that $0 \neq xJ \subset R$. In case S is a Y-rational extension of R, we call S a ring of Y-fractions of R. It is always true that every ring of Y-fractions of R is also a ring of Y-quotients of R. The converse is true if R has zero Y-singular ideal. Note that if W is another D-system of right ideals of R^1 containing Y, then every ring of Y-quotients (resp. Y-fractions) of R is also a ring of W-quotients (resp. W-fractions) of R. In fact it should be mentioned that Findlay and Lambek's rings of right quotients of R are precisely rings of $L(R^1)$-fractions of R. Thus every ring of Y-fractions of R is a Findlay and Lambek's ring of right quotients of R.

11.1. Lemma. An overring S of R is a ring of Y-fractions of R if and only if for every finite set of elements $x_1, \ldots, x_n$ of S with $x_1 \neq 0$, there exists J in Y such that $x_1J, \ldots, x_nJ \subset R, x_1J \neq 0$.

Proof. Suppose S is a ring of Y-fractions of R. We prove by induction on n that for every finite set of elements $x_1, \ldots, x_n$ of S with $x_1 \neq 0$, there exists J in Y such that $x_1J, \ldots, x_nJ \subset R$ and $x_1J \neq 0$. This is clearly true for n = 1. Suppose it is true for any finite set of n-1 elements and let $x_1, \ldots, x_n \in S$ and $x_1 \neq 0$. Then there exists J in Y such that $x_1J, \ldots, x_{n-1}J \subset R$ and $x_1J \neq 0$. For each j in J, if $x_1j \neq 0$, then $0 \neq x_1I_j \subset R$ and $x_nI_j \subset R$ for some I_j in Y. If $x_1j = 0$, we may choose I_j in Y such that $x_nI_j \subset R$. Let I to be the sum of jI_j (j in J). Then I belongs to Y and $x_1I, \ldots, x_nI \subset R$, $x_1I \neq 0$. This completes the induction.

Let S and T be two overrings of R. By a homomorphism of S into T over R we mean a homomorphism of S into T which leaves every element of R invariant.

11.2. Theorem. Let Y be a D-system of right ideals of R. Then R possesses a maximal ring Q^Y of Y-fractions which is unique up to isomorphism over R. Every ring of Y-fractions of R is isomorphic over R to a unique subring of Q^Y.

Proof. Let $Q^Y = F(R^Y_R(R^1), R)$. Then Q^Y is a ring containing an isomorphic image of $\mathrm{Hom}_R(R^1, R) = R$. By Proposition 10.1, Q^Y regarded as an R-module, is the unique maximal Y-rational extension of R. If S is any ring of Y-fractions of R, then S is a Y-rational extension of R. By Theorem 7.15, there exists an R-isomorphism φ (which is unique since Q^Y is a Y-rational extension of R) of S into Q^Y which leaves every element of R invariant. We shall show that φ is a ring isomorphism. Suppose the contrary, $\varphi(st) - \varphi(s)\varphi(t) \neq 0$ for some s and t in S. Since S is a Y-rational extension of R, there exists J in Y such that $tJ \subset R$ and $[\varphi(st) - \varphi(s)\varphi(t)]J \neq 0$. But for each j in J, $[\varphi(st) - \varphi(s)\varphi(t)]j = \varphi(st)j - \varphi(s)\varphi(t)j$ $= \varphi(st)j - \varphi(s)\varphi(tj) = \varphi(stj) - \varphi(s)tj = \varphi(stj) - \varphi(stj) = 0$.

This contradicts the choice of J. Hence φ is a ring isomorphism over R of S into Q^Y.

Since every ring of Y-fractions of R is a Findlay and Lambek's ring of right quotients of R, by Theorem 11.2, the unique maximal ring Q^Y of Y-fractions of R (and hecne every ring of Y-fractions of R) can be regarded as a subring of the maximal Findlay and Lambek's ring Q of right quotients of R. If Y and W are D-systems of right ideals of R^1 such that $Y \subset W$, then we may suppose $Q^Y \subset Q^W$. It might happen that $Q^Y = Q^W$ although $Y \neq W$.

Now let us consider the case where R is left faithful. The mapping $r \to r_L$, where r_L is the left multiplication of R by r, is an isomorphism of the ring R into $\mathrm{Hom}_R(R,R)$. We may identify R with the isomorphic image in $\mathrm{Hom}_R(R,R)$. If Y is a D-system of right ideals of R^1, then $R < R^1$ (R;Y) and hence $R < R$ (R;Y). Thus $\mathfrak{R}^Y_R(R)$ is nonempty. In fact $\mathfrak{R}^Y_R(R)$ consists of all A in $\mathfrak{R}^Y_R(R^1)$ that are contained in R. By Proposition 8.6, the additive group $F(\mathfrak{R}^Y_R(R),R)$ is isomorphic to Q^Y. It is not difficult to see that they are isomorphic as rings.

11.3. Theorem. Assume R is left faithful. Let the R-module M be a Y-rational extension of R. Then the rings $F(\mathfrak{R}^Y_M(M),M)$ and $F(\mathfrak{R}^Y_R(R),R)$ are isomorphic and are maximal rings of Y-fractions of R.

Proof. It remains to show that the ring $F(\mathfrak{R}^Y_M(M),M)$ and $F(\mathfrak{R}^Y_R(R),R)$ are isomorphic. Since $\mathfrak{R}^Y_R(R)$ consists of all A in $\mathfrak{R}^Y_M(M)$ that are contained in R, by Proposition 8.6, the additive groups (and hence the rings) $F(\mathfrak{R}^Y_M(M),R)$ and $F(\mathfrak{R}^Y_R(R),R)$ are isomorphic. Furthermore since M is a Y-rational extension of R, $\varphi^{-1}R$ belongs to $\mathfrak{R}^Y_M(M)$ for every φ in $\mathfrak{F}(\mathfrak{R}^Y_M(M),M)$. By Proposition 8.7, the groups (and hence the rings) $F(\mathfrak{R}^Y_M(M),M)$ and $F(\mathfrak{R}^Y_M(M),R)$ are isomorphic.

It is evident that every Utumi's right quotient ring of the left faithful ring R is a ring of $L(R^1)$-fractions of R. The converse is also true as can be seen from the following proposition.

11.4. Proposition. Let R be a left faithful ring and let Y be a D-system of right ideals of R^1. Then every ring S of Y-fractions of R is also a Utumi's right quotient ring of R.

Proof. Let s and t be two elements of S with $t \neq 0$. Then by Lemma 11.1, there exists J in Y such that $sJ, tJ \subset R$ and $tJ \neq 0$. Suppose $j \in J$ such that $0 \neq tj \in R$. Then since R is left faithful $tjR \neq 0$. There exists r in R such that $tjr \neq 0$. Now $jr \in R$, $s(jr) \in R$ and $t(jr) \neq 0$. Thus S is a Utumi's right quotient ring of R.

It follows from the above proposition that the maximal Findlay and Lambek's ring Q of right quotients of R is the maximal Utumi's right quotient ring of R.

If R is a ring with an identity 1, then there is an alternative way to look at the maximal ring Q^Y of Y-fractions of R which is, in essence, due to Lambek [26]. Let $\hat{R}^Y$ be the Y-injective envelope of R and let H be the ring of all R-endomorphisms of $\hat{R}^Y$. Then $\hat{R}^Y$ becomes a (H,R)-module if we write the R-endomorphisms of $\hat{R}^Y$ on the left of their arguments. The ring of all H-endomorphisms of $\hat{R}^Y$ will be designated by $\overline{R}$. We write the H-endomorphisms of $\hat{R}^Y$ on the right of their arguments and obtain a $(H,\overline{R})$-module $\hat{R}^Y$. By an argument similar to that given in [26], we can prove the following lemma.

11.5. Lemma. (1) The ring R is isomorphic to a subring of $\overline{R}$ under the mapping $\varphi : r \to r_R$, where r_R is the right multiplication of $\hat{R}^Y$ by r.

(2) $\overset{\Delta Y}{R}$ is a H-homomorphic image of H under the canonical mapping $\mu : h \to hl$.

(3) $\lambda : q \to lq$ is an R-isomorphism of $\overline{R}$ into $\overset{\Delta Y}{R}$. Moreover $\lambda\overline{R} = \cap \{ \ker \omega \mid \omega \in H \text{ and } \omega R = 0 \}$.

11.6. Theorem. Let R be a ring with an identity and let Y be a D-system of right ideals of R^1. Then $\overline{R} = \mathrm{Hom}_H(\overset{\Delta Y}{R}, \overset{\Delta Y}{R})$ is a maximal ring of Y-fractions of R, where $H = \mathrm{Hom}_R(\overset{\Delta Y}{R}, \overset{\Delta Y}{R})$.

Proof. By Theorem 7.15 and Lemma 11.5(3).

Let X be a D-system of right ideals of R. Then the family $X^{\#}$ of all right ideals of R that contain some members of X is a D-system of right ideals of R^1. A ring of $X^{\#}$-fractions (resp. $X^{\#}$-quotients) of R will be called simply a ring of X-fractions (resp. X-quotients) of R. If R has zero X-singular ideal, then every ring of X-quotients of R is also a ring of X-fractions. Hence there is no need to distinguish rings of X-quotients and rings of X-fractions. We proved in Theorem 11.3 that $F(\mathfrak{R}^X_{\overset{\Delta X}{R}}(\overset{\Delta X}{R}), \overset{\Delta X}{R})$ is a maximal ring of X-fractions of R. Under the assumption that R has zero X-singular ideal, $\mathfrak{R}^X_{\overset{\Delta X}{R}}(\overset{\Delta X}{R}) = \mathfrak{E}^X(\overset{\Delta X}{R})$, the family of all X-essential submodules of the R-module $\overset{\Delta X}{R}$. Furthermore, since $\overset{\Delta X}{R}$ is X-injective $F(\mathfrak{E}^X(\overset{\Delta X}{R}), \overset{\Delta X}{R}) \cong \mathrm{Hom}_R(\overset{\Delta X}{R}, \overset{\Delta X}{R})$. Hence

11.7. Theorem. Let X be a D-system of right ideals of R such that R has zero X-singular ideal. Then the rings $F(\mathfrak{E}^X(R), R)$ and $\mathrm{Hom}_R(\overset{\Delta X}{R}, \overset{\Delta X}{R})$ are isomorphic and are maximal rings of X-quotients of R.

If X is a D'-system of right ideals of R, then R has zero X-singular ideal and $X = \mathfrak{E}^X(R)$. It follows from the above theorem

11.8. Corollary. Let X be a D'-system of right ideals of R. Then $F(X, R)$ is the unique maximal ring of X-quotients of R.

Consider now the lattice $\Delta'(R)$ of all D'-systems of right ideals of the left faithful ring R. For each X in $\Delta'(R)$, there corresponds a ring F(X,R) which may be regarded as the ring of all congruence classes of X-full R-homomorphisms of R into itself or as the ring of all irreducible X-full R-homomorphisms of R into itself. In the following discussions we shall adopt the second alternative. Recall that the operations in F(X,R) are defined by

$$\alpha + \beta = (\alpha+\beta)^* \quad \text{and} \quad \alpha\beta = (\alpha\beta)^*$$

where $(\alpha+\beta)^*$ and $(\alpha\beta)^*$ are unique irreducible extensions of $\alpha+\beta$ and $\alpha\beta$ respectively.

It has been shown in Section 2 that the lattice $\Delta'(R)$ has the Utumi's D-system U as its unique maximal element. Utumi [36] has shown that F(U,R) is the unique maximal Utumi's right quotient ring of R. Each subring of F(U,R) that contains R is a Utumi's right quotient ring of R and every Utumi's right quotient ring of R can be isomorphically imbedded in F(U,R). We may regard those subrings of F(U,R) that contain R as the only Utumi's right quotient rings of R. The family of all such subrings form a complete modular lattice.

Now if X and W are two D'-systems of right ideals of R such that $X \subset W$, then evidently F(X,R) is a subset of F(W,R). Moreover the operations in F(X,R) agree with the corresponding operations in F(W,R). Thus F(X,R) is a subring of F(W,R). In particular F(X,R) is a subring of the maximal Utumi's right quotient ring of R.

Note that two D'-systems of right ideals of R may give rise to the same subring of F(U,R). Given X in $\Delta'(R)$, let X' be the intersection of all W in $\Delta'(R)$ such that F(W,R) = F(X,R). Then by the following lemma, X' belongs to $\Delta'(R)$ and F(X',R) = F(X,R).

11.9. Lemma. Let $\{X_i\}$ be a nonempty subclass of $\Delta'(R)$ and let X be the intersection of X_i's. Then $F(X,R) = \cap F(X_i,R)$.

Proof. By the above remark $F(X,R) \subset \cap F(X_i,R)$. To show the reverse inclusion, let α be an irreducible X_i-full R-homomorphism of R into itself for each i. Then dom α belongs to every X_i and hence to X, i.e., α belongs to $F(X,R)$.

For each Utumi's right quotient ring T of R, we shall denote by X_T the D-system generated by the set of all domains of irreducible semi R-homomorphisms of R into itself contained in T. Then X_T belongs to $\Delta'(R)$ and $F(X_T,R) \supset T$.

Consider now the following four mappings:

$$\delta : X \to F(X,R); \qquad i : X \to X';$$

$$\lambda : T \to X_T; \qquad c : T \to F(X_T,R)$$

where X is a D'-system of right ideals of R and T is a Utumi's right quotient ring of R. It is clear that $\delta\lambda = c$. Since $F(X',R) = F(X,R)$, $\delta i = \delta$. Moreover $\lambda\delta = i$ which follows from the following lemma.

11.10. Lemma. $X' = X_{F(X,R)}$ for every X in $\Delta'(R)$.

Proof. Let $X_{F(X,R)} = W$. Since $F(X',R) = F(X,R)$, the domains of irreducible X-full R-homomorphisms of R into itself are contained in X' and so $W \subset X'$. It follows that $F(W,R) \subset F(X',R)$. On the other hand $F(W,R) \supset F(X,R) \supset F(X',R)$. Thus $F(W,R) = F(X,R)$ and so $W \supset X'$. Hence $W = X'$.

Now $i^2 = (\lambda\delta)i = \lambda(\delta i) = \lambda\delta = i$ and $c^2 = (\delta\lambda)(\delta\lambda) = \delta(\lambda\delta)\lambda = \delta i\lambda = \delta\lambda = c$. Since δ and λ are order preserving, i and c are order preserving. This proves the following

11.11. Proposition. (1) Corresponding to each X in $\Delta'(R)$ is a unique minimal D'-system X' such that $F(X,R) = F(X',R)$. X' can also be defined as the D*system generated by the set of domains of all irreducible X-full R-homomorphisms of R into itself.

(2) $c : T \to F(X_T,R)$ is a closure operation on the lattice of all Utumi's right quotient rings of R.

(3) $i : X \to X'$ is an anti-closure operation on the lattice $\Delta'(R)$ of all D'-systems of all right ideals of R in the sense that it satisfies C2, C3 and

C11. $i(X) \subset X$ for each X in $\Delta'(R)$.

An element in $\Delta'(R)$ which is invariant under i is called an invariant D'-system of right ideals of R. A Utumi's right quotient ring of R which is invariant under c is called an invariant Utumi's right quotient ring of R. Note that the invariant Utumi's right quotient rings of R are precisely those $F(X,R)$, $X \in \Delta'(R)$. The set of all invariant D'-systems of right ideals of R (resp. all invariant Utumi's right quotient rings of R) when ordered by inclusion, becomes a complete lattice which is not in general a sublattice of the lattice $\Delta'(R)$ (resp. the lattice of all subrings of $F(U,R)$).

11.12. Lemma. Let X and W be two invariant D'-systems of right ideals of R. Then $F(X,R) \subset F(W,R)$ if and only if $X \subset W$.

Proof. The "if" part is clear. Suppose $F(X,R) \subset F(W,R)$. Then the set of domains of all irreducible X-full R-homomorphisms of R into itself is a subset of the set of domains of all irreducible W-full R-homomorphisms of R into itself. Thus $X = X' \subset W' = W$.

11.13. Proposition. The lattice of all invariant D'-systems of right ideals of R is isomorphic to the lattice of all invariant Utumi's right quotient rings of R under the correspondence

$$X \to F(X,R).$$

By an argument similar to that given by Utumi [36], we can prove the following two propositions

11.14. Proposition. Assume that $\{R_i\}$ is a nonempty family of left faithful rings and for each i, X_i is a D'-system of right ideals of R_i. Let $\oplus X_i$ be the D'-system of right ideals of the discrete direct sum $\sum \oplus R_i$ defined in Proposition 2.12. Then $F(\oplus X_i, \sum \oplus R_i)$ is isomorphic to the complete direct sum $\prod F(X_i, R_i)$. If, in addition, each X_i is the Utumi's D-system of right ideals of R_i, then $F(\oplus X_i, \sum \oplus R_i)$ is the unique maximal Utumi's right quotient ring of $\sum \oplus R_i$.

11.15. Proposition. Let R be a ring with an identity, let X be a D'-system of right ideals of R and let $[X]_n$ be the D'-system of right ideals of $[R]_n$ as defined in Proposition 2.15. Then $F([X]_n, [R]_n)$ is isomorphic to the matrix ring $[F(X,R)]_n$. If, in addition, X is the Utumi's D-system of right ideals of R, then $F([X]_n, [R]_n)$ is the unique maximal Utumi's right quotient ring of $[R]_n$.

Now let X be a D'-system of right ideals of R and let Q be the overring F(X,R) of R. It is evident that R is an x-dense submodule of the R-module Q. By Proposition 1.10, the lattice $L^x(R)$ of all x-closed right ideals of R is isomorphic to the lattice $L^x(Q)$ of all x-closed submodules of the R-module Q under the correspondence $A \to A^{x(Q)}$, where

$$A^{x(Q)} = \{ \alpha \mid \alpha \in Q \text{ and } \alpha J \subset A \text{ for some } J \text{ in } X \}.$$

Observe that $A^{x(Q)} = F(X,A)$ and the latter is clearly a right ideal of Q. Hence every x-closed submodule of the R-module Q is a right ideal of Q. The ideal structure of quotient rings of R will be discussed in the next section. To conclude this section we mention a few special cases.

In case $L^X(R)$ is complemented, X satisfies the condition D4 by Theorem 4.2 and Lemma 9.1. In fact X is the Johnson's D-system of right ideals of R and Q is the unique maximal Johnson's right quotient ring of R. Thus Q is a regular ring and hence semi-simple in Jacobson's sense. Each right ideal of Q belonging to $L^X(Q)$ is a principal right ideal.

If X is an Asano's D-system of right ideals of R corresponding to a f-set G of regular elements of R, then for each x-closed right ideal A of R,

$$F(X,A) = \{ ag^{-1} \mid a \in A \text{ and } g \in G \}.$$

(See [1].) Let I be any right ideal of Q, If α belongs to $I^{x(Q)}$, then $\alpha A \subset I$ for some A in X. Suppose g is an element of $A \cap G$. Then $\alpha g \in I$. Since I is a right ideal of Q, $\alpha \in I$. Thus I is x-closed. Hence every right ideal of Q is x-closed. Consequently, the mapping $A \to F(X,A)$ defines a one-to-one correspondence between the lattice $L^X(R)$ of all x-closed right ideals of R and the lattice $L_r(Q)$ of all right ideals of Q with inverse $J \to J \cap R$.

It has been proved (see Corollary 5.14) that if X is a D'D*-system of right ideals of R such that $L^X(R)$ satisfies the ascending chain condition, then X is an Asano's D-system corresponding to the set $G_X(R)$ of regular elements of R. In this case, $L^X(Q)$ satisfies the ascending chain condition and consists of all right ideals of Q. If, in addition, $L^X(R)$ is complemented, then Q is

a semi-simple ring and $L^x(Q)$ is complemented. Hence Q satisfies descending chain condition on right ideals. By Wedderburn-Artin theorem, Q is a direct sum of a finite number of ideals each of which is isomorphic to the ring of linear transformations of a finite dimensional vector space over a division ring. In particular if R is a prime ring, then so is Q. It follows that Q is isomorphic to a ring of linear transformations of a finite dimensional vector space over a division ring. (See [11] and [12].)

12. Ideal structure. Prime ideals and primary decomposition of ideals in general rings have been studied by N. H. McCoy [30], D. C. Murdoch [31] and others. Since we are dealing with right ideals of a ring, generalizations of the notions of prime and primary ideals to the one-sided case seem called for. A right ideal P of R is said to be prime if $IJ \subset P$ implies either $I \subset P$ or $J \subset P$, where I and J are right ideals of R. A set M of elements of R is called an m-system if for any two elements m and n of M, $mRn \cap M$ is nonempty. The empty set is defined to be an m-system too.

12.1. Lemma. A right ideal P of R is prime if and only if its (set theoretical) complement $\sim P$ in R is an m-system.

Proof. Suppose P is a prime right ideal and let a and b be two elements of R not in P. If $aRb \subset P$, then $(a)R(b) \subset P$ and so either $(a) \subset P$ or $(b) \subset P$. This contradicts the choice of a and b. Thus $aRb \not\subset P$ and so $\sim P$ is an m-system. Conversely suppose that $\sim P$ is an m-system and let I and J be two right ideals of R such that $IJ \subset P$. If $I \not\subset P$ and $J \not\subset P$, then there exist $a \in I$ and $b \in J$ such that $a, b \notin P$. Now $aRb \subset IJ \subset P$. This contradicts the fact that $\sim P$ is an m-system. Hence $I \subset P$ or $J \subset P$ and P is a prime right ideal.

For any right ideal A of R we define the radical of A, denoted by rad A, to be the set of all elements r of R such that $M \cap A \neq \emptyset$ for every m-system M containing r. A prime right ideal P is a minimal prime divisor of A if $A \subset P$ and there exists no prime right ideal P' of R such that $A \subset P' \subsetneq P$. By an analogous argument given in the proof of Theorem 2 in [30], we can show that

12.2. Proposition. The radical of A is the intersection of all minimal prime divisors of A.

12.3. Corollary. rad A is a right ideal of R.

A right ideal Q of R is said to be primary if $IJ \subset Q$ implies $I \subset Q$ or $J \subset$ rad Q, where I and J are right ideals of R. It is easily seen that Q is primary if and only if $aRb \subset Q$ implies either $a \in Q$ or $b \in$ rad Q, where a and b are elements of R. In an exactly similar manner, we can prove the following analogue of Theorems 10 and 11 of [31].

12.4. Proposition. Assume R satisfies the ascending chain condition on right ideals. Then

(1) if $P_1, \ldots, P_n$ are minimal prime divisors of a right ideal A, then $P_{i_1} R P_{i_2} R \cdots R P_{i_m} \subset A$, where $i_1, \ldots, i_m$ is some finite permutation of the integers 1, .., n with repetitions allows;

(2) the radical P of a primary right ideal Q is prime.

[In this case, Q is said to be P-primary.]

Let M be any m-system. Associated with every ideal A of R are the upper M-component $u_M(A)$ and the lower M-component $l_M(A)$ of A. (See [31], [32] and [3].) We can extend the definitions of upper and lower M-components word for word, to the case of right ideals. Let A be a right ideal of R. If $M = \emptyset$, then we define the upper and lower M-components of A to be A itself. If $M \neq \emptyset$, the upper M-component $u_M(A)$ of A is defined to be the intersection of all right ideals B of R having the following properties:

(1) $B \supset A$; and

(2) if $rRm \subset B$ with r in R and m in M, then $r \in B$.

The lower M-component $l_M(A)$ of A is defined to be the right ideal of R consisting of all elements r of R such that $rRm \subset A$ for some m in M.

12.5. Proposition. Let M be an m-system of the ring R. Then

(1) u_M is a closure operation on L(R); and

(2) $\iota_M(A) \subset u_M(A)$ for every right ideal A of R, the equality holds if R satisfies the ascending chain condition on right ideals. (See [32], Theorem 12.)

If P is a prime right ideal of R, then $\sim P$ is an m-system. We shall call upper $\sim P$-component (resp. lower $\sim P$-component) of a right ideal A simply upper (resp. lower) P-components and denote by $u_P(A)$ (resp. $\iota_P(A)$).

Given a right ideal A of R, Lesieure and Croisot [27] have defined the tertiary radical of A to be the ideal t(A) consisting of all elements r of R such that in every principal right ideal generated by an element not in A, there exists an element $b \notin A$ for which $bR^1r \subset A$. A right ideal T is called tertiary if $aRb \subset T$ implies either $a \in T$ or $b \in t(T)$. It is worth mentioning that T is tertiary if and only if $AB \subset T$ implies either $A \subset T$ or $B \subset t(T)$, where A and B are two right ideals of R. In case R satisfies the ascending chain condition on right ideals, the tertiary radical P of a tertiary right ideal T is prime, (T is said to be P-tertiary) and every right ideal A of R can be represented as an irredundant intersection $A = T_1 \cap \cdots \cap T_n$ of a finite number of tertiary right ideals T_i's of R such that $t(T_i) \neq t(T_j)$ if $i \neq j$; $t(T_i)$'s are called associated primes of A.

Consider now a ring R which possesses a nonempty m-system M ($0 \notin M$) satisfying the following conditions:

(0)' For each m in M and each r in R there exist m' in M and r' in rR such that $mr' = rm'$.

(0)" For each m in M there exists n in M such that $mR \supset Rn$.

12.6. Proposition. Let A be a right ideal of R. Then the upper M-component $u_M(A)$ and the lower M-component $\ell_M(A)$ of A coincide with the right ideal

$$A^c = \{ r \mid r \in R \text{ and } rm \in A \text{ for some } m \text{ in } M \}.$$

[We shall call $u_M(A) = \ell_M(A)$ simply M-component of A.]

Proof. Suppose r is an element of A^c. Then $rm \in A$ for some m in M. By (0)", there exists n in M such that $mR \supset Rn$. Now $rRn \subset rmR \subset AR \subset A$, i.e., r belongs to $\ell_M(A)$. This shows that $A^c \subset \ell_M(A)$. Conversely let r be an element of $\ell_M(A)$. Then $rRm \subset A$ and hence $rmRm \subset A$ for some m in M. Since M is an m-system $r \in A^c$. Thus $A^c = \ell_M(A)$ and in particular, A^c is a right ideal of R. To complete the proof of the proposition, it remains to show that A^c has the property: If $rRm \subset A^c$ for some r in R and m in M, then $r \in A^c$. Since M is an m-system, there exists x in R such that $m' = mxm \in M$. Now $rm' = rmxm \in rRm \subset A^c$. There is an element n in M such that $rm'n \in A$. By (0)", $nR \supset Rn'$ for some n' in M. Thus $rm'Rn' \subset rm'nR \subset AR \subset A$. Since $m'Rn' \cap M \neq \emptyset$, $r \in A^c$.

12.7. Proposition. The mapping $A \to A^c$ is a modular closure operation on L(R).

Proof. Since $c = u_M$ and u_M is a closure operation on L(R), c is a closure operation which satisfies C4. To prove C5, it is enough to show that $(AR)^c \supset A^c$ for any right ideal A of R. Let r be an element of A^c. Then $rm \in A$ for some $m \in M$ and so $rmRm \subset AR$. Since $mRm \cap M \neq \emptyset$, $r \in (AR)^c$.

By Proposition 2.1, the family X of all c-dense right ideals of R forms a D-system of right ideals of R. The following proposition gives another characterization of c-dense right ideals of R.

12.8. Proposition. A right ideal A of R is c-dense if and only if $A \cap M \neq \emptyset$.

Proof. Suppose A is c-dense. Let m be any element of M. Then there exists n in M such that $mRn \subset A$. Now $A \cap M \supset mRn \cap M \neq \emptyset$. Conversely if A contains some element m of M, then by $(0)'$ for each r in R there exists r' in rR and m' in M such that $rm' = mr'$ which is in A. Thus r belongs to A^c. This hsows that A is c-dense.

12.9. Proposition. If Q is a primary (resp. tertiary) right ideal of R such that rad Q (resp. t(Q)) is disjoint from M, then Q is c-closed.

Proof. If $rRm \subset Q$ for some r in R and m in M, then since Q is primary (resp. tertiary) and $m \notin$ rad Q (resp. $m \notin t(Q)$), $r \in Q$. Thus Q is c-closed.

12.10. Corollary. A proper prime right ideal P of R is c-closed if and only if P is disjoint from M.

Proof. Follows from Propositions 12.8 and 12.9 and the fact that rad P = P.

12.11. Corollary. Assume that R satisfies the ascending chain condition on right ideals. Then

(1) a proper primary right ideal Q of R is c-closed if and only if Q is disjoint from M; and

(2) if $A = Q_1 \cap \cdots \cap Q_k$ is a primary decomposition of a right ideal A of R such that $M \cap \text{rad } Q_i = \emptyset$ for $i = 1, \ldots, j$ and $M \cap \text{rad} Q_i \neq \emptyset$ for $i = j+1, \ldots, k$, then $A^c = Q_1 \cap \cdots \cap Q_j$.

Proof. If Q is c-closed, then by Proposition 12.8, $Q \cap M = \emptyset$. Conversely if $Q \cap M = \emptyset$, then since R satisfies the ascending chain condition on right ideals and Q is primary, P = rad Q is

prime and so is the only minimal prime divisor of Q. By Proposition 12.4, $PRP^{\circ\circ}RP \subset Q$. Thus $P \cap M = \emptyset$. By Proposition 12.9, Q is c-closed. (2) follows easily from (1) and Proposition 12.8.

Now the M-component O^c of O is an ideal of R. Let λ be the canonical homomorphism of R onto the difference ring R/O^c. We shall show that λm is a regular element of R/O^c for each m in M. If $(\lambda x)(\lambda m) = 0$ for some x in R, then $\lambda(xm) = 0$ and so $xm \in O^c$. Thus $x \in O^{cc} = O^c$ and $\lambda x = 0$. This shows that λm is left regular. Next if $(\lambda m)(\lambda x) = 0$ for some x in R, then $mx \in O^c$ and $mxn = 0$ for some n in M. By $(0)''$, there exists n' in M such that $nR \supset Rn'$. Moreover by $(0)'$, there exists x' in xR and m' in M such that $xm' = mx'$. Now $xm'n' = mx'n' \in mxRn' \subset mxnR = OR = O$, i.e., $xm'n' = 0$ and $x \in O^{cc} = O^c$. Hence λm is also right regular.

12.12. Proposition. In the difference ring R/O^c the canonical images of elements of M are regular and form an m-system satisfying conditions $(0)'$ and $(0)''$.

In view of the above proposition, we may suppose for the time being the elements in M are regular in R and return to the general case later. By Proposition 12.8, the family X of all c-dense right ideals of R is a D'-system. We shall denote the overring $F(X,R)$ by $\hat{R}$. (Recall that R becomes a subring of $F(X,R)$ when we identify r with $\overline{r_L}$.) Let m be an element of M. Then since $mR \supset mRm$, mR contains some element of M and so belongs to X. Since m is regular, the mapping $m^* : mx \to x$ is a well-defined R-homomorphism of mR onto R. It is easily seen that $m^* m_L$ and $m_L m^*$ are identity maps on R and mR respectively. Hence $\overline{m^*} \circ m = m \circ \overline{m^*} = 1$ and $\overline{m^*} = m^{-1}$.

Now suppose $\overline{\alpha}$ is an element of $\hat{R}$. Then dom α contains some element m of M and α coincides with $(\alpha m)_L m^*$ on $mR \in X$. It follows that $\overline{\alpha} = \overline{(\alpha m)_L} \cdot \overline{m^*} = (\alpha m)m^{-1}$. Hence every element of $\hat{R}$ is of the form rm^{-1} where r in R and m in M. The converse is clear. This shows that the ring $\hat{R}$ consists precisely of elements of the form rm^{-1}, r in R and m in M.

12.13. Lemma. If $m_1, \cdots, m_i$ are elements of M, then there exist m in M and $r_1, \cdots, r_i$ in R such that $m = m_1 r_1 = \cdots = m_i r_i$.

Proof. Since $m_j R \in X$ for $j = 1, \cdots, i$, $\bigcap_{j=1}^{i} m_j R$ belongs to X. There exists m in M such that m belongs to every $m_j R$.

If $\alpha_1 m_1^{-1}, \cdots, \alpha_i m_i^{-1}$ are elements of R, then by the above lemma, there exist m in M and $r_1, \cdots, r_i$ in R such that $m_j^{-1} = r_j m^{-1}$. Thus $\alpha_1 m_1^{-1} + \cdots + \alpha_i m_i^{-1} = (\alpha_1 r_1 + \cdots + \alpha_i r_i)m^{-1}$.

12.14. Corollary. If A is a right ideal of R, then

$$AR = \left\{ am^{-1} \mid a \text{ in } A \text{ and } m \text{ in } M \right\}.$$

12.15. Lemma. If rm^{-1} and sn^{-1} are any two elements of $\hat{R}$, then there exist x in R and w in M such that $(rm^{-1})(sn^{-1}) = (rsx)w^{-1}$.

Proof. By (O)', there exist sy in sR and n' in M such that $msy = sn'$. It follows that $(rm^{-1})(sn^{-1}) = rsyn'^{-1}n^{-1}$. But since $n'^{-1}n^{-1}$ is an element of R, $n'^{-1}n^{-1} = x'w^{-1}$ for some x' in R and w in M. If we write $x = yx'$, then $(rm^{-1})(sn^{-1}) = (rsx)w^{-1}$.

12.16. Corollary. If A and B are two right ideals of R, then $(AB)\hat{R} = (A\hat{R})(B\hat{R})$.

Now let A be a c-closed right ideal of R. Then F(X,A) is a right ideal of $\hat{R}$. By a similar argument as given above we can show that F(X,A) consists precisely of elements of the form am^{-1}, where

$a \in A$ and $m \in M$. Thus by Corollary 12.14, $F(X,\Lambda) = A\hat{R}$ and so $A \to A\hat{R}$ is an isomorphism of the lattice $L^c(R)$ of all c-closed right ideals of R onto the lattice of all right ideals of $\hat{R}$ with inverse $J \to J \cap R$. (See the final portion of Section 11.) For any right ideal A of R, $A^c = R \cap A\hat{R}$ and for every right ideal J of $\hat{R}$, $J = (R \cap J)\hat{R}$.

12.17. Theorem. The mapping $A \to A\hat{R}$ induces a one-to-one correspondence between the prime right ideals of R which do not meet M and the proper prime right ideals of $\hat{R}$.

Proof. Let P be a prime right ideal of R which does not meet M. Then by Corollary 12.10, P is c-closed. If I and J are right ideals of $\hat{R}$ such that $IJ \subset P\hat{R}$, then $(R \cap I)(R \cap J) \subset IJ \cap R \subset R \cap (P\hat{R}) = P^c = P$. Since P is prime, $R \cap I \subset P$ or $R \cap J \subset P$. Thus either $I = (R \cap I)\hat{R} \subset P\hat{R}$ or $J \subset P\hat{R}$. Hence $P\hat{R}$ is a prime right ideal of $\hat{R}$. On the other hand if P' is a proper prime right ideal of $\hat{R}$, then since $P' = (R \cap P')\hat{R}$, $R \cap P'$ does not meet M. If A and B are right ideals of R such that $AB \subset R \cap P'$, then by Corollary 12.16, $(A\hat{R})(B\hat{R}) = (AB)\hat{R} \subset (R \cap P')\hat{R} = P'$. Since P' is prime, $A\hat{R} \subset P'$ or $B\hat{R} \subset P'$. Thus either $A \subset R \cap A\hat{R} \subset R \cap P'$ or $B \subset R \cap P'$. This shows that $R \cap P'$ is a prime right ideal of R not meeting M.

12.18. Theorem. Assume that R satisfies the ascending chain condition on right ideals. Let P be a prime right ideal of R not meeting M and let $P' = P\hat{R}$. Then $\hat{R}$ satisfies ascending chain condition on right ideals and the mapping $A \to A\hat{R}$ induces a one-to-one correspondence between the P-primary right ideals of R and the P'-primary right ideals of $\hat{R}$.

Proof. Let Q be a P-primary right ideal of R. Then since $P \cap M = \emptyset$, Q is c-closed by Proposition 12.9. If I and J are right

ideals of $\hat{R}$ such that $IJ \subset Q\hat{R}$, then $(R\cap I)(R\cap J) \subset R \cap IJ \subset R \cap (Q\hat{R})$ $= Q^c = Q$. Since Q is P-primary $R \cap I \subset Q$ or $R \cap J \subset P$. Thus either $I = (R \cap I)\hat{R} \subset Q\hat{R}$ or $J \subset P\hat{R} = P'$. By Theorem 12.17, P' is the unique prime divisor of QR and so rad $(Q\hat{R}) = P'$. This shows that QR is P'-primary. Conversely suppose Q' is a P'-primary right ideal of $\hat{R}$. Then $Q' \cap R$ is a c-closed right ideal of R with radical $P' \cap R$. If A and B are two right ideals of R such that $AB \subset R \cap Q'$, then $(A\hat{R})(B\hat{R}) = (AB)\hat{R} \subset (R \cap Q')\hat{R} = Q'$. Since Q' is P'primary, either $A\hat{R} \subset Q'$ or $B\hat{R} \subset P'$. Thus either $A \subset R \cap A\hat{R} \subset R \cap Q'$ or $B \subset R \cap P'$. Hence $R \cap Q'$ is $(R \cap P')$-primary.

12.19. Theorem. Assume that R satisfies the ascending chain condition on right ideals. If $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$, where Q_i are P_i-primary, $P_i \cap M = \emptyset$ for $i = 1, \ldots, m$ and $P_i \cap M \neq \emptyset$ for $i = m+1, \ldots, n$. Then

$$A\hat{R} = Q_1\hat{R} \cap Q_2\hat{R} \cap \cdots \cap Q_m\hat{R}$$

is a primary decomposition of $A\hat{R}$. Moreover if the decomposition of A is irredundant, then the decomposition of $A\hat{R}$ is also irredundant.

Proof. By Theorems 12.17 and 12.18, $P_i\hat{R}$ is a prime right ideal of R and $Q_i\hat{R}$ is $(P_i\hat{R})$-primary. Let $A' = Q_1\hat{R} \cap \cdots \cap Q_m\hat{R}$. Then $A' \cap R = Q_1 \cap \cdots \cap Q_m$. By Corollary 12.11, $R \cap A\hat{R} = A^c = Q_1 \cap \cdots \cap Q_m$ $= A' \cap R$. Hence $A\hat{R} = A' = Q_1 \cap \cdots \cap Q_m$. Suppose the decomposition of A is irredundant. Then the decomposition of AR must be irredundant. For otherwise if, for example, $Q_1\hat{R} \supset Q_2\hat{R} \cap \cdots \cap Q_m\hat{R}$, then $Q_1 = R \cap Q_1\hat{R} \supset Q_2 \cap \cdots \cap Q_m$ and this is a contradiction.

We return now to the general case where R has an m-system M ($0 \notin M$) satisfying the conditions (0)' and (0)". Elements in M are not necessarily regular elements of R. We shall assume

henceforth that R satisfies the ascending chain condition on right ideals. Note that the canonical homomorphism λ of R onto the difference ring $R/0^c$ induces a one-to-one correspondence $A \to A^\lambda$ of right ideals of R containing 0^c and the right ideals of $R/0^c$. It follows that the difference ring $R/0^c$ also satisfies the ascending chain condition on right ideals.

12.20. Lemma. Let Q be a primary right ideal of R disjoint from M. Then $0^c \subset Q$.

Proof. Let P be the radical of Q. Then by the proof of Corollary 12.11, P does not meet M. If $r \in 0^c$, then $rm = 0$ for some m in M. By (0)" there exists n in M such that $mR \supset Rn$. Now $rRn \subset rmR = 0 \subset Q$. Since Q is P-primary and $n \notin P$, $r \in Q$. Thus $0^c \subset Q$.

The proof of the following two propositions is straight forward and will be omitted.

12.21. Proposition. The mapping $A \to A^\lambda$ induces a one-to-one correspondence between the prime right ideals of R which do not meet M and the prime right ideals of $R/0^c$ which do not meet $M^\lambda = \{\lambda m \mid m \in M\}$.

12.22. Proposition. Let P be a prime right ideal of R which does not meet M. Then the mapping $A \to A^\lambda$ induces a one-to-one correspondence betweem the P-primary right ideals of R and the P^λ-primary right ideals of $R/0^c$.

By Proposition 12.12, elements in M^λ are regular and form an m-system satisfying conditions (0)' and (0)". We shall denote the overring $R^\lambda_{M^\lambda}$ of R^λ by R*. For every right ideal A of R, the right ideal $A^\lambda R^*$ will be designated by A*. Combining Theorems 12.17 -

12.19 and Propositions 12.21 - 12.22, we obtain the following theorem.

12.23. Theorem. Assume that R satisfies the ascending chain condition on right ideals and let M ($0 \notin M$) be an m-system of R satisfying the conditions (0)' and (0)". Then

(1) there exists a one-to-one correspondence between the prime right ideals of R which do not meet M and the proper prime right ideals of R* such that if P and P* correspond, then $P^* = P^{\lambda}R^*$ and $P = \lambda^{-1}(P^* \cap R^{\lambda})$;

(2) if P and P* are corresponding prime right ideals (as in (1)) of R and R*, then there exists a one-to-one correspondence between the P-primary right ideals of R and P*-primary right ideals of R* such that if Q and Q* correspond, then $Q^* = Q^{\lambda}R^*$ and $Q = \lambda^{-1}(Q^* \cap R^{\lambda})$; and

(3) corresponding to an (irredundant) primary decomposition $A = Q_1 \cap \cdots \cap Q_n$ of a right ideal A of R, where Q_i are P_i-primary, $P_i \cap M = \emptyset$ for $i = 1, .., m$ and $P_i \cap M \neq \emptyset$ for $i = m+1, .., n$, is the (irredundant) primary decomposition $A^* = Q_1^* \cap \cdots \cap Q_m^*$ of A*.

If R contains a prime right ideal P such that $M = \sim P$ satisfies conditions (0)' and (0)", then by the above theorem, the extension P* of P is a unique maximal prime right ideal of R*. This gives a partial solution to the noncommutative localization problem. We have to admit that this solution is far from being satisfactory because conditions (0)' and (0)" suggest high degree of commutativity.

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