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THESIS

FILTERS WITH MINIMUM AMPLIFIER SENSITIVITY

Submitted by

Dashamir M. Petrela

In partial fulfillment of the requirements

for the Degree of Doctor of Philosophy

Colorado State University

Fort Collins, Colorado

August, 1971

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COLORADO STATE UNIVERSITY

August, 1971

WE HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER OUR SUPERVISION BY DASHAMIR M. PETRELA ENTITLED FILTERS WITH MINIMUM AMPLIFIER SENSITIVITY BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

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ABSTRACT

FILTERS WITH MINIMUM AMPLIFIER SENSITIVITY

The use of silicon integrated amplifiers and tantalum thin film RC networks has made active filters competitive with passive LC filters at pole frequencies up to 100 kHz. However, the application of active filters at higher pole frequencies has been limited by both the frequency dependence and the temperature dependence of the open loop gain of silicon integrated amplifiers. As a result, active filters have serious frequency and temperature limitations.

This thesis considers n-th order single amplifier filters and shows that their frequency and temperature limitations are caused by the dependence of the filter poles on the finite, temperature-dependent gain-bandwidth product of the amplifier. A new pole sensitivity measure, M_p, is defined. This new sensitivity measure takes into account both the frequency and temperature dependence of the amplifier and thus leads to a quantitative figure of merit, $|M_p|$, for integrated active filters. The sensitivity measure, M_p, is shown to depend only on the leading coefficient and the zeros of a specific driving point admittance of the passive RC network. Expressions for practical lower bounds for $|M_p|$ and for the minimum gain-bandwidth product required to realize a given transfer function are determined.

The identification of the specific passive network parameters which affect the sensitivity measure, M_p, results in a new synthesis procedure for the realization of a given transfer function by an active filter with specified sensitivity properties (e.g. zero Q-sensitivity).

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Moreover, the use of the new sensitivity measure, M_p, and the expressions for the lower bounds for the figure of merit, $|M_p|$, lead directly to the modification of existing design procedures to achieve active filters with minimum $|M_p|$. These filters have minimum amplifier sensitivity. Finally, it is pointed out that the new sensitivity measure, M_p, and the figure of merit, $|M_p|$, may also be applied to multi-amplifier filters.

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CHAPTER I

INTRODUCTION

In 1955 Sallen and Key's landmark paper launched modern active filter synthesis (Sallen and Key 1955). In the same year Truxal (1955) popularized the concept of sensitivity (originally due to Bode (1945)). The two have been inseparable ever since.

An active filter may be defined (Huelsman 1968) as a circuit consisting of time invariant resistors and capacitors and some form of active element(s). The main reason for using an active filter is the desire to realize a voltage transfer function with complex conjugate poles (p and $\bar{p}$) without using inductors. For $p = -\omega_0/2Q+j\omega_0\sqrt{1-1/4Q^2}$ ($\omega_0 = 2\pi f_0$, $Q > \frac{1}{2}$), the Q versus f_0 spectrum presently realizable by active filters is shown in Fig. I-1, due to Moschytz (1970a). Note the frequencies covered only by active filters ($f_0 < 200$ Hz). Ten years ago active filters were used almost solely at these low frequencies and were, moreover, limited to Q's less than 100. Since that time active filters have evolved into a viable, and even preferred, alternative to LC filters for $f_0 < 100$ kHz, and to crystal filters for $Q < 50$. The main reasons for this rapid development have been the widespread application of sensitivity concepts to active filter design and the very rapid growth of silicon integrated circuit technology.

In the late 1950's and early 1960's, a great many active filter realizations were proposed in the literature. In practice, however, the effects of manufacturing tolerances on the specified element values (as well as the presence of parasitic elements) made the physical realization of many of these proposed filters extremely difficult.

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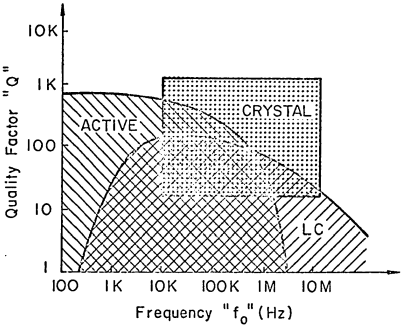

Fig. I-1

Filter Spectrum
(Moschytz 1970a)

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A measure of merit was needed to separate those proposed realizations that were practical to implement from those that were not. The classical sensitivity function popularized by Truxal filled this need, as researchers soon discovered that the poor practical performance of many designs could be accounted for by their relatively high ω_0 and Q sensitivities to the passive elements in the network. In 1959 Horowitz proposed a more direct sensitivity measure – namely, the pole sensitivity (Horowitz 1959) which in many instances was easier to calculate, and hence use, than the classical ω_0- and Q-sensitivities. The widespread application of both the classical and the pole sensitivity measures soon resulted in network realizations that were practical and could be built with little difficulty. However, these sensitivity measures had one serious limitation. No lower bounds on these sensitivity measures were known and the only standard of comparison was the classical LC filter. New schemes with lower sensitivities were being continually proposed as researchers kept searching for better filters.

This search, and the generally increased application of active filters were given tremendous impetus in the mid 1960's by the relatively sudden availability of low cost, off-the-shelf, high gain silicon integrated operational amplifiers. Soon research was concentrated on active filters that used the operational amplifier either directly or (with resistive feedback) as a positive or negative finite gain voltage amplifier. Moreover, multi-amplifier schemes were no longer prohibitively costly. As a result attention was focused on the realization of an overall n-th order transfer function by a cascade of second order building blocks (as discussed in Chapter IV), and also on the use of more than one amplifier to realize each second order

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building block. This resulted in network realizations — such as the Biquad (Tow 1968) — which have passive element sensitivities as good, or better, than those of LC filters and are, moreover, much more easily tuned. At about the same time tantalum thin film technology was applied to the realization of the passive RC network (Moschytz et al 1969; Moschytz 1967; Orr 1966). By 1970, hybrid-integrated filter building blocks, using silicon operational amplifiers for the active element and tantalum thin film realizations for the passive RC network had become a reality (Moschytz and Thelen 1970; Moschytz 1970b; Thomas 1971a,b). This made active filters competitive with LC filters at higher frequencies (and with crystal filters at lower Q's) with respect to the overall system cost and performance (Thomas 1971b). The result is the Q versus f_0 spectrum of Fig. I-1.

However, the application of integrated circuit technology has required two basic changes in the design considerations for active filters and, as a result, has raised several new problems.

The first change is due to the fact that the sensitivities of the poles to the passive elements no longer represent a valid figure of merit for an active filter. This is because in integrated circuit realizations the RC products that determine the ω_0 of the pole can be automatically trimmed to very close tolerances (Hintzman 1969) and held to these tolerances over wide temperature ranges. This is also true of the resistance and capacitance ratios that determine the pole Q. As a result, even very large passive element sensitivities can be tolerated. On the other hand, the sensitivity of the poles to the open loop gain A of the amplifier has become critical. This is because A is very temperature dependent — a 40% variation over a 50°C temperature range

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is quite common. As a result, some networks that were considered quite impractical (because of very poor sensitivities with respect to the passive elements) when discrete realizations had to be used, suddenly are not only feasible but even offer great advantages for integrated circuit realization (Moschytz 1971). Consequently, active network designers began to consider the pole sensitivity with respect to the amplifier open loop gain A as far back as 1966 (e.g. Gaash 1966) and have continued to use it as a figure of merit to the present time (e.g. Horowitz 1968; Moschytz 1971). However, in practice their efforts have met with relatively little success. Except at low frequencies, the classical ω_0- and Q-sensitivities (Truxal 1955) and the conventional pole sensitivity (Horowitz 1959) with respect to A have not given good quantitative results. It is shown in this thesis that this is so because these sensitivity functions cannot properly take into account the second major change caused by the use of integrated circuit realizations.

The second change in the design consideration for active filters is that the frequency limitations of the amplifier can no longer be ignored. The same integrated circuit technology that permitted the extension of active filters to higher frequencies by making them cost-competitive with LC filters, also places important frequency limitations on their use. This is because silicon integrated operational amplifiers are always strongly frequency compensated to ensure stability, and often have an open loop bandwidth of the order of only 20 Hz (Moschytz 1970a). This places severe frequency limitations on the active filter. These limitations are just beginning to be appreciated. Budak and Petrela (1969), Budak and Aronhime (1970), Tarmy and Ghausi

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(1970), and Thomas (1971a) have analyzed certain specific active filter networks (by assuming a one-pole frequency characteristic for the amplifier) and have shown that the dominant pole of the amplifier plays a major role in determining the filter pole shifts due to the finite open loop gain of the amplifier. Furthermore, Petrela and Budak (1971a,b) have shown that any sensitivity measure that is to be meaningful must include the effect of this amplifier pole, and that the frequency limitations and sensitivity considerations for active filters are closely interrelated.

As a result, there are at present five important questions to be answered. They are:

1. What are the frequency limitations of active filters?
2. What is a good sensitivity measure and figure of merit for integrated active filters?
3. How do the frequency limitations and sensitivity properties of the filter depend on the passive RC network and on the type of amplifier that are used?
4. What sensitivity properties with respect to the amplifier (e.g. zero Q-sensitivity) are achievable?
5. How can networks with desirable sensitivity properties actually be synthesized?

This thesis considers, and successfully answers, all five of these questions for active filters of the general form of Fig. I-2 where the amplifier K represents an operational amplifier, or a

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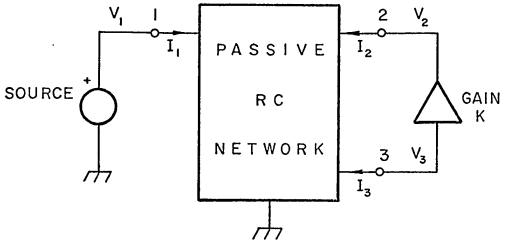

Fig. I-2

General Single Amplifier RC Network

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positive or negative finite gain voltage amplifier (obtained by using an operational amplifier with resistive feedback). The practical importance of such single amplifier filters is duscussed in Chapter II.

Chapter II also presents a critical review of the important properties of integrated circuit realizations, and of the methods used to answer question 1 for particular networks. Improvements in previously used methods are suggested, and the frequency limitations of the most commonly used single amplifier filters are presented.

Chapter III considers the general n-th order active filter of the form of Fig. I-2 and presents a new approach to the sensitivity problem. A new pole sensitivity measure is defined that takes into account both the frequency limitations and temperature variations of the open loop gain, and leads to a quantitative figure of merit for integrated active filters. The new pole sensitivity measure is related to the leading coefficient and zeros of a specific driving point admittance of the passive RC network. Expressions for practical lower bounds on the new figure of merit are also derived. Chapter III thus answers questions 1, 2 and 3 for the general n-th order active filter of the form of Fig. I-2. Chapter III also shows that question 4 can only be fully answered if the type of transfer function to be realized is specified.

Because of the widespread use of second order filters, Chapter IV applies the general results of Chapter III to the second order case. Some interesting and useful interpretations are presented and question 4 is answered for second order filters.

Chapter V considers the design of active filters with specified sensitivity properties. Section A gives a critical review of

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previous work in this area. Section B then presents a new synthesis procedure for realizing an active filter with prescribed sensitivity properties (e.g. zero Q-sensitivity), and thus answers question 5. Moreover, an additional section (C) is included to show how the new sensitivity measure of Chapter III may also be used to optimize existing filter networks, and to give specific design procedures for the optimization of the commonly used single amplifier filters.

Chapter VI summarizes the main achievements of this thesis and suggests future work in this area. In addition, Chapter VI also points out that the new sensitivity measure and general approach of Chapter III may be applied to multi-amplifier networks as well.

The International System of Units and standard IEEE recommended symbols, prefixes, and abbreviations are used throughout this work. References are listed under Literature Cited and a modified Cambridge system is used throughout. No general bibliography is given. Those readers who desire general background material are advised to consult the works of Newcomb (1968) and Mitra (1969a,b) for a review of the extensive literature on active filters.

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CHAPTER II

THE CRITICAL PARAMETER

The objective of this chapter is to provide the justification for the main viewpoint taken in this thesis — namely, that the effect of the finite gain-bandwidth product of the amplifier on the poles of the transfer function is the critical consideration in integrated active filters. This objective is achieved in three steps.

The first step is to review those properties of integrated active filters that cause the sensitivity of the poles of the transfer function to the open loop gain of the amplifier to be of critical importance. This is done in section A.

The second step is to demonstrate the importance of including the dominant pole of the amplifier in the analysis of active filters using silicon-integrated operational amplifiers. Section B does this by critically reviewing what has been done in the literature regarding the frequency limitations of specific active filter networks. In the course of this review certain generalizations are drawn and modifications of past procedures are suggested. Moreover, section B also presents the frequency limitations of many commonly used single amplifier networks — partly because they illustrate the discussion and partly because they should be of practical value to many practicing engineers.

The third and final step is accomplished in section C, which interrelates the major points brought out in sections A and B and proceeds to provide the justification for the approach taken in the remainder of this thesis.

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A. Sensitivity Characteristics of Integrated Filters

Consider an active filter consisting of "r" resistors, "c" capacitors, and "a" voltage amplifiers. Angelo (1955) has shown that the classical sensitivity function (Truxal 1955) of any network function F with respect to x; namely,

$$S_x^F = (x/F)(dF/dx) \tag{II-1}$$

can be used to characterize the network to finite as well as to differential changes in the parameter x. It follows that for a network with "m" parameters, x_ℓ, the total variation of any network function F is given by

$$(\Delta F/F) = \sum_{\ell=1,m} (\Delta x_\ell/x_\ell) S_{x_\ell}^F \tag{II-2}$$

Hence, for

$$\begin{aligned} p &= -\alpha + j\beta \\ &= -\omega_0/2Q + j\omega_0\sqrt{1-1/4Q^2} \end{aligned} \tag{II-3}$$

A pole (or zero) of the active filter,

$$(\Delta\omega_0/\omega_0) = \sum_{i=1,r} (\Delta R_i/R_i) S_{R_i}^{\omega_0} + \sum_{j=1,c} (\Delta C_j/C_j) S_{C_j}^{\omega_0} + \sum_{k=1,a} (\Delta A_k/A_k) S_{A_k}^{\omega_0} \tag{II-4}$$

and

$$(\Delta Q/Q) = \sum_{i=1,r} (\Delta R_i/R_i) S_{R_i}^{Q} + \sum_{j=1,c} (\Delta C_j/C_j) S_{C_j}^{Q} + \sum_{k=1,a} (\Delta A_k/A_k) S_{A_k}^{Q} \tag{II-5}$$

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For hybrid-integrated networks using tantalum thin-film resistors and capacitors (Berry et al 1968; Moschytz et al 1969) the resistors and capacitors track closely among themselves (e.g. ± 5 ppm/°C for resistors on a substrate; and ± 10 ppm/°C for capacitors on a substrate (Moschytz 1971)). With regard to temperature variations, therefore, it may be assumed that for all i=1,r

$$(\Delta R_i/R_i) = (\Delta R/R) \tag{II-6}$$

and for all j=1,c

$$(\Delta C_j/C_j) = (\Delta C/C) \tag{II-7}$$

Hence (II-4) and (II-5) become

$$(\Delta\omega_0/\omega_0) = (\Delta R/R)\sum_{i=1,r} S_{R_i}^{\omega_0} + (\Delta C/C)\sum_{j=1,c} S_{C_j}^{\omega_0} + \sum_{k=1,a} (\Delta A_k/A_k) S_{A_k}^{\omega_0} \tag{II-8}$$

and

$$(\Delta Q/Q) = (\Delta R/R)\sum_{i=1,r} S_{R_i}^{Q} + (\Delta C/C)\sum_{j=1,c} S_{C_j}^{Q} + \sum_{k=1,a} (\Delta A_k/A_k) S_{A_k}^{Q} \tag{II-9}$$

Moreover (because ω_0 has the dimension $(RC)^{-1}$) it can be shown that (Horowitz 1968; Belove 1966)

$$\sum_{i=1,r} S_{R_i}^{\omega_0} = \sum_{j=1,c} S_{C_j}^{\omega_0} = -1 \tag{II-10}$$

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Similarly (because Q is dimensionless) it follows that

$$\sum_{i=1,r} S^{Q}_{R_i} = \sum_{j=1,c} S^{Q}_{C_j} = 0 \qquad \text{(II-11)}$$

Substituting (II-10) and (II-11) into (II-8) and (II-9) yields

$$(\Delta\omega_0/\omega_0) = -\{(\Delta R/R) + (\Delta C/C)\} + \sum_{k=1,a} (\Delta A_k/A_k) S^{\omega_0}_{A_k} \qquad \text{(II-12)}$$

$$(\Delta Q/Q) = \sum_{k=1,a} (\Delta A_k/A_k) S^{Q}_{A_k} \qquad \text{(II-13)}$$

The first term in (II-12) can clearly be minimized by selecting the temperature coefficient α_r of the resistors and the temperature coefficient α_c of the capacitors such that $\alpha_r \simeq -\alpha_c$. Using tantalum thin-film networks, this can be achieved to better than 40 ppm/^{o}C (Parisi 1969). If this is done, then

$$\{(\Delta R/R) + (\Delta C/C)\} \simeq 0 \qquad \text{(II-14)}$$

Equations II-12 and II-13 then show that any temperature changes will affect the Q and the ω_0 of any pole p only by causing changes in the open loop gains A_k of the amplifiers. For silicon-integrated operational amplifiers these variations are generally quite high (e.g. Moschytz (1971) reports typical variations in the dc open loop gain of 20 to 40 per cent over a 50°C temperature range). Consequently, the sensitivities of p with respect to the open loop gains A_k of the amplifiers become the primary consideration as far as temperature variations are concerned.

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Equations II-4 and II-5 also apply to variations due to manufacturing tolerances. In hybrid-integrated networks, the capacitance and resistance ratios that determine Q can be held to close tolerances because of the nature of the manufacturing process (Berry 1968; Moschytz et al 1969). Rough adjustment of the RC time constants that determine ω_0 can be accomplished by scribing (or selective film removal) techniques. Trim anodization can then be used to provide a vernier adjustment of resistance values which may thus be increased up to 20% and set to an accuracy of 0.01%. This allows very accurate fine tuning of both the resistance ratios that affect the Q and the RC time constants that determine ω_0. Moreover, this procedure can be automated (Hintzman 1969). It is at present not possible to achieve such control over the dc open loop gain of the amplifiers. Typically, manufacturing tolerances of 20% or greater in the dc open loop gain must be accepted for high gain silicon-integrated operational amplifiers. As a result, (from equations II-2, 12, 13 and 14), the sensitivities of p with respect to the open loop gain of the amplifier must be the primary sensitivity consideration with regard to the dependence of the active filter on both manufacturing tolerances and temperature variations.

B. Frequency Limitations on Active Filters

The frequency limitations of active filters are just beginning to be appreciated. Budak and Petrela (1969) and Budak (1970) analyzed the Biquad network due to Tow (1968) by assuming a finite dc open loop gain $-|A_0|$ and a one-pole roll-off frequency characteristic

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for the operational amplifiers. That is,

$$A = \frac{-|A_0|}{s + \omega_b} \qquad \text{(II-15)}$$

(where ω_b is the open loop 3db bandwidth) is used instead of the usual infinite gain assumption. Direct analysis of the network yields the actual transfer function T(s). The denominator polynomial of T(s) is then compared to the denominator polynomial of the nominal ($|A_0| \to \infty$) second order transfer function $T_2(s)$. Attention is focused on the complex pole pair (p', $\bar{p}'$) of the actual transfer function that corresponds to the complex poles (p, $\bar{p}$) of the nominal second order transfer function. Specifically, on the pole

$$p' = -\omega_0'/2Q + j\omega_0'\sqrt{1-1/4Q'^2} \qquad \text{(II-16)}$$

of T(s) that approaches the pole

$$p = -\omega_0/2Q + j\omega_0\sqrt{1-1/4Q^2} \qquad \text{(II-17)}$$

of $T_2(s)$ as $|A_0| \to \infty$. First order estimates of ω_0' and Q' in terms of ω_0, Q, $|A_0|$, and ω_b are then obtained and used to estimate the shifts $\Delta Q = (Q'-Q)$ and $\Delta\omega_0 = (\omega_0'-\omega_0)$ due to the finite nature of the open loop dc gain $-|A_0|$. Experimental verification of these results was provided by Waltz (1969), and more recently by Thomas (1971a), who analyzes the Biquad under the same assumptions and obtains the same results. Tarmy and Ghausi (1970) in analyzing their new three-amplifier second order filter use the same approach to obtain first order estimates of the ω_0 and Q shifts due to the finite gain of the

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amplifier. They also present experimental evidence that supports their results.

For single amplifier networks, Petrela and Budak (1971a) have shown that for any n-th order single voltage amplifier filter with a nominal transfer function

$$T_n(s) = \frac{-N_{31}(s)}{V(s)} \qquad \text{(II-18)}$$

(where V(s) is defined to have unity leading coefficient) the actual transfer function obtained by using (II-15) always has the form

$$T(s) = \frac{-N_{31}(s)}{U(s)} = \frac{-N_{31}(s)}{V(s) + \{(s+\omega_b)/|\omega_a|\}|k_{33}|N_{33}(s)} \qquad \text{(II-19)}$$

where $|\omega_a| = \omega_b|A_o|$ is the finite gain bandwidth product of the amplifier. The derivation of (II-19) is fully discussed in Chapter III where the reasons for the notation used in (II-19) will be evident. Furthermore, it can be shown (Chapter III) that for the second order case V(s) and U(s) reduce to

$$V(s) = s^2 + (\omega_o/Q)s + \omega_o^2 \qquad \text{(II-20)}$$

and

$$U(s) = V(s) + \{(s+\omega_b)/|\omega_a|\}|k_{33}|(s^2 + q_1 s + q_o) \qquad \text{(II-21)}$$

These results can be used to find general expressions for ΔQ and $\Delta\omega_o$ for any single amplifier second order filter.

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Since the primary interest is in the pole p' of T(s), the $(s + \omega_b)$ term in (II-21) will have to be evaluated at $|s| = |p'| = \omega_0'$. For small pole shifts ω_0' will certainly be of the same order of magnitude as ω_0. If high gain silicon-integrated operational amplifiers are used, $|\omega_a| = \omega_b|A_0|$ is usually five orders of magnitude greater than ω_b. Hence, it is reasonable to assume that the second term of (II-21) will be important (and give rise to appreciable pole shifts) only when ω_0 approaches to within two or three orders of magnitude of $|\omega_a|$ and is therefore much greater than ω_b. But then $\omega_0' >> \omega_b$ also, and ω_b may be neglected. Then (II-21) becomes

$$U(s) = V(s) + (s/|\omega_a|)|k_{33}|(s^2 + q_1 s + q_0) \qquad \text{(II-22)}$$

and the only amplifier parameter that affects the pole position p' is the gain-bandwidth product of the amplifier. For $(1/|\omega_a|) = 0$, the poles $(p, \bar{p})$ of T(s) are the zeros of V(s), but for $(1/|\omega_a|) = (1/|\omega_a'|)$, a finite non-zero value, the poles $(p', \bar{p}')$ of T(s) are the zeros of U(s). To estimate the Q and ω_0 shifts $(\Delta Q = Q'-Q \; ; \; \Delta\omega_0 = \omega_0'-\omega_0)$ due to a finite value $|\omega_a'|$ for the amplifier gain-bandwidth product, (II-20) and s = p' are substituted into (II-22) to give

$$U(p') = V(p') + (p'/\omega_a')|k_{33}|\{V(p') + (q_1-\omega_0/Q)p' + (q_0-\omega_0^2)\} = 0 \qquad \text{(II-23)}$$

Series expansions may now be used to obtain a first order estimate of Q' and ω_0'. However, a much more direct procedure (that can be

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justified by a formal series expansion) can be used by recognizing that p is a zero of V(s) and hence for p' close to p (i.e. $|\Delta Q/Q|$ and $|\Delta\omega_0/\omega_0|$ both sufficiently small):

$$V(p') + (q_1-\omega_0/Q)p' + (q_0-\omega_0^2)$$

$$\simeq (q_1-\omega_0/Q)p' + (q_0-\omega_0^2) \qquad \text{(II-24)}$$

By direct substitution of (II-16), (II-20) and (II-24), equation II-23 then becomes

$$\{1 + |k_{33}/\omega_a|(q_1-\omega_0/Q)\}p'^2$$

$$+ \{1 + |k_{33}/\omega_a|(Q/\omega_0)(q_0-\omega_0^2)\}(\omega_0/Q)p' + \omega_0^2 \simeq 0 \qquad \text{(II-25)}$$

This may be solved directly for p'. Moreover, since it has already been assumed that p' is close to p, it follows that both $|k_{33}/\omega_a|(q_1-\omega_0 Q)$ and $|k_{33}/\omega_a|(Q/\omega_0)(q_0-\omega_0^2)$ have magnitudes much less than unity. As a result, the approximations

$$(1 + x)^{-1} \simeq 1 - x \qquad \text{(II-26a)}$$

$$(1 + x)^{\frac{1}{2}} \simeq 1 + x/2 \qquad \text{(II-26b)}$$

$$(1 + x)(1 + y) \simeq 1 + x + y \qquad \text{(II-26c)}$$

valid for $|x|, |y| \ll 1$, may be used to give

$$p'^2 + (\omega_0'/Q')p' + \omega_0'^2 = 0 \qquad \text{(II-27)}$$

where

$$\omega_0'^2 \simeq \omega_0^2\{1 - |k_{33}/\omega_a'|(q_1-\omega_0/Q)\} \qquad \text{(II-28)}$$

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and

$$(\omega_0'/Q') \simeq (\omega_0/Q)[1 + |k_{33}/\omega_a'|\{(Q/\omega_0)(q_0-\omega_0^2) - (q_1-\omega_0/Q)\}] \tag{II-29}$$

So that (again using (II-27)):

$$\Delta\omega_0/\omega_0 = (\omega_0'-\omega_0)/\omega_0$$

$$\simeq -(1/2)|k_{33}/\omega_a'|(q_1-\omega_0/Q) \tag{II-30}$$

and

$$\Delta Q/Q = (Q'-Q)/Q$$

$$\simeq -\tfrac{1}{2}|k_{33}/\omega_a'|\{2(Q/\omega_0)(q_0-\omega_0^2) - (q_1-\omega_0/Q)\} \tag{II-31}$$

Budak (1970) has derived the actual transfer function (II-19) for many commonly used second order filters, and has used basically this procedure to derive expressions corresponding to (II-30) and (II-31) for each case. Figures II-1 through II-10 give these results for the most commonly used single amplifier filters. Note that expressions for the $|k_{33}|$, q_1 and q_0 parameters introduced in the preceding general treatment of the second order case have been included in the figures. These q_1 and q_0 parameters are always proportional to ω_0 and ω_0^2 respectively, so that the relative ω_0 and Q shifts are inversely proportional to the relative gain-bandwidth product ($|\omega_a'|/\omega_0$) of the amplifier. Note also that, except for the positive finite gain amplifier networks of Figures II-3, 6 and 9, all these filters have a $q_1|k_{33}|$ product proportional to Q. Petrela and Budak (1971a) have shown that this always results in a relative ω_0 shift proportional to $(Q\omega_0/|\omega_a|)$; so that even if $\omega_0 << |\omega_a|$, the pole shifts may be far

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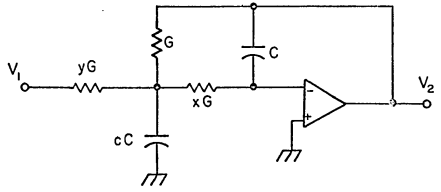

$$x = y = 1$$

$$c = 9Q^2$$

$$G/C = 3\omega_0 Q \qquad H = -\omega_0^2$$

$$|k_{33}| = 1$$

$$q_1 = (\omega_0/Q) + 3\omega_0 Q \qquad q_0 = 2\omega_0^2$$

$$(\Delta\omega_0/\omega_0) \simeq -(3/2)Q(\omega_0/|\omega_a'|)$$

$$(\Delta Q/Q) \simeq (1/2)Q(\omega_0/|\omega_a'|)$$

Fig. II-1

Operational Amplifier Low-Pass Filter

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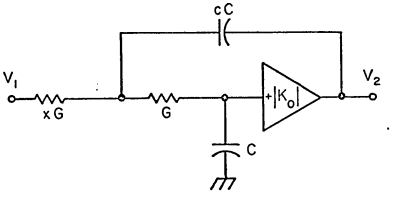

$$x = 1$$

$$c = 4Q^2$$

$$K_0 = 1$$

$$G/C = 2\omega_0 Q \qquad H = \omega_0^2$$

$$|k_{33}| = 1$$

$$q_1 = (\omega_0/Q) + 2\omega_0 Q \qquad q_0 = \omega_0^2$$

$$(\Delta\omega_0/\omega_0) \simeq -(\Delta Q/Q) \simeq Q(\omega_0/|\omega_a'|)$$

Fig. II-2

Positive Unity-Gain Amplifier Low-Pass Filter

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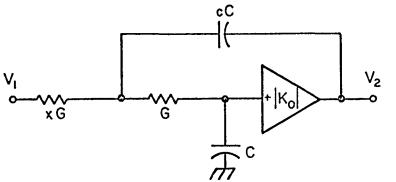

$$x = c = 1$$

$$K_0 = (3 - 1/Q) \qquad H = \omega_0^2 (3 - 1/Q)$$

$$|k_{33}| = (3 - 1/Q)$$

$$q_1 = 3\omega_0 \qquad q_0 = \omega_0^2$$

$$(\Delta\omega_0/\omega_0) \simeq -(\Delta Q/Q) \simeq -(K_0^2/2)(\omega_0/|\omega_0'|)$$

Fig. II-3

Positive Finite Gain Amplifier Low-Pass Filter with Unity-Element-Ratios

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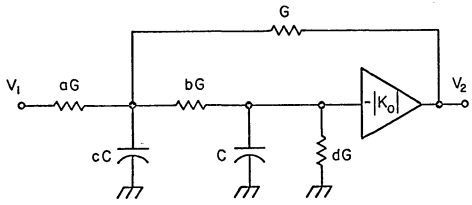

$$a = b = c = d = 1$$

$$|K_0| = 25Q^2 - 5$$

$$G/C = \omega_0/5Q \qquad H = -\omega_0^2(1 - 1/5Q^2)$$

$$|k_{33}| = 25Q^2 - 5$$

$$q_1 = (\omega_0/Q) \qquad q_0 = (\omega_0^2/5Q^2)$$

$$(\Delta\omega_0/\omega_0) \simeq 0$$

$$(\Delta Q/Q) \simeq |K_0|(Q - 1/5Q)(\omega_0/|\omega_0'|)$$

Fig. II-4

Negative Finite Gain Amplifier Low-Pass Filter

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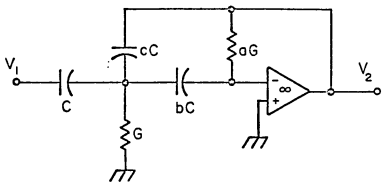

$$a = 1/9Q^2$$

$$b = c = 1$$

$$G/C = 3\omega_0 Q \qquad H = -1$$

$$|k_{33}| = 2$$

$$q_1 = (3/2)\omega_0 Q(1+1/3Q^2) \qquad q_0 = (\omega_0^2/2)$$

$$(\Delta\omega_0/\omega_0) \simeq -(3Q/4)(1-1/3Q^2)(\omega_0/|\omega_a'|)$$

$$(\Delta Q/Q) \simeq (5Q/4)(1-1/25Q^2)(\omega_0/|\omega_a'|)$$

Fig. II-5

Operational Amplifier High-Pass Filter

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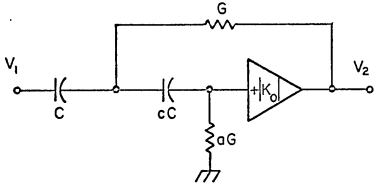

$$a = c = 1$$

$$K_0 = 3 - 1/Q$$

$$G/C = \omega_0 \qquad H = -(3 - 1/Q)$$

$$|k_{33}| = (3 - 1/Q)$$

$$q_1 = 3\omega_0 \qquad q_0 = \omega_0^2$$

$$(\Delta\omega_0/\omega_0) \simeq -(\Delta Q/Q) \simeq -(K_0^2/2)(\omega_0/|\omega_a'|)$$

Fig. II-6

Positive Finite Gain Amplifier High-Pass Filter

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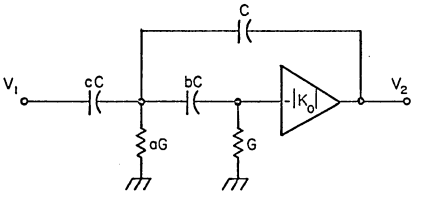

$$a = b = c = 1$$

$$|K_0| = 16Q^2 - 2$$

$$G/C = 4\omega_0 Q \qquad H = -(1 - 1/8Q^2)$$

$$|k_{33}| = 2(1 - 1/8Q^2)$$

$$q_1 = 8\omega_0 Q \qquad q_0 = 8Q^2\omega_0^2$$

$$(\Delta\omega_0/\omega_0) \simeq -8Q(1 - 1/8Q^2)(1 - 1/16Q^2)(\omega_0/|\omega_a'|)$$

$$(\Delta Q/Q) \simeq -16Q^3(1 - 1/2Q^2)(1 - 1/8Q^2)(1 - 1/16Q^2)(\omega_0/|\omega_a'|)$$

Fig. II-7

Negative Finite Gain Amplifier High-Pass Filter

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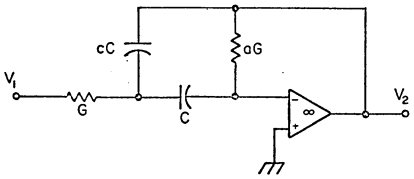

$$a = 1/4Q^2$$

$$c = 1$$

$$G/C = 2\omega_0 Q \qquad H = -2\omega_0 Q$$

$$|k_{33}| = 1$$

$$q_1 = (\omega_0/Q) + 2\omega_0 Q \qquad q_0 = \omega_0^2$$

$$(\Delta\omega_0/\omega_0) \simeq -(\Delta Q/Q) \simeq -Q'\omega_0/|\omega_a'|)$$

Fig. II-8

Operational Amplifier Band-Pass Filter

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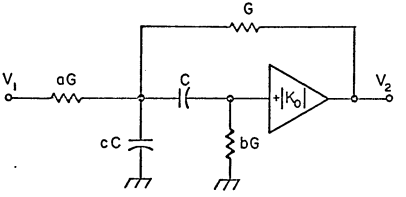

$$a = b = c = 1$$

$$K_0 = 4 - \sqrt{2}/Q$$

$$G/C = \omega_0/\sqrt{2} \qquad H = \omega_0(2\sqrt{2} - 1/Q)$$

$$|k_{33}| = 4 - \sqrt{2}/Q$$

$$q_1 = \omega_0 2\sqrt{2} \qquad q_0 = \omega_0^2$$

$$(\Delta\omega_0/\omega_0) = -(\Delta Q/Q) \simeq -(K_0^2/2\sqrt{2})(\omega_0/|\omega_a'|)$$

Fig. II-9

Positive Finite Gain Amplifier Band-Pass Filter

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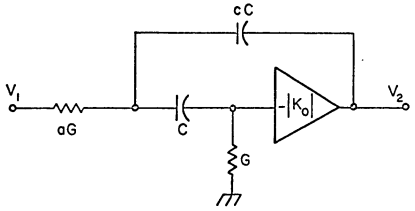

$$a = c = 1$$

$$|K_0| = 9Q^2 - 1$$

$$G/C = 3\omega_0 Q \qquad H = -3\omega_0 Q(1 - 1/9Q^2)$$

$$|k_{33}| = (1 - 1/9Q^2)$$

$$q_1 = 9\omega_0 Q \qquad q_0 = 9Q^2\omega_0^2$$

$$\Delta\omega_0/\omega_0 \simeq -(9Q/2)(1 - 1/9Q^2)(\omega_0/|\omega_a'|)$$

$$\Delta Q/Q \simeq -9Q^3(1 - 1/9Q^2)(1 - 1/2Q^2)(\omega_0/|\omega_a'|)$$

Fig. II-10

Negative Finite Gain Amplifier Band-Pass Filter

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from negligible if a high value of Q is used. This points out the fallacy in the intuitive reasoning of many designers who have assumed that if the bandwidth of the amplifier is much greater than ω_0, the frequency limitations imposed by the amplifier may be ignored. In the same vein, note that Figures II-2 and II-3 show the same positive gain amplifier filter network. The difference is that in Fig. II-3 unity element ratios are used, thus requiring a gain $K_0 = (3 - 1/Q)$. In Fig. II-2, on the other hand, the element ratios are adjusted to permit the use of a unity gain amplifier. If the same operational amplifier is used to realize both, the unity gain amplifier of Fig. II-2 has a closed loop bandwidth approximately three times that of the amplifier of Fig. II-3. Yet for Q = 9, and the same maximum permissible pole shift $|\Delta p/p|$, the unity-element-ratio network of Fig. II-3 can be used at frequencies twice as high as those permitted by the unity-gain network of Fig. II-2.

The frequency limitations of the filter are basically those values of ω_0 for which the relative ω_0 and Q shifts given by (II-30) and (II-31) become appreciable. The term "appreciable" is a relative one, and its meaning must be specified for a particular application. Generally, variations in Q of more than 5%; and in ω_0 of more than (5%/2Q) are considered "appreciable". The filter can, however, still be used beyond these limits by adjusting the passive component values to compensate for the pole shift. But if this is done, the pole locations will be strongly dependent on the actual gain-bandwidth product $|\omega_a'| = \omega_b|A_0|$ of the amplifier. Because the dc open loop gain $|A_0|$ has very poor manufacturing tolerances and temperature stability, this

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is a very undesirable situation and, as a result, limits the maximum value of ω_0 at which the filter may be used.

It may, however, be necessary to exceed these frequency limitations in certain cases. Then large pole shifts have to be considered and the first order approximations used to derive (II-30) and (II-31) are no longer valid. Rather (II-23) must be solved directly. Budak (1970) has done this for many commonly used second order filters and displayed the results graphically by plotting the actual pole locations (p'/ω_0) — normalized with respect to frequency — versus the relative gain-bandwidth product ($|\omega_a|/\omega_0$) for various values of Q. Of course, a different graph must be used for each set of the three circuit parameters ($|k_{33}|$, q_1 and q_0) that define equation II-23. Figures II-11 through II-13 illustrate this by giving the curves for the actual upper half plane pole location (p'/ω_0), as a function of the relative gain-bandwidth product ($|\omega_a'|/\omega_a$), for Q = 0.50, 0.60, 0.75, 1.00, 1.75, 3.00 and 10.0. The values of ($|\omega_a'|/\omega_0$) are marked along each curve adjacent to the corresponding actual pole location (p'/ω_0). The nominal pole locations (p/ω_0) correspond to the point denoted by (ω). Lehmkuhl et al (1971) have experimentally verified Fig. II-11, which corresponds to

$$|k_{33}| = 1 \qquad q_1 = (\omega_0/Q) + 2\omega_0 Q \qquad q_0 = \omega_0^2$$

and, as such, represents both the unity-gain low-pass filter of Fig. II-2 and the operational amplifier band-pass filter of Fig. II-8. Similarly, Fig. II-12 corresponds to both the low-pass and high-pass unity-element-ratio positive gain filters of Figures II-3 and II-6.

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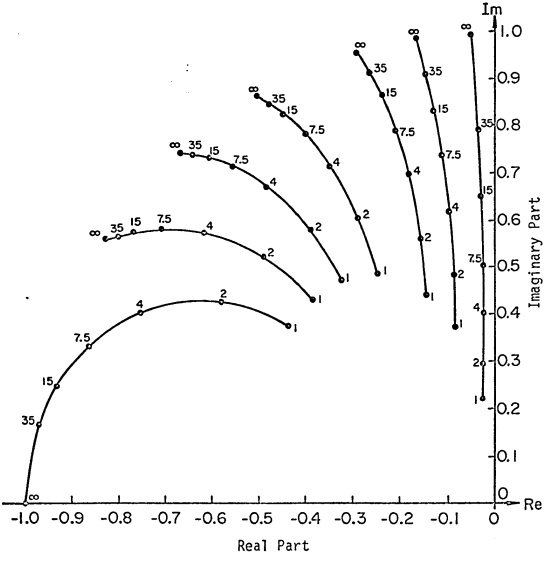

Fig. II-11

Pole Shifts due to Finite Gain-Bandwidth Product for $\{|k_{33}| = 1;\ q_1 = 2\omega_0 Q + Q/\omega_0;\ q_0 = \omega_0^2\}$

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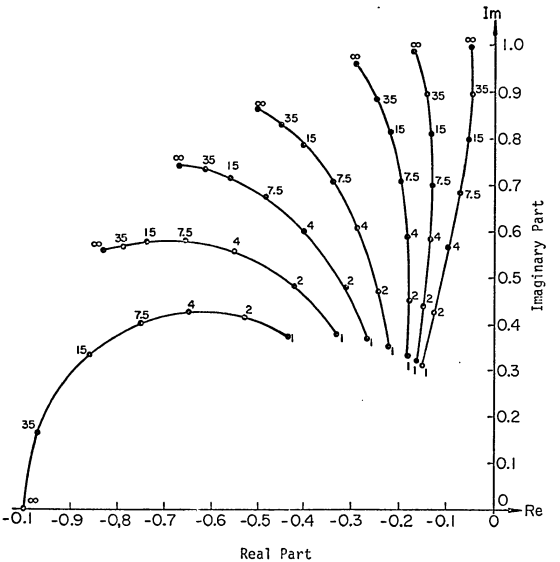

Fig. II-12

Pole Shifts due to Finite Gain-Bandwidth Product for $\{|k_{33}| = (3-1/Q);\ q_1 = 3\omega_0;\ q_0 = \omega_0^2\}$

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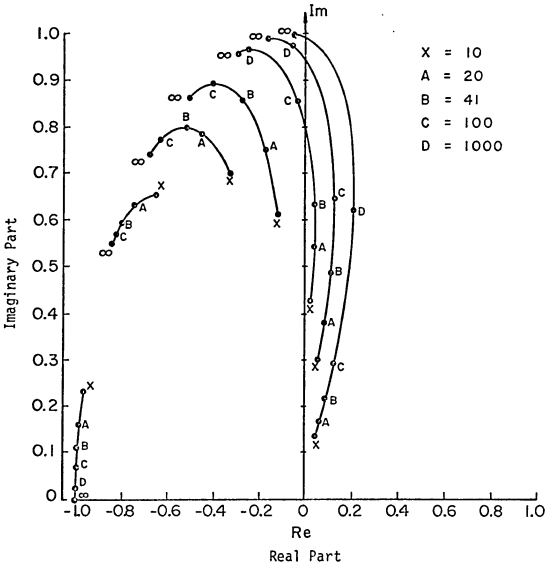

Fig. II-13

Pole Shifts due to Finite Gain-Bandwidth Product for $\{|k_{33}| = 25Q^2-4;\ q_1 = (\omega_0/Q);\ c_0 = (\omega_0^2/5Q^2)\}$

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Fig. II-13, on the other hand, corresponds only to the negative finite gain low-pass filter of Fig. II-4.

Fig. II-13 is included to illustrate that the negative finite gain filters, when used with Q's greater than unity, have the most severe frequency limitations (and general dependence on $|\omega_a|$). These negative filters were long regarded to be the best of all the filters shown because they have the lowest passive element sensitivities. However, Fig. II-13 certainly illustrates the need to reconsider this measure of merit. Moreover, all the curves show considerable pole shift for $(|\omega_a'|/\omega_0) = 35$; that is, ω_0 approximately 3% of $|\omega_a'|$. The gain-bandwidth product $|f_a'| = |\omega_a'|/2\pi$ of available silicon-integrated operational amplifiers is in the 2 - 5 MHz range (though experimental models with $|f_a'| = |\omega_a'|/2\pi \simeq 50$ MHz have recently been reported — Thomas (1971a)). This shows why many present day active filter networks are limited to (pole) frequencies less than 100 kHz.

C. The Gain-Bandwidth Product

Section A shows that in hybrid-integrated active filters the sensitivity of the poles (or zeros) to the open-loop gain of the amplifier is the paramount sensitivity consideration as far as the effects of manufacturing tolerances and temperature changes are concerned.

Section B shows that the frequency limitations of active filters also depend only on the open-loop gain of the amplifier. Moreover, the experimental verification of the pole shifts predicted by assuming a one-pole roll-off frequency characteristic for the amplifier implies that (II-15) represents a good model for the amplifier —

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at least as far as first order effects are concerned. Section B also shows that as far as frequency limitations are concerned, ω_b generally has negligible effect and only the value (and the variation) of the gain-bandwidth product of the amplifier is of importance. In addition, of the two parameters ω_b and $|\omega_a|$ in (II-15), ω_b is generally determined by the time constant of the frequency compensating network. However, $|\omega_a| = \omega_b|A_o|$ depends on $|A_o|$, which is the quantity most affected by changes in temperature and poor manufacturing tolerances.

As a result, the gain-bandwidth product of the amplifier is the critical parameter as far as integrated active filters are concerned. Specifically, the major concern of this work will be the determination of a good sensitivity measure with respect to the gain-bandwidth product and the general problem of how to minimize the sensitivity of single voltage amplifier active filters to the gain-bandwidth product of the amplifier.

With the exception of Chapter VI, only single amplifier filters will be considered in the remainder of this thesis. The importance of single amplifier filters is demonstrated by the following three facts:

1. In practice, the active realization of a given transfer function is nearly always in the form of a cascade of second order filters. The reasons for this "building block" approach are discussed in Chapter IV.
2. For pole Q's in the $0.5 < Q \leq 15$ range, even presently used single-amplifier second-order filters can achieve as wide a frequency

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range and as good a temperature stability as their more expensive multi-amplifier counterparts (Moschytz 1970b).

3. Communications systems operating below 500 kHz typically have filter requirements that demand the realization of poles with Q's less than 10 just about as often as they require the realization of poles with Q's greater than 10 (Moschytz 1970b).

As a result, it is reasonable to expect that at least 50% of all active filter applications will use single amplifier networks. Moreover, it is likely that the future use of totally integrated second-order "building blocks" will again give preference to single amplifier networks, for the power dissipation of the amplifiers used in each second order section will probably become the limiting factor on the physical size of the filter.

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CHAPTER III

A GENERAL SENSITIVITY MEASURE

The effect of the finite nature of the gain-bandwidth product of the amplifier on the performance of several specific networks is discussed in Chapter II. In this chapter attention is focused on the general case of a single voltage amplifier embedded in an RC network in order to realize an n-th order voltage transfer function $T_n(s)$. To avoid loading problems, it is assumed that the filter output is taken at the output terminal of the amplifier. Fig. III-1 then represents this general case with $T_n(s) = V_2(s)/V_1(s)$.

The basic problem solved in this chapter is the determination of the nature of the dependence of the transfer function on the gain-bandwidth product of the amplifier, and the isolation of those characteristics of the passive RC network that affect this dependence. The importance of this problem has been recognized. Thomas (1971a) and Tarmy and Ghausi (1970) have considered it for specific active realizations of certain second order transfer functions. However, no one has heretofore treated this problem in a general context. The approach used by both Thomas (1971) and Tarmy and Ghausi (1970) consists of first assuming a one-pole roll-off frequency characteristic for the amplifier, and then analyzing the specific network under consideration to come up with a transfer function that includes the effect of the amplifier poles. This transfer function is compared to the nominal second order transfer function the filter is designed to realize, and first order estimates of the shifts of the nominal transfer function poles are derived. This approach is basically the same as that used

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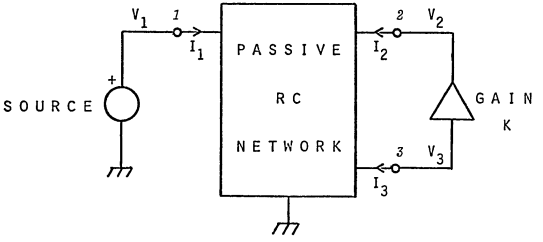

Fig. III-1

General Single Amplifier RC Network

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in Chapter II, except that Chapter II generalizes this procedure and also shows how the actual new pole locations may be found and plotted as a function of both the nominal pole locations and the gain-bandwidth product of the amplifier. Although this approach gives a great deal of valuable information on the dependence of the transfer function of a specific network on the gain-bandwidth product of the amplifier, it is by its very nature restricted to the consideration of specific network realizations. As a result, this direct approach cannot provide the insight needed to understand the general nature of the dependence of the transfer function on the gain-bandwidth product of the amplifier; nor can it pinpoint those specific characteristics of the passive RC network that affect this dependence.

The new approach presented in this chapter does both these things, and more. It is based upon a new general sensitivity measure applicable to all network realizations of a given transfer function that have the general form of Fig. III-1. In order to make the analysis independent of the basic type of amplifier that is used — that is, equally applicable to positive finite gain, negative finite gain, or operational amplifier networks — the amplifier is characterized in a manner that includes all three amplifier types as special cases. The passive RC network is characterized by its short-circuit admittance parameters (Weinberg 1962a). This characterization allows the definition of a new sensitivity measure that not only describes the dependence of the transfer function on the amplifier gain-bandwidth product, but also pinpoints the specific characteristics of the passive network that affect this dependence. The result is a new figure of merit applicable to all network realizations of the form of Fig. III-1 and

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thus extremely useful in comparing various realizations of a given transfer function. Furthermore, the knowledge of the precise RC network characteristics that affect the sensitivity measure permits the derivation of lower bounds for this figure of merit. These bounds not only enhance the usefulness of this measure of merit in comparing different network realizations of a given transfer function, but also allow it to be used as an absolute figure of merit for a particular realization. Finally, these lower bounds lead to a simple procedure for testing a particular transfer function to see if it is practical to even attempt to realize it by a network of the form of Fig. III-1.

A. Characterization of the Amplifier

The amplifier shown in Fig. III-1 may be one of three basic types:

1. an operational amplifier
2. a positive finite gain amplifier
3. a negative finite gain amplifier

Regardless of which type of amplifier is used, it is assumed that its gain K has a one-pole roll-off frequency characteristic. It is convenient to express this by first relating the amplifier gain K to an open-loop gain A by

$$K = \frac{A}{1 + A/K_0} \tag{III-1}$$

where the open-loop gain A is given by

$$A = \frac{A_0}{1 + s/\omega_b} = \frac{\omega_a}{s + \omega_b} \tag{III-2}$$

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In these expressions A_0, which may be positive or negative, is the open-loop dc gain and ω_b, which is always positive, is the open-loop 3db bandwidth of the amplifier. Thus, the magnitude of $\omega_a = A_0\omega_b$ is the open-loop gain-bandwidth product of the amplifier, and the sign of ω_a is the same as the sign of A_0.

If an operational amplifier is used, $(1/K_0) = 0$ (by definition) and the gain K is just the open-loop gain A. Moreover, A_0 and ω_a are negative.

For a finite gain amplifier, direct substitution of (III-2) into (III-1) yields

$$K = \left[\frac{K_0}{(1 + K_0/A_0)}\right] \Bigg/ \left[1 + \frac{s}{(\omega_a/K_0)(1 + K_0/A_0)}\right]$$

As a result, the closed-loop dc gain K_{dc} and 3db bandwidth ω_{3db} are given by

$$K_{dc} = \frac{K_0}{(1 + K_0/A_0)}$$

and

$$\omega_{3db} = (\omega_a/K_{dc})$$

Thus, K_0 is just the nominal ($|A_0| \to \infty$) closed-loop dc gain of the amplifier, and $|\omega_a|$ is the closed-loop gain-bandwidth product as well as the open-loop gain-bandwidth product. However, ω_a, A_0, and K_0 now must be of the same sign as K_{dc} which is positive for a positive finite gain amplifier and negative for a negative finite gain amplifier.

Table III-1 shows some networks commonly used to realize the amplifier of Fig. III-1 by using an operational amplifier either directly or in conjunction with some feedback scheme to obtain either a

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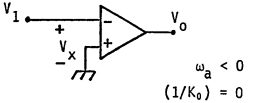
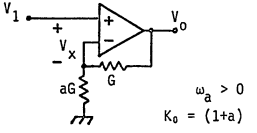
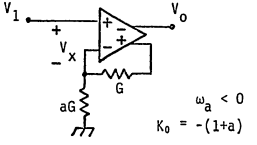
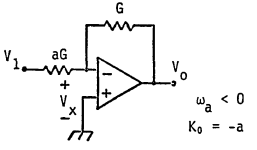
SCHEME	SAMPLE AMPLIFIER REALIZATION
-1- OPERATIONAL AMPLIFIER	
-2- POSITIVE FINITE GAIN AMPLIFIER	
-3- NEGATIVE FINITE GAIN AMPLIFIER	
-4- SPECIAL CASE OF -1-	

Table III-1

Some Practical Amplifier Realizations

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positive or a negative finite gain. Since the operational amplifier itself may be characterized by a dominant single pole frequency characteristic (Moschytz 1970a), it follows that all of the schemes of Table III-1 meet this assumption in practice.

In the discussion that follows it is also assumed that the amplifier input conductance and output resistance are negligibly small and may be considered to be zero. The effect of a non-zero input conductance and a non-zero output resistance is considered in Appendix A. However, the assumption that the input conductance and output resistance are effectively zero is usually justified in practice and has the advantage that it greatly simplifies the analysis because the amplifier constraints for the network of Fig. III-1 become just:

$$I_3 = 0 \qquad \text{(III-3)}$$

and

$$V_2 = KV_3 \qquad \text{(III-4)}$$

Of the schemes presented in Table III-1, only Scheme 4 does not meet this assumption in that its input conductance (aG) is generally not negligibly small. However, by lumping the two conductances (G and aG) with the RC network of Fig. III-1, Scheme 4 can be considered to be just a special case of Scheme 1. Consequently, all of the schemes of Table III-1 can be considered to meet the stated assumptions and to be effectively characterized by equations III-1, 2, 3 and 4.

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B. Characterization of the RC Network

The passive RC network used in Fig. III-1 may be described in terms of its short-circuit admittance matrix $\left[y_{ij}\right]$; namely,

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \left[y_{ij}\right] \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \qquad \text{(III-5)}$$

For a passive RC network (Weinberg 1962b) $y_{ij} = y_{ji}$ where the y_{ij}'s are ratios of polynomials (with real coefficients) in the complex frequency variable "s" and can be written as

$$y_{ij} = \frac{\pm N_{ij}(s)}{D(s)} \qquad \text{(III-6)}$$

where the plus sign is used for $i = j$ and the minus sign is used for $i \neq j$. Note that all the y_{ij}'s have the same denominator polynomial, $D(s)$, but different numerator polynomials, $\pm N_{ij}(s)$.

Fig. III-2 illustrates the definition of those y-parameters of interest in this chapter. Note that y_{33} is a driving point admittance of a passive RC network. Consequently (Weinberg 1962b),

$$N_{33}(s) = k_{33} \prod_{i\,=\,1,n_{33}} (s + \sigma_i) \qquad \text{(III-7)}$$

and

$$D(s) = k_d \prod_{i\,=\,1,n_d} (s + d_i) \qquad \text{(III-8)}$$

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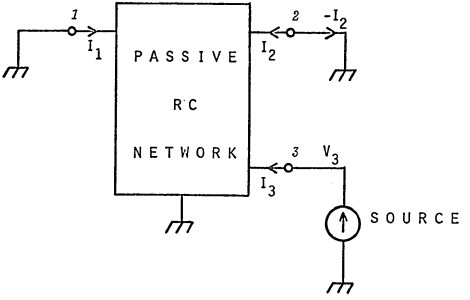

$$y_{13} = y_{31} = \frac{I_1}{V_3} = \frac{-N_{31}(s)}{D(s)}$$

$$y_{23} = y_{32} = \frac{I_2}{V_3} = \frac{-N_{32}(s)}{D(s)}$$

$$y_{33} = \frac{I_3}{V_3} = \frac{N_{33}(s)}{D(s)}$$

Fig. III-2

Definition of the Relevant
y-Parameters of the Passive Network

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where:

1. the leading coefficients k_{33} and k_d are of the same sign
2. the zeros $-\sigma_i$ of $N_{33}(s)$ must be real, non-positive, and distinct
3. the zeros $-d_i$ of $D(s)$ must be real, negative, and distinct
4. the degree n_{33} of $N_{33}(s)$ must be equal to, or one greater than, the degree n_d of $D(s)$
5. the zeros of $N_{33}(s)$ must alternate with the zeros of $D(s)$ such that the zero nearest, or at, the origin belongs to $N_{33}(s)$

Fig. III-2 also shows that

$$\left.\frac{-I_2}{I_3}\right|_{V_1=V_2=0} = \frac{-y_{32}}{y_{33}} = \frac{N_{32}(s)}{N_{33}(s)} \qquad \text{(III-9)}$$

where the polynomial $N_{32}(s)$, of degree n_{32}, can be written:

$$N_{32}(s) = k_{32}\left[s^{n_{32}} + \sum_{i=1,n_{32}} a_{i-1}s^{i-1}\right] \qquad \text{(III-10)}$$

Now define L as

$$L = \underset{s \to \infty}{\text{Limit}}\left[\left.\frac{-I_2}{I_3}\right|_{V_1=V_2=0}\right] \qquad \text{(III-11)}$$

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Using III-9, L becomes

$$L = \lim_{s \to \infty} \left[\frac{N_{32}(s)}{N_{33}(s)} \right] \qquad \text{(III-12)}$$

Substituting (III-7) and (III-10) into (III-12) gives

$$L = \lim_{s \to \infty} \left[\frac{k_{32}}{k_{33}} s^{-(n_{33}-n_{32})} \right] \qquad \text{(III-13)}$$

However, from (III-11), L represents the high frequency current gain of the passive RC network when it is driven as shown in Fig. III-2. As such, L must be a positive real number in the range

$$0 \leq L \leq 1 \qquad \text{(III-14)}$$

Thus, (III-13) implies that k_{32} must be of the same sign as k_{33} and that the degree of $N_{33}(s)$ must equal or exceed that of $N_{32}(s)$. That is,

$$n_{33} \geq n_{32} \qquad \text{(III-15)}$$

Equations III-7, III-14 and III-15 are extremely important in that they lead to lower bounds on the sensitivity measures to be considered in the next section.

C. The General Sensitivity Measure

The transfer function T(s) of the general network of Fig. III-1 is obtained by combining the RC network description (III-5) with the amplifier constraints, (III-3) and (III-4). The result is

$$T(s) = \frac{V_2(s)}{V_1(s)} = \frac{-y_{31}}{y_{32} + \frac{1}{K} y_{33}} \qquad \text{(III-16)}$$

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Substitution of the amplifier relationships (III-1) and (III-2), and the definition of the N_{ij} polynomials, (III-6), yields

$$T(s) = \frac{-N_{31}(s)}{\left[N_{32}(s) - \frac{1}{K_0}N_{33}(s)\right] - \left[\frac{s+\omega_b}{\omega_a}\right]N_{33}(s)} \qquad \text{(III-17)}$$

Note that the zeros of the transfer function are independent of the amplifier gain which affects only the poles (and magnitude) of T(s). The nominal ($|A_0| \to \infty$ and hence $|\omega_a| \to \infty$) poles of the transfer function are the zeros of

$$V(s) = N_{32}(s) - \frac{1}{K_0}N_{33}(s) \qquad \text{(III-18)}$$

However, because of the finite nature of the gain-bandwidth product $|\omega_a|$ of the amplifier, the actual pole locations are the zeros of

$$U(s) = V(s) - \left[\frac{s+\omega_b}{\omega_a}\right]N_{33}(s) \qquad \text{(III-19)}$$

and thus depend upon both the gain-bandwidth product $|\omega_a|$ and the open loop bandwidth ω_b of the amplifier. In practice, however, ω_b is relatively small (of the order of 50 rad/s for many commercially available operational amplifiers such as the LM201 or μa741), whereas $|\omega_a|$ is quite large (typically in the 5-50 Mrad/s range). As a result, the effect of a finite open-loop gain and its variation becomes particularly important when $|s| >> \omega_b$ and the open-loop gain $A = \omega_a/(s + \omega_b)$ is $A \simeq \omega_a/s$; that is, when the open loop gain element (usually an operational amplifier) behaves as an integrator. Consequently, ω_a is the primary parameter of interest. Since a sensitivity function can be

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used to characterize a network with respect to finite as well as differential parameter changes (Angelo 1955), it is desirable to describe the dependence of the poles of the transfer function on the amplifier in terms of their sensitivity with respect to the ω_a parameter.

Because $(1/\omega_a)$ is often given a nominal value of zero, the definition of any pole-sensitivity function chosen should not preclude a finite non-zero value when evaluated at $(1/\omega_a) = 0$. This is necessary if this function is to be used to compare different network realizations of a given transfer function, or to estimate the effect of a finite value of $(1/\omega_a)$ on the nominal $(1/\omega_a = 0)$ pole locations. Unfortunately, for most networks the conventional (Horowitz 1959) pole sensitivities

$$S^{p'}_{(1/\omega_a)} = (1/\omega_a)\cdot\frac{dp'}{d(1/\omega_a)} \qquad \text{and} \qquad S^{p'}_{\omega_a} = \omega_a\cdot\frac{dp'}{d\omega_a}$$

are either zero or infinite when they are evaluated at $(1/\omega_a) = 0$. For this reason it is necessary to define a new pole sensitivity function. Let

$$p' = -\alpha' + j\beta' = -\omega_0'/2Q' + j\omega_0'(1-1/4Q^{12})^{\frac{1}{2}}$$

denote the actual pole of T(s) corresponding to the nominal $(|A_0| \to \infty)$ pole

$$p = -\alpha + j\beta = -\omega_0/2Q + j\omega_0(1-1/4Q^2)^{\frac{1}{2}}$$

of T(s) (that is, $p' = p$ for $(1/\omega_a) = 0$). Then the new sensitivity function "$M_{p'}$," defined by

$$M_{p'} = \frac{1}{p'\omega_0'}\cdot\frac{dp'}{d(1/\omega_a)} \qquad \text{(III-20)}$$

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will be used to describe the dependence of the actual pole p' of T(s) on $(1/\omega_a)$.

In this thesis all derivatives of p', ω_0', Q', α', or β' are evaluated at the nominal, $(1/\omega_a) = 0$ and p' = p, pole location. This is denoted by dropping the prime in the expressions for the sensitivity functions and the derivatives. Thus, the value of $M_{p'}$ at $(1/\omega_a) = 0$ is denoted by M_p and $dp'/d(1/\omega_a)$ evaluated at $(1/\omega_a) = 0$ is denoted by $dp/d(1/\omega_a)$. That is,

$$M_p = M_{p'}\Big|_{(1/\omega_a)=0} = \frac{1}{p\omega_0} \cdot \frac{dp'}{d(1/\omega_a)}\Big|_{(1/\omega_a)=0} = \frac{1}{p\omega_0} \cdot \frac{dp}{d(1/\omega_a)} \tag{III-21}$$

Since extensive use will be made of this M_p parameter, it is desirable to give it a name. Henceforth, M_p will be referred to as the *pole sensitivity measure*. The pole sensitivity measure of the general network of Fig. III-1 is found by applying the following theorem (Huelsman 1968):

For p a simple zero of U(s) = V(s) + x·W(s), where V and W are polynomials in "s" independent of the parameter "x".

$$\frac{dp}{dx} = -\frac{W(s)}{U'(s)}\Big|_{s=p} \tag{III-22}$$

For $(1/\omega_a) = 0$, p' = p is, from (III-17), a pole of the nominal transfer function

$$T_n(s) = \frac{-N_{31}(s)}{V(s)} \tag{III-23}$$

where, from (III-18)

$$V(s) = N_{32}(s) - (1/K_0)N_{33}(s) \tag{III-24}$$

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It is assumed that the pole p is simple. Then, application of (III-22) to (III-19) and the use of $(1/\omega_a) = 0$ yields

$$\frac{dp}{d(1/\omega_a)} = (p + \omega_b) \cdot \frac{N_{33}(p)}{V'(p)} \qquad \text{(III-25)}$$

As a result, from (III-21), for p a pole of the nominal transfer function, the pole sensitivity measure M_p is given by

$$M_p = \left[\frac{p + \omega_b}{p}\right]\left[\frac{N_{33}(p)}{\omega_0 V'(p)}\right] \qquad \text{(III-26)}$$

From their definition (equations III-6, 7, 8, 10 and 18) the leading coefficients of $N_{32}(s)$, $N_{33}(s)$ and $D(s)$ can always be chosen so as to make the leading coefficient of $V(s)$ unity. Thus, $V(s)$ can be, and is, defined to have a leading coefficient of unity. Then in (III-26) the quantities p, ω_0 and $V'(p)$ are all fixed by the desired nominal pole locations of $T_n(s)$, and are thus independent of the particular network realization used; only the quantity $N_{33}(p)$ depends on the particular network realization. Thus, the pole sensitivity measure M_p depends only on the zeros and the leading coefficient, k_{33}, of $N_{33}(s)$ - see Fig. III-2. This pinpoints the characteristics of the RC network that affect the pole sensitivity. Different network realizations of a given transfer function will have different pole sensitivities if, and only if, the passive RC networks used give rise to different $N_{33}(p)$'s.

In order to evaluate M_p for a particular network realization of a given transfer function, only the zeros and the leading coefficient k_{33} of $N_{33}(s)$ must be evaluated for the particular passive RC network used. Since the zeros of $N_{33}(s)$ are the zeros of the driving

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point admittance function $y_{33}(s)$ of the passive RC network when it is driven as shown in Fig. III-2, they can often be determined by inspection - particularly for the commonly used second order case (Thornton et al 1965). To determine the leading coefficient k_{33}, use is made of the result of section B. There, it is shown that the degree n_{33} of $N_{33}(s)$ must equal or exceed the degree n_{32} of $N_{32}(s)$. Hence, from (III-18), n_{33} must also equal or exceed n, the degree of V(s); that is,

$$n_{33} \geq n \tag{III-27}$$

However, from the definition of V(s) the case $n_{33} > n$ can only occur when $(1/K_0) = 0$; that is, only when an operational amplifier is used. Moreover, V(s) is the nominal denominator polynomial of $T_n(s)$ and has a leading coefficient of unity. Consequently,

$$(1/k_{33}) = \underset{s \to \infty}{\text{Limit}} \left[\frac{V(s)}{N_{33}(s)} \cdot s^{(n_{33}-n)} \right] \tag{III-28}$$

Then, if $n_{33} = n$, the definition (III-18) of V(s) yields

$$(1/k_{33}) = \underset{s \to \infty}{\text{Limit}} \left[\frac{N_{32}(s)}{N_{33}(s)} \right] - (1/K_0) \tag{III-29}$$

which, from (III-12), may be written

$$(1/k_{33}) = L - (1/K_0) \tag{III-30}$$

In this equation (valid for $n_{33} = n$) L is the "high frequency" current gain $(-I_2/I_3)$ between ports 2 and 3 of the passive RC network when it is driven as shown in Fig. III-2. As such L, and hence k_{33}, can often be evaluated by inspection.

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Now consider the case $n_{33} > n$. From (III-24) this can only occur if an operational amplifier is used so that $(1/K_0) = 0$ and $V(s) = N_{32}(s)$. This implies that the leading coefficient k_{32} of $N_{32}(s)$ is unity and that the degree of $N_{32}(s)$ equals the degree of $V(s)$. Thus, from (III-13), the high frequency current gain L is

$$L = \lim_{s \to \infty} \left[(1/k_{33}) \cdot s^{-(n_{33}-n)}\right] \qquad \text{(III-31)}$$

This allows the determination of k_{33} from the asymptotic nature of the current gain L. Thus k_{33} can often be determined by inspection for the case $n_{33} > n$ as well as for the case $n_{33} = n$.

Since M_p is easily evaluated, it is readily used to compare different network realizations of a given transfer function. Of course, a lower magnitude pole sensitivity measure means a better sensitivity performance in the sense that smaller pole shifts result for given changes in the amplifier's gain-bandwidth product $|\omega_a|$ which, as discussed previously, is the main amplifier characteristic that affects the poles of the transfer function. Thus, $|M_p|$ is a natural choice for a figure of merit to be used to compare various network realizations. The practical utility of $|M_p|$ as a figure of merit is greatly enhanced by the fact that it is possible to determine the minimum possible value of $|M_p|$ that is obtainable with any network realization of the general form of Fig. III-1. This is the topic of the next section.

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D. Lower Bounds on $|M_p|$

From (III-26), $|M_p|$ is given by

$$|M_p| = \left|\frac{p + \omega_b}{p}\right| \left(\frac{|N_{33}(p)|}{\omega_o |V'(p)|}\right) \qquad \text{(III-32)}$$

For a given nominal transfer function $T_o(s) = -N_{31}(s)/V(s)$, and a given amplifier open-loop bandwidth ω_b, all the quantities in (III-32), with the exception of $|N_{33}(p)|$, are fixed. The problem then is to determine the minimum possible value of $|N_{33}(p)|$. From (III-7)

$$|N_{33}(p)| = |k_{33}| \prod_{i = 1, n_{33}} |(p + \sigma_i)| \qquad \text{(III-33)}$$

In section B it is pointed out that the σ_i must be real, non-negative and distinct. For $p = -\alpha + j\beta$ and σ_i real

$$|p + \sigma_i| \geq \beta \qquad \text{(III-34)}$$

The equality in this expression can only hold true for $\sigma_i = \alpha$. Thus, for σ_i distinct and $n_{33} > 1$,

$$\prod_{i = 1, n_{33}} |(p + \sigma_i)| > \beta^{n_{33}} \qquad \text{(III-35)}$$

From (III-29), $n_{33} \geq n$ where n is the degree of V(s). Since only the case $n \geq 2$ is of interest, it follows that

$$\prod_{i = 1, n_{33}} |(p + \sigma_i)| > \beta^n \qquad \text{(III-36)}$$

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Hence, from (III-36),

$$|M_p| > |k_{33}| \cdot \left[\frac{p + \omega_b}{p} \left| \left(\frac{\beta^n}{\omega_0 |V'(p)|} \right) \right| \right] \qquad \text{(III-37)}$$

It is important to note that in the above it is possible to make

$$\left| \beta^n - \prod_{i=1,n_{33}} |(p + \sigma_i)| \right| < \varepsilon \qquad \text{(III-38)}$$

for any $\varepsilon > 0$, by choosing a network realization with $n_{33} = n$ and

$$\underset{i=1,n_{33}}{\text{Max.}} \{|\sigma_i - \alpha|\} < \sqrt[n]{\varepsilon} \qquad \text{(III-39)}$$

Thus, (III-37) not only expresses a lower bound on $|M_p|$, but represents the greatest lower bound on $|M_p|$ for a network realization of the general form of Fig. III-1. However, it is still necessary to determine the lower bounds for the quantity $|k_{33}|$.

The case $n_{33} > n$ occurs very seldom in practice, and its inclusion would greatly complicate the following discussion. Hence, this case ($n_{33} > n$) is treated separately in Appendix B, and only the case $n_{33} = n$ is considered in this chapter. Then (III-30) applies; namely,

$$(1/k_{33}) = L - (1/K_0) \qquad \text{(III-40)}$$

From (III-14), the "high frequencv" current gain L must lie in the range $0 \leq L \leq 1$. This leads directly to some bounds on k_{33} in terms of K_0, which is the nominal dc gain of the amplifier. To determine

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these bounds, any constraints on K_0 are combined with the constraint on L; then equation (III-40) is used to find the resultant constraints on $(1/k_{33})$; finally the constraints on $(1/k_{33})$ are transformed into constraints on k_{33}.

1. The Operational Amplifier Case

For the operational amplifier case $(1/K_0) = 0$. The resulting constraints on $(1/k_{33}) = L$ and k_{33} are shown in Fig. III-3. For the purpose of this section, the result of interest is that

$$|k_{33}| \geq 1 \tag{III-41}$$

for any operational amplifier realization of the form of Fig. III-1. As a result, from (III-37),

$$|M_p| > \left|\frac{p + \omega_b}{p}\right| \left(\frac{\beta^n}{\omega_0 |V'(p)|}\right) \tag{III-42}$$

for any operational amplifier realization of the general form of Fig. III-1.

2. The Positive Finite Gain Amplifier Case

The case of a positive finite gain amplifier, $K_0 > 0$, is somewhat more complicated in that four possibilities must be considered in turn. They are

(a) $1 > K_0 > 0$

(b) $K_0 = 1$

(c) $K_0 \geq (1/L) > 1$

(d) $(1/L) \geq K_0 > 1$

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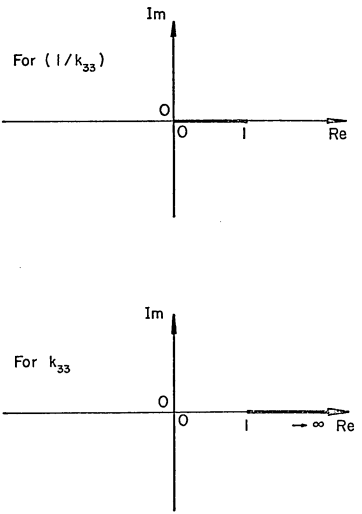

Fig. III-3

Possible Ranges of $(1/k_{33})$ and k_{33} for an Operational Amplifier Realization

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Fig. III-4 gives the resultant ranges of $(1/k_{33})$ and k_{33} for possibilities (a) and (b), while Fig. III-5 gives the corresponding results for possibilities (c) and (d). A study of these two figures leads to the conclusion that $|k_{33}|$ must always be greater than either K_0 or unity - whichever is less; that is,

$$|k_{33}| \geq \text{Min.}(1, K_0) \tag{III-43}$$

Thus, any positive finite gain amplifier realization of the general form of Fig. III-1 (and with a nominal closed-loop dc gain K_0) must have

$$|M_p| > [\text{Min.}(1, K_0)] \left[\left| \frac{p + \omega_b}{p} \right| \left(\frac{\beta^n}{\omega_0 |V'(p)|} \right) \right] \tag{III-44}$$

3. The Negative Finite Gain Amplifier Case

For a negative finite gain amplifier $K_0 < 0$. This leads to the relatively simple constraints on k_{33} shown in Fig. III-6, and hence

$$|k_{33}| \geq \frac{|K_0|}{|K_0| + 1} \tag{III-45}$$

Therefore, from (III-37), any negative finite gain amplifier realization of the general form of Fig. III-1 (and with a nominal closed-loop dc gain $-|K_0|$) must have

$$|M_p| > \left[\frac{|K_0|}{|K_0| + 1} \right] \cdot \left[\left| \frac{p + \omega_b}{p} \right| \cdot \left(\frac{\beta^n}{\omega_0 |V'(p)|} \right) \right] \tag{III-46}$$

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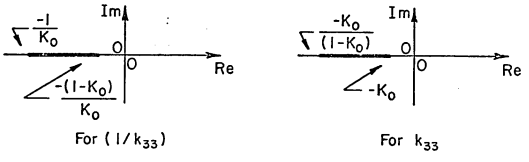
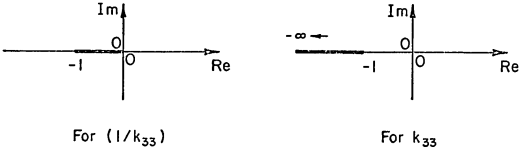


(a)

$0 < K_0 < 1$



(b)

$K_0 = 1$

Fig. III-4

Possible Ranges for $(1/k_{33})$ and k_{33} for a Positive Finite Gain Amplifier Realization with $K_0 \leq 1$

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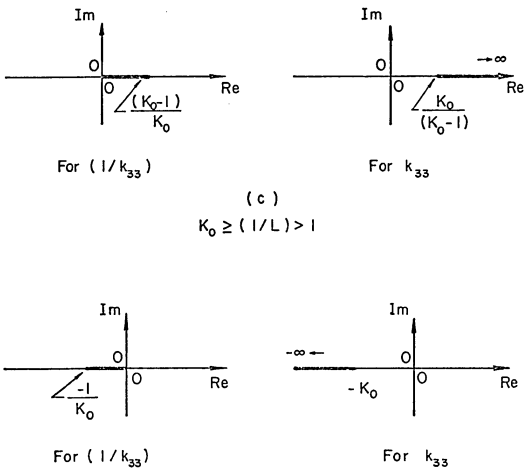

Fig. III-5

Possible Ranges of $(1/k_{33})$ and k_{33} for a Positive Finite Gain Amplifier Realization with $K_0 > 1$

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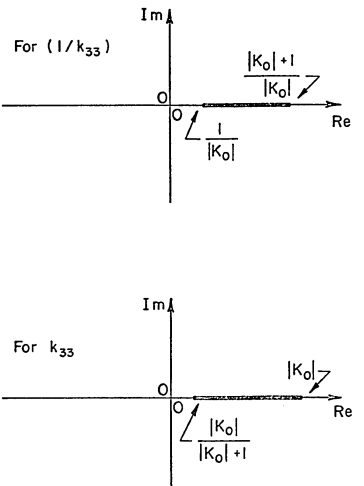

Fig. III-6

Possible Ranges of $(1/k_{33})$ and k_{33} for a Negative Finite Gain Amplifier Realization

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4. A Special Finite Gain Amplifier Case

Because a great many of the finite gain amplifier realization schemes proposed in the past decade (Moschytz 1967) use RC networks for which the degree n_{32} of $N_{32}(s)$ is less than the degree n of V(s), it is useful to consider this special case separately. Now $n_{32} < n$ implies, from (III-18), $n_{33} = n$. As a result, $n_{32} < n_{33}$ and the high frequency current gain L of (III-12) must be zero. Equation (III-40) then yields

$$k_{33} = -K_0 \tag{III-47}$$

irrespective of whether a positive or negative finite gain amplifier is used. From (III-37), this implies that any realization of the form of Fig. III-1 that uses an RC network with an $N_{32}(s)$ of lower degree than V(s) must have

$$|M_p| > |K_0| \cdot \left|\frac{p + \omega_b}{\omega_a}\right| \left(\frac{\beta^n}{\omega_0 |V'(p)|}\right) \tag{III-48}$$

Equations (III-42), (III-44), (III-46), and (III-48) represent lower bounds on $|M_p|$ for any network realization of the form of Fig. III-1. Since (III-37) represents a greatest lower bound on $|M_p|$, it is reasonable to expect these lower bounds to be quite practical. In subsequent chapters it is shown by example that this is indeed so; there do exist network realizations for which $|M_p|$ approaches these lower bounds to within a factor of two or less. Furthermore, either operational amplifier schemes or finite gain amplifier schemes which have a nominal closed-loop dc gain magnitude $|K_0|$ greater or equal to unity are used in practice. Consequently, for the positive finite

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gain case

$$|k_{33}| \geq \text{Min.}\{1, K_0\} = 1$$

and for the negative finite gain case

$$|k_{33}| \geq \frac{|K_0|}{|K_0| + 1} \geq \tfrac{1}{2}$$

As a result, schemes (1), (2) and (4) of Table III-1 must have, from (III-37),

$$|M_p| > \left|\frac{p + \omega_b}{p}\right| \cdot \left[\frac{\beta^n}{\omega_0 |V'(p)|}\right] \qquad \text{(III-49)}$$

whereas for scheme (3) of Table III-1

$$|M_p| > \tfrac{1}{2} \left|\frac{p + \omega_b}{p}\right| \cdot \left[\frac{\beta^n}{\omega_0 |V'(p)|}\right] \qquad \text{(III-50)}$$

5. A Special Sensitivity Case

When $N_{33}(s)$ does not have a zero at the origin, it is sometimes useful to choose the zeros $-\sigma_i$ of $N_{33}(s)$ such that the constant coefficient $k_{33}q_0$ of $N_{33}(s)$ is less than the constant coefficient W_0^2 of $V(s)$*. When $q_0 < W_0^2$ the previously derived lower bounds on $|k_{33}|$ may be raised by considering the current gain of (III-11) and (III-12) as $s \to 0$ rather than as $s \to \infty$.

* *The notation $(k_{33}q_0, W_0^2)$ is used because this case is of importance for second order transfer functions for which this notation is used in Chapter IV.*

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From the definition (III-18) of V(s)

$$\frac{V(s)}{N_{33}(s)} = \frac{N_{32}(s)}{N_{33}(s)} - \frac{1}{K_0}$$

hence

$$\underset{s \to 0}{\text{Limit}} \left[\frac{V(s)}{N_{33}(s)} \right] = \underset{s \to 0}{\text{Limit}} \left[\frac{N_{32}(s)}{N_{33}(s)} \right] - \frac{1}{K_0} \qquad \text{(III-51)}$$

The left side of (III-51) is just the coefficient ratio $W_0^2/(k_{33}q_0)$; whereas from (III-9) and Fig. III-2, the limit on the right hand side of (III-51) is just the dc current gain

$$L_0 = \underset{s \to 0}{\text{Limit}} \left[\frac{-I_2}{I_3} \bigg|_{V_1=V_2=0} \right] \qquad \text{(III-52)}$$

of the passive RC network of Fig. III-2. As a result, (III-51) becomes

$$\frac{W_0^2}{k_{33}q_0} = L_0 - \frac{1}{K_0} \qquad \text{(III-53)}$$

Equation (III-53) is of the same form as (III-40). Moreover the dc current gain L_0 of the passive network must also be a positive real number in the range $0 \leq L_0 \leq 1$. Consequently, the bounds on k_{33} shown in Figures III-3, 4, 5 and 6 must also apply to the quantity $(k_{33}q_0/W_0^2)$. This in turn means that the lower bounds on $|k_{33}|$ and $|M_p|$ tabulated in Table III-2 can (and should) be increased by the factor W_0^2/q_0 whenever this factor is greater than unity; that is, whenever $q_0 < W_0^2$.

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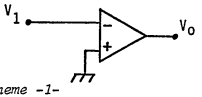
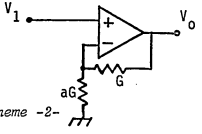
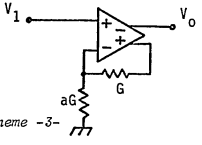
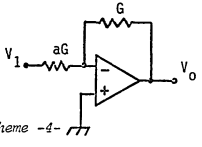
AMPLIFIER TYPE	LOWER BOUND ON $\lvert M_p \rvert$
 *Scheme -1-*	$\left\lvert \frac{p+\omega_b}{p} \right\rvert \frac{\beta^n}{\omega_0 \lvert V'(p) \rvert}$
 *Scheme -2-*	$\left\lvert \frac{p+\omega_b}{p} \right\rvert \frac{\beta^n}{\omega_0 \lvert V'(p) \rvert}$
 *Scheme -3-*	$\frac{1}{2} \left\lvert \frac{p+\omega_b}{p} \right\rvert \frac{\beta^n}{\omega_0 \lvert V'(p) \rvert}$
 *Scheme -4-*	$\left\lvert \frac{p+\omega_b}{p} \right\rvert \frac{\beta^n}{\omega_0 \lvert V'(p) \rvert}$

Table III-2

Lower Bounds on the Magnitude $\lvert M_p \rvert$ of the Pole Sensitivity Measure for the Amplifier Types shown in Table III-1

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Table III-2 lists these lower bounds on $|M_p|$ for the four amplifier schemes of Table III-1.

Equations (III-49) and (III-50) then represent lower bounds for the magnitude of the pole sensitivity measure (for the pole $p = -\alpha + j\beta$) of any active RC network having the form of Fig. III-1 and using one of the amplifier schemes of Table III-1 to realize the nominal transfer function $T_n(s) = -N_{31}(s)/V(s)$. Nearly all the single amplifier schemes proposed in the past decade satisfy the conditions for (III-49) and (III-50). This greatly enhances the usefulness of $|M_p|$ as a figure of merit, for it can now be used not only to compare two or more alternate network realizations, but also serves as an absolute measure of merit for a particular network realization of a given transfer function.

Since a sensitivity function may be used to give a first order estimate of finite changes (Angelo 1955; Belove 1966), the pole sensitivity measure M_p can be used to estimate the relative pole shift caused by a finite gain-bandwidth product $|\omega_a'|$. From (III-21),

$$(\Delta p/p) \simeq (\omega_0 M_p)\cdot\Delta(1/\omega_a) \qquad \text{(III-54)}$$

Since the nominal value of $(1/\omega_a)$ is zero, $\Delta(1/\omega_a)$ is the actual value $(1/\omega_a')$ of $(1/\omega_a)$. Thus, if p' denotes the actual pole location and p the nominal pole location, $(p'-p)/p = (\Delta p/p)$ is given by

$$(\Delta p/p) \simeq (\omega_0/\omega_a')\cdot M_p \qquad \text{(III-55)}$$

As a result, the lower bounds on the figure of merit $|M_p|$ can be used to estimate the minimum amplifier gain-bandwidth product

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required to realize a given transfer function $T_n(s)$ by a network of the form of Fig. III-1. Let the n poles of $T_n(s)$ be denoted by p_j (j = 1,n) and the lower bound of $|M_p|$ evaluated at each of these poles be denoted by $|M_{p_j}|_{min}$. Then for a maximum relative pole shift $|(\Delta p/p)|_{max}$, the minimum required gain-bandwidth product of the amplifier is (since $\omega_0 = |p|$)

$$|\omega_a'| > \max_{j=1,n} \left[\frac{|p_j| \cdot |M_{p_j}|_{min}}{|(\Delta p/p)|_{max}} \right] \qquad \text{(III-56)}$$

If p and $\bar{p}$ are two complex conjugate poles of the same transfer function their respective pole sensitivity measures (see (III-21)) must also be complex conjugates of one another. Hence, such complex conjugate poles have the same $|M_p|$. Moreover, from (III-37), any real axis pole p_j has a lower bound $|M_{p_j}|_{min} = 0$. As a result only the upper half-plane poles of $T_n(s)$ need be considered when (III-56) is used to estimate the minimum required gain-bandwidth product of the amplifier.

E. Other Related Sensitivity Measures

The pole sensitivity measure M_p defined by (III-21) is a general sensitivity measure not only because it is applicable to all active networks of the form of Fig. III-1, but also because it contains information on the magnitude and direction of the pole shift due to the finite nature of the gain-bandwidth product $|\omega_a|$. In the preceding section, attention is focused on the magnitude of $|M_p|$ and thus on the magnitude of the pole shift. Frequently, however, certain directivity

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characteristics of these pole shifts are of greater interest than $|M_p|$. In particular, for

$$\begin{aligned} p &= -\alpha + j\beta \\ &= -\omega_0/2Q + j\omega_0\sqrt{1-(1/4Q^2)} \end{aligned} \tag{III-57}$$

a nominal pole of the transfer function $T_n(s)$, the relative changes in α, β, ω_0 or Q are often of interest. For this reason it is convenient to define the following four additional sensitivity measures.

(1) The α-Sensitivity Measure

$$M_\alpha = \frac{1}{\alpha\omega_0} \cdot \frac{d\alpha}{d(1/\omega_a)} \tag{III-58}$$

(2) The β-Sensitivity Measure

$$M_\beta = \frac{1}{\beta\omega_0} \cdot \frac{d\beta}{d(1/\omega_a)} \tag{III-59}$$

(3) The ω_0-Sensitivity Measure

$$M_{\omega_0} = \frac{1}{\omega_0^2} \cdot \frac{d\omega_0}{d(1/\omega_a)} \tag{III-60}$$

(4) The Q-Sensitivity Measure

$$M_Q = \frac{1}{Q\omega_0} \cdot \frac{dQ}{d(1/\omega_a)} \tag{III-61}$$

Note that these definitions are analogous to that used for M_p (III-21). Because α, β, ω_0 and Q are related to p by (III-57), their respective sensitivity measures are related to M_p. The derivation of these relationships, though cumbersome, is relatively straightforward and for

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that reason is omitted. The results are:

(1) For the α-Sensitivity Measure

$$M_{\alpha} = -(1/\alpha)\mathrm{Re}\{pM_p\} \qquad \text{(III-62)}$$

(2) For the β-Sensitivity Measure

$$M_{\beta} = (1/\beta)\mathrm{Im}\{pM_p\} \qquad \text{(III-63)}$$

(3) For the ω_0-Sensitivity Measure

$$M_{\omega_0} = \mathrm{Re}\{M_p\} \qquad \text{(III-64)}$$

(4) For the Q-Sensitivity Measure

$$M_Q = -2Q\sqrt{1-(1/4Q^2)}\cdot\mathrm{Im}\{M_p\} \qquad \text{(III-65)}$$

Note that these relationships clearly indicate the conditions on M_p needed to make any of these sensitivities zero. For example, M_{ω_0} will be zero if, and only if, M_p is purely imaginary. Depending upon the particular transfer function under consideration, making any of these sensitivities zero might, however, preclude achieving the minimum value of $|M_p|$ otherwise attainable. In the next chapter it is shown that for a second order transfer function the requirement for $M_{\omega_0} = 0$ implies minimum $|M_p|$; however, the requirement for $M_Q = 0$, implies a value for $|M_p|$ that is at least twice the minimum value otherwise attainable.

It should also be stressed that all the sensitivity measures used in this chapter were evaluated at $(1/\omega_a) = 0$. This is the value

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normally assumed in practice, because the gain-bandwidth product $|\omega_a|$ is usually not known to any great degree of accuracy. Manufacturing tolerances, as well as aging and temperature variations, can easily cause $|\omega_a|$ to vary by 20% or more (Moschytz 1970b). Even if $|\omega_a|$ is known, the sensitivity measures then depend on the particular pole locations desired. This is illustrated in (Petrela and Budak 1971b) where it is shown that in order to achieve zero Q-sensitivity at a particular value of $|\omega_a|$ one must custom design the network for the particular set of pole locations desired. Though this may sometimes be desirable, it is more useful to come up with network realizations that minimize $|M_p|$ (or one of the other sensitivity functions) over a broad range of the gain-bandwidth product $|\omega_a|$. To achieve this, the use of the sensitivity measures evaluated at $(1/\omega_a) = 0$ is to be preferred because the evaluation of these measures at a non-zero nominal value of $(1/\omega_a)$ does not offer any significant advantage to offset the greatly increased complexity of the resulting analytical expressions. The practical usefulness of this approach is illustrated in the next two chapters which are concerned with the application of the general concepts introduced in this one.

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CHAPTER IV

THE SECOND ORDER CASE

There is good reason to consider the second order case in detail: by far the most common application of active RC networks is the realization of a second order transfer function with a pair of complex conjugate poles. This predominance of the second order case is itself due to two reasons.

The first is that any n-th order transfer function can be decomposed into the product of an m-th order transfer function which may be realized by a passive RC network, and (n-m)/2 second order transfer functions which must be actively realized. A discussion of the manner in which this decomposition should be performed is outside the scope of this thesis and the reader is referred to the article by Lueder (Lueder 1970) which discusses certain aspects of the problem and also gives further references. The fact of interest is that such a decomposition can always be carried out, and each of the resulting second order transfer functions can then be realized by an active RC network of the form of Fig. III-1.

In Fig. III-1 the output for the transfer function is taken at the low impedance output terminal of the voltage amplifier. This avoids loading problems and allows the realization of the (decomposed) overall transfer function by a cascade of non-interacting, second order, active "building blocks" (and, if necessary, a passive RC section). This building block approach is the second reason for the predominance of the second order case. The use of a few basic building blocks to realize any desired transfer function presents great cost

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advantages, particularly for multipurpose filtering systems (Moschytz 1967, 1970b; Thomas 1971b). The building block approach also greatly simplifies the tuning problem. If it were not used each high order filter realized (e.g. 6-th order low-pass, 8-th order low-pass, 8-th order band-pass, etc.) would require a different, complicated tuning procedure of its own. However, the use of non-interactive building blocks reduces the problem of tuning the overall filter to that of independently tuning each second order building block - a relatively simple procedure. Moreover, since only a few building blocks are needed to realize any transfer function, it becomes feasible to develop automated tuning procedures such as those described by Hintzman (Hintzman 1969).

The nominal transfer function under consideration in this chapter has the form

$$T_2(s) = \frac{N(s)}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2} \qquad \text{(IV-1)}$$

where $Q > 1/2$, and $N(s)$ is any real polynomial of degree two or less. Since the two poles of $T_2(s)$ are complex conjugates of one another, it is only necessary to consider the effect of the finite gain-bandwidth nature of the amplifier on the upper half plane pole at

$$\begin{aligned} p &= -\alpha + j\beta \\ &= -\omega_0/2Q + j\omega_0\sqrt{1-(1/4Q^2)} \end{aligned} \qquad \text{(IV-2)}$$

The sensitivity measures introduced in Chapter III depend both on ω_a and ω_b. However, the open-loop bandwidth, ω_b, of the

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amplifier only appears in the factor

$$\left[\frac{p + \omega_b}{p}\right] = \left[\frac{-(\alpha-\omega_b) + j\beta}{-\alpha + j\beta}\right]$$

If $\alpha = \omega_0/2Q$ is much greater than ω_b this factor is practically unity and the sensitivity measures (and pole shifts) therefore are independent of ω_b. The assumption $\alpha >> \omega_b$ is, thus, equivalent to assuming that the open-loop gain element (with gain $A = \{\omega_a/(s + \omega_b)\}$) is acting as an integrator (with gain $A \simeq \omega_a/s$) in the frequency range of interest. In order to determine whether or not the assumption $\alpha >> \omega_b$ is usually valid, note that in Chapter III it is shown that the relative pole shift (from the nominal pole locations) has a magnitude

$$\left|\frac{\Delta p}{p}\right| \simeq \frac{\omega_0 |M_p|}{|\omega_a|}$$

In this chapter it is shown that well designed second order networks have $|M_p|$ well below 10. Hence, the effect of the finite amplifier bandwidth $|\omega_a|$ on the pole locations only becomes appreciable if ω_0 approaches $|\omega_a|$ to within two or three orders of magnitude. However, $|\omega_a| = \omega_b|A_0|$ is usually five or six orders of magnitude greater than ω_b. As a result, if the pole Q's are much less than 500, the assumption $\alpha = \omega_0/2Q >> \omega_b$ is a good one. Since (as stated in the Introduction), most applications of single amplifier RC networks involve the realization of poles with Q's less than 50 (and usually less than 10), the assumption $\alpha >> \omega_b$ is usually a good one.

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The reason for assuming $\alpha >> \omega_b$ is that the expressions to be derived in this chapter are greatly simplified if $\{(p + \omega_b)/\omega_b\} \simeq 1$, and this facilitates their interpretation. However, it should be emphasized that the methods used do not depend on this assumption; that is, if ω_b is not negligible, exactly the same procedures can be used to derive the corresponding expressions that include the effect of ω_b. However, the algebraic manipulations involved (as well as some of the resultant expressions) become quite cumbersome. For these reasons the condition $\{(p + \omega_b)/p\} \simeq 1$, which certainly holds if $\alpha >> \omega_b$, will be assumed throughout this chapter.

A. The Pole Sensitivity Measure

Since both the nominal denominator polynomial V(s) of the transfer function and the polynomial $N_{33}(s)$ (of the passive RC network - shown in Fig. III-2) are of degree two, they may be written in the following forms:

$$\begin{aligned} V(s) &= (s + \alpha)^2 + \beta^2 \\ &= s^2 + \frac{\omega_0}{Q} s + \omega_0^2 \end{aligned} \tag{IV-3}$$

and

$$\begin{aligned} N_{33}(s) &= k_{33}(s + \sigma_1)(s + \sigma_2) \\ &= k_{33}(s^2 + q_1 s + q_0) \end{aligned} \tag{IV-4}$$

Substituting this into (III-28) and using $\{(p + \omega_b)/p\} \simeq 1$, the pole sensitivity measure (for the pole $p = -\alpha + j\beta$) is

$$M_p = k_{33}\left[\frac{p^2 + q_1 p + q_0}{2(p + \alpha)}\right] \tag{IV-5}$$

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To simplify (IV-5), let the quantities d_1, d_0, and d_3 be defined by:

$$d_1 = \frac{1}{2\omega_0}(2\alpha - q_1)$$

$$d_0 = \frac{1}{2\omega_0}(\omega_0^2 - q_0) \qquad \text{(IV-6)}$$

$$d_3 = \frac{1}{\beta}(\alpha d_1 - d_0)$$

Substitution of these relationships, and $p = -\alpha + j\beta$, into equation IV-5, then yields the simple expression

$$M_p = - k_{33}(d_1 + jd_3) \qquad \text{(IV-7)}$$

From (III-40), k_{33} is

$$(1/k_{33}) = L - (1/K_0) \qquad \text{(IV-8)}$$

where, as discussed in Chapter III, the high frequency current gain L of the passive RC network is readily evaluated; similarly, q_1 and q_0 are readily evaluated from the short-circuit and open-circuit time constants of the passive network. As a result, the pole sensitivity measure of a particular second order realization is easy to find. Consider the following example.

Example 1

Fig. IV-1(a) shows a low-pass filter that uses a positive finite gain amplifier to realize the transfer function

$$T(s) = \frac{V_2(s)}{V_1(s)} = \frac{K_0\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

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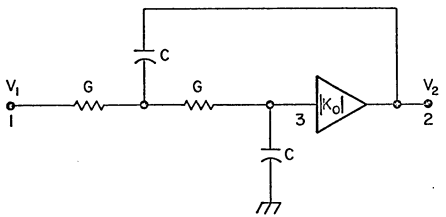
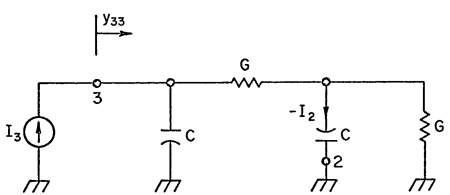


(a)



(b)

Fig. IV-1

Low-Pass Filter for Examples 1 and 4

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where

$$\omega_0 = \frac{G}{C} \qquad K_0 = 3 - \frac{1}{Q}$$

Fig. IV-1(b) shows the passive RC network driven as per Fig. III-2. Inspection of Fig. IV-1(b) immediately yields the high frequency current gain

$$L = \underset{s \to \infty}{Limit} \left[\frac{-I_2}{I_3}\right] = 0$$

Hence, from (IV-8)

$$k_{33} = -K_0 = -(3 - 1/Q)$$

The coefficients q_1 and q_0 of (IV-4) can be expressed in terms of the short-circuit time constants (τ_{js}) and the open-circuit time constants (τ_{jo}) of the passive network (Thornton et al 1965). Thus, from Fig. IV-1(b)

$$q_1 = \Sigma \frac{1}{\tau_{js}} = \left(\frac{1}{1} + \frac{1}{(1/2)}\right)\frac{G}{C} = 3\omega_0$$

and

$$q_0 = \frac{q_1}{\Sigma\tau_{jo}} = \frac{3\omega_0}{(2 + 1)(C/G)} = \omega_0^2$$

Substitution of these into (IV-6) yields

$$d_1 = -\tfrac{1}{2}(3 - 1/Q) \quad ; \quad d_0 = 0$$

$$d_3 = \frac{-(3 - 1/Q)}{4Q\sqrt{1-(1/4Q^2)}}$$

As a result, from (IV-7),

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$$M_p = -\frac{1}{2}\left[3 - \frac{1}{Q}\right]^2\left[1 + j\,\frac{1}{2Q\sqrt{1-(1/4Q^2)}}\right]$$

Note that for $Q \gg \frac{1}{2}$, $|M_p| \simeq 4.5$.

The advantage of the definitions (IV-6) is that they not only lead to a very simple expression for M_p, but with (IV-4) also imply a one-to-one correspondence between the zeros (σ_1, σ_2) of $N_{33}(s)$ and the quantity ($d_1 + jd_3$) which (in conjunction with k_{33}) determines the pole sensitivity measure M_p. Consequently, any constraints on σ_1 and σ_2 may be translated into constraints on d_1 and d_3 and, hence, on M_p. More important, the converse is also true. Any desirable property of M_p is readily translated into constraints on d_1 and d_3 and, hence, on σ_1 and σ_2.

In particular, σ_1 and σ_2 must be real and distinct. This requires, from (IV-4), that q_1^2 exceed $(4q_0)$. Using (IV-6), it is readily shown that this will be so if, and only if,

$$d_3 < \frac{\omega_0}{2\beta}\,(d_1^2 - \beta^2/\omega_0^2) \qquad \text{(IV-9)}$$

Furthermore, σ_1 and σ_2 must be non-negative. From (IV-4) this requires both $q_0 \geq 0$, and $q_1 > 0$. From (IV-6) it follows that these conditions are met if, and only if, both

$$d_3 > (\alpha/\beta)d_1 - (\omega_0/2\beta) \qquad \text{(IV-10a)}$$

and

$$d_1 < (\alpha/\omega_0) \qquad \text{(IV-10b)}$$

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Conditions (IV-9) and (IV-10) are illustrated by Fig. IV-2. The shaded region - including the solid border line representing (IV-10a) but excluding the dotted border lines representing (IV-9) and (IV-10b) - represents those (d_1, d_3) combinations that correspond to σ_1, σ_2 real, non-negative, and distinct. As such, this shaded region represents (d_1, d_3) combinations that are realizable by a passive RC network. The unshaded region represents (d_1, d_3) combinations that are impossible to realize by a passive RC network.

From Fig. IV-2 it is seen that any second order filter (of the general form of Fig. III-1) must have

$$|(d_1 + jd_3)| > \tfrac{1}{2}(\beta/\omega_0) \qquad \text{(IV-11)}$$

and, thus, from (IV-7), its pole sensitivity measure must have a magnitude

$$|M_p| > \tfrac{1}{2}|k_{33}|(\beta/\omega_0) \qquad \text{(IV-12)}$$

Note that this is just the special case ($n = 2$, $\alpha >> \omega_b$) of the greatest lower bound on $|M_p|$ given by (III-37). This very simple result, when combined with the general bounds on $|k_{33}|$ derived in Chapter III, readily yields the lower bounds on $|M_p|$ listed in Table IV-1.

Fig. IV-2 also shows that to approach these lower bounds, it is necessary to have $|d_1| << \tfrac{1}{2}(\beta/\omega_0)$ and $d_3 \simeq -\tfrac{1}{2}(\beta/\omega_0)$. Equations (IV-6) and (IV-4) can be used to express these constraints in terms of q_1 and q_0 or σ_1 and σ_2. The resulting condition on σ_1 and σ_2 for

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TYPE OF AMPLIFIER USED	LOWER BOUND on $\lvert M_p \rvert = \lvert k_{33} \rvert \cdot \lvert d_1 + j d_3 \rvert$
OPERATIONAL	$\frac{1}{2}\sqrt{1 - \frac{1}{4Q^2}}$
POSITIVE FINITE GAIN $\lvert K_o \rvert \geq 1$	$\frac{1}{2}\sqrt{1 - \frac{1}{4Q^2}}$
NEGATIVE FINITE GAIN $\lvert K_o \rvert \geq 1$	$\frac{1}{4}\sqrt{1 - \frac{1}{4Q^2}}$

Table IV-1

Some Lower Bounds on the Figure of Merit of Any Second Order Filter

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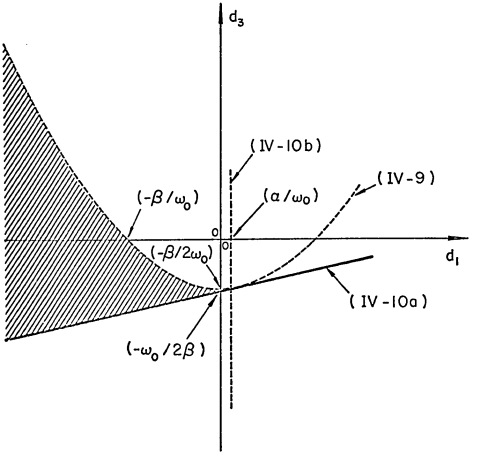

Fig. IV-2

Region of Realizable (d_1, d_3) Combinations Corresponding to (σ_1, σ_2) Real, Non-Negative, and Distinct

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minimum $|M_p|$ is

$$|\sigma_i - \alpha| << \beta \qquad i = 1,2 \tag{IV-13}$$

For the case $Q >> 1$ (so that $\alpha << \beta$) this may, from (IV-4), be simplified to the single condition

$$q_1 << \omega_0 \tag{IV-14}$$

This minimizes $|d_1 + jd_3|$. However, $|k_{33}|$ must also be minimized if minimum $|M_p|$ is to be achieved. Consider the following example.

Example 2

Suppose that the network of Fig. IV-1 and Example 1 is to be used to realize a Q of 5. From the expression for M_p found in Example 1, $|M_p| \simeq 4.3$ when $Q = 5$. Yet, from Table IV-1, the lower bound on $|M_p|$ for a positive finite gain amplifier used to realize a Q of 5 is $|M_p| \simeq \frac{1}{2}$. This factor of 10 difference indicates that the network of Fig. IV-1 is a very poor realization. The main reason for this is the choice of unity element ratios. The topology used forces L to be zero so that $|k_{33}| = K_0$, but the choice of unity element ratios requires $K_0 = (3 - 1/Q) = 2.8$. Thus $|k_{33}|$ is 2.8 times its optimum value of 1. Secondly, the topology forces $q_0 = \omega_0^2$ so that $d_0 = 0$ and for $Q >> \frac{1}{2}$, $|d_3| << d_1$. As a result, from Fig. IV-2, the lowest achievable $|M_p|$ is approximately twice the minimum otherwise obtainable. To achieve even this requires (from Fig. III-1) $d_1 \simeq -(\beta/\omega_0) \simeq -1$ which, from (IV-6), implies $q_1 \simeq 2\omega_0$. However, in Example 1 the use of unity element ratios forces $q_1 = 3\omega_0$. Hence, though the topology is

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responsible for a factor of 2 increase in $|M_p|$, *the use of unity element ratios is the main reason that* $|M_p|$ *for Example 1 exceeds its lower bound by a factor of 10. Consequently, different element ratios should be chosen. The techniques for doing this are the subject of the next chapter; for now just consider the same network with the element ratios shown in Fig. IV-3.*

The basic design equations now are

$$K_0 = 1 + \frac{1+r}{c} - \frac{1}{Q}\sqrt{\frac{r}{c}} \qquad \frac{G}{C} = \omega_0\sqrt{\frac{c}{r}}$$

Using the same techniques as in Example 1, the pole sensitivity measure is found in three steps:

(a) from Fig. IV-3(b) and (IV-8)

$$q_1 = \omega_0\left[\frac{1}{Q} + K_0\sqrt{\frac{c}{r}}\right] \qquad q_0 = \omega_0^2 \qquad k_{33} = -K_0$$

(b) from (IV-6)

$$d_1 = -\frac{K_0}{2}\sqrt{\frac{c}{r}} \qquad d_3 = \frac{d_1}{2Q\sqrt{1-(1/4Q^2)}}$$

(c) from (IV-7)

$$M_p = -\frac{K_0^2}{2}\sqrt{\frac{c}{r}}\left[1 + j\,\frac{1}{2Q\sqrt{1-(1/4Q^2)}}\right]$$

For the element ratios r = 10 and c = 30, these expressions yield $K_0 \simeq 1.25$, $q_1 \simeq 2.4\omega_0$ *and* $|M_p| \simeq 1.38$, *a significant improvement over the* $|M_p| \simeq 4.3$ *case of Example 1, and fairly close to the* $|M_p| \simeq 1$ *limit dictated by the topology used.*

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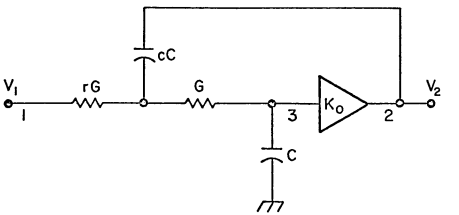
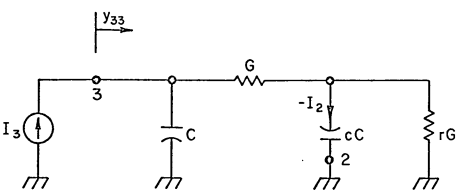


(a)



(b)

Fig. IV-3

Low-Pass Filter for Example 2

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Example 2 illustrates that the minimum pole sensitivity magnitude that can be achieved by a particular configuration is generally limited by both the topology of the RC network used and the maximum allowable element ratios. In Example 2, the topology forced both $L = 0$ and $q_0 = \omega_0^2$. This last condition occurs quite frequently in commonly used networks and is worth considering in some detail.

From (IV-6) the condition $q_0 = \omega_0^2$ requires

$$d_1 = d_3\left[2Q\sqrt{1-(1/4Q^2)}\right] \tag{IV-15}$$

As a result, from (IV-7),

$$M_p = -k_{33}d_1\left[1 + j\frac{1}{2Q\sqrt{1-(1/4Q^2)}}\right] \tag{IV-16}$$

and

$$|M_p| = \frac{|k_{33}|\cdot|d_1|}{\sqrt{1-(1/4Q^2)}} \tag{IV-17}$$

Furthermore, if (IV-15) is plotted on Fig. IV-2, the minimum permissible magnitude of d_1 (and hence M_p) is readily seen to occur at a value of d_1 just less than $-\{1-(1/2Q)\}$. Consequently,

$$|M_p| > |k_{33}|\cdot\left[\frac{1-(1/2Q)}{\sqrt{1-(1/4Q^2)}}\right] \tag{IV-18}$$

For $Q >> \frac{1}{2}$, this means that the lower bound on M_p is approximately twice what it would be if q_0 were not restricted to be equal to ω_0^2.

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B. The ω_0- and Q-Sensitivity Measures

In many practical applications, the relative changes in the ω_0 and Q parameters are of special interest. For this reason it is useful to discuss the ω_0- and Q-sensitivity measures defined in Chapter III for the second order case. Equations (III-64) and (III-65), when combined with (IV-7), yield

$$M_{\omega_0} = -k_{33}d_1 \tag{IV-19}$$

and

$$M_Q = \left[2Q\sqrt{1-(1/4Q^2)}\right]k_{33}d_3 \tag{IV-20}$$

Thus, the achievement of zero Q-sensitivity is possible. It simply requires $d_3 = 0$. However, Fig. IV-2 shows that $d_3 = 0$ is realizable only if $d_1 < -\beta/\omega_0$. This in turn requires

$$|M_p| = |M_{\omega_0}| > |k_{33}|\sqrt{1-(1/4Q^2)} \tag{IV-21}$$

where the signs of M_p (which is now real) and M_{ω_0} must be the same as the sign of k_{33}. As a result, requiring zero Q-sensitivity not only forces $M_p = M_{\omega_0}$ but, from a comparison of (IV-12) and (IV-21), also implies a lower bound on $|M_p|$ that is twice the value otherwise attainable. Furthermore, it is clear from Fig. IV-2 that the attainment of both zero Q-sensitivity and as low an ω_0-sensitivity as possible, requires both $d_3 = 0$ and $d_1 \simeq -\beta/\omega_0$.

Note that because of the 2Q factor in equation IV-20, the condition $d_3 \ll d_1$ which gives $|M_p| \simeq |M_{\omega_0}|$ does not necessarily imply a low Q-sensitivity. For example, networks (such as those of Fig. IV-3) which have $q_0 = \omega_0^2$ will, from (IV-15), have $d_3 \ll d_1$ and

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$|M_{\omega_0}| \simeq |M_p|$ if $Q >> \frac{1}{2}$. However, from (IV-15), (IV-19) and (IV-20), the condition $q_0 = \omega_0^2$ implies

$$M_Q = -M_{\omega_0} = k_{33}d_1 \qquad \text{(IV-22)}$$

As a result, such networks cannot achieve zero Q-sensitivity, but rather must have, from (IV-20),

$$M_Q > |k_{33}| \cdot \{1-(1/2Q)\} \qquad \text{(IV-23)}$$

For this reason the low-pass filter of Fig. IV-3 cannot be used if zero Q-sensitivity is desired. Rather, a network topology that allows $d_3 = 0$ must be chosen. However, this might also result in a very large ω_0-sensitivity. Consider the following example.

Example 3

The nominal transfer function of the low-pass filter network of Fig. IV-4(a) is

$$T(s) = \frac{V_2(s)}{V_1(s)} = \frac{-y\omega_0^2}{s^2 + \frac{\omega_0}{Q}\, s + \omega_0^2}$$

where

$$c = Q^2\{2(1+y) + x + (1+y)^2/x\}$$

$$\frac{G}{C} = \omega_0\sqrt{\frac{c}{x}}$$

Fig. IV-4(b) and the methods of Example 1 give:

$$k_{33} = 1 \; ; \qquad d_1 = -\tfrac{1}{2}\sqrt{xc}$$

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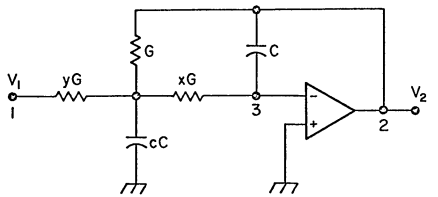
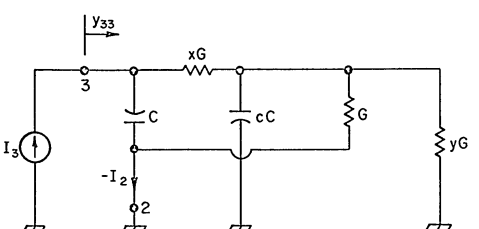


(a)



(b)

Fig. IV-4

Low-Pass Filter for Example 3

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and

$$d_3 = \frac{-(1 + x - y)}{4\sqrt{1-(1/4Q^2)}}$$

Zero Q-sensitivity is achieved if, and only if, y = (x+1) so that $d_3 \equiv 0$. *However, this also gives*

$$c = 4Q^2\left[\frac{y}{y - 1}\right] = \frac{4Q^2(x + 1)^2}{x}$$

$$\frac{G}{C} = 2Q\omega_0\left[\frac{y}{y - 1}\right] = 2Q\left[\frac{x + 1}{x}\right]\omega_0$$

and

$$d_1 = - 2Q^2(x + 1)$$

Thus $|d_1|$ *and* $|M_p|$ *will approach their minimum values if* $x << 1$. *Then*

$$M_p = M_{\omega_0} \simeq 2Q^2$$

For Q = 5 this gives $M_p \simeq 50$. *However, from (IV-21), the minimum* $|M_p|$ *of an operational amplifier network with zero Q-sensitivity is approximately unity. Consequently, it might be advisable to seek other realizations that achieve zero Q-sensitivity or even just accept one with a low Q-sensitivity. The network of Fig. IV-3, for example, is shown in Example 2 to have (for Q = 5, r = 10, and c = 30)* $k_{33} = -1.21$, $d_1 \simeq -1.05$, *and* $d_3 \simeq -0.104$; *so that it has* $|M_p| \simeq 1.28$, $M_{\omega_0} \simeq -1.27$ *and* $M_Q \simeq + 1.27$. *Hence, though the network of Fig. IV-3 cannot achieve zero Q-sensitivity it might be preferable to that of Fig. IV-4 because of its much smaller* ω_0*-sensitivity.*

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In order to achieve zero ω_0-sensitivity, it is necessary, from (IV-19), to have $d_1 = 0$. From Fig. II-2 this is only possible for d_3 in the narrow range

$$-(\omega_0/2\beta) \leq d_3 < -(\beta/2\omega_0)$$

or in terms of the pole Q,

$$\frac{-(1/2)}{\sqrt{1-(1/4Q^2)}} \leq d_3 < -(1/2)\sqrt{1-(1/4Q^2)} \qquad \text{(IV-24)}$$

Furthermore, from (IV-7) and (IV-20), this implies that a network with zero ω_0-sensitivity has

$$-k_{33}Q \leq M_Q < -k_{33}Q\sqrt{1-(1/4Q^2)} \qquad \text{(IV-25a)}$$

and

$$(k_{33}/2)\sqrt{1-(1/4Q^2)} < |M_p| \leq \frac{(k_{33}/2)}{\sqrt{1-(1/4Q^2)}} \qquad \text{(IV-25b)}$$

For the common $Q >> \frac{1}{2}$ case, these inequalities become the approximate relations:

$$M_Q \simeq -k_{33}Q \qquad \text{(IV-26)}$$

$$M_p \simeq j(k_{33}/2) \qquad \text{(IV-27)}$$

Note that the constraint $d_1 = 0$ for $M_{\omega_0} = 0$ implies the constraints $|d_1| << \beta/2\omega_0$ and $d_3 \simeq -\beta/2\omega_0$ for minimum $|M_p|$. The converse is, of course, not true. Approximately minimum $|M_p|$ may be obtained without necessarily achieving zero ω_0-sensitivity. It is also important to note that, from (IV-6), the constraint $d_1 = 0$ (for zero

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ω_0-sensitivity) requires that q_1 equal 2α; but the need for real distinct roots for $N_{33}(s)$ - see (IV-4) - then requires that q_0 be less than α^2. This means that q_0 is less than ω_0^2 and the discussion accompanying (III-53) applies. Hence, the lower bounds on $|k_{33}|$ given by equations III-41, 43 and 45 must be increased by the factor $(\omega_0/q_0) > (\omega_0/\alpha^2) = 4Q^2$. Consequently, any zero ω_0-sensitivity realization that uses either an operational amplifier or a positive finite gain amplifier (with $K_0 \geq 1$) must have $|k_{33}| > 4Q^2$; and thus, from (IV-25a)

$$|M_Q| > 4Q^3\sqrt{1-(1/4Q^2)} \qquad \text{(IV-28a)}$$

Similarly, any zero ω_0-sensitivity realization that uses a negative finite gain amplifier (with $|K_0| \geq 1$) must have

$$|M_Q| > 2Q^3\sqrt{1-(1/4Q^2)} \qquad \text{(IV-28b)}$$

Table IV-2 summarizes the requirements on d_1 and d_3 for achieving a minimum figure of merit $|M_p|$, a minimum (magnitude) ω_0-sensitivity, or a minimum (magnitude) Q-sensitivity. The corresponding sensitivity measures for each of these parameters are also shown in Table IV-2.

C. Estimating Pole Shifts

In the preceding discussion p, ω_0 and Q represented the values of these parameters at the nominal pole location for which $(1/\omega_a) = 0$. If $|\omega_a|$ is finite, however, $1/\omega_a$ is not zero and the new pole position will be at $p' = p + \Delta p$ where p' may be expressed as:

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QUANTITY TO BE MINIMIZED	REQUIRED CONDITION on d_1	REQUIRED CONDITION on d_3
$\lvert M_p \rvert$	$\lvert d_1 \rvert \ll \beta/2$	$d_3 \simeq -\beta/2$
$\lvert M_{\omega_0} \rvert$	$d_1 = 0$	$d_3 \simeq -\beta/2$
$\lvert M_Q \rvert$	$d_1 < -\beta/\omega_0$	$d_3 = 0$
$\lvert M_Q \rvert$ for $M_{\omega_0} = 0$	$d_1 = 0$	$d_3 \simeq -\beta/2$
$\lvert M_{\omega_0} \rvert$ for $M_Q = 0$	$d_1 \simeq -\beta/\omega_0$	$d_3 = 0$
SENSITIVITY MEASURES $M_p = -k_{33}(d_1 + jd_3)$; $M_{\omega_0} = -k_{33}d_1$; $M_Q = 2Q(\beta/\omega_0)k_{33}d_3$		

Table IV-2

Conditions on d_1 and d_3 for Various Sensitivity Minima

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$$p' = -\alpha' + j\beta'$$
$$= -\omega_0'/2Q' + j\omega_0'\sqrt{1-(1/4Q'^2)} \qquad \text{(IV-29)}$$

Here $Q' = Q + \Delta Q$ and $\omega_0' = \omega_0 + \Delta\omega_0$ are the values of these parameters at the new pole location p'. If the changes (Δp, ΔQ, $\Delta\omega_0$) are sufficiently small (relative to the nominal values: p, Q, ω_0), the sensitivity measures evaluated at $(1/\omega_a) = 0$ may be used to estimate these changes (Angelo 1955). Letting x represent any one of the above parameters (p, Q, etc.) the definition of the sensitivity measure M_x always has the form

$$M_x = \left[\frac{1}{\omega_0 x}\right]\left[\frac{dx}{d(1/\omega_a)}\right] \qquad \text{(IV-30)}$$

hence, for $|\Delta x/x| \ll 1$,

$$(\Delta x/x) \simeq M_x\{\omega_0\Delta(1/\omega_a)\} \qquad \text{(IV-31)}$$

Since the nominal value of $(1/\omega_a)$ is zero, $\Delta(1/\omega_a) = (1/\omega_a')$ where $|\omega_a'|$ is the actual gain-bandwidth product of the amplifier and ω_a' has the same sign as the dc open loop gain A_0. Thus,

$$(\Delta x/x) \simeq M_x(\omega_0/\omega_a') \qquad \text{(IV-32)}$$

Hence, for $|M_x(\omega_0/\omega_a')| \ll 1$, the actual value x' of x (where x represents whatever parameter is of interest; i.e. p, ω_0, or Q) due to the finite nature of the gain-bandwidth product $|\omega_a'|$ of the amplifier is

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given by

$$x' \simeq x\{1 + M_x(\omega_0/\omega_a')\} \qquad \text{(IV-33)}$$

When designing a network, however, it is the actual (x') values of the parameters that are specified. Yet the design equations usually available are for the nominal case $\{(1/\omega_a) = 0\}$ and as such are in terms of the nominal (x) values of these parameters. The problem then is to recast these design equations in terms of the actual (x') values that are specified. For this, x must be expressed in terms of x'. A direct approach, similar to that used to derive (IV-33), yields that for $|M_x'(\omega_0'/\omega_a')| \ll 1$

$$x \simeq x'\{1 - M_x'(\omega_0'/\omega_a')\} \qquad \text{(IV-34)}$$

where M_x' is the sensitivity measure evaluated at $(1/\omega_a) = (1/\omega_a')$. This last represents a bit of a problem, for it has been shown (Petrela and Budak 1971b) that it is considerably more complicated to evaluate the sensitivity measures for a non-zero value of $(1/\omega_a)$ than it is to evaluate them for $(1/\omega_a) = 0$. Fortunately, this step (though possible) is rarely necessary because, from (III-22), the sensitivity measures are continuous functions of $(1/\omega_a)$. As such, the expression for M_x' can be approximated by the expression for M_x if the changes involved are sufficiently small for all the parameters i.e. if $|M_x'\{(\omega_0'/\omega_a')\}| \ll 1$ for all x parameters. If this is so, then (IV-34) may be used with M_x' replaced by M_x to calculate the nominal values of p, ω_0, and Q around which the network should be designed. The

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resulting expressions for the second order case are:

$$p \simeq p'\{1 + k_{33}(d_1 + jd_3)(\omega_0'/\omega_a')\} \qquad \text{(IV-35a)}$$

$$\omega_0 \simeq \omega_0\{1 + k_{33}d_1(\omega_0'/\omega_a')\} \qquad \text{(IV-35b)}$$

$$Q \simeq Q'\{1 - 2Qk_{33}d_3(\omega_0'/\omega_a')\sqrt{1-(1/4Q^2)}\} \qquad \text{(IV-35c)}$$

where the quantities k_{33}, d_1, and d_3 are evaluated (as in Example 1) for a pole quality factor Q'. Consider the following example.

Example 4

Suppose that the network of Fig. IV-1 and Example 1 is to realize a low-pass filter with $\omega_0' = 100$ Krad/s and Q' = 5, and that an amplifier with a gain-bandwidth product $|\omega_a'| \simeq 5$ Mrad/s and an open-loop bandwidth $\omega_b \simeq 50$ rad/s is to be used. Since $\alpha' = \omega_0'/2Q' = 10$ Krad/s is much greater than ω_b, (IV-35) may be used to estimate the values of $\omega_0 = (G/C)$ and $Q = 1/(3-K_0)$ that will result in the desired pole locations. This estimate will be a good one if the relative changes $|(Q' - Q)/Q|$ and $|(\omega_0' - \omega_0)/\omega_0|$ are much less than 1.

From Example 1:

$$K_0 = (3-1/Q') = 2.8$$

$$k_{33} = -K_0 = -2.8$$

$$d_1 = -\tfrac{1}{2}(3-1/Q') = -1.4$$

$$d_3 = -(3-1/Q')/\{2Q'\sqrt{1-1/4Q'^2}\} = -0.141$$

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Hence, from (IV-35);

$$\omega_0 = 108 \text{ Krad/s}$$

and

$$Q = 4.60$$

Thus, the network should be designed with

$$(G/C) = \omega_0 = 108 \text{ Krad/s}$$

and

$$K_0 = (3-1/Q) = 2.78$$

Note that the relative changes in ω_0 and Q are both less than 8% so that the assumption of small changes is still justified. Furthermore, the assumption made in deriving (IV-35) (that the expressions for the sensitivity measures at $(1/\omega_a) = 0$ hold at $(1/\omega_a) = (1/\omega_a')$) may be checked by using (IV-33) to find ω_0' and Q' given $\omega_0 = 108$ Krad/s and $Q = 4.60$. The result is $\omega_0' = 99.5$ Krad/s and $Q' = 4.97$. Considering that the value of ω_a' is rarely known to better than ± 10%, the assumption made in deriving (IV-35) is clearly justified.

Example 4 illustrates the use of (IV-35) to take into account small relative changes in ω_0 and Q due to the finite gain-bandwidth product $|\omega_a'|$. If the relative changes are greater than 10%, the first order approximation implied in (IV-33) and (IV-35) is no longer valid and the graphical approach of Chapter II must be used. Generally, however, if (IV-35) indicates relative shifts greater than 10%, the network designer should either choose an amplifier with a greater gain-bandwidth product or a network realization

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with lower sensitivity measures. Only if neither of these courses is possible should the graphical approach of Chapter II be used. This is because large relative changes usually imply poor sensitivities, not only at $(1/\omega_a) = 0$, but also at $(1/\omega_a)$ not equal to zero (Petrela and Budak 1971b). As a result, although the graphical procedure of Chapter II yields a network that has the desired pole locations at a given value of $|\omega_a|$, these pole locations can be expected to be extremely sensitive to changes in $|\omega_a|$ due to temperature variations, etc. The graphical procedure of Chapter II is really useful only when the maximum available gain-bandwidth product coupled with the minimum achievable sensitivity measures still results in relative changes of ω_0 or Q greater than 10%.

Fortunately, the sensitivities of many existing network realizations can be considerably reduced by a proper use of any degrees of freedom that are present. The procedures for this, and some resulting network designs, are considered in section C of Chapter V.

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CHAPTER V

APPLICATION: MINIMUM SENSITIVITY FILTERS

The preceding two chapters are concerned primarily with the definition, properties, and evaluation of the new sensitivity measures introduced in Chapter III. This chapter considers the application of these concepts to the design of minimum sensitivity filters. There are two basic ways in which this can be done. The first is to use the sensitivity properties developed in Chapters III and IV to modify an existing network so as to achieve the desired sensitivity properties. This is possible because the element ratios of any network realization are not completely specified by the required transfer function. In the past these element ratios have been chosen to suit one's convenience; for example, equal capacitors, equal resistors or unity gain amplifiers were often chosen - sometimes for practical reasons and sometimes just because such choices simplified the algebraic expressions involved. For second order networks, however, these element ratios can be chosen to minimize the new figure of merit $|M_p|$ introduced in Chapter III. This is the subject of section C of this chapter.

The results of Chapters III and IV not only permit the modifications of an existing network design, but lead to a procedure for the direct synthesis of an active network that realizes a given n-th order transfer function and also has certain prescribed sensitivity properties. This is the other method presented in this chapter. These two methods are both based on the new results of Chapters III and IV, but the synthesis method is more general and will therefore be considered first. However, some design methods for achieving certain

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sensitivity properties have already been proposed in the literature. Section A reviews these and discusses their chief limitations. Section B then presents the new procedure for realizing both a given transfer function and certain desired sensitivity properties.

A. The Design of Networks with Prescribed Sensitivity Properties

Many investigators have tackled the problem of synthesizing an active filter with both a given transfer function and certain sensitivity properties. They have met with relatively little success - to quote Thomas (Thomas 1971a):

In fact, one might argue that the most important result of a decade of study was the development of the concept of sensitivity, about which much was written but little could be done.

Specifically, as far as the sensitivity of the poles of the transfer function to the active gain element is concerned, only two basic approaches that deserve serious consideration have been proposed.

The first is due to Hakim (Hakim 1965). Hakim's design procedure uses an amplifier with a finite gain K and infinite bandwidth to obtain both the desired transfer function T(s) and certain desired properties for the derivative (dp/dK) where p is a specified pole of T(s). Hakim's work is interesting because it was the first to provide significant insight into the dependence of the poles on the amplifier gain; however, his results are not very useful because of his choice of (dp/dK) as the sensitivity parameter of interest, and because of his assumption of infinite bandwidth. Even if the amplifier bandwidth is assumed infinite the poles of the transfer function T(s) are the

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roots of

$$U(s) = V(s) - (1/A_0) \cdot N_{33}(s) \qquad (V\text{-}1)$$

where $V(s)$ is the nominal denominator of $T(s)$ and A_0 is the dc open loop gain of the amplifier ((III-2) and (III-17)). Since A_0 is the quantity that actually varies in practice, the pole sensitivity with respect to A_0 or $(1/A_0)$ must be considered. From (III-1) and (III-2) the closed loop gain for an infinite bandwidth amplifier is

$$K = \frac{A_0}{1 + (A_0/K_0)} = \frac{1}{(1/A_0) + (1/K_0)} \qquad (V\text{-}2)$$

so that $d(1/A_0)/dK = -1/K^2$; hence

$$\frac{dp}{dK} = \frac{-1}{K^2} \cdot \frac{dp}{d(1/A_0)} \qquad (V\text{-}3)$$

Since $dp/d(1/A_0)$ is itself a function of K, Hakim's minimization of $|dp/dK|$ does not yield networks with minimum $|dp/d(1/A_0)|$.

Moschytz (1971) has recently drawn further attention to the need to consider the sensitivities with respect to the open loop gain by calling for the use of the magnitude of the (gain)·(Q-sensitivity) product

$$|K \cdot S_K^Q| = |K \cdot \left(\frac{K}{Q} \cdot \frac{dQ}{dK}\right)| \qquad (V\text{-}4)$$

as the figure of merit to be minimized in integrated active filters. But $|K \cdot S_K^Q|$ is the same as $|(1/Q)dQ/d(1/A_0)|$ because from (V-2),

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$$\frac{dQ}{dK} = \frac{-1}{K^2} \frac{dQ}{d(1/A_0)} \qquad (V\text{-}5)$$

so that from (V-4),

$$|K \cdot S_K^Q| = (1/Q)|dQ/d(1/A_0)| \qquad (V\text{-}6)$$

Hence the use of $|K \cdot S_K^Q|$ as a figure of merit is just an indirect way to minimize $|dQ/d(1/A_0)|$. Moschytz (1971) recognizes this, and his proposal of $|K \cdot S_K^Q|$ as a figure of merit is probably due to the fact that S_K^Q has been considered a significant sensitivity parameter for many years, and expressions for S_K^Q have been derived and tabulated (e.g. Mitra 1969a) for many second order filters. However, the use of $|dQ/d(1/A_0)|$ - or equivalently $|K \cdot S_K^Q|$ - as a figure of merit not only assumes an infinite bandwidth amplifier but also presupposes either the use of a network configuration for which $|d\omega_0/d(1/A_0)|$ is zero, or the use of a compensation scheme that can somehow cancel the change in ω_0 produced by a change in A_0. Moschytz (1971) assumes the use of a network configuration for which $d\omega_0/dK$, and hence $|d\omega_0/d(1/A_0)|$, is zero. On the other hand, Gaash et al (1966) proposed the use of a compensation scheme to offset the change in ω_0 due to a change in A_0.

Gaash's primary contribution - used by Horowitz (1968) - was the recognition that in integrated circuit realizations the resistors and capacitors all track well with temperature, and the temperature coefficient of the RC products can be closely controlled. Because the resistor ratios and the capacitor ratios (that together determine

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the nominal Q of the pole) track, any change ΔQ in the pole Q due to a temperature variation ΔT is due solely to the corresponding change ΔA in the amplifier open loop gain; that is, from Chapter I,

$$\frac{\Delta Q}{Q} = \frac{1}{Q} \cdot \frac{dQ}{dA} \cdot \Delta A \qquad \text{(V-7)}$$

On the other hand, the change in the ω_0 of the pole depends on both the temperature variation of the amplifier open loop gain A and the temperature variation of the RC products that determine the nominal ω_0 of the pole. Again, from Chapter I,

$$\frac{\Delta\omega_0}{\omega_0} = -\left(\frac{\Delta R}{R} + \frac{\Delta C}{C}\right) + \frac{1}{\omega_0} \cdot \frac{d\omega_0}{dA} \cdot \Delta A \qquad \text{(V-8)}$$

From this Gaash et al (1966) proceeded to design temperature insensitive networks by first assuming an infinite bandwidth amplifier so that $A = A_0$ and then designing a network with zero Q-sensitivity (i.e. $dQ/dA_0 = 0$). The variations in ω_0 due to the finite ω_0-sensitivity (i.e. finite $d\omega_0/dA_0$) were compensated for by controlling the temperature coefficients of the RC products to obtain

$$\left(\frac{\Delta R}{R} + \frac{\Delta C}{C}\right) \simeq \frac{1}{\omega_0} \cdot \frac{d\omega_0}{dA} \cdot \Delta A \qquad \text{(V-9)}$$

and hence, from (V-8), $(\Delta\omega_0/\omega_0) \simeq 0$. Horowitz (1968) later extended this approach to achieve either zero temperature sensitivity at more than one temperature or the same Q and ω_0 at two different specified temperatures.

The basic idea of Gaash and Horowitz - namely, the use of a network with zero Q-sensitivity coupled with a compensation scheme

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to offset the effect of a non-zero ω_0-sensitivity - is sound. However, the design procedures they propose to achieve a network with zero Q-sensitivity are limited by their assumption of infinite bandwidth for the amplifier. This is shown by the following consideration of the general limitations of the infinite bandwidth assumption.

It is important to consider these limitations in some detail because (with the exception of Petrela and Budak (1971a)) no one has as yet considered the problem of minimizing the ω_0, Q, or pole sensitivities with respect to the open loop gain of the amplifier without making the infinite bandwidth assumption by neglecting the pole of the amplifier.

The effect of neglecting the pole of the amplifier is most clearly shown by considering the new general sensitivity measures M_p, M_{ω_0} and M_Q introduced in Chapter III, under two conditions:

1. For $\omega_b >> \omega_0$ so that $\{(p + \omega_b)/p\} \simeq \omega_b/p$. This condition represents the infinite bandwidth assumption.
2. For $\omega_b << \omega_0$ so that $\{(p + \omega_b)/p\} \simeq 1$. This condition (discussed in Chapter IV) represents the situation most commonly encountered in practice.

These two conditions can be compared by assuming that the changes in the gain-bandwidth product are due solely to changes in the dc open loop gain so that for any parameter x,

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$$dx/d(1/\omega_a) = dx/d(1/\omega_b A_0) = \omega_b\{dx/d(1/A_0)\} \tag{V-10}$$

Then from (III-21) and (III-26):

$$M_p\Big|_{\omega_b>>\omega_0} = \frac{\omega_b}{p\omega_0}\cdot\frac{dp}{d(1/A_0)}\Big|_{\omega_b>>\omega_0} = \left(\frac{\omega_b}{p}\right)\left[\frac{N_{33}(p)}{\omega_0 V'(p)}\right] \tag{V-11}$$

But

$$M_p\Big|_{\omega_b<<\omega_0} = \frac{\omega_b}{p\omega_0}\cdot\frac{dp}{d(1/A_0)}\Big|_{\omega_b<<\omega_0} = \left[\frac{N_{33}(p)}{\omega_0 V'(p)}\right] \tag{V-12}$$

Hence

$$\frac{dp}{d(1/A_0)}\Big|_{\omega_b<<\omega_0} = \left(\frac{p}{\omega_b}\right)\frac{dp}{d(1/A_0)}\Big|_{\omega_b>>\omega_0} \tag{V-13}$$

This means that the minimization of the total pole variation by minimizing $|dp/d(1/A_0)|$ and neglecting the amplifier pole, is equivalent to minimizing $|M_p|$ - which includes the effect of the amplifier pole. Consequently, any design procedures that minimize the pole sensitivity magnitude by minimizing $|dp/d(1/A_0)|$ under the assumption of infinite bandwidth are still valid even when the assumption itself does not hold.

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This is not so for the ω_0- or Q-sensitivities. Rather, if the amplifier pole is taken into account and $\omega_b << \omega_0$, the actual pole shift Δp due to a change in A_0 is related to the apparent pole shift $\Delta p'$ found by neglecting the amplifier pole by (V-13), with the result that:

$$\Delta p \simeq (p/\omega_b)\Delta p' \tag{V-14}$$

This relationship is illustrated by Fig. V-1. Since $\omega_b << \omega_0 = |p|$, the actual pole shift Δp has a much greater magnitude than the apparent pole shift $\Delta p'$. More important, the direction of Δp is radically different from that of $\Delta p'$. As illustrated in Fig. V-1, multiplication by (p/ω_0) rotates the $\Delta p'$ vector counterclockwise by the angle of p itself. From

$$\begin{aligned} p &= -\alpha + j\beta \\ &= -\omega_0/2Q + j\omega_0\sqrt{1-(1/4Q^2)} \end{aligned} \tag{V-15}$$

it is readily shown that (V-13) implies that the actual changes $\Delta\omega_0$ and ΔQ (corresponding to Δp) are related to the apparent changes $\Delta\alpha'$ and $\Delta\beta'$ (corresponding to $\Delta p'$) by:

$$(\Delta\omega_0/\omega_0) \simeq -(\alpha/\omega_b)\cdot(\Delta\alpha'/\alpha) \tag{V-16}$$

and

$$(\Delta Q/Q) \simeq -(\beta/\omega_b)\cdot(\Delta\beta'/\beta) \tag{V-17}$$

This has serious implications. For example, suppose a network is to have a very small Q-sensitivity, but a very high ω_0-sensitivity is permitted; if this is apparently achieved by a method that neglects

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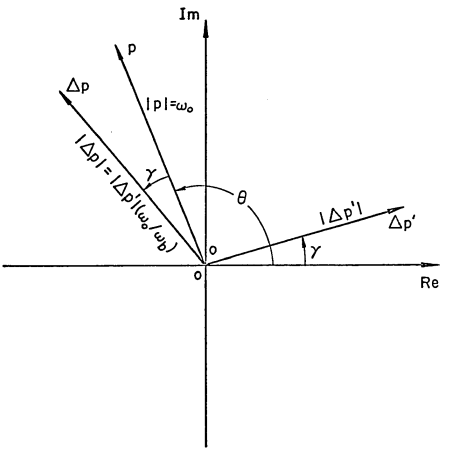

Fig. V-1

Illustration of the Relationship

$\Delta p = (p/\omega_0)\Delta p'$

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the amplifier pole, then for $\omega_b << \omega_0$ the effect of that amplifier pole is to give the actual network a low ω_0-sensitivity but a high negative Q-sensitivity. Another common specification (especially for high Q networks) is to require a negative ΔQ to insure stability. However, if this is achieved by neglecting the amplifier pole, the presence of that pole might in fact result in a positive ΔQ - that could easily cause instability - when $\omega_b << \omega_0$.

For this reason the design methods of Gaash and Horowitz are only fully applicable when the open loop bandwidth ω_b is sufficiently large to make the factor $\{(p + \omega_b)/p\}$ approach ω_b rather than unity. Then $\Delta p \simeq \Delta p'$ and the amplifier pole presents no problem. However, this requires $\omega_b >> \omega_0$, which, as discussed in Chapter IV, is not the usual situation of interest. Rather, in practice $\{(p + \omega_b)/p\} \simeq 1$, so that a method that takes the amplifier pole into account is needed. The new procedure proposed in the next section does this. Moreover, it has the additional advantage of being both simpler and more general than the other methods heretofore proposed.

B. A New Design Procedure

The design method presented in this section depends on the new results of Chapter III. In particular, equations III-7, 11, 12, 21, 23, 24, 26, 30, 60, 61, 64 and 65 will be used. Note that these equations simply state:

1. That the nominal transfer function is

$$T_n(s) = \frac{-N_{31}(s)}{V(s)} \qquad \text{(V-18)}$$

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where

$$V(s) = N_{32}(s) - \frac{1}{K_0} N_{33}(s) \qquad \text{(V-19)}$$

is defined to have unity leading coefficient, and for the operational amplifier case $(1/K_0) = 0$ while for the finite amplifier case K_0 is the nominal dc gain of the amplifier.

2. That the sensitivity measures for a nominal pole at $p = -\alpha + j\beta = -\omega_0/2Q + j\omega_0\sqrt{1-(1/4Q^2)}$ are:

$$M_p = \frac{1}{p\omega_0}\cdot\frac{dp}{d(1/\omega_a)} = \left(\frac{p+\omega_b}{p}\right)\left(\frac{N_{33}(p)}{\omega_0 V'(p)}\right) \qquad \text{(V-20a)}$$

$$M_{\omega_0} = \frac{1}{\omega_0^2}\cdot\frac{d\omega_0}{d(1/\omega_a)} = \mathrm{Re}\{M_p\} \qquad \text{(V-20b)}$$

and

$$M_Q = \frac{1}{\omega_0 Q}\cdot\frac{dQ}{d(1/\omega_a)} = -2Q\sqrt{1-(1/4Q^2)}\,\mathrm{Im}\{M_p\} \qquad \text{(V-20c)}$$

Moreover, let it be assumed that the degree n_{33} of $N_{33}(s)$ equals the degree n of V(s) so that

$$N_{33}(s) = k_{33}\prod_{i=1,n}(s+\sigma_i) \qquad \text{(V-21)}$$

and

$$(1/k_{33}) = L - (1/K_0) \qquad \text{(V-22)}$$

For convenience, Fig. III-2 which defines the functions $N_{31}(s)$, $N_{32}(s)$ and $N_{33}(s)$ as well as the high frequency current gain

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$$L = \lim_{s \to \infty} \left[\frac{-I_2}{I_3} \right] \qquad \text{(V-23)}$$

of the passive RC network, is reproduced as Fig. V-2. Equations (V-20) and (V-21) show that the pole sensitivity depends only on the leading coefficient k_{33} and the zeros $-\sigma_i$ of $N_{33}(s)$. Hence, from (V-22) and Fig. V-2, a proper choice of the high frequency current gain L and the zeros $(-\sigma_i)$ of the driving point admittance y_{33} of the passive RC network will yield any desired sensitivity property that is possible. In particular, the direction of the pole shift is controlled solely by the zeros $-\sigma_i$ of y_{33}, for k_{33} just affects the magnitude of the pole sensitivity measure M_p — see (V-20) and (V-21). For example, if the σ_i are chosen to make M_p real, the pole p will have zero ω_0-sensitivity; on the other hand, if the σ_i are chosen to make M_p purely imaginary, the pole p will have zero Q-sensitivity.

Moreover, (V-18) and (V-19) show that if an operational amplifier ($1/K_0 = 0$) is used, the poles and zeros of the nominal transfer function $T_n(s)$ do not depend on either the coefficient k_{33} or the zeros $-\sigma_i$ (which are the only quantities that affect the sensitivity measures (V-20)). Rather, the zeros of the transfer function $T_n(s)$ are the zeros of the transfer admittance $y_{31}(s)$, whereas the poles of $T_n(s)$ are the zeros of the transfer admittance $y_{32}(s)$ of the passive RC network — see Fig. V-2.

To realize both the desired transfer function (to within a gain constant) and any desired directivity properties of the pole sensitivity measure, it is therefore only necessary to realize an RC

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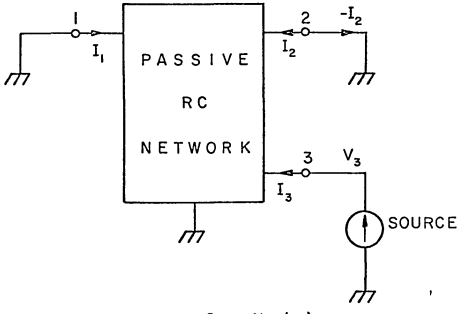

$$y_{13} = y_{31} = \frac{I_1}{V_3} = \frac{-N_{31}(s)}{D(s)}$$

$$y_{23} = y_{32} = \frac{I_2}{V_3} = \frac{-N_{32}(s)}{D(s)}$$

$$y_{33} = \frac{I_3}{V_3} = \frac{N_{33}(s)}{D(s)}$$

$$L = \lim_{s \to \infty} \left[\frac{-I_2}{I_3} \right]$$

Fig. V-2

Definition of the Relevant y-Parameters of the Passive Network

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network with specified zeros for its input admittance y_{33} and its transfer admittances y_{32} and y_{31}. One simple way of doing this is to replace the original passive RC network (T) by two subnetworks (A and B) as shown in Fig. V-3. Then the y-parameters (y_{ijT}) of the overall network (T) are related to the y-parameters (y_{ijA} and y_{ijB}) of the two subnetworks (A and B) by

$$y_{33T} = y_{33A} + y_{33B} \tag{V-24a}$$

$$y_{32T} = y_{32A} \tag{V-24b}$$

$$y_{31T} = y_{31B} \tag{V-24c}$$

If these y-parameters are all chosen to have the same denominator polynomial D(s), (V-24) becomes

$$N_{33T} = N_{33A} + N_{33B} \tag{V-25a}$$

$$N_{32T} = N_{32A} \tag{V-25b}$$

$$N_{31T} = N_{31B} \tag{V-25c}$$

Thus, to realize both the desired transfer function $T_n(s)$ and the desired sensitivity property, the desired poles of $T_n(s)$ are assigned to be the zeros of N_{32A}; the desired zeros of $T_n(s)$ are assigned to be the zeros of N_{31B}; and the desired zeros of N_{33T} (which are to produce the desired sensitivity properties) are either assigned to be both the zeros of N_{33A} and the zeros of N_{33B}, or the coefficients of N_{33A} and N_{33B} are so chosen that the zeros of $(N_{33A} + N_{33B})$ are the desired zeros of N_{33T}.

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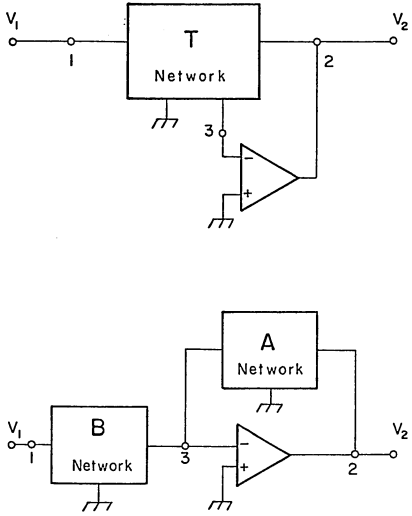

Fig. V-3

A Possible Realization of the Passive Network T in Terms of Two Passive Subnetworks A and B

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Once this is done, existing methods for synthesizing a passive RC network with both prescribed input admittance and transfer admittance zeros (Guillemin 1957; Thorp 1965; Hazony and Lagerlof 1966; Halpern 1969) may be used to synthesize the networks A and B. In simple cases network B may not really be necessary since the zeros of $T_n(s)$ can often be realized by the "pliers" insertion (Guillemin 1957) of the input voltage source in series with one or more grounded elements of network A. Consider the following example.

Example 1

Suppose that it is desired to realize the band-pass transfer function

$$T(s) = \frac{Hs}{s^2 + 0.2s + 1}$$

where H is an arbitrary constant, by an active filter with zero ω_0-sensitivity with respect to the amplifier gain-bandwidth product. Assuming $\{(p + \omega_b)/p\} \simeq 1$, then (see Table IV-2) the filter must have $d_1 = 0$. But if

$$N_{33T} = k_{33}(s^2 + q_1 s + q_0)$$

then from (IV-6), d_1 will be zero if, and only if,

$$q_1 = 2\alpha = 0.2$$

Since the zeros of N_{33T} must be distinct, q_0 must be less than $(q_1/2)^2 = 0.01$. Let

$$q_0 = 0.005$$

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The denominator polynomial D(s) of y_{33T} can be chosen to be any polynomial of degree two or one with real negative zeros interlaced with those of N_{33T} as discussed in section B of Chapter III. Let

$$D(s) = (s + 0.1)$$

The zeros of N_{33T} are assigned to N_{33A} — in the hope that network B is not required. If network B is required to realize the zeros of T(s), then the zeros of N_{33T} can be simply assigned to N_{33B} as well, so that network A will not have to be modified. As a result, network A must have:

$$y_{33A} = k_{A3} \frac{(s^2 + 0.2s + .005)}{(s + 0.1)}$$

$$y_{32A} = -k_{A2} \frac{(s^2 + 0.2s + 1)}{(s + 0.1)}$$

where k_{A3} and k_{A2} are arbitrary positive coefficients. In order to synthesize the network, Guillemin's parallel ladder technique (Guillemin 1957) may be used. This can be done with one ladder (X) being used to realize

$$y_{33AX} = k_{A3X} \frac{(s^2 + 0.2s + 0.005)}{(s + 0.1)}$$

and

$$y_{32AX} = -k_{A2} \frac{(s^2)}{(s + 0.1)}$$

while the other ladder (Y) is used to realize

$$y_{33AY} = k_{A3Y} \frac{(s^2 + 0.2s + 0.005)}{(s + 0.1)}$$

$$y_{32AY} = -k_{A2} \frac{(0.2s + 1)}{(s + 0.1)}$$

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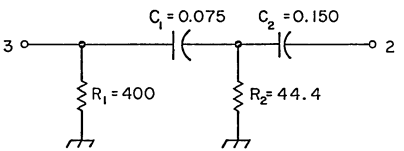
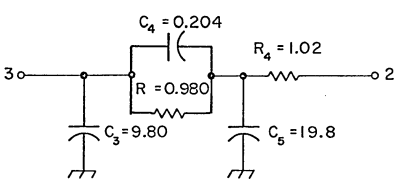


(a)
X - Ladder



(b)
Y - Ladder

Fig. V-4

Ladder Networks for Example 1

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The resultant ladder networks are shown in Fig. V-4. Their parallel connection has the specified y-parameters

$$y_{33T} = y_{33A} = y_{33AX} + y_{33AY}$$

and

$$y_{32T} = y_{32A} = y_{32AX} + y_{32AY}$$

Furthermore, inspection of these ladders shows that the desired numerator polynomial can be realized by connecting the input voltage source between R_2 *and ground to yield:*

$$y_{31T} = (y_{32AX})/(sC_2R_2)$$

$$= \frac{-(k_{A2}/C_2R_2)s}{(s + 0.1)}$$

The overall zero ω_0*-sensitivity network is shown in Fig. V-5 (the element values are as given in Fig. V-4). Note that the choice* $q_0 = 0.005$ *forces* $d_3 = -0.50$ *(approximately its minimum possible value). The high frequency current gain L of the resultant network is by inspection*

$$L = \frac{\{C_1C_2/(C_1 + C_2)\}}{\{C_1C_2/(C_1 + C_2)\} + C_3 + \{C_4C_5/(C_4 + C_5)\}} = \frac{1}{201}$$

As a result $k_{33} = \frac{1}{L} = 201$ *and the* ω_0*- and Q-sensitivities are, from Table IV-2,*

$$M_{\omega_0} = -k_{33}d_1 = 0$$

and

$$M_Q = 2Q(B/\omega_0)k_{33}d_3 \simeq -1000$$

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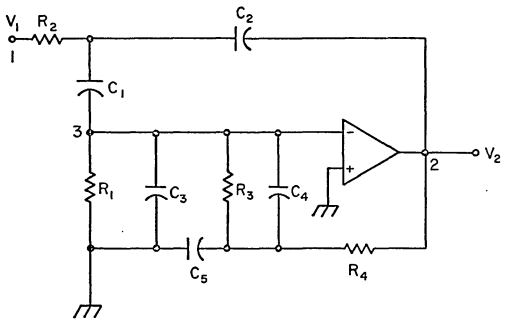

Fig. V-5

Zero ω_0-Sensitivity Band-Pass Filter of Example 1

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Although $|M_Q| \simeq 1000$ is very high, it is not too far off from the optimum value given by (IV-28), which states that the lowest $|M_Q|$ possible for any operational amplifier filter with zero ω_0-sensitivity is

$$|M_Q| \geq (Q\omega_0^2/q_0) > 4Q^3\sqrt{1-(1/4Q^2)}$$

For the given transfer function $Q = 5$ and therefore any zero ω_0-sensitivity operational amplifier network must have $|M_Q| > 500$. In this particular case the choice $q_0 = 0.005$ causes $(Q\omega_0^2/q_0) = 1000$. A higher choice of q_0 (close to its upper limit, α^2) would cause the $|M_Q|$ of this network to approach the lower limit of 500; however, if this is done the element ratios (which are large — see Fig. V-4) become so large as to make the network impractical.

Example 1 illustrates a shortcoming that is characteristic of all presently available methods for directly synthesizing an RC network with both prescribed input admittance and transfer admittance zeros. All methods proposed have the characteristic that during the synthesis procedure very little control is available on the element ratios. Not that high element ratios are peculiar to ladder structures; they seem to arise in all RC networks whenever the distance between two zeros of the input admittance becomes small compared to the magnitudes of the (complex) zeros of the transfer admittance; that is, in terms of Fig. IV-1, whenever $|d_1| << 1$. The nature of this dependence is as yet not understood; however, when designing simple RC networks (such as those of the filters presented in Chapter II) by

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algebraic manipulation, one can keep an "eye" on what is happening to the element ratios and thus control the tradeoffs necessary to keep the element ratios within bounds. This is not possible in the numerical direct synthesis schemes. There the zeros must be chosen a priori, and it is not known until after the procedure is completed how good, or bad, the resultant element ratios are. If needed, iterative procedures can be used to overcome this in practice. Fortunately, such iterative schemes are rarely necessary because the network under consideration is usually a second-order one. As a result, the necessary tradeoffs can often be identified by inspection, or at worst require some algebraic manipulation. Consider the following example.

Example 2

Suppose it is desired to improve the very high resistance ratios of the network of Fig. V-5 and yet maintain its zero ω_0*-sensitivity property.*

Inspection of the element values given in Fig. V-4 shows that the element ratios within each ladder are quite reasonable; the overall element ratios are high primarily because the two ladders require widely different impedance levels to properly match the coefficients of their individual transfer admittances. In order to vary the coefficients of the transfer admittance of ladder X without affecting its input admittance, it is only necessary to introduce an additional capacitor C_6 *in parallel with* R_2*, as shown in Fig. V-6. As far as the input admittance is concerned,* C_6 *is in parallel with* C_2*; but for the transfer admittance,* C_6 *and* C_2 *now form a current divider that allows*

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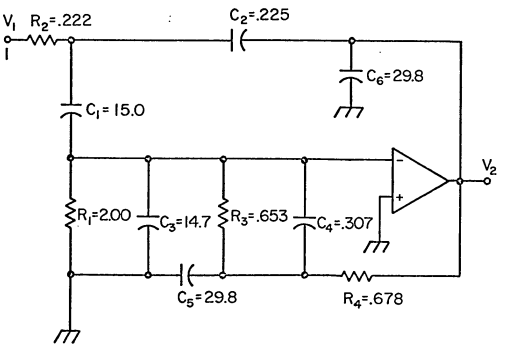

Fig. V-6

Zero ω_0-Sensitivity Band-Pass Filter

of Example 2

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the reduction of the transfer admittance by a constant factor. Fig. V-6 shows the resulting element ratios obtained by taking advantage of the additional degree of freedom afforded by C_6. Unfortunately, C_6 also reduces the high frequency current gain L and thus increases k_{33} and $|M_Q|$ which are now respectively

$$k_{33} = 333 \qquad M_Q = -1665$$

However, the reduction in the maximum resistance ratio by a factor of 40 is probably worth the increase in $|M_Q|$ by a factor of 0.6.

The results of Examples 1 and 2 should not be taken to imply that ladder realizations must be used. To synthesize a filter with both a prescribed transfer function and prescribed sensitivity properties only requires the realization of a passive RC network with specified driving point and transfer admittances. The general realization schemes of Guillemin (1957), Hazony and Lagerlof (1966), and others cited, all involve the use of ladder networks and Examples 1 and 2 are intended to illustrate the problems associated with such ladder realizations. In particular, ladder realization techniques which use only grounded capacitors are to be avoided. The use of only grounded capacitors forces the high frequency current gain L in (V-22) to be zero, with the result that the k_{33} coefficient (and hence $|M_p|$) becomes infinitely large. In the paper by Petrela and Budak (1971b) it is shown that for such grounded-capacitor networks the pole sensitivity measure M_p, when evaluated at a non-zero value of $(1/\omega_a)$, is proportional to $|\omega_a|$ thus causing $|M_p|$ to be very large when $|\omega_a|$ is large. For this

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reason grounded-capacitor synthesis schemes that use an operational amplifier as the active element (e.g. Giguere and Swami (1970)) result in networks that have very poor sensitivities with respect to $(1/\omega_a)$.

Fortunately, for second order transfer functions, a great many RC configurations have been analyzed and tabulated (e.g. Mitra 1969a). This means that for given specified second order input and transfer admittances, the network topology is often known and it is only necessary to determine the element values that will give both the specified transfer function and the desired sensitivity property. The following example illustrates this by realizing a band-pass configuration that can be simply adjusted for zero Q-sensitivity.

Example 3

It is desired to realize the band-pass transfer function

$$T(s) = \frac{Hs}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2}$$

by an active filter with zero Q-sensitivity with respect to $(1/\omega_a)$, using an amplifier with $\omega_b \ll \omega_0$ (so that $\{(p + \omega_b)/p\} \simeq 1$). As in Example 1, let it be assumed that the zero of T(s) will be realizable by the proper insertion of the input into network A of Fig. V-7(a). Then the y-paramaters of network A must be

$$y_{33A} = \frac{k_{A3}(s^2 + q_1 s + q_0)}{(s + d_0)}$$

$$-y_{32A} = \frac{k_{A2}(s^2 + \frac{\omega_0}{Q} s + \omega_0^2)}{(s + d_0)}$$

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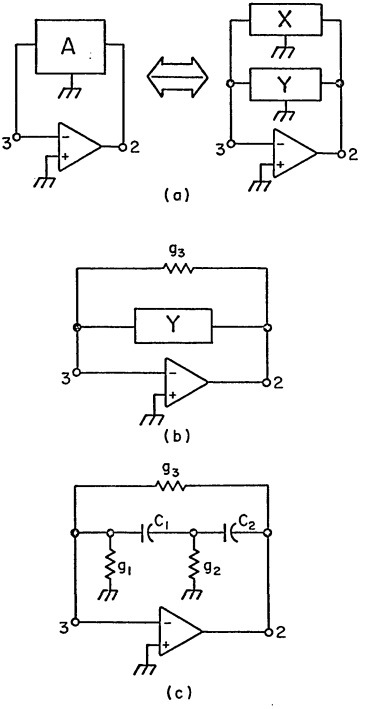

Fig. V-7

Steps in the Realization of the Passive Network of the Zero Q-Sensitivity Band-Pass Filter of Example 3

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where, for simplicity, the denominator polynomial has been assigned a degree of one. Zero Q-sensitivity is obtained if, and only if, q_1 and q_0 are so chosen that $d_3 = 0$ (Table IV-2). From (IV-6), this will be so if, and only if,

$$(q_0 - \omega_0^2) = (\omega_0/2Q)(q_1 - \omega_0/Q)$$

In order to realize it, network A is split into two parallel subnetworks X and Y with admittance parameters

$$-y_{32AX} = \frac{k_{A2}s^2}{s + d_0}$$

$$-y_{32AY} = \frac{\frac{\omega_0}{Q}(s + \omega_0 Q)}{(s + d_0)}$$

and

$$y_{33A} = y_{33AX} + y_{33AY}$$

Now the condition for zero Q-sensitivity allows a great deal more latitude in the choice of the zeros ($-\sigma_1$, $-\sigma_2$) of y_{33A} — and hence in the choice of d_0 ($\sigma_1 < d_0 < \sigma_2$) — than the zero ω_0-sensitivity condition required in Example 1. In particular, the choice $d_0 = \omega_0 Q$, which makes the transfer admittance y_{32} a constant, is now possible. For $d_0 = \omega_0 Q$, $y_{32} = -k_{A2}(\omega_0/Q)$ can be realized by a conductance

$$g_3 = k_{A2}(\omega_0/Q)$$

as shown in Fig. V-7(b). However, y_{33AY} is then also equal to g_3, so that y_{33AX} must be:

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$$y_{33AX} = y_{33A} - g_3$$

$$= \frac{k_{A3}(s^2 + \{q_1 - (k_{A2}/k_{A3})(\omega_0/Q)\}s + \{q_0 - (k_{A2}/k_{A3})\omega_0^2\})}{s + \omega_0 Q}$$

where q_1 and q_0 must be chosen to satisfy the condition for zero Q-sensitivity and yield nonpositive real zeros ($-\sigma_{1X}$, $-\sigma_{2X}$) such that network X is realizable ($\sigma_{1X} < \omega_0 Q < \sigma_{2X}$). For example, this can always be done if $k_{A2} = k_{A3}$ as was the case for the X network in Example 1 and shown in Fig. V-4. Thus, the X network of Fig. V-4 can be used as shown in Fig. V-7(c) to realize the required zero Q-sensitivity filter. The zero of T(s) is realized by inserting the input between conductance g_2 and ground, thus yielding the filter shown in Fig. V-8. Straightforward analysis of the network shown in Fig. V-8 yields that if q_1 and q_0 are constrained by the zero Q-sensitivity requirement, then the resistance ratio is minimized by choosing $C_1 = C_2 = C$. For this choice the other expressions of interest are found to be:

$$Q = 1/2\sqrt{g_2/g_3}\,; \qquad \omega_0 = 2Q(g_3/C)$$

$$k_{33} = 1\,; \qquad q_1 = 2\omega_0 Q + \frac{\omega_0}{Q}(1 + g_1/g_3)\,; \qquad q_0 = \omega_0^2(1 + g_1/g_3)$$

where (g_1/g_3) can be chosen to achieve the constraint for zero Q-sensitivity. The result is

$$(g_1/g_3) = 1/(1 - 1/2Q^2)$$

and yields an ω_0-sensitivity (from Table IV-2 and (IV-6))

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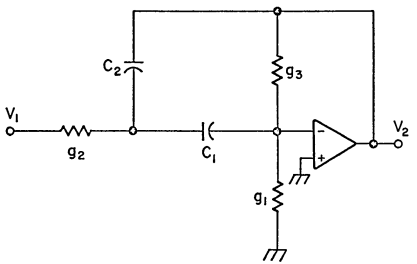

Fig. V-8

Zero Q-Sensitivity Filter of Example 3

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$$M_{\omega_0} = -k_{33}d_1 = \{-1\}\{(1/2\omega_0)(\omega_0/Q - q_1)\} = Q$$

The minimum value theoretically possible for any operational amplifier realization would be $|M_{\omega_0}| \simeq 1$ *(IV-21). Thus, though the resistance ratio needed* $(g_2/g_3 = 4Q^2)$ *is fairly high, this filter does achieve a fairly low* ω_0*-sensitivity as well as zero Q-sensitivity. This realization is also very easily tuned: first* ω_0 *can be adjusted using* g_3*; then* g_2 *can be used to set Q without affecting* ω_0*; and finally,* g_1 *can be used to adjust for zero Q-sensitivity without affecting either* ω_0 *or Q. It should be noted that the realization of Fig. V-8 is the standard operational amplifier band-pass filter discussed in Chapter II with one important difference: namely, the addition of* g_1*. It is easily shown (Table V-3) that if* g_1 *is not used* $|M_Q|$ *is approximately Q, whereas the inclusion of the conductance* g_1 *allows zero Q-sensitivity to be obtained without increasing the required maximum resistance ratio. Moreover, at its zero value, the Q-sensitivity is itself not very sensitive to changes in* g_1*. Note that (from Table IV-2, (IV-6), and the expressions for* g_1 *and* q_0*)*

$$M_Q = Q\{(1 - 1/2Q^2)(g_2/g_3) - 1\}$$

so that

$$\frac{\partial M_Q}{\partial g_2} = (Q/g_2)$$

As a result, even if g_1 *is 5% from its prescribed value,* M_Q *will only increase to approximately (.05)Q which is still very small compared to the magnitude (Q) obtained if* g_1 *is not used.*

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In the above the constant $H = (g_2/C_1)$ was assumed to be unspecified. The choice $C_1 = C_2 = C$ forces $H = 2\omega_0 Q$. If another value of H is specified, equal capacitors cannot be used and a higher resistance ratio becomes necessary.

Examples 1 and 3 illustrated the application of the new concepts introduced in Chapter III to the realization of a given transfer function by an active filter having specified sensitivity properties. For simplicity, the examples treated only second order transfer functions under the assumption $\{(p + \omega_b)/p\} \simeq 1$ of Chapter IV. However, it should be clear that this new synthesis method is applicable both for an n-th order transfer function and for any value of the open-loop bandwidth ω_b. In particular, for an n-th order transfer function with m complex upper half plane poles p_j (j = 1,m) it is only necessary to choose the n_{33} zeros $(-\sigma_i)$ of $N_{33}(s)$ such that, from (V-20) and (V-21),

$$M_{p_j} = \frac{p_j + \omega_b}{p_j} \cdot \frac{k_{33}}{\omega_0 V'(p_j)} \cdot \prod_{i=1,n_{33}} (p_j + \sigma_i) \qquad \text{(V-26)}$$

is real to achieve zero Q-sensitivity for the pole p_j. Since the only restriction on the σ_i is that they be non-negative, real, and distinct (and n_{33} is greater or equal to 2m) this can be done for all the upper half plane poles p_j (j = 1,m) simultaneously. The only requirement is that for an n-th order transfer function the σ_i be chosen to satisfy m simultaneous constraint equations (i.e. (V-26) for j = 1,m) instead of the single constraint equation that arises in the second order case.

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As a result, this new method can realize any transfer function by an active filter with the property that all the complex poles have zero Q-sensitivity with respect to the amplifier gain-bandwidth product $|\omega_a|$. As such, this method represents a significant improvement over the much more tedious design procedures of Gaash et al (1966) and Horowitz (1968) which not only neglect the pole of the amplifier but also achieve zero Q-sensitivity for only one of the complex-conjugate pole pairs of the transfer function.

Since the number of constraints for zero Q-sensitivity is still less than, or equal to, half the number of zeros ($-\sigma_i$) of $N_{33}(s)$, considerable freedom in the selection of these zeros still exists. This freedom can be used as in Example 3 to simplify the network topology, or it can be used, as mentioned in Example 1, to control the ω_0-sensitivities of the various poles. If the ω_0-sensitivities of the various poles are made equal, then Gaash's (1966) original idea of compensating for the temperature changes in the amplifier characteristics by controlling the temperature coefficient of the RC products of the passive network can be used (as discussed in section A) to achieve a filter with zero overall pole sensitivities with respect to temperature. To do this, however, requires that the temperature behaviour of ω_a be well known, and that considerable control be exercised in the manufacturing processes. The details of this are beyond the scope of this thesis, and the reader is referred to Moschytz (1969) and Parisi (1969) as well as Gaash (1966) and Horowitz (1968). The important point is that it can be done. However, because the value of M_{ω_0} depends on both Q and the passive network used, the required passive element temperature coefficients will depend on the Q, ω_0, and topology

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of the filter — as well as on the temperature characteristics of the amplifier. As a result it is rarely worth the trouble to adjust the resistor and capacitor temperature coefficients to compensate for the ω_0 shifts due to the amplifier unless a very large number of the same filter is required. Rather, it is usually more practical to always manufacture the passive elements so that the RC products have approximately zero temperature coefficient (Moschytz 1971) and then choose (or modify) the filter network so as to minimize the overall pole sensitivity magnitude $|M_p|$ with respect to the amplifier. For this reason it is useful to have available design procedures that minimize this figure of merit $|M_p|$ for a given realization, and the next section provides this for the important second order networks presented in Chapter II.

C. Minimization of the Figure of Merit $|M_p|$

The new figure of merit $|M_p|$ introduced in Chapter III is significant not only because it is a true measure of the pole shift due to a finite (and/or varying) amplifier gain-bandwidth product, but also because it balances the effect of a shift in ω_0 and the effect of a shift in Q on the filter performance. Note that from (V-20):

$$|M_p| = \sqrt{M_{\omega_0}^2 + \frac{M_Q^2}{(4Q^2-1)}} \qquad \text{(V-27)}$$

The factor $(4Q^2-1)$ compensates for the fact that in practice a change in the ω_0 of the pole causes a much more drastic change in the frequency response of a filter than the same percentage change in Q does. To show this it is only necessary to consider the effect of a change

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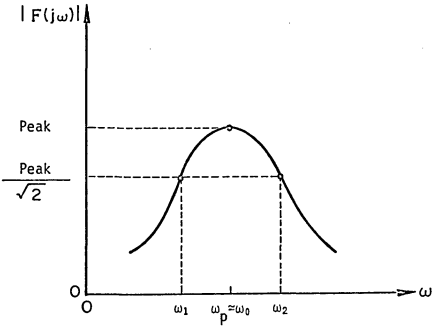

Fig. V-9

Frequency Response (Magnitude) for the Factor

$F(s) = 1/\{s^2 + (\omega_0/Q)s + \omega_0^2\}$

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in ω_0 or Q on the factor

$$F(s) = \frac{1}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2} \qquad (V\text{-}28)$$

of the overall transfer function T(s) (because complex poles always occur as complex conjugate pairs). The frequency response (magnitude only) of this factor is shown in Fig. V-9. The change ΔF_{ω_0} in $|F(j\omega)|$ due to a small relative change in $(\Delta\omega_0/\omega_0)$ in ω_0 is

$$\Delta F_{\omega_0} \simeq \omega_0(\partial|F(j\omega)|/\partial\omega_0)(\Delta\omega_0/\omega_0) \qquad (V\text{-}29)$$

Similarly, the change ΔF_Q in $|F(j\omega)|$ due to a small relative change in Q is

$$\Delta F_Q \simeq Q(\partial|F(j\omega)|/\partial Q)(\Delta Q/Q) \qquad (V\text{-}30)$$

If the changes in ω_0 and Q are both due to a change in the gain-bandwidth product $|\omega_a|$ of the amplifier, then from (V-20)

$$\frac{(\Delta\omega_0/\omega_0)}{(\Delta Q/Q)} \simeq (M_{\omega_0}/M_Q) \qquad (V\text{-}31)$$

From equations V-28, 29, 30 and 31 it then follows (by direct differentiation and substitution) that the changes ΔF_{ω_0} and ΔF_Q in $|F(j\omega)|$, due to changes in ω_0 and Q caused by a change in $(1/\omega_a)$ are related by

$$(\Delta F_{\omega_0})/(\Delta F_Q) = (M_{\omega_0}/M_Q)\{2Q^2(1-1/2Q^2-\omega_0^2/\omega^2)\} \qquad (V\text{-}32)$$

The frequencies ω of primary interest (as regards the effect of a change in ω_0 and Q) are those at the edge of the passband of the

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response of Fig. V-9. For a high value of Q, it can be readily shown that these frequencies are $\omega \simeq \omega_0 \pm \omega_0/2Q$ so that

$$(1 - 1/2Q^2 - \omega_0^2/\omega^2) \simeq \pm\frac{1}{Q}$$

and hence

$$(\Delta F_{\omega_0})/(\Delta F_Q) \simeq \pm(2QM_{\omega_0}/M_Q) \qquad \text{(V-33)}$$

Consequently, it is reasonable to require that any cost function that is used to optimize the network should assign M_{ω_0} approximately 2Q times the weight assigned to M_Q. From (V-27), the new figure of merit $|M_p|$ introduced in Chapter III does just that.

In order to minimize $|M_p|$ for any existing design, two things must be done:

1. $|M_p|$ must be expressed in terms of the element ratios of the network.
2. $|M_p|$ must then be minimized subject to the constraints imposed on the element ratios by the required transfer function and the maximum allowable element ratios permitted.

In general, the first of the above two steps is easily accomplished by using (V-20), (V-21) and (V-22). The second step, however, involves the constrained minimization of a function of several variables. For a high order filter this second step might have to be accomplished by numerical techniques. However, for many second order networks the relationships involved are sufficiently simple so that an

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analytical solution is possible; and a design procedure that will yield minimum $|M_p|$ (as well as the desired transfer function) can be formulated. Since integrated circuit realizations are presumed, care must be taken to formulate the design procedure in terms of a maximum allowable resistance ratio r (Moschytz 1970b). It might also be necessary to specify a maximum allowable capacitance ratio c; however, experience indicates that the capacitance ratios required do not present a problem and the figure of merit is limited by the maximum resistance ratio r. The actual minimization procedure is quite straightforward and is best illustrated by some examples. Consider therefore the following two sample minimizations.

Example 4

Fig. V-10(a) shows a standard operational amplifier realization of the band-pass transfer function

$$T(s) = \frac{Hs}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2}$$

Direct analysis (assuming an ideal amplifier) yields the design equation

$$H = (G/C)a \; ; \qquad \omega_0 = H\sqrt{ac} \; ; \qquad Q = \sqrt{ac}/\{a(1 + c)\}$$

If the gain constant H is fixed, these three relations completely specify the four elements to within an arbitrary impedance level. However, in many applications H is not specified, so some freedom in choosing a or c exists. This freedom can be used to minimize $|M_p|$.

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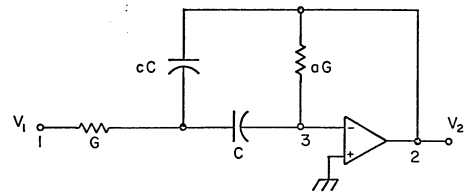
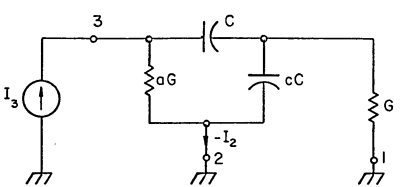


(a)



(b)

Fig. V-10

Band-Pass Filter Optimized in Example 4

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Note that from Fig. V-10(b) (by the method of Example 1 of Chapter IV):

$$q_1 = (\omega_0/Q + aG/C) = \omega_0/Q + \omega_0/\sqrt{ac}$$

$$q_0 = \omega_0^2 \; ; \qquad k_{33} = 1$$

Hence

$$|M_p| = \left| \frac{p + \omega_b}{\omega_b} \right| \left| \frac{N_{33}(p)}{\omega_0 V'(p)} \right|$$

is minimum if, and only if, (from IV-7),

$$\left| \frac{N_{33}(p)}{\omega_0 V'(p)} \right| = |k_{33}(d_1 + jd_3)|$$

is minimum. However, the values of k_{33}, q_0 and q_1 for this network yield (from (IV-6) and (IV-17) that

$$|k_{33}(d_1 + jd_3)| = \frac{1}{2\sqrt{1-1/4Q^2}} \cdot \frac{1}{a(1 + c)}$$

Hence $|M_p|$ will be minimum if, and only if, $a(1 + c)$ is maximum. The a and c ratios are related by

$$Q = \sqrt{ac}/\{a(1 + c)\}$$

Solving this for c yields

$$\sqrt{c} = \frac{1}{2Q\sqrt{a}} \left[1 \pm \sqrt{1-4Q^2a} \right]$$

Since $a(1 + c)$ is to be maximized, the positive root is chosen to obtain the maximum value of c for a given value of a. Then direct substitution gives

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$$a(1 + c) = \frac{1}{2Q^2}\left(1 + \sqrt{1-4Q^2a}\right)$$

This will be a maximum if, and only if, a is minimum. Therefore, if $r \geq 4Q^2$ *is the maximum permissible resistance ratio, choose*

$$a = 1/r$$

and then

$$c = \frac{r}{4Q^2}\left[1 + \sqrt{1-4Q^2/r}\right]^2$$

to obtain

$$H = aG/C = \omega_0/\{1 + \sqrt{1-4Q^2/r}\}$$

and

$$|M_p| = \left|\frac{p + \omega_b}{p}\right|\left[\frac{Q}{(1 + \sqrt{1-4Q^2/r})(\sqrt{1-1/4Q^2})}\right]$$

For $\{(p + \omega_b)/p\} \simeq 1$, $|M_p|$ *has a lower bound* $|M_p|_{min} \simeq \frac{1}{2}Q(1-1/4Q^2)^{-\frac{1}{2}}$ *obtained for* $r >> 4Q^2$ *(and* $c \simeq r/4Q^2$*). For* $r = 4Q^2$, $|M_p| \simeq Q$ *(which corresponds to* $c = 1$*).*

Example 5

Fig. V-11(a) shows the commonly employed positive gain realization of the band-pass transfer function

$$T(s) = \frac{Hs}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

The nominal transfer function (assuming the amplifier to have a dc gain

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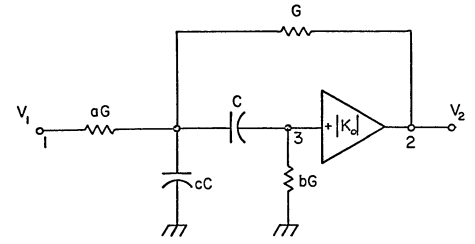
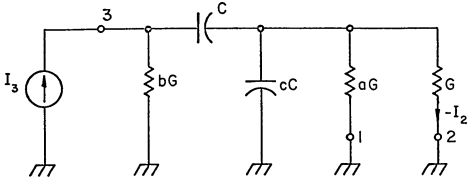


(a)



(b)

Fig. V-11

Band-Pass Filter Optimized in Example 5

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K_0 *and infinite gain-bandwidth product) gives rise to the following design equations:*

$$H = K_0(a/c)(G/C)$$

$$\omega_0 = G/C\sqrt{(b/c)(a + 1)}$$

$$Q = \sqrt{bc(a + 1)}/(1 + a + b + bc - K_0)$$

There are a total of six variables (a, b, c, K_0*, C and G) in this network. Hence, five independent constraints are needed to specify the values of these elements to within an arbitrary impedance level. If H is specified, the nominal transfer function provides three such constraints. However, if H is not specified, as is often the case, the nominal transfer function only provides two constraints (for* ω_0 *and for Q). Let it be assumed that H is not specified. Then three element ratios, or the gain and two element ratios, can be specified in a manner that minimizes the figure of merit* $|M_p|$.

Since in practice the maximum resistance ratio in a network is limited, the two resistance ratios a and b and the gain K_0 *are a natural choice for the variables to be specified so as to reduce* $|M_p|$*; whereas the* ω_0 *and Q constraints of the nominal transfer function are satisfied by, for Q,*

$$c = \frac{1}{4Q^2}\cdot\frac{1 + a}{b}\cdot\left[1 \pm \sqrt{1-4Q^2\{1 + (b-K_0)/(1 + a)\}}\right]^2$$

and, for ω_0*,*

$$(G/C) = \omega_0/\sqrt{(b/c)(a + 1)}$$

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The expression for c also implies (since c is real)

$$K_0 \geq b + (1 + a)(1-1/4Q^2)$$

From Fig. 11(b), the expressions for k_{33}, q_1 *and* q_0 *may be found (using the methods of Example 1, Chapter IV) to be*

$$k_{33} = -K_0 \; ; \qquad q_1 = \frac{\omega_0}{Q} + \frac{K_0}{c}\frac{G}{C} \; ; \qquad q_0 = \omega_0^2$$

From (V-20) and (IV-7)

$$|M_p| = \left|\frac{p + \omega_b}{p}\right|\left|\frac{N_{33}(p)}{\omega_0 V'(p)}\right|$$

will be minimum if, and only if,

$$\left|\frac{N_{33}(p)}{\omega_0 V'(p)}\right| = |k_{33}(d_1 + jd_3)|$$

is minimum. Because $q_0 = \omega_0^2$, d_3 *is proportional to* d_1*(IV-15) and* $|M_p|$ *is minimum if, and only if,* $|d_1|$ *is minimum. From (IV-6),* d_1 *is found, by direct substitution, to be:*

$$d_1 = -\left[\frac{K_0}{2\omega_0}\right]\left[\frac{G}{aC}\right] = \frac{-K_0}{2\sqrt{bc(a + 1)}}$$

Hence, substituting the expression for c,

$$|k_{33}d_1| = \frac{QK_0^2/(a + 1)}{1 \pm \sqrt{1-4Q^2\{1 + (b-K_0)/(1 + a)\}}}$$

To minimize $|d_1|$, *the positive root (in the expression for c) should be used. Moreover, direct differentiation of* $|k_{33}d_1|$ *with respect to*

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K_0 yields that $|k_{33}d_1|$ has its minimum value when

$$K_0 = (4/3)(1+a)\{x - 1/12Q^2\}\{1 + \sqrt{1-x(x-1/4Q^2)/(x-1/12Q^2)}\}$$

where $x = 1 + b/(1+a)$. For $Q > 1$ this value of K_0 is approximately

$$K_0 \simeq (4/3)(1 + a + b)$$

Substituting this value of K_0 into the expression for $|k_{33}d_1|$ yields

$$|k_{33}d_1| = \frac{(16/9)Qx^2(1+a)}{1 + \sqrt{1+(4Q^2/3)x}}$$

When $|k_{33}d_1|$ is considered as a function of x, or of $(1+a)$, it can be shown that

$$\partial|k_{33}d_1|/\partial x > 0 \qquad \text{and} \qquad \partial|k_{33}d_1|/\partial(1+a) > 0$$

and hence both $x = 1 + b/(1+a)$ and $(1+a)$ should be minimized. Moreover, for $Q >> 1$, $|k_{33}d_1|$ is proportional to

$$(1+a)x^{\frac{3}{2}} = (1+a+b)[1+b/(1+a)]^{\frac{1}{2}}$$

Consequently, for a maximum allowable resistance ratio r, the element ratios (a and b) and the gain (K_0) are chosen to be:

$$a = b = (1/r)$$

$$K_0 = (4/3)(1 + 2/r)$$

This yields:

$$c = (r+2)/3 + \{(r+1)/2Q^2\}\{1 + \sqrt{1+(4Q^2/3)(r+2)/(r+1)}\}$$

$$(G/C) = \omega_0 r\sqrt{c/(r+1)} \; ; \qquad H = \omega_0 K_0/\sqrt{c(r+1)}$$

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and

$$|M_p| = \left|\frac{p + \omega_b}{p}\right| \left(\frac{K_0^2\sqrt{r/c}}{2\sqrt{1-1/4Q^2}}\right)\sqrt{\frac{r}{r+1}}$$

For $r >> 2$ *and* $Q >> 1$, $K_0 \simeq 4/3$, $c \simeq r/3$ *and* $|M_p| \simeq (8/9)\sqrt{3} \simeq 1.54$ *when* $\{(p + \omega_b)/\omega_b\} \simeq 1$.

Examples 4 and 5 illustrate the minimization of the figure of merit $|M_p|$ for both an operational amplifier realization and a finite gain amplifier realization of a given transfer function. In both examples it is assumed that the gain constant H of the transfer function is not specified. In Example 4 this was necessary because for that particular realization, if H is specified, then so is $|M_p|$. On the other hand, for the finite gain amplifier realization of Example 5, if H is specified the minimization procedure can still be carried out by expressing K_0 in terms of H, a, and b and then minimizing $|M_p|$ with respect to a and b. Comparing the results, it is seen that the finite gain amplifier realization has the advantage that it can be used for resistance ratios much less than $4Q^2$ and yet (for high Q) still yields a much lower figure of merit than that attainable by the operational amplifier realization with a resistance ratio of $4Q^2$. It should also be noted that the requirement of minimum sensitivity in Example 5 yields $K_0 \simeq 4/3$ for $r >> 2$. This points out that the use of the unity gain configuration, though it does maximize the closed-loop bandwidth of the amplifier, does not necessarily minimize the dependence of the transfer function on the gain-bandwidth product.

The procedures illustrated in Examples 4 and 5 can be applied to the optimization of a great many currently used filter designs.

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Figures V-12 through V-21 give the results of this minimization procedure for the commonly used (e.g. Mitra 1969b) single amplifier low-pass, high-pass, band-pass and band-stop filters. In all cases it is assumed that the gain constant H of the transfer function is not specified, but that the maximum allowable resistance ratio r is specified.

Figures V-12, 13 and 14 give the design equations that yield the low-pass transfer function

$$T(s) = \frac{H}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2} \qquad \text{(V-34)}$$

and achieve a minimum figure of merit $|M_p|$ for the particular network used. The element ratios given yield minimum $|M_p|$ if $Q > 1$. The expression for $|M_p|$ is not given directly; rather, the expressions for d_1 and d_3 are given. It is then a relatively simple matter to obtain

$$M_p = -\{(p + \omega_b)/p\}k_{33}(d_1 + jd_3)$$

Once M_p is known, M_{ω_0} and M_Q can be found (from (V-20b) and (V-20c)). If $\{p + \omega_b/p\}$ is close to unity the simple expressions of Table IV-2 may be used to find M_p, M_{ω_0} and M_Q directly from k_{33}, d_1 and d_3. For comparison, Table V-1 gives approximate expressions for the figure of merit $|M_p|$ that each of the low-pass networks is capable of achieving if $\{(p + \omega_b)/p\} \simeq 1$ and a high resistance ratio is used. In addition, Table V-1 also lists the expressions for the corresponding ω_0- and Q-sensitivity measures.

Figures V-15, 16 and 17 and Table V-2 present the corresponding results for the single amplifier networks commonly used to realize

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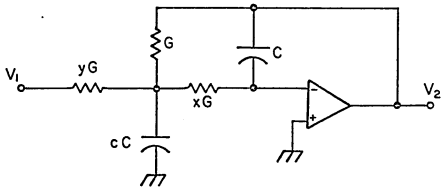

$$x = y = (1/r)$$

$$c = Q^2(r+2)^2/r$$

$$G/C = \omega_o r\sqrt{(c/r)}$$

$$H = -(\omega_o^2/r)$$

$$k_{33} = 1$$

$$d_1 = -(Q/2)\left[1+(2/r)\right]$$

$$d_3 = -1/\sqrt{1-1/4Q^2}$$

Fig. V-12

Operational Amplifier Low-Pass Filter Realization and Design Equations for Minimum Figure of Merit $|M_p|$

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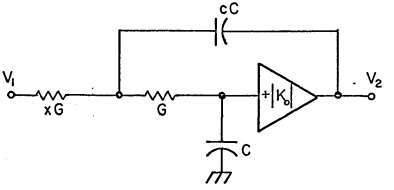

$$x = r$$

$$c = (3r-1) - (r/2Q^2)\left\{\left[1+12Q^2(r+1)/r\right]^{\frac{1}{2}} - 1\right\}$$

$$G/C = \omega_0\sqrt{(c/r)}$$

$$K_0 = 1 + (r+1)/c - (1/Q)\sqrt{(r/c)}$$

$$H = K_0\,\omega_0^2$$

$$k_{33} = -K_0$$

$$d_1 = -\tfrac{1}{2}K_0\sqrt{(c/r)}$$

$$d_3 = d_1/(2Q\sqrt{1-1/4Q^2})$$

Fig. V-13

Positive Finite Gain Amplifier Low-Pass Filter Realization and Design Equations for Minimum Figure of Merit $|M_p|$

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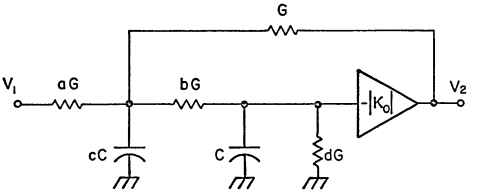

$$a = d = (1/r)$$

$$\text{for } r > 1.6181,\ b = (\sqrt{1+r})/r$$

$$\text{for } r \le 1.6181,\ b = 1$$

$$c = (1+a+b)/(b+d)$$

$$G/C = (\omega_0/2Q)/(b+d)$$

$$|K_0| = (4Q^2-1)(1+a+b)(1+d/b)+b$$

$$H = -(\omega_0^2/r)(1-1/4Q^2)\left[|K_0|/(|K_0|-b)\right]$$

$$k_{33} = |K_0|$$

$$d_1 = 0$$

$$d_3 = -\frac{1}{2}\sqrt{1-1/4Q^2}\left[|K_0|/(|K_0|-b)\right]$$

Fig. V-14

Negative Finite Gain Amplifier Low-Pass Filter Realization and Design Equations for Minimum Figure of Merit $|M_p|$

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Low-Pass Filter Network	Assumed Resistance Ratio r and Q	$\lvert M_p \rvert$	M_{ω_0}	M_Q
Operational Amplifier Fig. V-12	$r \gg 1$ $Q \gg \frac{1}{2}$	$\frac{1}{2}Q$	$\frac{1}{2}Q$	$-\frac{1}{2}Q$
Positive Gain Amplifier Fig. V-13	$r \gg 1$ $Q \gg \frac{1}{2}$	1.5	-1.5	1.5
Negative Gain Amplifier Fig. V-14	$\sqrt{r} \gg 1$ $Q \gg \frac{1}{2}$	$2Q^2$	0	$-4Q^3$

Table V-1

Approximate Figure of Merit $\lvert M_p \rvert$
and Corresponding Sensitivity Measures M_{ω_0} *and* M_Q
for Low-Pass Filters Optimized for Minimum $\lvert M_p \rvert$

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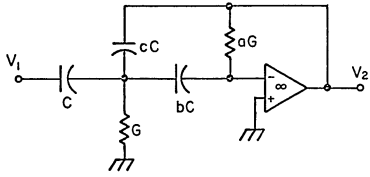

$$a = (1/r) \text{ for } r > 4Q^2$$

$$\text{Choose } c \geq \frac{1}{r/4Q^2 - 1} >> 1$$

$$b = (2cr/4Q^2)\left[1 - \sqrt{1 - 4Q^2(1+c)/cr}\right] - (1+c)$$

$$G/C = (\omega_0/Q)(rbc)/(1+b+c)$$

$$H = -(1/c)$$

$$k_{33} = (c+1)/c$$

$$d_1 = -\left[\frac{(rcb)/(1+b+c) - 1}{2Q(c+1)}\right]$$

$$d_3 = -\left[\frac{2Q^2 + (rbc/(1+b+c)) - 1}{4Q^2\sqrt{1 - 1/4Q^2}(1+c)}\right]$$

Fig. V-15

Operational Amplifier High-Pass Filter Realization and Design Equations for Minimum Figure of Merit $|M_p|$

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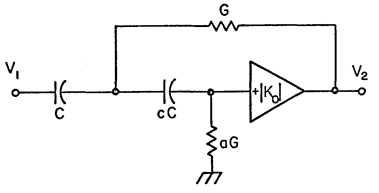

$$a = 1/r$$

$$c = \frac{3}{(r+1)} - \left[\frac{r}{2Q^2(r+1)^2}\right]\left[\sqrt{1 + 12Q^2(r+1)/r} - 1\right]$$

$$K_0 = 1 + (1/r) + (1/rc) - (1/Q\sqrt{rc})$$

$$G/C = \omega_0\sqrt{rc}$$

$$H = K_0$$

$$k_{33} = -K_0$$

$$d_1 = -\frac{1}{2}K_0\sqrt{rc}$$

$$d_3 = d_1 / (2Q\sqrt{1 - 1/4Q^2})$$

Fig. V-16

Positive Finite Gain Amplifier High-Pass Filter Realization and Design Equations for Minimum Figure of Merit $|M_p|$

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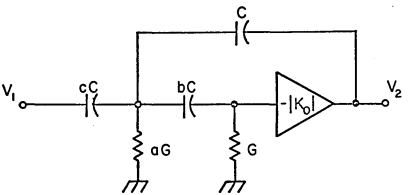

$$a = r$$

Choose $c << 1$

$$b = (c+1)/r$$

$$G/C = \omega_0 Q(2 + 1/r)\left[(c+1)/r\right]$$

$$|K_0| = (c+1)\left[Q^2(2+1/r)^2 - 1\right]$$

$$H = -(c/(c+1))(|K_0|) \;/\; \left[1 + |K_0|/(1+c)\right]$$

$$k_{33} = |K_0|/\left[1 + |K_0|/(1+c)\right]$$

$$d_1 = -|K_0|/\left[2Q(c+1)\right]$$

$$d_3 = -d_1(Q - 1/2Q) \;/\; \sqrt{1 - 1/4Q^2}$$

Fig. V-17

Negative Finite Gain Amplifier High-Pass Filter Realization and Design Equations for Minimum Figure of Merit $|M_p|$

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High-Pass Filter Network	Assumed Resistance Ratio r and Q	$\lvert M_p \rvert$	M_{ω_0}	M_Q
Operational Amplifier Fig. V-15	$r = 8Q^2$ $Q \gg \frac{1}{2}$ $c = 1$	2Q	2Q	-3Q
Positive Gain Amplifier Fig. V-16	$r \gg 1$ $Q^2 \gg 12$	1.5	-1.5	1.5
Negative Gain Amplifier Fig. V-17	$r \gg 1$ $Q \gg \frac{1}{2}$	$2Q^2$	2Q	$4Q^3$

Table V-2

Approximate Figure of Merit $\lvert M_p \rvert$
and Corresponding Sensitivity Measures M_{ω_0} *and* M_Q
for High-Pass Filters Optimized for Minimum $\lvert M_p \rvert$

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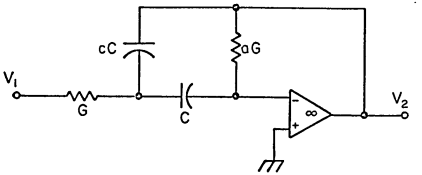

$$a = (1/r) \leq (1/4Q^2)$$

$$c = (r/2Q^2)\left[1 + \sqrt{1 - 4Q^2/r}\right] - 1$$

$$G/C = \omega_0\sqrt{rc}$$

$$H = \omega_0\sqrt{r/c}$$

$$k_{33} = 1$$

$$d_1 = -\frac{1}{2}\sqrt{r/c}$$

$$d_3 = d_1/(2Q\sqrt{1 - 1/4Q^2})$$

Fig. V-18

Operational Amplifier Band-Pass Filter Realization and Design Equations for Minimum Figure of Merit $|K_p|$

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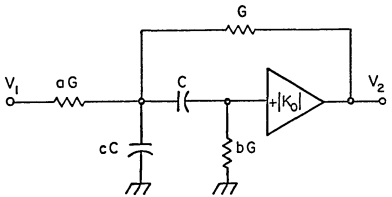

$$a = b = (1/r)$$

$$c = \left(\frac{r+2}{3}\right) + \left(\frac{r+1}{2Q^2}\right)\left[\sqrt{1+\left(\frac{4Q^2}{3}\right)\left(\frac{r+2}{r+1}\right)} + 1\right]$$

$$K_0 = (4/3)(1+2/r)$$

$$G/C = \omega_0 r \sqrt{c/(r+1)}$$

$$H = \omega_0 K_0 / \sqrt{c(r+1)}$$

$$k_{33} = -K_0$$

$$d_1 = -(K_0/2)\sqrt{(r/c)\left\{r/(r+1)\right\}}$$

$$d_3 = d_1 / \left(2Q\sqrt{1-1/4Q^2}\right)$$

Fig. V-19

Positive Finite Gain Amplifier Band-Pass Filter Realization and Design Equations for Minimum Figure of Merit $|M_p|$

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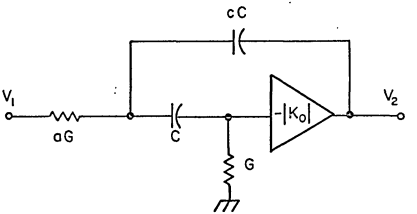

$$a = c = r$$

$$|K_0| = Q^2(2 + 1/r)^2 - 1$$

$$G/C = \omega_0 Q (2 + 1/r)$$

$$H = -(|K_0|/|K_0|+1)(\omega_0 Q)(2 + 1/r)$$

$$k_{33} = |K_0|/(|K_0|+1)$$

$$d_1 = -|K_0|/2Q$$

$$d_3 = \frac{1}{2}|K_0|(1 - 1/2Q^2)/\sqrt{1 - 1/4Q^2}$$

Fig. V-20

Negative Finite Gain Amplifier Band-Pass Filter Realization and Design Equations for Minimum Figure of Merit $|M_p|$

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Band-Pass Filter Network	Assumed Resistance Ratio r and Q	$\lvert M_p \rvert$	M_{ω_0}	M_Q
Operational Amplifier Fig. V-18	$r = 4Q^2$ $Q >> \frac{1}{2}$	Q	Q	-Q
Positive Gain Amplifier Fig. V-19	$r >> 1$ $Q >> \frac{1}{2}$	1.5	-1.5	1.5
Negative Gain Amplifier Fig. V-20	$r >> 1$ $Q >> \frac{1}{2}$	$2Q^2$	2Q	$4Q^3$

Table V-3

Approximate Figure of Merit $\lvert M_p \rvert$
and Corresponding Sensitivity Measures M_{ω_0} *and* M_Q
for Band-Pass Filters Optimized for Minimum $\lvert M_p \rvert$

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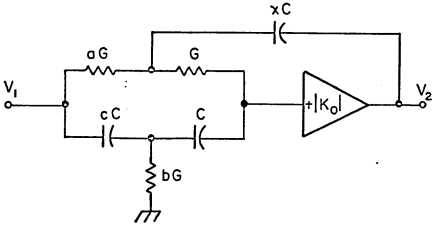

$$a = b = r$$

$$c = \left(\frac{9r}{4Q^2}\right)\left(\frac{r}{r+8}\right)^2\left(\frac{r}{r+1}\right)\left\{\left[1+\frac{4Q^2}{3}\left(\frac{r+1}{r}\right)\left(\frac{r+8}{r}\right)\left(1-\frac{r+8}{3r}\right)\right]^{1/2}-1\right\}^2$$

$$K_0 = (4/3)(1+2/r)$$

$$x = (c+1)(r+1)/r$$

$$(G/C) = (\omega_0/r)\sqrt{c(r+1)}$$

$$H = K_0$$

$$k_{33} = -K_0$$

$$d_1 = -\frac{1}{2}K_0\left\{(c+1)/r\right\}\sqrt{(r+1)/c}$$

$$d_3 = d_1/(2Q\sqrt{1-1/4Q^2})$$

Fig. V-21

Positive Finite Gain Amplifier Band-Stop Filter Realization and Design Equations for Minimum Figure of Merit $|M_p|$

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the high-pass transfer function

$$T(s) = \frac{Hs^2}{s + \frac{\omega_0}{Q} s + \omega_0^2} \qquad (V\text{-}35)$$

In a similar manner, Figures V-18, 19 and 20 and Table V-3 are used to present the results for the commonly used band-pass networks for which

$$T(s) = \frac{Hs}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2} \qquad (V\text{-}36)$$

Figure V-21 represents the only single amplifier network that is commonly used to realize the band-stop transfer function

$$T(s) = \frac{H(s^2 + \omega^2)}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2} \qquad (V\text{-}37)$$

Figure V-21 stands alone because band-stop filters are usually realized by first realizing a band-pass filter with the desired poles and then using a summing amplifier to add the input (properly scaled) to the output of the band-pass filter. The summing amplifier has no effect on the poles of the resulting band-stop transfer function, but only affects its zeros. As a result the use of a band-pass filter with a low figure of merit is all that is required if good pole sensitivities are desired. The effect of the finite gain-bandwidth products of both amplifiers on the zeros of the overall transfer function is outside the scope of this thesis, but can be handled in a manner similar to that introduced in Chapter III for determining the pole sensitivities. The figure of merit $|M_p|$ of the realization of Fig. V-21 can, however, be compared to the values of $|M_p|$ achieved by the other filters (Tables

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V-1, 2 and 3). Specifically, if $r >> 8$; $Q >> 1$, the sensitivity measures of the band-stop filter of Fig. V-21 become

$$M_Q = -M_{\omega_0} \simeq |M_p| \simeq 1.5 \qquad \text{(V-38)}$$

Thus, all the positive finite gain networks can have $|M_p| \simeq 1.5$. If $Q \geq 1$, Tables V-1, 2 and 3 clearly show that the positive finite gain realizations in general achieve either the lowest figure of merit for the same resistance ratio, or a comparable figure of merit for a much lower resistance ratio. As a result, the positive finite gain realizations are to be preferred over both the negative finite gain realizations (which, however, offer an advantage for $Q < 1$) or the operational amplifier realizations which, except for the low-pass case, require much higher resistance ratios to achieve a comparable figure of merit even for low Q. These conclusions are borne out by the experimental results of Moschytz (1971), who (although working with standard, non-optimum, element ratios) also concluded that the positive finite gain schemes yield the best performance.

Since the direct synthesis method presented in section B requires both some familiarity with passive synthesis techniques and some work, Figures V-12 through V-21 and Tables V-1 through V-3 should be of particular use to the circuit designer who does not wish to use the new method of section B, but rather desires to pick an existing design that will give good performance with very little design effort.

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CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

The purpose of this chapter is to discuss the importance (and limitations) of the major results of this thesis, and to point out some new areas of investigation suggested by these results. This is done in two parts. Section A presents a brief summary of the major contributions of this thesis and states the overall conclusions. Section B then discusses the significance and limitations of the results summarized in section A. During this discussion several new problems suggested by these results are identified and specific suggestions for future work in this area are made.

A. Summary of Results and General Conclusions

The major contributions of this thesis may be divided into two categories. The first consists of those results that are general enough to be directly applicable (as discussed in section B) to multi-amplifier filters as well as single amplifier filters. The second category consists of those results that are (as yet) only applicable to single amplifier filters. Consequently, the contributions of this thesis may be summarized as follows.

1. General Results:

   a. The characterization of the amplifier.

   b. The characterization of the RC network.

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c. The identification of the gain-bandwidth product of the amplifier as the critical parameter.

d. The new sensitivity measure M_p.

e. The new figure of merit $|M_p|$.

2. Specific Results for Single Amplifier Filters:

a. The practical lower bounds on $|M_p|$.

b. The determination of practical lower bounds on the minimum gain-bandwidth product required to realize a given transfer function.

c. The new synthesis procedure for the realization of a given transfer function by a filter with specified sensitivity properties.

d. The practical design procedures for the optimization of existing filter networks to achieve minimum $|M_p|$ and, hence, minimum amplifier sensitivity.

Two general conclusions may be drawn from these results. The first is that the frequency and temperature dependence problem (as defined by the five questions listed in Chapter I) has been solved for single amplifier filters. The second is that the general results listed above represent a general approach to the temperature and frequency dependence problem. The application of this approach to multi-amplifier

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filters (as discussed in section B) is the major recommendation for future work in this area.

B. Discussion of Results and Specific Suggestions for Future Work

The pole sensitivity measure M_p introduced in Chapter III is a quantitative measure of the effect of variations in the finite gain-bandwidth product $|\omega_a|$ on the performance of the filter. As such, it is an effective measure of both the frequency limitations and temperature stability of an integrated active filter. The use of M_p is based on the assumption that the amplifier's frequency dependence can be characterized in terms of a single dominant pole, and that ω_a is the critical parameter. These assumptions have strong experimental support. Among others, Tarmy and Ghausi (1970) and Thomas (1971a) have also assumed a one-pole roll-off frequency characteristic, and have used it to predict experimentally-verified frequency limitations on specific active filters. The choice of ω_a as the critical parameter has also been substantiated by experimental results. The first order pole shifts predicted by M_p are identical to those predicted by the approximate approach described in Chapter II (see Figures II-1 through II-10, and Figures IV-12 through IV-20). The latter have been checked against those given by an exact solution of the equation for the transfer functions (Budak 1970), and Lehmkuhl et al (1971) have experimentally verified the pole shifts predicted by Fig. II-11 and the first order approximations of Fig. II-13. Thus

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M_p is an effective measure of the amplifier-sensitivity of any single amplifier active filter using silicon-integrated operational amplifiers. It is well established (Gaash et al 1966; Horowitz 1968; Moschytz 1971) that the change in A_0 (and hence $\omega_a = \omega_b A_0$) basically determines the temperature performance of an integrated filter. Hence, the choice of ω_a as the critical parameter with regard to temperature variations needs no further justification. As a result, $|M_p|$ is a good measure of merit for any single amplifier integrated active filter.

However, the fact that M_p is an accurate measure of the performance of the filter does not by itself make M_p important. This new sensitivity measure is important because it is simple. Simple to evaluate (e.g. Example IV-1), simple to use (e.g. Examples IV-4 and IV-5), and, above all, simple to interpret. This is because M_p only depends on the leading coefficient and zeros of the numerator ($N_{33}(s)$) of a particular driving-point admittance of the passive RC network. The proper interpretation of this fact — especially for the important second order case (Chapter IV) — leads to the other results of this thesis.

One of the most useful of these is the set of practical lower bounds for $|M_p|$ derived in Chapter III (Table III-2). These lower bounds and the ease with which M_p may be evaluated greatly enhance the usefulness of $|M_p|$ as a figure of merit. By setting limits on what can be achieved with single amplifier filters — not only with regard to $|M_p|$ but also with regard to the trade-offs between the ω_0- and Q-sensitivities (Chapter IV) — these lower bounds make

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it possible to make a judgement as to whether the possible improvement in performance is worth any further effort for any particular design that is available.

Moreover, for the second order case — which is the case of most practical importance — the use of $|M_p|$ as a figure of merit is so simple that it leads directly to the optimization of existing networks by a proper use of the degrees of freedom present in these networks (e.g. Example V-5). This allows certain conclusions to be drawn regarding which of these networks are most suited for integrated realization. As discussed in Chapter V, these conclusions are supported by experimental data.

Another general result of the fact that M_p only depends on the constant coefficient and zeros of $N_{33}(s)$ is that the direction of this pole shift can be controlled by suitably selecting these zeros — particularly for operational amplifier realizations for which these zeros may be specified independent of the nominal pole locations of the filter. This leads directly to the general synthesis procedure of Chapter V which allows the realization of any possible directional sensitivity property, as well as a specified nominal transfer function. Not only is this procedure new, but since all other such methods have ignored the pole of the amplifier (e.g. Horowitz 1968), it is the only such procedure that is practical for integrated active filters in the new frequency ranges of interest ($\omega_o >> \omega_b$) where, as shown in Chapter V, the pole of the amplifier has a great effect on the direction of the pole shifts due to temperature changes. The results of the application

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of these concepts to achieve zero Q-sensitivity in a particular case have also been experimentally verified (Petrela and Budak 1971b).

Before considering the specific new areas of investigation suggested by these results, one further point regarding the usefulness of the new sensitivity measure M_p should be noted; namely, that M_p may be applied to multi-amplifier networks as well as single amplifier networks. Consider an RC network containing "a" (a > 1) voltage amplifiers with respective gain-bandwidth products $|\omega_{aj}|$, j = 1,a. Each amplifier can be considered, in turn, as shown in Fig. VI-1, where the other amplifiers are considered to be ideal and part of the RC network. The only difference between Fig. VI-1 and Fig. III-1 is that in Fig. III-1 the RC network is passive, whereas in Fig. VI-1 the RC network contains ideal voltage amplifiers and is therefore active. Clearly,

$$\Delta p \simeq \sum_{j=1,a} \{\partial p / \partial(1/\omega_{aj})\}\Delta(1/\omega_{aj}) \qquad \text{(VI-1)}$$

and one can also define the partial pole sensitivity measure $M_{p|_j}$ with respect to amplifier "j" to be

$$M_{p|_j} = \frac{1}{p\omega_0} \cdot \frac{\partial p}{\partial(1/\omega_{aj})}\bigg|_{(1/\omega_{ai}) = 0 \;;\; i \neq j} \qquad \text{(VI-2)}$$

If the amplifiers are assumed to have the same gain-bandwidth product (and temperature dependence) so that

$$(1/\omega_{aj}) = (1/\omega_a) \qquad \text{for all } j \qquad \text{(VI-3)}$$

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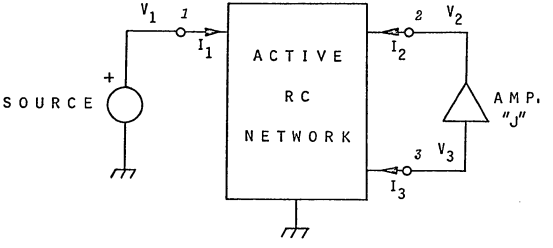

Fig. VI-1

General Multi-Amplifier RC Network

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then the overall pole sensitivity measure of the filter is simply

$$M_p = \sum_{j=1,a} M_{p_j} \qquad \text{(VI-4)}$$

where the M_{p_j} can be found by using the general approach of Chapter III.

The assumption that the gain-bandwidth product is the same for all the amplifiers has already been shown to give good results in practice. Both Budak and Petrela (1969) and Thomas (1971a) have used it to predict pole shifts for the Biquad network due to Tow (1968), while Tarmy and Ghausi (1970) have used it for a different three-amplifier configuration. In all cases the predicted pole shifts have agreed well with experimental measurements. As a result, $|M_p|$ can be used as an effective figure of merit for all integrated active filters. However, the other results of Chapter III (e.g. the lower bounds on $|M_p|$) that depend on the properties of a passive RC network no longer apply. This, of course, immediately suggests the general investigation of multi-amplifier networks in an attempt to achieve zero overall amplifier-sensitivity. Specifically, the following approach, which has already given some signs of success, should be investigated.

For a single operational amplifier filter the nominal transfer function depends only on the y_{31} and y_{32} (III-16) transfer admittances of the RC network, while the sensitivity measure M_p depends only on the y_{33} driving point admittance of the same RC network. As a result, one possible approach to achieving a zero sensitivity filter

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would be to take a single operational amplifier filter, and then use additional active elements to make $y_{33} \equiv 0$ without affecting y_{32}. Theoretically, the scheme of Fig. VI-2 accomplishes this for any finite non-zero gain α for the current controlled current source ICIS. In practice, second order effects may limit the usefulness of such a scheme. In any case it, and similar approaches, deserve further consideration.

Another problem of considerable interest is the extension of the work of Chapter III to the determination of greatest, rather than just practical, lower bounds on $|M_p|$. In Chapter III a greatest lower bound is only found for the

$$\prod_{i=1,n_{33}} |(p + \sigma_i)| \geq \beta^n$$

factor of $N_{33}(p)$, so that what is needed is a greatest lower bound on the magnitude of its leading coefficient k_{33}. The lower bounds on $|k_{33}|$ determined in Chapter III have been shown to be so practical as to suggest that greatest lower bounds on $|k_{33}|$ do exist. Hence, this would be a good problem to investigate.

The results of Appendix A, which considers the dependence of the pole shifts on the amplifier input conductance G and output resistance R, could also be extended. Normally R and G are chosen to be negligibly small; however, both G and R may be deliberately increased and controlled by the addition of external resistors to, as shown in Appendix A, affect the direction of the pole shifts. This may have practical applications and should be investigated further.

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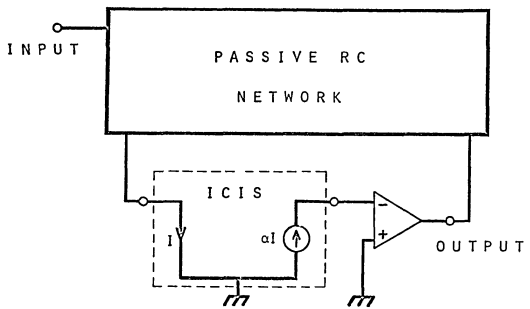

Fig. VI-2

A Zero Pole-Sensitivity Scheme

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Actually, the most vexing problem that arises is really in the area of passive RC network synthesis. This is the problem (discussed in Chapter V) of the dependence of the element ratios in an RC network on the relative location of the zeros of its driving point admittance y_{33} and its transfer admittance y_{32}. This problem is not new (Halpern 1969) and no suggestions, just motivation, towards its solution can be given by this author. Its solution would give great insight on the practical limitations of active filters in general, and would probably also lead to a solution for the previous problem (on the greatest lower bounds on $|k_{33}|$ and $|M_p|$) which is, as discussed in Chapter V, related to this one.

Finally, this thesis has applied the new sensitivity measure M_p to the minimization of the amplifier-sensitivity of single-amplifier filters. Undoubtedly, the indicated application of M_p to multi-amplifier filters will itself suggest many further problems for future work.

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APPENDIX A

EFFECT OF NON-ZERO AMPLIFIER INPUT CONDUCTANCE AND OUTPUT RESISTANCE

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EFFECT OF NON-ZERO AMPLIFIER INPUT CONDUCTANCE AND OUTPUT RESISTANCE

The effect of a non-zero output resistance R and a non-zero input conductance G is to cause equations III-3 and III-4 to become

$$I_3 = -GV_3 \tag{A-1}$$

and

$$V_2 = KV_3 - I_2R \tag{A-2}$$

Combining these two amplifier constraints and the gain expressions (III-1) and (III-2) with the description of the passive RC network ((III-5) and (III-6)) yields an expression for the general transfer function T(s) that includes the effect of R and G. The result is

$$T(s) = \frac{V_2}{V_1} = \frac{-y_{31} + \{y_{21}R(y_{33} + G)/(K-Ry_{32})\}}{y_{32} + \{(y_{33} + G)(1 + Ry_{22})/(K-Ry_{32})\}} \tag{A-3}$$

instead of the much simpler expression (III-16). This may be rewritten (in a manner similar to that used to obtain (III-17)) as

$$T(s) = -\frac{N_{31}D + (R/K)\{(N_{33} + GD)(N_{21} + N_{32}N_{31})\}}{N_{32}D - (1/K)\{(N_{33} + GD)(D + RN_{22}) - RN_{32}^2\}} \tag{A-4}$$

where, for convenience, $N_{ij}(s)$ and $D(s)$ have been simply denoted by N_{ij} and D; and, from (III-1) and (III-2),

$$(1/K) = (1/K_o) + (s+\omega_b)/\omega_a$$

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Equation III-22 may then be used to find the quantity $dp/d(1/\omega_a)$. This is greatly simplified if small pole shifts are assumed, so that the denominator polynomial in (A-4) is approximately D(s)V(s) and for p a pole of T(s) (since $D(p) \neq 0$)

$$V(p) \simeq 0$$

Then direct application of (III-22) yields

$$\frac{dp}{d(1/\omega_a)} \simeq \frac{(p+\omega_b)N_{33}(p)}{V'(p)} \left[\left(1 + \frac{G}{y_{33}(p)} \right) \left(1 + Ry_{22}(p) \right) - \frac{Ry_{33}(p)}{K_\delta^2} \right] \tag{A-5}$$

and since R and G are assumed small

$$\frac{dp}{d(1/\omega_a)} \simeq \frac{(p+\omega_b)N_{33}(p)}{V'(p)} \left[1 + \frac{G}{y_{33}(p)} + R\{y_{22}(p) - y_{33}(p)/K_\delta^2\} \right] \tag{A-6}$$

Comparing this to (III-25), it is seen that a first order correction for non-zero R and G is to simply multiply the expression for M_p, $dp/d(1/\omega_a)$ or Δp in the main body of the thesis by the correction term

$$C = 1 + \{G/y_{33}(p)\} + R\{y_{22}(p) - y_{33}(p)/K_\delta^2\} \tag{A-7}$$

Of course, if this correction factor is not close to unity, the approximations used to derive it are no longer valid. The main usefulness of C is that it indicates when G and R become important — or, conversely, what values of R and G are needed to make their effect negligible.

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APPENDIX B

BOUNDS ON $|M_p|$ FOR n_{33} GREATER THAN n

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BOUNDS ON $|M_p|$ FOR n_{33} GREATER THAN n

In Chapter III the lower bounds on k_{33} (and hence M_p) were derived for the practical case $n_{33} = n$. It is, however, possible to derive lower bounds on $|M_p|$ without this restriction by using (III-53) (rather than (III-40) which is only valid for $n_{33} = n$).

In (III-53), the dc current gain L_0 must lie in the range $0 \leq L_0 \leq 1$. Moreover, since n_{33} can only exceed n if an operational amplifier realization is used, this discussion is restricted to operational amplifier realizations. Thus, $(1/K_0) = 0$, and $V(s) = N_{32}(s)$ is of degree n and has unity leading coefficient. The constant coefficient of V(s) is denoted by W_0^2. The term q_0 in (III-53) represents the constant coefficient of $N_{33}(s)$, which is of degree n_{33} and has a positive leading coefficient k_{33}. Thus, equation III-53 yields

$$k_{33} \geq (W_0^2/q_0) \tag{B-1}$$

for

$$q_0 = \prod_{i=1,n_{33}} (\sigma_i) > 0 \tag{B-2}$$

where (from (III-33))

$$N_{33}(p) = k_{33} \prod_{i=1,n_{33}} (p + \sigma_i) \tag{B-3}$$

Substituting (B-1) into (B-3) gives

$$|N_{33}(p)| \geq W_0^2 \prod_{i=1,n_{33}} |(p + \sigma_i)/\sigma_i| \tag{B-4}$$

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For

$$p = -\alpha + j\beta$$

$$= -\omega_0/2Q + j\omega_0(1-1/4Q^2)^{\frac{1}{2}} \qquad \text{(B-5)}$$

the quantity $|(p + \sigma_i)/\sigma_i|$ is readily shown to have a minimum value at $\sigma_i = 2\omega_0 Q$. This value is

$$|(p + \sigma_i)/\sigma_i|_{min} = (1-1/4Q^2)^{\frac{1}{2}} \qquad \text{(B-6)}$$

Since the σ_i must be distinct and $n_{33} \geq 2$, (B-4) becomes

$$|N_{33}(p)| > W_0^2(1-1/4Q^2)^{n_{33}/2} \qquad \text{(B-7)}$$

Thus, from (III-26),

$$|M_p| > \left|\frac{p + \omega_b}{p}\right| \frac{W_0^2(1-1/4Q^2)^{n_{33}/2}}{\omega_0|V'(p)|} \qquad \text{(B-8)}$$

This expression (B-8) represents a lower bound on M_p for operational amplifier realizations regardless of whether n_{33} equals or exceeds n. On the other hand, the lower bound for operational amplifier realizations with $n_{33} = n$, derived in Chapter III, is

$$|M_p| > \frac{1}{2}\left|\frac{p + \omega_b}{p}\right| \frac{\beta^n}{\omega_0|V'(p)|} \qquad \text{(B-9)}$$

To compare these two lower bounds, consider the second order case. Then $W_0^2 = \omega_0^2$ and $|V'(p)| = 2\beta$. Hence (B-8) becomes

$$|M_p| > \frac{1}{2}\left|\frac{p + \omega_b}{p}\right|(1-1/4Q^2)^{(n_{33}/2-1)} \qquad \text{(B-10)}$$

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whereas (B-9) becomes

$$|M_p| > \frac{1}{2}\left|\frac{p + \omega_b}{p}\right|(1-1/4Q^2)^{\frac{1}{2}} \qquad \text{(B-11)}$$

Clearly, for $n_{33} = n + 1 = 3$ (B-10) and (B-11) are identical. However, for $n_{33} > 3$, (B-10) does yield a smaller lower bound on M_p than is possible for $n_{33} = n$. But this does not mean that a lower $|M_p|$ can actually be realized if $n_{33} > 3$ is chosen, for neither (B-10) nor (B-11) represent greatest lower bounds on $|M_p|$.

The only greatest lower bound derived so far has been the bound (III-35)

$$\prod_{i=1,n_{33}} |(p + \sigma_i)| > \beta^{n_{33}} \qquad \text{(B-12)}$$

which increases as n_{33} increases. However, from (III-26)

$$|M_p| = \left|\frac{p + \omega_b}{p}\right|\left(\frac{k_{33}}{\omega_0|V'(p)|}\right)\prod_{i=1,n_{33}} |(p + \sigma_i)| \qquad \text{(B-13)}$$

and it is conveivable that the use of $n_{33} > n$ might allow a reduction in k_{33} more than sufficient to offset the increase in

$$\prod_{i=1,n_{33}} |(p + \sigma_i)| > \beta^{n_{33}}$$

However, this is unlikely to be the case. The examples of Chapters IV and V show that the lower bound of (B-9) can be approached for $n_{33} = n$. A reduction in this lower bound for $n_{33} > n$ can only be achieved by placing all the zeros $(-\sigma_i)$ close to the point $-2\omega_0 Q$.

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However, the examples of Chapter V indicate that close groupings of the zeros ($-\sigma_i$) of $N_{33}(s)$ generally result in poorer element ratios and higher values of k_{33}. Furthermore, the pole shifts are originally due to the term $\{(s + \omega_b)/s\}N_{33}(s)$ in (III-17). This term generally becomes important at high pole frequencies ω_0, which would indicate that the degree of $N_{33}(s)$ should be kept as low as possible. Experience seems to justify this viewpoint, and leads to the conclusion that the use of $n_{33} > n$ can be expected to result in a higher value for $|M_p|$.

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