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ON GENERALIZATIONS OF CONTINUITY

by

David Alon Rose

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics in the University of South Florida

March, 1977

Major Professor: You-Feng Lin

Graduate Council
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CERTIFICATE OF APPROVAL

PH.D. THESIS

This is to certify that the Ph.D. dissertation of

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with a major in Mathematics has been approved by the Examining Committee on March 1, 1977 as satisfactory for the dissertation requirement for the Ph.D. degree.

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ON GENERALIZATIONS OF CONTINUITY

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An Abstract

Of a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Mathematics in the University of South Florida

March, 1977

Major Professor: You-Feng Lin

ABSTRACT

Generalized continuity conditions for functions between topological spaces have been investigated almost from the beginning of the century. Recently, an increased interest has been taken in these conditions weaker than continuity both to explore the conditions themselves with their innate properties and interrelationships and to use these conditions as tools for characterizing certain classes of topological spaces.

Two of the prominent independent generalized continuity conditions are weak continuity and almost continuity for which new characterizations are found. Conditions for interdependence of these two generalizations of continuity are also found. A mapping condition is sought which when combined with almost continuity will yield continuity. This leads to the notion of strong semi-continuity. It was known that weak continuity plus weak* continuity is equivalent to continuity. Here weak* continuity is replaced by local weak* continuity whereby a pure topological version of the closed graph theorem is obtained for a function into a strongly locally compact space. Many theorems treated individually in the literature are shown to have a common basis from which they stem. Thus

a unification of some of the heretofore disjointed results is established.

Shwu-Yeng T. Lin and You-Feng Lin recently found the following mapping characterization of Baire spaces.

THEOREM (L-L)1. A topological space X is a Baire space if and only if every function $f:X\to Y$ on X into a second countable infinite regular Hausdorff space Y has a dense set of points of almost continuity D(f) in X.

Theorem (L-L)1 has a common origin with the recent results on Blumberg spaces. In fact this theorem provides a new proof of the known result that the Blumberg spaces are Baire spaces. The notion of a C1 (C2) space is introduced and investigated for the purpose of improving Theorem (L-L)1. The improved version states that a topological space X is a Baire space if and only if every function on X into a second countable infinite Hausdorff space Y has a dense set of points of almost continuity.

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INTRODUCTION

Early this century in the mathematical literature conditions weaker than continuity for functions between certain spaces were being investigated. Henry Blumberg, following Denjoy, introduced a notion of almost continuity for real-valued functions [5] though the name "almost continuity" was not introduced (or rediscovered) until forty years later by Husain [32]. In fact, some of the earlier conditions studied which are weaker than continuity have stimulated research right up until the present. Casper Goffman and Daniel Waterman [24] defined a notion called "approximate continuity" for functions from Euclidean space into an arbitrary topological space in terms of metric density of inverse images of open sets. Their approximate continuity appears very similar to a mapping condition described by Denjoy [18].

Upper and lower semi-continuity for real-valued functions are examples of complementary conditions weaker than continuity in the sense that a function satisfies both conditions if and only if it is continuous. Norman Levine introduced two complementary conditions weaker than continuity for functions between arbitrary topological spaces and called these conditions weak continuity and

weak* continuity [39]. Later, Y.-F. Lin and Leonard Soniat defined a different weak continuity [47] for functions between Hausdorff spaces. Thus in the literature the same names have been used to denote very different conditions. There are at least four pairwise non-equivalent definitions of almost continuity including Husain's definition ([7], [61], [63]). Not only can different mapping conditions be given the same name in the literature but occasionally the same concept has been given different names depending on the authors and context. The concept of near continuity used by B. J. Pettis [58] particularly in the setting of topological algebra and the closed graph theorem, is the same as the almost continuity of Husain. In the same context with reference to the open mapping theorem, Pettis refers to a dual mapping condition as "near openness", as does Kung-Fu Ng [53]. However, L. G. Brown refers to this same mapping condition as "almost openness" when obtaining an open mapping theorem for topologically complete groups [9]. The large number of applications of almost continuity in the sense of Husain and of its dual almost openness not only in topological algebra but in point-set topology as well demonstrates the importance of this particular generalized continuity condition. Chapter II of this work will deal with a recent characterization of Baire spaces by Shwu-Yeng T. Lin and You-Feng Lin [45] in which Husain's almost

continuity plays a chief role. Throughout the remainder of this work, unless otherwise specified, the term "almost continuity" will always be used in the sense of Husain.

Recently, generalized continuity conditions have become objects of investigation in a number of papers. Interrelationships and comparisons between various generalized continuity conditions have been explored by Paul E. Long and Donald A Carnahan [48], Paul Long and Earl E. McGehee, Jr. [51], Paul Long and Michael D. Hendrix [49], Larry L. Herrington ([27], [28]), M. K. Singal and Asha Rani Singal [61], Takashi Noiri ([54], [55]), and R. V. Fuller [20], to name only a few. In Chapter I further interrelationships will be found for some of the more prominent generalized continuity conditions. A mapping approach is employed to obtain new and recent results.

Among the functions satisfying a generalized continuity condition would also be the connectivity, connected, and semi-connected mappings ([63], [57], [33]). Finally, since every continuous function into a Hansdorff space has a closed graph, the closure of the graph of a function could be considered a generalized continuity condition and thus perhaps the most important of all generalized continuity conditions.

A MAPPING APPROACH TO NEW AND RECENT RESULTS

The following theorem is well known (p. 226, [19]).

THEOREM (D)1. Every continuous bijection from a compact space onto a Hausdorff space is a homeomorphism.

The usual method of proof is to show that such a function is closed. Variations of Theorem (D)1 are obtained by relaxing some of the hypotheses and yet showing that the function is closed. While working with Theorem (D)1, Karl R. Gentry and Hughes B. Hoyle III conceived of the idea of a c-continuous function as defined below [21].

DEFINITION 1. (Gentry and Hoyle) Let X and Y be topological spaces, let $f:X\to Y$ be a function, and let p be an element of X. Then f is said to be c-continuous at p provided if V is an open subset of Y containing f(p) such that Y-V is compact, then there is an open subset U of X containing p such that $f(U) \subseteq V$. The function f is said to be c-continuous (on X) provided f is c-continuous at each point of X.

An easy consequence of this definition is that if $f:X\to Y$ is a function into a Hausdorff space Y then f is

c-continuous if and only if $f^{-1}(C)$ is closed for each compact subset C of Y (Theorem 1, [21]). Paul E. Long and Michael D. Hendrix improved this result by showing that a function $f:X\to Y$ is c-continuous if and only if $f^{-1}(C)$ is closed for each closed compact subset C of Y (Theorem 2, [49]). Evidently continuity implies c-continuity. In general the converse is not true. Gentry and Hoyle found the following sufficient condition for a c-continuous function to be continuous (Theorem 5, [21]).

THEOREM (G-H)1. If $f:X\to Y$ is a c-continuous function with f(X) contained in a compact subset of Y, then f is continuous.

Thus Theorem (D)1 becomes a corollary of the following result (Theorem 2, [21]).

THEOREM (G-H)2. If $f:X\to Y$ is a continuous bijection from a topological space X onto a Hausdorff space Y, then $f^{-1}:Y\to X$ is c-continuous.

For if $f:X\to Y$ is a continuous bijection from a compact space X onto a Hausdorff space Y then $f^{-1}(Y)$ is contained in a compact subset of X so that f^{-1} is continuous. Long and Hendrix strengthened Theorem (G-H)2 by showing that continuity could be replaced by almost continuity (S and S) (Theorem 13, [49]). A function $f:X\to Y$ is almost continuous (S and S) if and only if f is almost continuous in the sense of M. K. Singal and Asha Rani Singal [61] as defined below.

DEFINITION 2. (Singal and Singal) A function $f:X\to Y$ is said to be almost continuous (S and S) at a point $x \in X$, if for every neighborhood V of $f(x)$ there is a neighborhood U of x such that $f(U) \subseteq \underline{\text{Int}}\ \underline{\text{Cl}}\ V$. The function $f:X\to Y$ is said to be almost continuous (S and S) if f is almost continuous (S and S) at each point $x \in X$.

THEOREM (L-MH)1. If $f:X\to Y$ is an almost continuous (S and S) bijection from a topological space X onto a Hausdorff space Y then $f^{-1}:Y\to X$ is c-continuous.

A new method of proof can be used not only to prove Theorem (L-MH)1 but to strengthen this result. Long and Hendrix proved the following result (Theorem 7, [49]).

THEOREM (L-MH)2. Every function $f:X\to Y$ with a closed graph, $G(f) \subseteq X\times Y$, is c-continuous.

In light of Theorem (G-H)1, Theorem (L-MH)2 generalizes the well-known result that every function into a compact space which has a closed graph is continuous.

If $f:X\to Y$ is a bijection, then $G(f^{-1}) \subseteq Y\times X$ is closed if and only if $G(f) \subseteq X\times Y$ is closed. Further, Paul E. Long and Larry L. Herrington have shown that continuity can be replaced by almost continuity (S and S) in the well-known theorem that every continuous function into a Hausdorff space has a closed graph.

THEOREM (L-LH). Every almost continuous (S and S) function $f:X\to Y$ from a topological space X into a Hausdorff space Y has a closed graph $G(f) \subseteq X\times Y$.

Thus it is easy to deduce that if $f:X\to Y$ is an almost continuous (S and S) bijection onto a Hausdorff space then f and f^{-1} are c-continuous functions since G(f) is closed in $X\times Y$. Also this proves a claim by Long and Hendrix that an almost continuous function (S and S) into a Hausdorff space is c-continuous [49].

Almost continuity (S and S) can be replaced by weak continuity to strengthen Theorem (L-MH)1 by first strengthening Theorem (L-LH) in the same way. Unless otherwise indicated, a function is said to be weakly continuous if and only if it is weakly continuous in the sense of Norman Levine [39] as defined below.

DEFINITION 3. (Levine) A function $f:X\to Y$ is said to be weakly continuous at $x \in X$ if for each open set V containing f(x), there is an open set U containing x such that $f(U) \subseteq \underline{Cl}\ V$. The function $f:X\to Y$ is said to be weakly continuous if it is weakly continuous at each point x of X.

Almost immediate from the definition is the following result of Levine (Theorem 1, [39]). A function $f:X\to Y$ is weakly continuous if and only if $f^{-1}(V) \subseteq \underline{Int}\ f^{-1}(\underline{Cl}\ V)$ for each open subset V of Y. Note that almost continuity

(S and S) implies weak continuity since a function $f:X\to Y$ is almost continuous (S and S) if and only if $f^{-1}(V) \subseteq \underline{\text{Int}}\ f^{-1}(\underline{\text{Int}}\ \underline{\text{Cl}}\ V)$ for each open subset V of Y (Theorem 2.2 (e), [61]). So Theorem (L-LH) is implied by the following result known by Noiri (Theorem 10, [55]).

THEOREM 1. (Noiri) Every weakly continuous function $f:X\to Y$ from a topological space X into a Hausdorff space Y has a closed graph $G(f) \subseteq X\times Y$.

Noiri used standard point-set methods to demonstrate that each point of $(X\times Y)-G(f)$ is an interior point of $(X\times Y)-G(f)$ under the hypotheses of Theorem 1. A different proof will be given using mapping properties and the concept of a strongly closed subset of a product space.

DEFINITION 4. Let $\{Y_a:a\in A\}$ be a family of topological spaces and let $B \subseteq A$ be a subset of the index set. If $Y = P\{Y_a:a\in A\}$ is the product space then a subset $F \subseteq Y$ is said to be strongly closed with respect to B (or with respect to the family $\{Y_a:a\in B\}$) if and only if whenever $\{y_a\} \in Y-F$ then there exists a basic open set $V = P\{V_a:a\in A\}$ containing $\{y_a\}$ such that each $V_a \subseteq Y_a$ is open and $V_a = Y_a$ for all but finitely many $a \in A$ and such that $P\{W_a:a\in A\} \cap F = \emptyset$ where $W_a = \underline{\text{Cl}}\ V_a$ for $a \in B$ and $W_a = V_a$ for $a \in A-B$. If $F \subseteq Y$ is strongly closed with respect to A then F is said to be fully strongly closed.

Clearly, $F \subseteq P\{Y_a : a \in A\}$ is strongly closed with respect to $B \subseteq A$ implies that F is strongly closed with respect to any subset C of B. Furthermore, F is strongly closed with respect to $\emptyset \subseteq A$ if and only if F is closed. Thus every strongly closed set is closed. This generalizes the result of Veličko [66] that θ-closed sets are closed for it will be seen that the θ-closed subsets of a space are precisely the fully strongly closed subsets. In particular, if $F \subseteq P\{Y_a : a \in A\} = Y$ and A is a singleton set, then F is fully strongly closed if and only if F is a θ-closed subset of Y in the sense of Veličko. This definition of a strongly closed set generalizes a definition by Larry L. Herrington and Paul E. Long [29] by which the graph G(f) of a function $f: X \to Y$ is said to be strongly closed if and only if, in the sense of Definition 4 above, $G(f) \subseteq X \times Y$ is strongly closed with respect to the second coordinate space Y. Note that a subset F of a product space may be strongly closed with respect to each coordiate space without being fully strongly closed. The following theorems show that the diagonal D of $Y \times Y$ is strongly closed with respect to each coordinate space but not fully strongly closed if and only if Y is a Hausdorff non-Urysohn topological space. Larry Herrington and Paul Long (p. 422, [30]) have remarked that the identity function $i: Y \to Y$ has a strongly closed graph (with respect to the second coordinate space) if Y is a Hausdorff space. The following theorem generalizes this fact.

THEOREM 2. A topological space Y is a Hausdorff space if and only if the diagonal of Y×Y, $D = \{(y,y):y\varepsilon Y\}$, is strongly closed with respect to each coordinate space.

Proof. If Y is a Hausdorff space and $(y_1,y_2)\ \varepsilon\ (Y\times Y)-D$ then $y_1 \neq y_2$ and there exist disjoint open subsets V_1 and V_2 of Y containing y_1 and y_2 respectively. Then $(\underline{Cl}\ V_1) \cap V_2 = V_1 \cap \underline{Cl}\ V_2 = \phi$, and $[(\underline{Cl}\ V_1)\times V_2] \cap D = (V_1 \times \underline{Cl}\ V_2) \cap D = \phi$. Thus D is strongly closed with respect to each coordinate space.

Conversely, if D is strongly closed with respect to each coordinate space then D is closed so that Y is a Hausdorff space.

Theorem 2 strengthens the well-known characterization of Hausdorff spaces as those spaces Y having closed diagonals in Y×Y. The following result is a similar characterization for Urysohn topological spaces.

THEOREM 3. A topological space Y is a Urysohn space if and only if the diagonal of Y×Y, $D = \{(y,y):y\varepsilon Y\}$, is fully strongly closed.

Proof. If Y is a Urysohn space and if $(y_1,y_2)\ \varepsilon\ (Y\times Y)-D$ then $y_1 \neq y_2$ and there exist open sets V_1 and V_2 containing y_1 and y_2 respectively such that $\underline{Cl}\ V_1 \cap \underline{Cl}\ V_2 = \phi$. Thus $(\underline{Cl}\ V_1 \times \underline{Cl}\ V_2) \cap D = \phi$ and D is fully strongly closed.

Conversely, if D is fully strongly closed and if

$y_1, y_2 \in Y$ such that $y_1 \neq y_2$, then $(y_1, y_2) \notin D$ and there exist open subsets V_1 and V_2 of Y containing y_1 and y_2 respectively such that $(\underline{Cl}\ V_1 \times \underline{Cl}\ V_2) \cap D = \phi$. Thus $\underline{Cl}\ V_1 \cap \underline{Cl}\ V_2 = \phi$ and Y is a Urysohn space.

THEOREM 4. Let $\{(f_a: X_a \to Y_a): a \in A\}$ be a family of weakly continuous functions. Let $X = P\{X_a: a \in A\}$ and $Y = P\{Y_a: a \in A\}$ be the product spaces and let $f: X \to Y$ be the product function $P\{f_a: a \in A\}$. If $f_a: X_a \to Y_a$ is continuous for each $a \in A-B$ for some subset B of A and if $F \subseteq Y$ is strongly closed with respect to B then $f^{-1}(F)$ is a closed subset of X.

Proof. If $x = \{x_a\} \in X - f^{-1}(F)$, then $f(x) = \{f_a(x_a)\} \in Y-F$. Then for each $a \in A$, there is an open set $V_a \subseteq Y_a$ with $f_a(x_a) \in V_a$ and such that $V_a = Y_a$ for all but finitely many $a \in A$ and such that $P\{W_a: a \in A\} \cap F = \phi$ where $W_a = V_a$ for $a \in A-B$ and $W_a = \underline{Cl}\ V_a$ for $a \in B$. For each $a \in A$, let $U_a = f_a^{-1}(V_a)$ if $a \in A-B$ and let $U_a = \underline{Int}\ f_a^{-1}(W_a)$ if $a \in B$ and let $U = P\{U_a: a \in A\}$. Then U is an open subset of X containing x. Further, if $W = P\{W_a: a \in A\}$, then $U \subseteq f^{-1}(W) \subseteq X - f^{-1}(F)$ so that x is an interior point of $X - f^{-1}(F)$ hence $f^{-1}(F)$ is closed.

The following notion of a θ-closed set was studied by N. V. Veličko [66].

DEFINITION 5. If X is a topological space, a point $x \varepsilon X$ belongs to the θ-closure of a subset $K \subseteq X$ if and only if $(\underline{Cl}\ V) \cap K \neq \phi$ for each open set V containing x. This would be notated $x \varepsilon \theta\text{-}\underline{Cl}\ K$. A subset $K \subseteq X$ is θ-closed if and only if $\theta\text{-}\underline{Cl}\ K \subseteq K$.

DEFINITION 6. If X is a topological space, a subset $U \subseteq X$ is called θ-open if and only if X-U is θ-closed.

If (X,T) is a topological space, let T_θ be the collection of all θ-open subsets of X. Then T_θ is a subtopology of T for X since Veličko [66] showed that every θ-closed set is closed and the collection of θ-closed sets includes ϕ and X and is closed under the operations of finite unions and arbitrary intersections. It is shown in Chapter II (Theorem 13) that a topological space (X,T) is regular if and only if $T = T_\theta$.

A subset U of a space X is called regular open (p. 92, [19]) if and only if $U = \underline{Int}\ \underline{Cl}\ U$. Similarly a set E is called regular closed if and only if $E = \underline{Cl}\ \underline{Int}\ E$. Thus ϕ and X are regular open sets in a space X and U is a regular open if and only if X-U is regular closed. A finite intersection of regular open sets is a regular open set (p. 92, [19]). Thus the collection of regular open subsets of a topological space (X,T) form a base for a subtopology $T_S \subseteq T$ for X. A topological space is said to be semiregular if and only if its topology has a basis consisting of regular open sets. Thus a topological

space (X,T) is semiregular if and only if $T = T_S$. Note that T_S is generated by the collection of all sets of the form $\underline{\text{Int}}\ \underline{\text{Cl}}\ U$ where $U \in T$ and the closure and interior operations are taken with respect to (X,T). Furthermore, the regular open subsets of (X,T_S) are precisely the regular open subsets of (X,T) so that $(T_S)_S = T_S$ and (X,T_S) is semiregular. In fact, T_S is the maximal (finest) semiregular subtopology of T in the sense that every semiregular subtopology of T is contained in T_S. Note that $\{\phi,X\}$ is always a semiregular subtopology of T and $T_S = \{\phi,X\}$ if and only if each nonempty open subset of X is dense in X.

For any topological space (Y,T), T_θ is always a subtopology of T_S so that every regular space is semiregular. That is, if $T_\theta = T$ then $T_S = T$. If $Y = (Y,T)$, $Y_S = (Y,T_S)$, and $Y_\theta = (Y,T_\theta)$ and if $f:X\to Y$, $f_S:X\to Y_S$, and $f_\theta:X\to Y_\theta$ are functions which agree at each $x \in X$, then f is almost continuous (S and S) if and only if f_S is continuous [50]. If f is weakly continuous then f_θ is continuous. So if $f:X\to Y$ is a product of functions, $f = P\{(f_a:X_a\to Y_a):a\in A\}$, and if $X = P\{X_a:a\in A\}$ and $Y = P\{Y_a:a\in A\}$ are product spaces, then $f_\theta:X\to Y_\theta$ is continuous if each f_a is weakly continuous. For by Theorem 4, if $F \subseteq Y_\theta$ is closed then F is fully strongly closed in Y and $f_\theta^{-1}(F) = f^{-1}(F)$ is closed in X. The following stronger result can be obtained.

THEOREM 5. If $f_a:X_a \to Y_a$ is weakly continuous for each $a \in A$ and if $X = P\{X_a:a \in A\}$ and $Y = P\{Y_a:a \in A\}$ and if $f:X \to Y$ is defined by $f(\{x_a\}) = \{f_a(x_a)\}$ then f is weakly continuous.

An easy proof of Theorem 5 uses the following supporting theorem.

THEOREM 6. If Y is a topological space whose topology has an open basis B and if $f:X \to Y$ is a function such that $f^{-1}(V) \subseteq \underline{\text{Int}}\ f^{-1}(\underline{\text{Cl}}\ V)$ for all $V \in B$, then f is weakly continuous.

Proof. If W is an open subset of Y then $W = \bigcup\{V_a:a \in A\}$ where $V_a \in B$ for each $a \in A$. Therefore,

$$\begin{aligned} f^{-1}(W) &= \bigcup\{f^{-1}(V_a):a \in A\} \\ &\subseteq \bigcup\{\underline{\text{Int}}\ f^{-1}(\underline{\text{Cl}}\ V_a):a \in A\} \\ &\subseteq \underline{\text{Int}}\ \bigcup\{f^{-1}(\underline{\text{Cl}}\ V_a):a \in A\} \\ &= \underline{\text{Int}}\ f^{-1}(\bigcup\{\underline{\text{Cl}}\ V_a:a \in A\}) \\ &\subseteq \underline{\text{Int}}\ f^{-1}(\underline{\text{Cl}} \bigcup\{V_a:a \in A\}) \\ &= \underline{\text{Int}}\ f^{-1}(\underline{\text{Cl}}\ W). \end{aligned}$$

Proof of Theorem 5. Let $V = P\{V_a:a \in A\}$ be a basic open set in Y where $V_a \subseteq Y_a$ is open for each $a \in A$ and $V_a = Y_a$ for all but finitely many $a \in A$. Then

$$\begin{aligned} f^{-1}(V) &= P\{f_a^{-1}(V_a):a \in A\} \\ &\subseteq P\{\underline{\text{Int}}\ f_a^{-1}(\underline{\text{Cl}}\ V_a):a \in A\} \\ &\subseteq \underline{\text{Int}}\ P\{f_a^{-1}(\underline{\text{Cl}}\ V_a):a \in A\} \\ &= \underline{\text{Int}}\ f^{-1}(\underline{\text{Cl}}\ V). \end{aligned}$$

Theorem 5 is analogous to the result of Singal and Singal (Theorem 2.10, [61]) that a product of almost continuous (S and S) functions is almost continuous (S and S). The converse, that each factor function is almost continuous (S and S) if the product function is almost continuous (S and S) was obtained by Long and Herrington (Theorem 1, [50]). The analogous result for weakly continuous functions also holds. First observe that if X and Y are topological spaces and if $A\times B \subseteq X\times Y$, then $\underline{\text{Int}}(A\times B) = (\underline{\text{Int}}\ A)\times(\underline{\text{Int}}\ B)$. This result yields the following lemma.

LEMMA. If $X = P\{X_a : a\in A\}$ and $U = P\{U_a : a\in A\}$ where $U_a \subseteq X_a$ for each $a \in A$ and $U_a = X_a$ for all but finitely many $a \in A$, then $\underline{\text{Int}}\ U = P\{\underline{\text{Int}}\ U_a : a\in A\}$.

Proof. Let $B = \{a_1, a_2, \ldots, a_n\}$ be a finite subset of A such that $U_a = X_a$ for $a \in A-B$. Let $h:X \to P\{X_a : a \in A-B\} \times P\{X_a : a\in B\}$ be the natural homeomorphism. Then

$$\begin{aligned}\underline{\text{Int}}\ U &= h^{-1}(\underline{\text{Int}}\ h(U))\\ &= h^{-1}(\underline{\text{Int}}\ P\{X_a : a \in A-B\}\times \underline{\text{Int}}\ P\{U_a : a\in B\})\\ &= h^{-1}(P\{X_a : a \in A-B\}\times P\{\underline{\text{Int}}\ U_a : a\ B\})\\ &= P\{\underline{\text{Int}}\ U_a : a\in A\}.\end{aligned}$$

THEOREM 7. Let $X = P\{X_a : a\in A\}$, $Y = P\{Y_a : a\in A\}$, and let $f_a:X_a \to Y_a$ be a function for each $a \in A$. If $f = P\{f_a : a\in A\}$ is weakly continuous then f_a is weakly continuous for each $a \in A$.

Proof. Fix $b \in A$ and let V_b be an open subset of Y_b. Let $V = P\{V_a : a \in A\}$ where $V_a = Y_a$ for $a \neq b$. Let $U_a = X_a$ for $a \neq b$ and $U_b = f_b^{-1}(V_b)$. Then

$$\begin{aligned} P\{U_a : a \in A\} &= f^{-1}(V) \\ &\subseteq \underline{\text{Int}}\ f^{-1}(\underline{\text{Cl}}\ V) \\ &= \underline{\text{Int}}\ P\{f_a^{-1}(\underline{\text{Cl}}\ V_a) : a \in A\}. \end{aligned}$$

Thus $f_b^{-1}(V_b) \subseteq \underline{\text{Int}}\ f_b^{-1}(\underline{\text{Cl}}\ V_b)$ and f_b is weakly continuous.

Theorems 5 and 7 combine to assert that a product of functions is weakly continuous if and only if each factor function is weakly continuous. Two other results analogous to those obtained for almost continuous (S and S) functions by Long and Herrington (Theorems 2 and 3, [50]) are obtained for weakly continuous functions using Theorem 5 and the following interesting result.

THEOREM 8. If $f:X \to Y$ and $g:Y \to Z$ are weakly continuous functions, then $gf:X \to Z$ is weakly continuous if either f or g is continuous.

Proof. If $f:X \to Y$ is continuous and if W is an open subset of Z, then

$$\begin{aligned} (gf)^{-1}(W) &= f^{-1}(g^{-1}(W)) \\ &\subseteq f^{-1}(\underline{\text{Int}}\ g^{-1}(\underline{\text{Cl}}\ W)) \\ &\subseteq \underline{\text{Int}}\ f^{-1}(g^{-1}(\underline{\text{Cl}}\ W)) \\ &= \underline{\text{Int}}\ (gf)^{-1}(\underline{\text{Cl}}\ W), \end{aligned}$$

so that gf is weakly continuous. On the other hand, if g is continuous and W is an open subset of Z then

$$(gf)^{-1}(W) = f^{-1}(g^{-1}(W))$$
$$\subseteq \underline{\text{Int}}\ f^{-1}(\underline{\text{Cl}}\ g^{-1}(W))$$
$$\subseteq \underline{\text{Int}}\ f^{-1}(g^{-1}(\text{Cl } W))$$
$$= \underline{\text{Int}}\ (gf)^{-1}(\underline{\text{Cl}}\ W)$$

so that gf is weakly continuous.

Norman Levine observed that weak continuity is equivalent to continuity when the range space is regular [39]. This result is embodied in the following theorem for which a different proof using properties of functions will be given.

THEOREM (L)1. If $f:X\to Y$ is a weakly continuous function into a regular space (Y,T), then f is continuous.

Proof. Let $f_\theta:X\to(Y,T_\theta)$ be the function which agrees with f at each $x \in X$. Then f_θ is continuous since f is weakly continuous but $T_\theta = T$ since Y is regular so that $f = f_\theta$ is continuous.

Theorem (L)1 allows the following corollary to Theorem 8.

COROLLARY TO THEOREM 8. If $f:X\to Y$ and $g:Y\to Z$ are weakly continuous functions, then 1) gf is weakly continuous if Y is a regular space and 2) gf is continuous if Z is regular.

Proof. In case (1) f is continuous and in case (2) g is continuous so that gf is weakly continuous into a regular space and is therefore continuous.

It appears that Theorem 8 is not true when weak continuity is replaced by almost continuity (S and S). Thus by using Theorem 8, the following results are obtained more easily than the analogous results for almost continuous (S and S) functions were obtained [50].

THEOREM 9. Let $Y = P\{Y_a : a \in A\}$ be a product space and let $p_a : Y \to Y_a$ be the projection to the a^{th} coordinate space for each $a \in A$. If $f : X \to Y$ is any function, then f is weakly continuous if and only if $p_a f$ is weakly continuous for each $a \in A$.

Proof. Since each p_a is continuous, $p_a f$ is weakly continuous.

Conversely, if each $p_a f : X \to Y_a$ is weakly continuous, then $F = P\{p_a f : a \in A\}$ is weakly continuous by Theorem 5. Let $X_a = X$ for each $a \in A$ and define the function $g : X \to P\{X_a : a \in A\}$ by $g(x) = \{x_a\}$ where $x_a = x$ for each $a \in A$. Then g is the continuous "diagonal" function and $Fg = f$. Thus f is weakly continuous by Theorem 8.

COROLLARY TO THEOREM 9. If $f_a : X \to Y_a$ is a function for each $a \in A$ and if $f : X \to Y$ is defined by $f(x) = \{f_a(x)\}$ where $Y = P\{Y_a : a \in A\}$, then f is weakly continuous if and only if each f_a is weakly continuous.

Proof. Let $p_a : Y \to Y_a$ be the projection function for each $a \in A$. Then $f_a = p_a f$ for each $a \in A$ and f is weakly continuous if and only if each f_a is weakly continuous by Theorem 9.

The following corollary to Theorem 4 resembles the above Corollary to Theorem 9.

COROLLARY 1 TO THEOREM 4. Let $Y = P\{Y_a : a \in A\}$ and let $f_a : X \to Y_a$ be a weakly continuous function for each $a \in A$. Define the function $f : X \to Y$ by $f(x) = \{f_a(x)\}$. Let B be a subset of A and let $E \subseteq Y$ be strongly closed with respect to B. If f_a is continuous for each $a \in A-B$ then $f^{-1}(E)$ is closed.

Proof. Let $F = P\{f_a : a \in A\}$. By Theorem 4, $F^{-1}(E)$ is a closed subset of $P\{X_a : a \in A\}$ where each $X_a = X$. If $g : X \to P\{X_a : a \in A\}$ is the diagonal function defined by $g(x) = \{x_a\}$ where $x_a = x$ for each $a \in A$, then g is continuous so that $g^{-1}(F^{-1}(E)) = f^{-1}(E)$ is closed in X.

A sequence of results obtained by Noiri (section 5, [54]) by point-set methods follows from Theorem 4 via mapping arguments.

COROLLARY 2 TO THEOREM 4. (Noiri) If $f : X \to Y$ is a weakly continuous injection into the Urysohn space Y, then X is a Hausdorff space.

Proof. By Theorem 3, the diagonal $D \subseteq Y \times Y$ is fully strongly closed and by Theorem 4, $(f \times f)^{-1}(D) = \{(x,x) : x \in X\}$ is the closed diagonal of $X \times X$. So X is a Hausdorff space. Alternatively, Y is a Urysohn if and only if D is θ-closed so that $(f \times f)^{-1}(D)$ is closed since $f \times f$ is weakly continuous by Theorem 5.

COROLLARY 3 TO THEOREM 4. (Noiri) If $f_1:X\to Y$ and $f_2:X\to Y$ are weakly continuous functions into a Urysohn space Y, then $\{x:f_1(x) = f_2(x)\}$ is a closed subset of X.

Proof. Define $f:X \to Y\times Y$ by $f(x) = (f_1(x),f_2(x))$ for each $x \in X$. By Corollary 1 to Theorem 4, if $D \subseteq Y\times Y$ is the diagonal, then $f^{-1}(D) = \{x:f_1(x) = f_2(x)\}$ is closed in X. Again alternatively, D is θ-closed and f is weakly continuous by Corollary to Theorem 9 so that $f^{-1}(D)$ is closed.

COROLLARY 4 TO THEOREM 4. (Noiri) If $f_1:X\to Y$ and $f_2:X\to Y$ are weakly continuous functions into the Urysohn space Y and if $f_1 = f_2$ on a dense subset $B \subseteq X$, then $f_1 = f_2$ on X.

Proof. By Corollary 3 to Theorem 4, $B = \underline{Cl}\ B = X$ so that $f_1 = f_2$ on X.

COROLLARY 5 TO THEOREM 4. If $f_1:X\to Y$ and $f_2:X\to Y$ are weakly continuous functions into the Hausdorff space Y and if either f_1 or f_2 is continuous, then 1) $B = \{x:f_1(x) = f_2(x)\}$ is a closed subset of X and 2) if a subset A of B is dense in X then $f_1 = f_2$ on X.

Proof. Define a function $f:X \to Y\times Y$ by $f(x) = (f_1(x),f_2(x))$ for each $x \in X$. By Corollary 1 to Theorem 4 and Theorem 2, $f^{-1}(D) = B$ is closed in X if $D \subseteq Y\times Y$ is the diagonal. Therefore, $B = \underline{Cl}\ B = \underline{Cl}\ A = X$ if $A \subseteq B$ is dense in X so that $f_1 = f_2$ on X.

Theorem 1, which stated that a weakly continuous function into a Hausdorff space has a closed graph, also follows from Theorem 4 as is shown below.

Proof of Theorem 1. Let $f:X\to Y$ be a weakly continuous function from a topological space X into a Hausdorff space Y, and let $G(f) \subseteq X\times Y$ be the graph of f. Let $i:Y\to Y$ be the continuous identity function and let $D \subseteq Y\times Y$ be the diagonal. By Theorem 2, D is strongly closed with respect to the first factor space. By Theorem 4, $G(f) = (f\times i)^{-1}(D)$ is closed in $X\times Y$.

Several results follow as corollaries to Theorem 1. For example, any theorem hypothesizing an almost continuous (S and S) function into a Hausdorff space which depends only upon the graph being closed is true more generally for a weakly continuous function into a Hausdorff space. Thus the following corollary improves Theorem (L-MH)1 since c-continuity is implied by closure of the graph (Theorem (L-MH)2).

COROLLARY 1 TO THEOREM 1. If $f:X\to Y$ is a weakly continuous bijection from a topological space X onto a Hausdorff space Y, then $f^{-1}:Y\to X$ is c-continuous.

The following corollary improves a similar result of Singal and Singal (Theorem 2.13, [61]) in that the range space is not required to be Urysohn.

COROLLARY 2 TO THEOREM 1. If $f:X\to Y$ is a weakly continuous bijection from a compact space X onto a Hausdorff space Y, then f is open and closed.

Proof. By Corollary 1 to Theorem 1, $f^{-1}:Y\to X$ is c-continuous and hence continuous by Theorem (G-H)1. Thus f is both open and closed.

COROLLARY 3 TO THEOREM 1. If $f:X\to Y$ is a weakly continuous function from a topological space X into a Hausdorff space Y, then f is c-continuous.

Note that in general any function $f:X\to Y$ with closed graph $G(f)\subseteq X\times Y$ is closed provided that X is compact. For if $V\subseteq X$ is open and $C = X-V$, then $f(C) = p_2((C\times Y)\cap G(f))$ is closed where $p_2:X\times Y\to Y$ is the projection parallel to the compact factor X. Thus f(V) is open if f is bijective.

Other corollaries to Theorem 1 follow from basic properties of closure of the graph of an arbitrary function $f:X\to Y$. Some of the properties are also true for multifunctions (multi-valued functions) $F:X\to Y$ with closed graph $G(F)\subseteq X\times Y$ or more generally for closed relations $R\subseteq X\times Y$. The following notation will be adopted if $R\subseteq X\times Y$ is a relation from X to Y. For any subsets $A\subseteq X$ and $B\subseteq Y$, $R[A] = p_2((A\times Y)\cap R)$ and $R^{-1}[B] = p_1((X\times B)\cap R)$ where $p_1:X\times Y\to X$ and $p_2:X\times Y\to Y$ are the coordinate projections. This notation is consistent with the notation $R^{-1} = \{(y,x):(x,y)\in R\}$ as the inverse relation of R. Further,

if $A = \{x\}$ and $B = \{y\}$, denote $R[A]$ by $R[x]$ and $R^{-1}[B]$ by $R^{-1}[y]$. Thus for $A \subseteq X$ and $B \subseteq Y$, $R[A] = \bigcup\{R[x]:x\in A\}$ and $R^{-1}[B] = \{x:R[x] \cap B \neq \phi\}$. Further let $R^{-1}(B) = \{x:R[x] \subseteq B\}$ for $B \subseteq Y$. Then $X = R^{-1}(B) \cup R^{-1}[Y-B]$ and $R^{-1}(B) \cap R^{-1}[Y-B] = \phi$ for all $B \subseteq Y$. Also, if $D = p_1(R)$ is the domain of R, $X-D \subseteq R^{-1}(B)$ for each $B \subseteq Y$ since $R[x] = \phi$ if $x \in X-D$, and $R^{-1}[B] \subseteq D$ for each $B \subseteq Y$ so that in general $R^{-1}(B)$ and $R^{-1}[B]$ are not related by set inclusions. However, for multifunctions $F:X\to Y$, since the domain of F is X, $F^{-1}(B) \subseteq F^{-1}[B]$ for each $B \subseteq Y$. Thus $F^{-1}(B)$ could be referred to as the upper inverse image of B under F and $F^{-1}[B]$ could be referred to as the lower inverse image of B under F. Following Ernest Michael [52] upper and lower semi-continuity for a multifunction is defined as follows.

DEFINITION 7. If $F:X\to Y$ is a multifunction from a topological space X to a topological space Y, F is lower (upper) semi-continuous if and only if $F^{-1}[V](F^{-1}(V))$ is open whenever $V \subseteq Y$ is open. Further, F is continuous if and only if F is both upper and lower semi-continuous.

Note that upper (lower) semi-continuity is equivalent to continuity for single-valued functions. Further, a multifunction is continuous if and only if upper and lower inverse images of closed (open) sets are closed (open).

The proof of the following result is left as an exercise in John L. Kelley's General Topology (Problem 6A, [37]).

THEOREM 10. If X and Y are topological spaces and $R \subseteq X \times Y$ is closed and if $A \subseteq X$ is compact then R[A] is a closed subset of Y.

Proof. If $y \notin R[A]$ then $A \times \{y\}$ is a subset of the open set $(X \times Y) - R = W$. Since A and $\{y\}$ are compact subsets of X and Y respectively, by a theorem of A. D. Wallace (p. 142, [37]) there are open sets $U \subseteq X$ and $V \subseteq Y$ with $A \subseteq U$, $y \in V$, and $U \times V \subseteq W$. Thus, $V \cap R[A] = \emptyset$ and R[A] is closed in Y.

Since $R^{-1} \subseteq Y \times X$ is the homeomorphic image of $R \subseteq X \times Y$ under the homeomorphism $h: X \times Y \to Y \times X$ defined by $h((x,y)) = (y,x)$, R^{-1} is closed if and only if R is closed so that the following Corollary to Theorem 10 is equivalent to Theorem 10.

COROLLARY 1 TO THEOREM 10. If $R \subseteq X \times Y$ is closed and $B \subseteq Y$ is compact, then $R^{-1}[B]$ is a closed subset of X.

Corollary 1 to Theorem 10 yields a generalization of Theorem (L-MH)2 for closed graph multifunctions.

DEFINITION 8. A multifunction $F: X \to Y$ is lower (upper) semi-c-continuous if and only if $F^{-1}[V](F^{-1}(V))$ is open whenever V is an open set in Y with a compact complement Y-V.

COROLLARY 2 TO THEOREM 10. If a multifunction $F: X \to Y$ has a closed graph $G(F) \subseteq X \times Y$, then F is upper semi-c-continuous.

Proof. If $V \subseteq Y$ is open and Y-V is compact, then $F^{-1}[Y-V] = X - F^{-1}(V)$ is closed so that $F^{-1}(V)$ is open and F is upper semi-c-continuous.

Y.-F. Lin and Leonard Soniat [47] characterized Hausdorff k-spaces using their notion of weak continuity (L and S) which they defined for functions between Hausdorff spaces. Without requiring the domain and range spaces to be Hausdorff, their definition is the following.

DEFINITION 9. (Lin and Soniat) A function $f:X\to Y$ is weakly continuous (L and S) if and only if $f^{-1}(y)$ is closed in X for each $y \in Y$.

Now the following corollaries to Theorem 1 result.

COROLLARY 4 TO THEOREM 1. Every weakly continuous function $f:X\to Y$ from a topological space X into a Hausdorff space Y is weakly continuous (L and S).

Proof. For each $y \in Y$, $\{y\}$ is compact so that $f^{-1}(y)$ is closed in X by Corollary 1 to Theorem 10.

The following result is similar to Corollary 2 to Theorem 4.

COROLLARY 5 TO THEOREM 1. If $f:X\to Y$ is a weakly continuous injection into a Hausdorff space Y, then X is a T_1-space.

Proof. By Corollary 4 to Theorem 1, if $x \in X$, $\{x\} = f^{-1}(f(x))$ is closed so that X is a T_1-space.

In comparing Corollary 3 to Theorem 1 with Corollary 4 to Theorem 1, it may be asked if c-continuity and weak continuity (L and S) are independent. An affirmative answer to this question would require an example of a c-continuous function $f:X\to Y$ into a non-T_1-space Y whose graph is not closed and an example of a weakly continuous (L and S) function without a closed graph. The following examples show the independence of c-continuity and weak continuity (L and S).

EXAMPLE 1. Let X = (X,T) be a non-T_1-space and let Y = (X,S) be the indiscrete space on the underlying set X. Let $f:X\to Y$ be the identity function on the set X. Then f is c-continuous since f is continuous but for some non-closed point $\{x\}$ in X, $\{x\} = f^{-1}(f(x))$ so that f is not weakly continuous (L and S).

EXAMPLE 2. Let X = [0,1] be the unit interval of real numbers with the usual subspace topology. Define $f:X\to X$ by $f(x) = x$ if $x \neq \frac{1}{2}$ and $f(\frac{1}{2}) = 1$. If $V = (\frac{1}{2},1]$ then V is open and has a compact complement. But $f^{-1}(V) = [\frac{1}{2},1]$ is not open so that f is not c-continuous. Yet $f^{-1}(x)$ is finite and hence closed for each $x \in X$. Thus f is weakly continuous (L and S).

It can be observed that a closed graph function from a space X into X has a closed set of fixed points. The following result generalizes this observation.

THEOREM 11. Let $X_1 = (X,T_1)$ and $X_2 = (X,T_2)$ be topological spaces with common underlying set X. Let $R \subseteq X_1 \times X_2$ be a closed relation from X_1 to X_2. Define $G = \{x : x \in R[x]\}$ and $F = \{x : x \in R^{-1}[x]\}$ to be the sets of fixed points with respect to R in X_2 and X_1 respectively. Then G is closed in X_2 if $T_1 \subseteq T_2$ and F is closed in X_1 if $T_2 \subseteq T_1$.

Proof. (Note that as subsets of X, G = F.) If $\{x_b\}$ is a net in G converging to $x_2 \in X_2$, then for each b, $x_b \in R[x_b]$ so that $(x_b, x_b) \in R$. If $T_1 \subseteq T_2$ and $x_2 \in U \in T_1$ then $\{x_b\}$ is eventually in U so that the net $\{(x_b, x_b)\}$ converges to (x_2, x_2) in R since R is closed. Thus $x_2 \in R[x_2]$ and $x_2 \in G$ showing that $G \subseteq X_2$ is closed. Similarly, if $T_2 \subseteq T_1$ then $F \subseteq X_1$ is closed.

An important corollary follows when $T_1 = T_2$.

COROLLARY 1 TO THEOREM 11. Let X be a topological space and let $R \subseteq X \times X$ be closed. If $F = \{x : x \in R^{-1}[x]\}$ is the set of fixed points under R, then F is closed in X.

This corollary applies directly to retractions having closed graphs. The following definition generalizes the notion of a retraction to include multifunction retractions.

DEFINITION 10. Let A be a subset of a topological space X. Then A is a multifunction retract of X if and only if there is a multifunction $F:X\to X$ with $F[X] \subseteq A$ and $x \in F^{-1}[x]$ for each $x \in A$. Then F is called a multifunction retraction from X onto A.

COROLLARY 2 TO THEOREM 11. If $A \subseteq X$ is a multifunction retract of a Hausdorff space X with a multifunction retraction having a closed graph, then A is a closed subset of X.

Proof. If $F:X\to X$ is a multifunction retraction of X onto A having a closed graph $G(F) \subseteq X\times X$, then $A = \{x:x \in F^{-1}[x]\}$ is closed by Corollary 1 to Theorem 11.

Since every (single-valued) retraction is a multifunction retraction, Corollary 2 to Theorem 11 supports the following corollary to Theorem 1 which Noiri obtained (Theorem 2, [54]) by standard point-set methods.

COROLLARY 6 TO THEOREM 1. (Noiri) Let A be a subset of a Hausdorff space X. If $f:X\to X$ is a weakly continuous retraction of X onto A, then A is closed in X.

Proof. By Theorem 1, $G(f) \subseteq X\times X$ is closed so that A is closed by Corollary 2 to Theorem 11.

Singal and Singal [61] showed that examples exist of weakly continuous functions which are almost continuous (S and S). The range space for such a function must fail

to be regular for otherwise the function is continuous. A special case of the equivalence of weak continuity and continuity when the range space is regular was discovered in an obscure way by J. D. Weston [67]. He discovered that one of two possible relationships that could be investigated between two topologies on a common set reduces to a trivial situation when one of the topologies is regular. If T_1 and T_2 are topologies on a set X, the two relationships between T_1 and T_2 considered by Weston are

1) $\underline{Cl}_1 V \subseteq \underline{Cl}_2 V$ for all $V \in T_2$, and

2) $\underline{Cl}_1 U \subseteq \underline{Cl}_2 U$ for all $U \in T_1$,

where $\underline{Cl}_i$ is the closure operator with respect to (X,T_i) for $i = 1,2$. Weston discovered that if (1) is satisfied then $T_2 \subseteq T_1$ if (X,T_2) is regular (Theorem 1, [67]). Thus Weston considered relationship (2) as the more interesting and proved several results about topologies satisfying (2) particularly if in addition $T_1 \subseteq T_2$, in which case Weston wrote $T_1 \leq T_2$. Since a bijection $f:X \to Y$ from a set X to a space Y induces a topology T_2 on X consisting of inverse images under f of open sets in Y and for which $f:(X,T_2) \to Y$ is a homeomorphism, Weston's relationships (1) and (2) can be expressed as properties of bijections from a space X onto a space Y. Thus relationship (1) is equivalent to $\underline{Cl}(f^{-1}(V)) \subseteq f^{-1}(\underline{Cl}\ V)$ for all $V \in S$ where $f:(X,T) \to (Y_1 S)$ is a bijection. Takashi Noiri has shown (Theorem 4, [54]) that all weakly continuous functions satisfy this condition. However, the converse is also true so that a new

characterization of weak continuity is obtained from which Weston's Theorem 1 [67] is realized as a special case of Levine's result that a weakly continuous function into a regular space is continuous.

THEOREM 12. Let $f:X\to Y$ be a function from a topological space X into a topological space Y. Then f is weakly continuous if and only if $\underline{Cl}\ f^{-1}(V) \subseteq f^{-1}(\underline{Cl}\ V)$ for all open subsets V of Y.

Proof. The necessity has been shown by Noiri (Theorem 4, [54]). For the sufficiency, let V be an open subset of Y and let $W = Y - \underline{Cl}\ V$. Then W is open and

$$\begin{aligned} X - \underline{Int}\ f^{-1}(\underline{Cl}\ V) &= \underline{Cl}[X - f^{-1}(\underline{Cl}\ V)] \\ &= \underline{Cl}\ f^{-1}(W) \\ &\subseteq f^{-1}(\underline{Cl}\ W) \\ &= f^{-1}(Y - \underline{Int}\ \underline{Cl}\ V) \\ &\subseteq f^{-1}(Y - V) \\ &= X - f^{-1}(V). \end{aligned}$$

Therefore, $f^{-1}(V) \subseteq \underline{Int}\ f^{-1}(\underline{Cl}\ V)$ and f is weakly continuous.

Weston's relationship (2) for two topologies on a set is equivalent to the condition $f(\underline{Cl}\ U) \subseteq \underline{Cl}\ f(U)$ for all $U \in T$ where $f:(X,T)\to(Y,S)$ is a bijection. Weston proved (Theorem 2, [67]) that for any bijection $f:X\to Y$ satisfying the condition $f(\underline{Cl}\ U) \subseteq \underline{Cl}\ f(U)$ for each open set U in X, if $x \in X$ and $f(x) \in V$ where V is open in Y, then

$x \in \underline{\text{Int}}\ \underline{\text{Cl}}\ f^{-1}(V)$. Thus such bijections are almost continuous in the sense of Husain [32], which will be defined below. Thus Weston's results can be interpreted as results for weakly continuous bijections, almost continuous bijections, or open almost continuous bijections in the case $T_1 \leq T_2$. Weston's results were obtained before weak continuity and almost continuity had been introduced by name or formulated in full generality.

DEFINITION 11. (T. Husain) The function $f:X\to Y$ is almost continuous at $x \in X$ if for each open set $V \subseteq Y$ containing $f(x)$, the closure of $f^{-1}(V)$ is a neighborhood of x. If f is almost continuous at each point of X, then f is called almost continuous.

Paul E. Long and Earl E. McGehee, Jr. noted (Theorem 1, [51]) that a function $f:X\to Y$ is almost continuous at $x \in X$ if and only if for each open set V containing $f(x)$ there is an open set U containing x such that $U \subseteq \underline{\text{Cl}}\ f^{-1}(V)$, or in other words, such that $f^{-1}(V)$ is dense in U. But if U is open, $U \subseteq \underline{\text{Cl}}\ f^{-1}(V)$ if and only if $U \subseteq \underline{\text{Int}}\ \underline{\text{Cl}}\ f^{-1}(V)$ and the following characterization of almost continuity is immediate.

THEOREM 13. The function $f:X\to Y$ is almost continuous if and only if $f^{-1}(V) \subseteq \underline{\text{Int}}\ \underline{\text{Cl}}\ f^{-1}(V)$ for each open subset V of Y.

Weston's result that condition (2) for topologies on a set implies almost continuity when interpreted for bijections can be generalized to obtain a new characterization of almost continuity.

THEOREM 14. The function $f:X\to Y$ is almost continuous if and only if $f(\underline{Cl}\ U) \subseteq \underline{Cl}\ f(U)$ for each open subset U of X.

Proof. For the sufficiency let $x \in X$ with $f(x) \in V$ for some open subset V of Y. If $x \notin \underline{Int}\ \underline{Cl}\ f^{-1}(V)$, then $x \in X - \underline{Int}\ \underline{Cl}\ f^{-1}(V) = \underline{Cl}\ U$ where $U = X - \underline{Cl}\ f^{-1}(V)$. Thus $f(x) \in \underline{Cl}\ f(U)$ by the hypothesis so that $V \cap f(U) \neq \emptyset$. But this is impossible since $U \subseteq X - f^{-1}(V) = f^{-1}(Y-V)$. Thus f is almost continuous at each $x \in X$. For the necessity, let U be an open subset of X so that $V = Y - \underline{Cl}\ f(U)$ is open in Y. Assuming $f:X\to Y$ is almost continuous,

$$\begin{aligned} X - f^{-1}(\underline{Cl}\ f(U)) &= f^{-1}(V) \\ &\subseteq \underline{Int}\ \underline{Cl}\ f^{-1}(V) \\ &= X - \underline{Cl}\ \underline{Int}\ f^{-1}(\underline{Cl}\ f(U)) \\ &\subseteq X - \underline{Cl}\ U, \end{aligned}$$

so that $\underline{Cl}\ U \subseteq f^{-1}(\underline{Cl}\ f(U))$ and $f(\underline{Cl}\ U) \subseteq \underline{Cl}\ f(U)$.

Theorems 12 and 14 show that Weston's two relationships for a pair of topologies on a set are special cases of weak continuity and almost continuity respectively. Theorem 13 shows that these two types of generalized

continuity conditions have similar formulations. Recall that a function $f:X\to Y$ is weakly continuous if and only if $f^{-1}(V) \subseteq \underline{\text{Int}}\ f^{-1}(\underline{\text{Cl}}\ V)$ for every open subset V of Y. Examples have been given in the literature to show that neither of these generalized continuity conditions implies the other. An example of Long and McGehee (Example 1, [51]) was given to show that almost continuity does not imply almost continuity (S and S). But in fact the almost continuous function of this example is a discontinuous function into a regular space so that it must fail to be weakly continuous. Singal and Singal gave an example [61] of an almost continuous (S and S) function which is not almost continuous but their function has neither a Hausdorff domain or range. That almost continuity (S and S) does not imply almost continuity in the setting of Hausdorff spaces without isolated points can be shown as follows. Let $f:(X,T)\to(X,S)$ be the identity function where $X = \{(x,y):x \text{ and } y \text{ are rational numbers and } y \geq 0\}$ is the upper rational half-plane in Euclidean space $R\times R$, and T is the usual Euclidean subspace topology and S is the irrational-slope topology (p. 93, [64]) as defined in Example 12 of Chapter II of this paper. Then (X,T) and (X,S) are Hausdorff spaces without isolated points and f is almost continuous (S and S) and hence weakly continuous and f^{-1} is almost continuous. But f is not almost continuous nor is f^{-1} weakly continuous. Therefore almost continuity is independent from almost continuity (S and S),

and with weak continuity, even for functions between Hausdorff spaces without isolated points.

Singal and Singal proved (Theorem 2.3, [61]) that every open weakly continuous function is almost continuous (S and S). Later, Long and Carnahan showed (Theorem 5, [48]) that every open almost continuous (S and S) function is almost continuous. Combining these results yields the following theorem for which direct proof will be given after a lemma.

THEOREM 15. If $f:X\to Y$ is an open weakly continuous function, then f is almost continuous.

The following lemma was noted by Long and Carnahan (Lemma, [48]).

LEMMA FOR THEOREM 15. (Long and Carnahan) If $f:X\to Y$ is an open function, then $f^{-1}(\underline{Cl}\ B) \subseteq \underline{Cl}\ f^{-1}(B)$ for every subset B of Y.

Proof of Theorem 15. Let V be an open subset of Y. By the Lemma for Theorem 15 and by Theorem 12, $f^{-1}(\underline{Cl}\ V) = \underline{Cl}\ f^{-1}(V)$. By Theorem 13, $f^{-1}(V) \subseteq \underline{Int}\ \underline{Cl}\ f^{-1}(V)$ so that f is almost continuous.

The following corollary gives another set of conditions with which weak continuity implies almost continuity. This corollary improves the similar result obtained by Long and Carnahan (Corollary, [48]) by neither requiring

f:X→Y to be almost continuous (S and S) nor requiring Y to be Urysohn.

COROLLARY TO THEOREM 15. If f:X→Y is a weakly continuous bijection from a compact space X onto a Hausdorff space Y, then f is almost continuous.

Proof. By Corollary 2 to Theorem 1, f is open and hence f is almost continuous by Theorem 15.

The following example shows that openness of the weakly continuous function f:X→Y does not imply continuity.

EXAMPLE 3. Let X = R be the usual space of real numbers and let $Q \subseteq R$ be the set of rational numbers. Let Y = R be the space of real numbers topologized with the smallest extension of the usual topology for which $Q \subseteq Y$ is open. Then Y is Urysohn but non-regular (Example 2, p. 141, [19]). Let f:X→Y be the identity function. Then $f^{-1}(V) \subseteq \underline{\text{Int}}\ f^{-1}(\underline{\text{Cl}}\ V)$ for each $V = (a,b) \cap Q$ with $a < b$. By Theorem 6, f is weakly continuous and clearly f is open. By Theorem 15, f is almost continuous but since $f^{-1}(Q) = Q$ is not open in X, f is not continuous. In fact, Q is the set of discontinuities in X for f.

Shwu-Yeng T. Lin and Y.-F. Lin posed the following question [45]. If f:X→Y is an almost continuous function from a Baire space X into a second countable space Y and if f has a closed graph $G(f) \subseteq X \times Y$, then is f necessarily

continuous? Example 3 above answers this question in the negative. Certainly, if X, Y, and f are as defined in this example, X is a Baire space. Furthermore, Y is a second countable space for X is second countable and if $\{V_n : n \geq 1\}$ is a countable basis for the usual topology then $\{V_n : n \geq 1\} \cup \{V_n \cap Q : n \geq 1\}$ is a countable basis for the topology on Y. As noted, f is almost continuous. Also, $G(f) \subseteq X \times Y$ is closed by Theorem 1. Finally, f is not continuous. Yet this counterexample raises a new question. Under the hypotheses of the Lins' question must f be continuous at each point of a dense subset of X? Or, if X is not assumed to be a Baire space, must the set of points of discontinuity for f be a first category (meager) subset of X?

Though Example 3 showed that openness is not strong enough to yield continuity of a weakly continuous function, is there a condition weaker than openness which can replace openness in Theorem 15 and still yield the conclusion? Generalized continuity conditions for bijective functions dually yield the following generalized openness and closedness conditions for the inverse functions.

DEFINITION 12. The function $f : X \to Y$ is almost open if and only if $f(U) \subseteq \underline{\text{Int}}\ \underline{\text{Cl}}\ f(U)$ for each open subset U of X.

B. J. Pettis ([58], [59]) defines a subset A of a topological space X to be nearly open if and only if

$A \subseteq \underline{\text{Int}}\ \underline{\text{Cl}}\ A$. For consistency with Definition 12 above, such subsets will be called almost open sets. Pettis also defines the function $f:X \to Y$ to be open (nearly continuous) if and only if $f(f^{-1})$ carries open sets onto nearly open sets. Clearly, near openness is almost openness and near continuity is almost continuity and a function $f:X \to Y$ is almost open (almost continuous) if and only if $f(f^{-1})$ carries open sets onto almost open sets.

Note that a function $f:X \to Y$ is almost continuous if and only if $\underline{\text{Cl}}\ \underline{\text{Int}}\ f^{-1}(C) \subseteq f^{-1}(C)$ for each closed subset C of Y. This motivates the following definition.

DEFINITION 13. The function $f:X \to Y$ is almost closed if and only if $\underline{\text{Cl}}\ \underline{\text{Int}}\ f(C) \subseteq f(C)$ for each closed subset C of X.

DEFINITION 14. (Singal and Singal, [61]) The function $f:X \to Y$ is almost open (S and S) if and only if f carries regular open sets onto open sets.

DEFINITION 15. (Singal and Singal [61]) The function $f:X \to Y$ is almost closed (S and S) if and only if f carries regular closed sets onto closed sets.

A function $f:X \to Y$ is almost open (S and S) if and only if $f(U) \subseteq \underline{\text{Int}}\ f(\underline{\text{Int}}\ \underline{\text{Cl}}\ U)$ for each open subset U of X and similarly $f:X \to Y$ is almost closed (S and S) if and only if $\underline{\text{Cl}}\ f(\underline{\text{Cl}}\ \underline{\text{Int}}\ C) \subseteq f(C)$ for each closed subset C of X. Furthermore, almost closedness (S and S) is the dual

of almost continuity (S and S) since a function $f:X\to Y$ is almost continuous (S and S) if and only if $\underline{Cl}\ f^{-1}(\underline{Cl}\ \underline{Int}\ C) \subseteq f^{-1}(C)$ for each closed subset C of Y.

DEFINITION 16. The function $f:X\to Y$ is weakly open if and only if $f(U) \subseteq \underline{Int}\ f(\underline{Cl}\ U)$ for each open subset U of X.

DEFINITION 17. The function $f:X\to Y$ is weakly closed if and only if $\underline{Cl}\ f(\underline{Int}\ C) \subseteq f(C)$ for each closed subset C of X.

That weak openness is dual to weak continuity is obvious and similarly so is weak closedness since the function $f:X\to Y$ is weakly continuous if and only if $\underline{Cl}\ f^{-1}(\underline{Int}\ C) \subseteq f^{-1}(C)$ for each closed subset C of Y.

The above definitions allow dualizations of some of the results for almost continuous, almost continuous (S and S), and weakly continuous functions. Clearly, almost openness (S and S) [almost closedness (S and S)] implies weak openness [weak closedness]. Further, for a bijection $f:X\to Y$, f is almost open (almost open (S and S)) [weakly open] if and only if f is almost closed (almost closed (S and S)) [weakly closed] if and only if f^{-1} is almost continuous (almost continuous (S and S)) [weakly continuous]. Since there exist bijective almost continuous (almost continuous (S and S)) [weakly continuous] functions which fail to be continuous, as is shown by Example 3, almost openness (almost openness (S and S)) [weak openness] is implied by but does not imply openness.

Similarly the generalized closedness conditions above are implied by but do not imply closedness. A partial dual to Theorem 14 is the following result.

THEOREM 16. If the function $f:X\to Y$ is almost open then $f^{-1}(\underline{Cl}\ V) \subseteq \underline{Cl}\ f^{-1}(V)$ for each open subset V of Y.

Proof. If $V \subseteq Y$ is open and $x \in f^{-1}(\underline{Cl}\ V)$ and if $U \subseteq X$ is an open set containing x, then $f(x) \in f(U) \subseteq \underline{Int}\ \underline{Cl}\ f(U)$. Since $f(x) \in \underline{Cl}\ V$, $V \cap \underline{Int}\ \underline{Cl}\ f(U) \neq \phi$. But then $V \cap f(U) \neq \phi$ so that $U \cap f^{-1}(V) \neq \phi$. Thus $x \in \underline{Cl}\ f^{-1}(V)$ showing that $f^{-1}(\underline{Cl}\ V) \subseteq \underline{Cl}\ f^{-1}(V)$.

It may be remarked that the converse of Theorem 16 and hence the complete dualization of Theorem 14 hold if f is an epimorphism between topological groups X and Y. For if $f:X\to Y$ is surjective, and $U \subseteq X$ is open, $f(U) \subseteq \underline{Int}\ \underline{Cl}\ f(U)$ if and only if $f^{-1}(f(U)) \subseteq f^{-1}(Cl[Y - \underline{Cl}\ f(U)])$. But

$$\begin{aligned} f^{-1}(Y - \underline{Int}\ \underline{Cl}\ f(U)) &= f^{-1}(\underline{Cl}[Y - \underline{Cl}\ f(U)]) \\ &\subseteq \underline{Cl}\ f^{-1}(Y - \underline{Cl}\ f(U)) \\ &= X - \underline{Int}\ f^{-1}(\underline{Cl}\ f(U)). \end{aligned}$$

Further, $f^{-1}(f(U)) \subseteq \underline{Int}\ f^{-1}(\underline{Cl}\ f(U))$. For if $x \in f^{-1}(f(U))$ then for some $x' \in U \subseteq \underline{Int}\ f^{-1}(\underline{Cl}\ f(U))$, $f(x) = f(x')$. Thus, if W is a neighborhood of the identity e for X with $x'W \subseteq U$ then $xW \subseteq f^{-1}(\underline{Cl}\ f(U))$ and $x \in \underline{Int}\ f^{-1}(\underline{Cl}\ f(U))$. Hence, $f^{-1}(f(U)) \subseteq f^{-1}(\underline{Int}\ \underline{Cl}\ f(U))$ and f is almost open establishing the following characterization of almost openness for surjective homomorphisms of topological groups.

THEOREM 17. Let X and Y be topological groups and let $f:X\to Y$ be a surjective homomorphism. Then f is almost open if and only if $f^{-1}(\underline{Cl}\ V) \subseteq \underline{Cl}\ f^{-1}(V)$ for each open subset V of Y.

The following result improves Theorem 15 by replacing openness with the weaker condition of almost openness.

THEOREM 18. If $f:X\to Y$ is an almost open weakly continuous function, then f is almost continuous.

Proof. Let $V \subseteq Y$ be open. Then Theorems 12 and 16 imply that $f^{-1}(\underline{Cl}\ V) = \underline{Cl}\ f^{-1}(V)$. Thus $f^{-1}(V) \subseteq \underline{Int}\ \underline{Cl}\ f^{-1}(V)$ and f is almost continuous.

Now consider the problem of finding conditions under which almost continuity implies weak continuity. T. Noiri attempted such a result (Theorem 5, [54]). He showed that if $f:X\to Y$ is an almost continuous function such that $\underline{Cl}\ f^{-1}(V) \subseteq f^{-1}(\underline{Cl}\ V)$ for each open subset V of Y, then f is weakly continuous. The proof is easy for if $f:X\to Y$ is almost continuous then $f^{-1}(V) \subseteq \underline{Int}\ \underline{Cl}\ f^{-1}(V) \subseteq \underline{Int}\ f^{-1}(\underline{Cl}\ V)$ for each open subset V of Y so that f is weakly continuous. However, Theorem 12 shows that the hypothesis of almost continuity is superfluous. Other theorems with unduly strong hypotheses are found in the article "Comparing almost continuous functions" by Long and Carnahan [48] which Noiri had used as a guide for his paper "On weakly continuous mappings" [54]. The

difficulty was apparently in not being aware of the characterization of weak continuity as given in Theorem 12. In their paper, Long and Carnahan referenced a paper by Singal and Singal [61] in which as a remark it noted that almost continuity (S and S) implies weak continuity and it was proven (Corollary 2.1, [61]) that an open function is almost continuous (S and S) if and only if it is weakly continuous. Yet Long and Carnahan proved (Theorem 6, [48]) that if $f:X\to Y$ is almost continuous (S and S) and if V is an open subset of Y then $\underline{Cl}\ f^{-1}(V) - f^{-1}(V) \subseteq f^{-1}(\underline{Cl}\ V)$. By Theorem 12, this result is no stronger than the remark by Singal and Singal since it states that every almost continuous (S and S) function satisfies a condition met by every weakly continuous function. Then Long and Carnahan showed (Theorem 7, [48]) that every open almost continuous (S and S) function satisfies the condition that $\underline{Cl}\ f^{-1}(V) \subseteq f^{-1}(\underline{Cl}\ V)$ for each open subset V of Y. The openness of the hypotheses is superfluous and Noiri proved (Theorem 4, [54]) that every weakly continuous function satisfies this condition. As a corollary to their Theorem 7 [48], it was shown that for every open almost continuous (S and S) function $f:X\to Y$, $\underline{Cl}\ f^{-1}(V) = f^{-1}(\underline{Cl}\ V)$ for each open subset of Y. In fact, as Theorems 12 and 16 show, the conclusion of this result still holds even for almost open weakly continuous functions. Finally, Long and Carnahan proved (Theorem 8, [48]) that an open almost continuous function $f:X\to Y$ is almost continuous (S and S)

if and only if $\underline{Cl}\ f^{-1}(V) = f^{-1}(\underline{Cl}\ V)$ for every open subset V of Y. Note that equivalently the theorem could have been stated with $\underline{Cl}\ f^{-1}(V) \subseteq f^{-1}(\underline{Cl}\ V)$ for every open subset V of Y since by Theorem 16 every almost open function and hence every open function satisfies the condition that $f^{-1}(\underline{Cl}\ V) \subseteq \underline{Cl}\ f^{-1}(V)$ for each open subset V of Y. Further, for open functions almost continuity (S and S) is equivalent to weak continuity. Hence the theorem could be equivalently restated as follows. An open almost continuous function $f:X\to Y$ is weakly continuous if and only if $\underline{Cl}\ f^{-1}(V) \subseteq f^{-1}(\underline{Cl}\ V)$ for each open subset V of Y. Indeed Noiri improved this result by showing (Corollary 1, [54]) that the hypothesis of openness in this version is inessential. But in fact Theorem 12 shows that almost continuity is also inessential. So, in fact, for an open function $f:X\to Y$, almost continuity (S and S) is equivalent to the condition that $\underline{Cl}\ f^{-1}(V) = f^{-1}(\underline{Cl}\ V)$ for each open subset V of Y. The question arises as to whether here openness can be replaced by almost openness. An almost open and almost continuous (S and S) function $f:X\to Y$ must be weakly continuous so that $\underline{Cl}\ f^{-1}(V) = f^{-1}(\underline{Cl}\ V)$ for each open subset V of Y. But the question of whether the almost open function $f:X\to Y$ and satisfying $\underline{Cl}\ f^{-1}(V) = f^{-1}(\underline{Cl}\ V)$ for all open subsets V of Y, must be almost continuous (S and S) is answerable in the affirmative if and only if weak continuity plus almost openness implies almost continuity (S and S). In light of Theorem 18,

a counterexample for the converse would require an almost continuous function f:X→Y which is weakly continuous but not almost continuous (S and S). Further, the counter-example function would necessarily be almost open but not open.

Each of the above theorems from the literature attempting to derive weak continuity from almost continuity plus added conditions failed because in each case the almost continuity was an inessential hypothesis. Long and McGehee found the following result (Theorem 10, [51]).

THEOREM (L-M)1. Let f:X→Y be an almost continuous function where Y is regular and locally connected. If $\underline{Cl}\ f^{-1}(C) \subseteq f^{-1}(\underline{Cl}\ C)$ for every connected subset C of Y, then f is continuous.

Since continuity is equivalent to weak continuity when the range space Y is regular by Theorem (L)1, Theorem (L-M)1 gives conditions when weak continuity can be derived from almost continuity. Apparently the full purpose of the hypothesis of regularity of Y is to derive continuity from weak continuity for the following more general result also holds.

THEOREM 19. Let f:X→Y be an almost continuous function where Y is a locally connected space. If $\underline{Cl}\ f^{-1}(C) \subseteq f^{-1}(\underline{Cl}\ C)$ for each connected subset C of Y then f is weakly continuous.

Proof. Let V be a connected open subset of Y. Then $f^{-1}(V) \subseteq \underline{\text{Int}}\ \underline{\text{Cl}}\ f^{-1}(V) \subseteq \underline{\text{Int}}\ f^{-1}(\underline{\text{Cl}}\ V)$ so that by Theorem 6 f is weakly continuous since the connected open sets form a basis for the topology on Y.

The need for the hypothesis of almost continuity in this theorem may be questioned. For indeed $\underline{\text{Cl}}\ f^{-1}(V) \subseteq f^{-1}(\underline{\text{Cl}}\ V)$ holds for all V belonging to an open basis for the topology on Y. However, alone, this condition does not imply weak continuity showing a contrast between Theorem 12 and Theorem 6. The following example shows that almost continuity of the hypotheses of Theorem 19 and of Theorem (L-M)1 is not superfluous.

EXAMPLE 4. Let X = (Z,T) be any non-discrete T_1-space. Then Z is a non-singleton set. Let Y = (Z,S) be a discrete space and let $f:X \to Y$ be the identity function on the set Z. Then Y is locally connected and regular and totally disconnected. Thus if $C \subseteq Y$ is connected and nonempty, then C = {y} is a singleton subset of Y. Therefore, $\underline{\text{Cl}}\ f^{-1}(C) = f^{-1}(\underline{\text{Cl}}\ C)$ for each connected subset C of Y. Clearly f is not continuous, weakly continuous, or almost continuous. By Theorem 12, there exists an open subset V of Y such that $\underline{\text{Cl}}\ f^{-1}(V) \not\subseteq f^{-1}(\underline{\text{Cl}}\ V)$, even though the connected open subsets of Y form a basis for the discrete topology. In particular, since X is a non-discrete T_1-space, there exists a non-isolated point $x_0 \in X$. Then $Z - \{x_0\} = V$ is clopen (closed and open)

in Y and open but not closed in X. Thus $\underline{Cl}\ f^{-1}(V) \not\subseteq f^{-1}(\underline{Cl}\ V)$.

Thus Theorem 19 gives conditions under which almost continuity implies weak continuity and the hypothesis of almost continuity is not superfluous. The proof of Theorem 19 shows a more general result. By Theorem 12 a weakly continuous function $f:X\to Y$ must satisfy the condition that $\underline{Cl}\ f^{-1}(V) \subseteq f^{-1}(\underline{Cl}\ V)$ for each V belonging to an open basis for the topology on Y. But Example 4 above shows that this condition does not imply weak continuity, even if equality of $\underline{Cl}\ f^{-1}(V)$ and $f^{-1}(\underline{Cl}\ V)$ holds and Y is regular.

THEOREM 20. If $f:X\to Y$ is an almost continuous function and if there is an open basis B for the topology on Y such that $\underline{Cl}\ f^{-1}(V) \subseteq f^{-1}(\underline{Cl}\ V)$ for each member V of B, then f is weakly continuous.

Since the open connected sets form a basis for the topology of a locally connected space, Theorem 20 implies Theorem 19. Excluding the hypothesis of almost continuity, the other hypotheses of Theorem 20 are stronger than the other hypotheses of Theorem 19. For example, if $D = \{0,1\}$ has the discrete topology and $I = (0,1)$ is the open interval of real numbers between zero and one with the usual subspace topology then by letting $Y = D + I$ be the topological sum of D and I and by letting $f:X\to Y$ be the

identity function where X = [0,1] is the closed unit interval with the usual subspace topology, f satisfies all the hypotheses of Theorem 20 other than almost continuity. Since f is not continuous and Y is regular, evidently f is not weakly continuous. Thus by Theorem 20 it is deduced that f is not almost continuous. This deduction cannot come directly from Theorem 19 or Theorem (L-M)1 since C = (0,1) is a connected subset of Y and yet $\underline{Cl}\ f^{-1}(C) = [0,1] \not\subseteq (0,1) = f^{-1}(\underline{Cl}\ C)$. This example does not conclusively prove that the hypotheses of Theorem 20 are satisfied by a larger class of functions than the hypotheses of Theorem 19 since the function of this example was not almost continuous.

COROLLARY TO THEOREM 20. If $f:X\to Y$ is an almost continuous injection into the Hausdorff space Y and if there is a basis B for the topology on Y such that $Cl\ f^{-1}(V) \subseteq f^{-1}(\underline{Cl}\ V)$ for each member V of B, then X is a Hausdorff space.

Proof. By Theorem 20, f is weakly continuous. By Theorem 1, G(f), the graph of f, is closed in $X\times Y$. Thus X is a Hausdorff space since Long and McGehee have shown that the domain of any almost continuous injection with a closed graph must be Hausdorff (Theorem 7, [51]).

The proof of this corollary shows that in particular if $f:X\to Y$ is an almost continuous and weakly continuous

function into a Hausdorff space Y, then X is Hausdorff. This can be compared with Corollary 2 to Theorem 4 and Corollary 5 of Theorem 1.

John Jones, Jr. [33] defined a function $f:X\to Y$ to be semiconnected if and only if inverse images under f of closed connected subsets of Y are closed connected subsets of X. Long and McGehee proved the following theorem (Theorem 12, [51]) in their paper "Properties of almost continuous functions" in which Theorem (L-M)1 appeared.

THEOREM (L-M)2. If $f:X\to Y$ is an almost continuous semiconnected function into a locally connected regular space Y, then f is continuous.

A different proof than that of Long and McGehee can be given showing Theorem (L-M)2 to be a corollary of Theorem (L-M)1. Further, as Theorem (L-M)1 was generalized by Theorem 19, Theorem (L-M)2 can be generalized similarly and is also a corollary of the more general Theorem 20.

Proof of Theorem (L-M)2. If $C \subseteq Y$ is connected then $\underline{Cl}\ C$ is connected and closed so that $f^{-1}(\underline{Cl}\ C)$ is closed and connected. Thus $\underline{Cl}\ f^{-1}(C) \subseteq f^{-1}(\underline{Cl}\ C)$ for all connected subsets C of Y and by Theorem (L-M)1, f is continuous.

THEOREM 21. If $f:X\to Y$ is an almost continuous semiconnected function into a locally connected space Y, then f is weakly continuous.

Proof. Apply Theorem 20 using the basis of open connected subsets of Y.

Conditions have been found which when combined with weak continuity imply almost continuity and other conditions have been noted which yield weak continuity when combined with almost continuity. These results and others show how continuity can be deduced from either weak continuity or almost continuity in certain cases. Can new or perhaps more general conditions be found under which either weak continuity or almost continuity will imply continuity? Two generalizations of continuity might be called complementary if each alone is weaker than continuity but jointly they imply continuity, in which case the pair of complementary conditions could be called a decomposition of continuity. Either member of the pair could be called a partial decomposition of continuity. It can be shown that weak continuity is a partial decomposition of continuity. Levine [39] showed that weak continuity has a complement which he called weak* continuity. In another paper [40] Levine defined a class of semi-open sets in a topological space (X,T) and used this class to define semi-continuity for functions $f:X \to Y$.

DEFINITION 18. (Levine) A subset A of a topological space (X,T) is semi-open if and only if for some open set U of X, $U \subseteq A \subseteq \underline{Cl}\ U$. The class of all semi-open subsets of (X,T) is denoted S.O.(X,T) or S.O.(X) if the topology

T on X with respect to which the semi-open sets are determined is understood.

Clearly $T \subseteq S.O.(X,T)$ and hence semi-openness and semi-continuity as defined below are generally weaker than openness and continuity respectively. Further, if $A \in S.O.(X,T)$ then $\underline{Cl}\ A$ is a regular closed set since $\underline{Cl}\ A = \underline{Cl}\ U$ for some open set $U \in T$.

DEFINITION 19. (Levine) A function $f:X \to Y$ is semi-continuous (semi-open) if and only if $f^{-1}(f)$ carries open sets onto semi-open sets.

Immediate from the definitions follows Levine's first result of his paper "Semi-open sets and semi-continuity in topological spaces" (Theorem 1, [40]).

THEOREM 22. (Levine) If X is a topological space, then $A \in S.O.(X)$ if and only if $A \subseteq \underline{Cl}\ \underline{Int}\ A$.

Proof. If $U \subseteq A \subseteq \underline{Cl}\ U$ for some open set U then $A \subseteq \underline{Cl}\ \underline{Int}\ U \subseteq \underline{Cl}\ \underline{Int}\ A$. If conversely, $A \subseteq \underline{Cl}\ \underline{Int}\ A$, then $\underline{Int}\ A \subseteq A \subseteq \underline{Cl}\ \underline{Int}\ A$ and $A \in S.O.(X)$.

From Theorem 22 the following result follows. The sufficiency was noted by Levine (Theorem 3, [40]).

THEOREM 23. (Levine) If $B \subseteq X$ then $B \in S.O.(X)$ if and only if $A \subseteq B \subseteq \underline{Cl}\ A$ for some $A \in S.O.(X)$.

Proof. The necessity is immediate by taking A = B. Now if A ε S.O.(X) and A $\subseteq$ B $\subseteq$ Cl A then by Theorem 22, Cl A = Cl Int A so that B $\subseteq$ Cl Int A $\subseteq$ Cl Int B and by Theorem 22 again, B ε S.O.(X).

COROLLARY OF THEOREM 23. Let A be a closed subset of the space X. Then A ε S.O.(X) if and only if A is a regular closed subset of X.

Proof. A subset A of a space X is regular closed if and only if A is the closure of an open subset U of X. Since U ε S.O.(X), Cl U = A ε S.O.(X) by Theorem 23. Conversely, if A = Cl A ε S.O.(X) then A $\subseteq$ Cl Int A $\subseteq$ Cl A so that A = Cl Int A is a regular closed subset of X.

The following result was noted by Levine (Theorem 2, [40]).

THEOREM 24. (Levine) If X is a space, then S.O.(X) is closed under arbitrary unions of its members.

Proof. If $\{A_b : b \varepsilon B\} \subseteq$ S.O.(X) then for each b ε B, $A_b \subseteq$ Cl Int A_b. If A = $\bigcup\{A_b : b \varepsilon B\}$, then A $\subseteq$ $\bigcup\{\text{Cl Int } A_b : b \varepsilon B\} \subseteq$ Cl $\bigcup\{\text{Int } A_b : b \varepsilon B\} \subseteq$ Cl Int A so that A ε S.O.(X).

COROLLARY 1 TO THEOREM 24. If f:X$\to$Y is a function such that $f^{-1}(V) \varepsilon$ S.O.(X) for each V belonging to an open basis for the topology on Y, then f is semi-continuous.

The proof is an immediate application of Theorem 24.

A pointwise definition of semi-continuity could be offered as follows.

DEFINITION 20. A function $f:X\to Y$ is semi-continuous at the point $x \in X$ if and only if for each open set V containing $f(x)$, $f(A) \subseteq V$ for some semi-open set A containing x.

The following result of Levine (Theorem 12, [40]) shows that semi-continuity is equivalent to semi-continuity at each point.

COROLLARY 2 TO THEOREM 24. (Levine) Let $f:X\to Y$ be a function. Then f is semi-continuous at each point $x \in X$ if and only if f is semi-continuous.

Proof. If f is semi-continuous and $x \in X$ with V an open subset of Y containing $f(x)$, then $x \in f^{-1}(V) \in S.O.(X)$ and $f(f^{-1}(V)) \subseteq V$ so that f is semi-continuous at x.

Conversely, if f is semi-continuous at each point of X and if V is an open subset of Y, then for each $x \in f^{-1}(V)$ there is a set $A(x) \in S.O.(X)$ with $x \in A(x)$ and $f(A(x)) \subseteq V$. Thus $A(x) \subseteq f^{-1}(V)$ and by Theorem 24, $f^{-1}(V) = \bigcup\{A(x):x \in f^{-1}(V)\} \in S.O.(X)$.

Another formulation of semi-continuity similar in appearance to the formulation of almost continuity given

in Theorem 13 follows now as an immediate consequence of Theorem 22.

THEOREM 25. The function $f:X\to Y$ is semi-continuous if and only if $f^{-1}(V) \subseteq \underline{Cl}\ \underline{Int}\ f^{-1}(V)$ for each open subset V of Y.

From Theorem 25, one might wonder if semi-continuity is a complement to almost continuity in a decomposition of continuity. But an example (Example 9) will be given to show that jointly semi-continuity and almost continuity is in general weaker than continuity. A condition stronger than semi-continuity can be found which when combined with almost continuity implies continuity. However, this stronger condition is not comparable with continuity in the sense that it neither implies nor is implied by continuity in general. Thus only a partial result toward finding a continuity decomposition complement of almost continuity is obtained in this way.

DEFINITION 21. If (X,T) is a topological space, the subset A of X is strongly semi-open if and only if $U \subseteq A \subseteq \underline{Cl}\ U$ for some regular open subset U of X. The collection of strongly semi-open subsets of (X,T) is denoted S.S.O.(X,T) or S.S.O.(X).

Clearly S.S.O.(X,T) $\subseteq$ S.O.(X,T) for every topological space (X,T). Yet in general T is not a subset of S.S.O.(X,T) even if T is a semiregular topology for X.

EXAMPLE 6. Let $X = [\bigcup\{(\frac{1}{n+1}, \frac{1}{n}) : n \text{ is a positive integer}\}] \cup \{0\}$ have the usual subspace topology. Then X is regular and hence semiregular. Let $V = X - \{0\}$. Then V is open but not regular open since $\underline{\text{Int}}\ \underline{\text{Cl}}\ V = X \neq V$. Further, if W is regular open with $W \subseteq V \subseteq \underline{\text{Cl}}\ W$ then $\underline{\text{Int}}\ V = V \subseteq \underline{\text{Int}}\ \underline{\text{Cl}}\ W = W$ so that $V = W$. Therefore, $V \notin$ S.S.O.(X) but $V \in$ S.O.(X), and in particular V is an open subset of X.

Example 6 actually shows the following result.

THEOREM 26. If (X,T) is a topological space and if R.O.(X,T) is the collection of regular open subsets of (X,T), then $T \cap$ S.S.O.(X,T) = R.O.(X,T).

Let R.C.(X,T) and C.(X,T) be the collections of regular closed subsets of the space (X,T) and of closed subsets of the space (X,T) respectively. By the Corollary to Theorem 23, S.O.(X,T) $\cap$ C.(X,T) = R.C.(X,T). But S.S.O.(X,T) $\subseteq$ S.O.(X,T) so that if R.C.(X,T) is contained in S.S.O.(X,T) then the following result also holds.

THEOREM 27. If (X,T) is a topological space, R.C.(X,T) = S.S.O.(X,T) $\cap$ C.(X,T).

Proof. By the Corollary to Theorem 23, it suffices to show that R.C.(X,T) $\subseteq$ S.S.O.(X,T). If $A = \underline{\text{Cl}}\ \underline{\text{Int}}\ A$ then $\underline{\text{Int}}\ A$ is regular open being the interior of a closed set, and $\underline{\text{Int}}\ A \subseteq A \subseteq \underline{\text{Cl}}\ \underline{\text{Int}}\ A$ so that $A \in$ S.S.O.(X,T).

THEOREM 28. If $B \subseteq X$, then $B \in S.S.O.(X,T)$ if and only if $A \subseteq B \subseteq \underline{Cl}\ A$ for some $A \in S.S.O.(X,T)$.

Proof. The necessity is immediate by choosing $A = B$. If $A \subseteq B \subseteq \underline{Cl}\ A$ with $A \in S.S.O.(X,T)$, then if U is regular open with $U \subseteq A \subseteq \underline{Cl}\ U$ then $U \subseteq \underline{Int}\ A \subseteq \underline{Cl}\ U$ so that $\underline{Int}\ A \in S.S.O.(X)$ and by Theorem 26, $\underline{Int}\ A$ is a regular open subset of X. But $\underline{Int}\ A \subseteq B \subseteq \underline{Cl}\ A \subseteq \underline{Cl}\ U = \underline{Cl}\ \underline{Int}\ A$. Thus $B \in S.S.O.(X)$.

THEOREM 29. If (X,T) is a topological space, $A \in S.S.O.(X,T)$ if and only if $\underline{Int}\ A$ is a regular open subset of (X,T) and $A \subseteq \underline{Cl}\ \underline{Int}\ A$.

Proof. The sufficiency is easy since $\underline{Int}\ A \subseteq A \subseteq \underline{Cl}\ \underline{Int}\ A$. For the necessity let $A \in S.S.O.(X,T)$. Then for some regular open set U, $U \subseteq A \subseteq \underline{Cl}\ U$ so that $U \subseteq \underline{Int}\ A \subseteq \underline{Int}\ \underline{Cl}\ U = U$. Thus $\underline{Int}\ A$ is a regular open set and since S.S.O.(X,T) is contained in S.O.(S,T) by Theorem 22, $A \subseteq \underline{Cl}\ \underline{Int}\ A$.

DEFINITION 22. The function $f:X \to Y$ is strongly semi-continuous if and only if there is an open basis B for the topology on Y such that $f^{-1}(V) \in S.S.O.(X)$ for each $V \in B$.

THEOREM 30. Every strongly semi-continuous function is semi-continuous.

Proof. Use Theorem 24 and the fact that $S.S.O.(X,T) \subseteq S.O.(X,T)$ for a strongly semi-continuous function $f:(X,T)\to(Y,S)$.

In general, $f^{-1}(V) \notin S.S.O.(X)$ if $V \in S$. Further, if T_S is the semiregular subtopology of T on X generated by the regular open subsets of (X,T), then the following result holds.

THEOREM 31. If (X,T) is a topological space and T_S is the maximal semiregular subtopology of T on X, then

$$\{A:A \in S.O.(X,T) \text{ and } \underline{Int}\ A \in T_S\} \subseteq S.O.(X,T_S).$$

Proof. If $A \subseteq X$ and $\underline{Int}\ A \in T_S$, then since $T_S \subseteq T$, $\underline{Int}_S\ A = \underline{Int}\ A$. If further $A \in S.O.(X,T)$, then $A \subseteq \underline{Cl}\ \underline{Int}_S\ A \subseteq \underline{Cl}_S\ \underline{Int}_S\ A$ so that $A \in S.O.(X,T_S)$.

COROLLARY TO THEOREM 31. If $f:(X,T)\to(Y,S)$ is strongly semi-continuous and $f^S:(X,T_S)\to Y$ agrees with f at each $x \in X$, then f^S is semi-continuous.

Proof. Let B be an open basis for S such that $f^{-1}(V) \in S.S.O.(X,T)$ for each $V \in B$. Then by Theorem 29, $\underline{Int}\ f^{-1}(V) \in T_S$ and by Theorem 31, $(f^S)^{-1}(V) \in S.O.(X,T_S)$ for each $V \in B$. By Corollary 1 to Theorem 24, f^S is semi-continuous.

S. Gene Crossley and S. K. Hildebrand noted the following result (Theorem 1.1, [15]).

THEOREM 32. (Crossley and Hildebrand) If $f:X\to Y$ is semi-continuous and $g:Y\to Z$ is continuous then $gf:X\to Z$ is semi-continuous.

The proof is immediate. The corresponding result for strongly semi-continuous functions does not seem to hold, since the members of a specified basis for the topology on Y may not be inverse images of open sets from Z even if g is open and continuous. If $f:X\to Y$ is open and continuous and $g:Y\to Z$ is semi-continuous then $gf:X\to Z$ is semi-continuous. In fact a stronger result can be stated.

THEOREM 33. If $f:X\to Y$ is almost open and continuous and if $g:Y\to Z$ is semi-continuous, then $gf:X\to Z$ is semi-continuous.

Proof. Let W be open in Z. Then $g^{-1}(W) \subseteq$ Cl Int $g^{-1}(W)$. Using Theorem 16 and continuity of f,

$$f^{-1}(g^{-1}(W)) \subseteq f^{-1}(\text{Cl Int } g^{-1}(W))$$
$$\subseteq \text{Cl } f^{-1}(\text{Int } g^{-1}(W))$$
$$\subseteq \text{Cl Int } f^{-1}(g^{-1}(W))$$

so that $(gf)^{-1}(W) \in S.O.(X)$.

A slight modification of the proof of Theorem 33 yields a similar result

THEOREM 34. If $f:X\to Y$ is almost open and almost continuous (S and S) and if $g:Y\to Z$ is strongly semi-continuous, then $gf:X\to Z$ is semi-continuous.

Proof. Let W be a basic open set in Z such that $g^{-1}(W) \subseteq \underline{Cl}\ \underline{Int}\ g^{-1}(W)$ and $\underline{Int}\ g^{-1}(W)$ is a regular open subset of Y. Then continue as in the proof of Theorem 33 and use Theorem 24 to obtain semi-continuity for gf. Alternatively, if Y_s is the set Y endowed with the maximal semiregular subtopology of the topology on the space Y, and if $f_s: X \to Y_s$ agrees with f at each $x \in X$, then f_s is continuous. Further if U is open in X, $f_s(U) = f(U) \subseteq \underline{Int}\ \underline{Cl}\ f(U)$ and $\underline{Int}\ \underline{Cl}\ f(U)$ is open in Y_s. Thus $f_s(U) \subseteq \underline{Int}_s\ \underline{Cl}_s\ f_s(U)$ so that f_s is almost open. Hence by the Corollary to Theorem 31 $g^s: Y_s \to Z$ is semi-continuous where g^s agrees with g at each $y \in Y$, so that by Theorem 33 $gf = g^s f_s$ is semi-continuous.

The following examples show that strong semi-continuity neither implies nor is implied by continuity.

EXAMPLE 7. Let $X = \{a,b,c\}$ be endowed with the topology $T = \{\emptyset,\{a\},\{b\},\{a,b\},X\}$ and let $Y = \{0,1\}$ be Sierpinski space with topology $S = \{\emptyset,\{0\},Y\}$. Define the function $f: X \to Y$ by $f(a) = f(c) = 0$ and $f(b) = 1$. Then $\{a\}$ is a regular open subset of X and $f^{-1}(0) = \{a,c\} \in$ S.S.O.(X,T), since $\{a\} \subseteq f^{-1}(0) = \underline{Cl}\ \{a,c\}$. Thus f is strongly semi-continuous but not continuous since $f^{-1}(0)$ is not open.

EXAMPLE 8. Let $X = \{0,1\}$ be Sierpinski space with topology $T = \{\phi,\{0\},X\}$. Then the collection of regular open sets in X is $\{\phi,X\}$. Clearly the identity function $f:X\to X$ is a homeomorphism but not strongly semi-continuous.

Though semi-continuity is implied by continuity, it fails to be a complement for almost continuity in a decomposition of continuity. The following example shows that a semi-continuous, almost continuous function need not be continuous. Recall that a set A is almost open in a space (X,T) if $A \subseteq \underline{\text{Int}}\ \underline{\text{Cl}}\ A$. Let the collection of all almost open subsets of (X,T) be denoted A.O.(X,T) or more simply A.O.(X).

EXAMPLE 9. Let $X = \{a,b,c\}$ have topology $T = \{\phi,\{a\},\{a,b\},X\}$ and let $Y = \{a,b,c\}$ have topology $S = \{\phi,\{a\},\{a,b\},\{a,c\},Y\}$. S. Gene Crossley and S. K. Hildebrand showed (Example 1.1, [14]) that S.O.(X,T) = S.O.(Y,S) = S. Further, it can be easily verified that A.O.(X,T) = A.O.(Y,S) = S. Thus if $f:X\to Y$ is the identity function, f is semi-continuous and almost continuous but not continuous since $f^{-1}(\{a,c\}) = \{a,c\}$ is not open in X. But f is an example of a semi-homeomorphism as defined by Crossley and Hildebrand [16] since f is a bijection and f and f^{-1} take semi-open sets onto semi-open sets.

T. R. Hamlett [25] used the spaces of Example 9 to show false the statement by Levine (Corollary to Theorem

10, [40]) that two topologies on a set are equal if they produce the same semi-open sets. Crossley and Hildebrand defined [16] two topologies T_1 and T_2 on a set X to be semi-correspondent if and only if $S.O.(X,T_1) = S.O.(X,T_2)$. They proved that semi-correspondence of topologies is an equivalence relation on the set of all topologies for X. The equivalence classes of semi-correspondent topologies are called semitopological classes [13]. Clearly, the semi-continuity of a function $f:X\to Y$ depends only on the semitopological class of the topology on X. Furthermore, Crossley and Hildebrand have shown (Theorem 1.12, [16]) that if [T] is the semitopological class of the topology T on X, then the collection of nowhere dense subsets of X remains the same for each representative from [T]. Therefore the first category subsets of X are also independent of the representative from [T] on X. Hamlett used this fact to improve the following result of Levine (Theorem 13, [40]).

THEOREM (L)2. If $f:X\to Y$ is a semi-continuous function from a topological space X into a second countable space Y and if P is the set of discontinuities of f, then P is a first category set.

Hamlett generalized Theorem (L)2 as follows (Theorem 2, [25]).

THEOREM (H). If T_1 and T_2 are semi-correspondent topologies on a set X and if $f:(X,T_1)\to Y$ is a semi-continuous function into a second countable space Y, then the set of discontinuities of f with respect to T_2 is a set of first category in (X,T_2) and in (X,T_1).

Proof. If $g:(X,T_2)\to Y$ is defined by $g(x) = f(x)$ for all $x \in X$, then g is semi-continuous and by Theorem (L)2 g has a first category set of discontinuities P. But then P is also a first category set in (X,T_1).

Crossley and Hildebrand showed (Example 1.6, [16]) that {U-N:U is open and N is nowhere dense} is the finest topology on the set of real numbers that is semi-correspondent to the usual topology. Later, Crossley showed (Theorem 2, [13]) that this result holds for any topological space.

THEOREM (C). Let (X,T) be a topological space. Then the finest topology in the semitopological class T containing T is $F(T) = \{U-N:U \in T$ and N is nowhere dense in $(X,T)\}$.

Note that $T \subseteq F(T)$ for every topology T such that $[T] = [F(T)]$.

As a corollary to Theorem (H), Hamlett found the following to be true.

COROLLARY (H). Let $f:R\to Y$ be a function from the usual space of real numbers into a second countable space Y. If for each $x \in R$ and each open neighborhood V of $f(x)$ there exists an open set U and a nowhere dense set N such that $x \in U-N$ and $f(U-N) \subseteq V$, then the set of discontinuities of f is a set of first category.

Proof. Let T_2 be the usual topology on R and let $T_1 = F(T_2)$. Let $g:(R,T_1)\to Y$ be defined by $g(x) = f(x)$ for each $x \in R$. By the characterization of $F(T_2)$, g is continuous and hence semi-continuous so that by Theorem (H) the set of discontinuities of $f:(R,T_2)\to Y$ is a first category set.

A stronger result can be stated since by Theorem (C) the domain of f need not be R and also semi-continuity of g is sufficient for f to be semi-continuous. If (X,T) is a topological space and if $T^* = F(T)$ is the finest topology in T , let Cl* A and Int* A represent the closure and interior respectively of an arbitrary subset A of $X^* = (X,T^*)$.

THEOREM 35. Let $f:(X,T)\to Y$ be a function from a topological space (X,T) into a space Y satisfying the second axiom of countability. If for each $x \in X$ and open set V in Y with $f(x) \in V$, there exists an open set $U \in T$ such that $x \in \underline{Cl}\ U$ and $U - f^{-1}(V)$ is a nowhere dense subset of (X,T), then the set of discontinuities of f is a first category set.

The proof will be given after a corollary to Theorem 24 and a lemma.

COROLLARY 3 TO THEOREM 24. A function $f:X\to Y$ is semi-continuous if and only if for each $x \in X$ and open set V containing $f(x)$, there exists an open set U in X with $x \in \underline{Cl}\ U$ and $U \subseteq f^{-1}(V)$.

Proof. If $f:X\to Y$ is semi-continuous at $x \in X$ and if V is an open set containing $f(x)$ then there is a semi-open set A containing x such that $A \subseteq f^{-1}(V)$. Let $U = \underline{Int}\ A$. Then $x \in A \subseteq \underline{Cl}\ \underline{Int}\ A = \underline{Cl}\ U$ and $U \subseteq f^{-1}(V)$.

Conversely, if $f(x) \in V$, an open set in Y, implies that for some open U in X, $x \in \underline{Cl}\ U$ and $U \subseteq f^{-1}(V)$, then by letting $A = U \bigcup \{x\}$, $U \subseteq A \subseteq \underline{Cl}\ U$ so that $A \in S.O.(X)$ with $x \in A$ and $A \subseteq f^{-1}(V)$. Thus f is semi-continuous at each point $x \in X$ and by Corollary 2 to Theorem 24, f is semi-continuous.

LEMMA. Let (X,T) be a topological space and let $X^* = (X,T^*)$ where $T^* = F(T)$. If $V \in T^*$ then $V = U-N$ where $U \in T$ and N is a nowhere dense set in X (and in X^*), and $\underline{Cl}^*\ V = \underline{Cl}\ U$ where $\underline{Cl}^*\ V$ is the closure of V in X^*.

Proof. Recall that nowhere dense sets are the same with respect to semi-correspondent topologies on X. By Theorem (C), $T \subseteq T^*$ and if $V \in T^*$ then $V = U-N$ where $U \in T$ and N is nowhere dense. So X-N is dense in X^* and $U \subseteq \underline{Cl}^*(U \cap (X-N)) = \underline{Cl}^*(V)$ (Exercise 1G, [37]). Thus

$\underline{Cl}^*$ U = $\underline{Cl}^*$ V. Since T $\subseteq$ T*, $\underline{Cl}^*$ U $\subseteq$ $\underline{Cl}$ U. If x ε $\underline{Cl}$ U and W ε T with x ε W, then W $\cap$ U $\neq \phi$ and for each nowhere dense set M, (W-M) $\cap$ U $\neq \phi$. For if (W-M) $\cap$ U = ϕ then W $\cap$ U is nowhere dense so that being open W $\cap$ U = ϕ. This contradiction shows that x ε $\underline{Cl}^*$ U since for each open set W* ε T*, W* = W-M for some W ε T and nowhere dense set M. Thus $\underline{Cl}^*$ V = $\underline{Cl}$ U.

Proof of Theorem 35. Let g:X*$\rightarrow$Y be the function defined on (X,T*) which agrees with f at each x ε X where T* = F(T). Let x ε X and let V be an open set containing g(x) = f(x). Then there is an open set U ε T with x ε $\underline{Cl}$ U and N = U - $f^{-1}(V)$ is a nowhere dense set. So U-N $\subseteq$ $f^{-1}(V)$. But U - N ε T* and x ε $\underline{Cl}^*$(U-N) by the Lemma. Therefore since U-N $\subseteq$ $g^{-1}(V)$ by Corollary 3 to Theorem 24, g is semi-continuous. By Theorem (H), the set of discontinuities of f is a first category set.

COROLLARY TO THEOREM (L)2. If f:X$\rightarrow$Y is a semi-continuous function from a Baire space X into a space Y satisfying the second axiom of countability, then f is continuous at each point of a dense subset of X.

Proof. By Theorem (L)2, f is continuous at each point of a comeager set. But every comeager subset of a Baire space is dense.

COROLLARY TO THEOREM 35. If $f:X\to Y$ is a function from a Baire space X into a space Y satisfying the second axiom of countability such that for each $x \in X$ and open set containing $f(x)$ there exists an open set U in X with $x \in \underline{Cl}\ U$ and $U - f^{-1}(V)$ nowhere dense, then f is continuous at each point of a dense subset of X.

Proof. Apply Theorem 35 and the fact that comeager subsets of Baire spaces are dense.

It will now be shown that strong semi-continuity in conjunction with almost continuity implies continuity.

THEOREM 36. If $f:X\to Y$ is strongly semi-continuous and almost continuous then f is continuous.

Proof. Let W be a basic open set in Y for which $f^{-1}(W) \subseteq \underline{Cl}\ \underline{Int}\ f^{-1}(W)$ and $\underline{Int}\ f^{-1}(W)$ is a regular open subset of X. Then by almost continuity of f,

$$\begin{aligned} f^{-1}(W) &\subseteq \underline{Int}\ \underline{Cl}\ f^{-1}(W) \\ &\subseteq \underline{Int}\ \underline{Cl}\ \underline{Int}\ f^{-1}(W) \\ &= \underline{Int}\ f^{-1}(W) \end{aligned}$$

and $f^{-1}(W)$ is an open subset of X. Thus $f^{-1}(V)$ is open for each open subset V of Y.

A result of Long and McGehee (Theorem 13, [51]) which follows from Theorem 36 is now stated.

THEOREM (L-M)3. Let $f:R\to Y$ be an almost continuous function from the usual space of real numbers into a locally connected space Y. If $f^{-1}(C)$ is connected for each connected subset C of Y, then f is continuous.

Proof. Let B be the basis consisting of the open connected subsets of Y. Then $f^{-1}(V)$ is connected for each $V \in B$, and $f^{-1}(V) = \emptyset$ or $f^{-1}(V)$ is an interval of positive length since $f^{-1}(V) \subseteq \underline{\text{Int}}\ \underline{\text{Cl}}\ f^{-1}(V)$. Thus $f^{-1}(V)$ is strongly semi-open for each $V \in B$ and f is strongly semi-continuous. By Theorem 36, f is continuous.

Theorem 36 provides a condition which when coupled with almost continuity yields continuity but this condition of strong semi-continuity is not a generalized continuity condition. So almost continuity has not been shown to be a partial decomposition of continuity. However, weak continuity is a partial decomposition of continuity with complement weak* continuity [39] which will now be defined.

DEFINITION 23. (Levine) The function $f:X\to Y$ is weak* continuous if and only if $f^{-1}(Fr(V))$ is closed in X for each open subset V of Y where $Fr(V) = (\underline{\text{Cl}}\ V) - \underline{\text{Int}}\ V = (\underline{\text{Cl}}\ V) - V$ is the frontier (or boundary) of V.

Clearly every continuous function is weak* continuous.

The following decomposition theorem will be given a different proof than that given by Levine (Theorem 3, [39]).

THEOREM (L)3. The function $f:X\to Y$ is continuous if and only if f is weakly continuous and weak* continuous.

Proof. Clearly continuity implies weak continuity and weak* continuity. If f is weakly continuous and weak* continuous and if V is open in Y, then $f^{-1}(Fr(V)) = f^{-1}(\underline{Cl}\ V) - f^{-1}(V)$ is closed so that $f^{-1}(V) = (X - f^{-1}(Fr(V)) \cap \underline{Int}\ f^{-1}(\underline{Cl}\ V)$ is open.

Requiring $f^{-1}(Fr(V))$ to be closed for each open subset V of Y is not essential to the proof of Theorem (L)3. This motivates the following definition.

DEFINITION 24. The function $f:X\to Y$ is locally weak* continuous if and only if there is an open basis B for the topology on Y such that $f^{-1}(Fr(V))$ is a closed subset of X for each $V \in B$.

Every weak* continuous function is locally weak* continuous but the converse is not true.

EXAMPLE 10. Let $X = Y = R$ be the space of real numbers with the usual topology. Let Q be the set of rational numbers and let Z be the set of integers. Let $g:Q\to Z$ be a set equivalence or bijection. Define $f:X\to Y$ by $f(x) = g(x)$ if $x \in Q$ and $f(x) = x$ if $x \notin Q$. The collection of all open intervals (a,b) where a and b are irrational numbers is a basis for the topology on Y and $f^{-1}(Fr(a,b)) = \{a,b\}$is closed in X. Yet $V = \bigcup\{(2n,2n+1):n\in Z\}$ is open

in Y and $f^{-1}(Fr(V)) = f^{-1}(Z) = Q$ is not closed in X. So f is locally weak* continuous but not weak* continuous.

THEOREM 37. The function $f:X \to Y$ is continuous if and only if it is weakly continuous and locally weak* continuous.

Proof. If f is weakly continuous and locally weak* continuous then $f^{-1}(V) = [X - f^{-1}(Fr(V))] \cap \underline{Int}\ f^{-1}(\underline{Cl}\ V)$ is open for each V in an open basis for the topology on Y. Thus f is continuous. The converse is immediate.

DEFINITION 25. A topological space Y is rim-compact if and only if there is an open basis for the topology on Y such that Fr(V) is compact for each $V \in B$.

Long and Herrington showed (Theorem 7, [50]) that every almost continuous (S and S) function into a rim-compact space and having a closed graph is continuous. This result can be strengthened by replacing the hypothesis of almost continuity (S and S) by weak continuity.

THEOREM 38. Let $f:X \to Y$ be a weakly continuous function with a closed graph G(f). If Y is rim-compact, then f is continuous.

Proof. Let B be an open basis for the topology on Y such that Fr(V) is compact for each $V \in B$. Then by

Corollary 1 to Theorem 10, $f^{-1}(Fr(V))$ is closed for each $V \in B$ and f is locally weak* continuous. By Theorem 37, f is continuous.

Note that for functions into a rim-compact Hausdorff space, by Theorem 1 and Theorem 38, weak continuity is equivalent to continuity. But this was already known since every rim-compact Hausdorff space is regular.

A locally compact space is one in which each point has a compact neighborhood [37]. A topological space is strongly locally compact if each point has a closed compact neighborhood [64]. Clearly a strongly locally compact space is locally compact and rim-compact. For if a space is strongly locally compact then each point is contained in an open set whose closure is compact. For Hausdorff or regular spaces, local compactness is equivalent to strong local compactness and thus a locally compact Hausdorff space must be regular since such a space is rim-compact. It is known (Examples 52, 62, and 73, [64]) that not every locally compact space is strongly locally compact and that non-Hausdorff strongly locally compact spaces exist which are either regular or non-regular.

THEOREM 39. If $f:X \to Y$ is an almost continuous function from a topological space X into a strongly locally compact space Y and if f has a closed graph G(f), then f is continuous.

Proof. Let B be an open basis for the topology on Y such that Cl V is compact for each V ϵ B. By Corollary 1 to Theorem 10, $f^{-1}(\text{Cl } V)$ is closed for each V ϵ B so that Cl $f^{-1}(V) \subseteq f^{-1}(\text{Cl } V)$ for each V ϵ B. Thus by Theorem 20, f is weakly continuous and hence by Theorem 38, f is continuous.

Theorem 39 generalizes the closed graph theorem for Hausdorff topological groups with locally compact range as stated in General Topology (p. 213, [37]) since the spaces are not required to be groups nor is the function required to be a homomorphism. Further since the spaces are not assumed to be Hausdorff, even if Theorem 39 is applied to groups and homomorphisms, this special case is not subsumed by any of the usual versions of the closed graph theorem.

Some of the results in this chapter have dual versions. For example, every weakly open function defined on a regular space is open. Also Corollary 1 to Theorem 10 is dual to Theorem 10. If dual versions could be found for Theorem 20 and Theorem 38, then an open mapping theorem dual to Theorem 39 could be obtained. A dual to Theorem 20 is now found.

THEOREM 40. If $f:X \to Y$ is an almost open function and if A is an open basis for the topology on X such that Cl $f(U) \subseteq f(\text{Cl } U)$ for each U ϵ A, then f is weakly open.

Proof. If $U \in A$ then $f(U) \subseteq \underline{\text{Int}}\ \underline{\text{Cl}}\ f(U) \subseteq \underline{\text{Int}}\ f(\underline{\text{Cl}}\ U)$. Thus if W is open in X, W is a union of members of A, and since f preserves unions, $f(W) \subseteq \underline{\text{Int}}\ f(\underline{\text{Cl}}\ W)$. Hence f is weakly open.

In general, a dualized version of a theorem concerning generalized continuity conditions may be obtainable in terms of generalized openness conditions or may be obtainable in terms of generalized closedness conditions. Another consideration is that for a function $f:X \to Y$, a generalized continuity condition for f usually involves a relationship between a class of subsets of X to a class of inverse images of certain subsets of Y. The dual condition for f may require a relationship between certain subsets of Y and images under f of certain subsets of Y. But f^{-1} has some advantages over f in the sense that f^{-1} is always surjective and preserves intersections of sets. Further f^{-1} is one-to-one in the sense that $(f^{-1})^{-1}$ is a function. Thus an open mapping dual to Theorem 39 can be obtained with some sacrifice of generality. In particular the domain space will be assumed regular.

THEOREM 41. If $f:X \to Y$ is an almost open function from a locally compact regular space X into a topological space Y and if f has a closed graph G(f), then f is open.

Proof. Let A be an open basis for the topology on X such that $\underline{\text{Cl}}\ U$ is compact for each $U \in A$. By Theorem 10,

$f(\underline{Cl}\ U)$ is closed for each $U \in A$ so that $\underline{Cl}\ f(U) \subseteq f(\underline{Cl}\ U)$ for each $U \in A$. By Theorem 40, f is weakly open. But X is regular so that f is open.

Theorem 41 generalizes the open mapping theorem for Hausdorff topological groups with locally compact domain space as stated in General Topology (p. 213, [37]). L. G. Brown proved [8] an open mapping theorem for continuous almost open homomorphism from a metrically complete topological group into a Hausdorff group. Later Brown proved [9] more generally that any continuous almost open homomorphism from a topologically complete group to Hausdorff group is open. Here a space is topologically complete (in the sense of Čech, [10]) if it is homeomorphic to a dense G_δ-set in a compact Hausdorff space. Locally compact Hausdorff spaces and metrically complete spaces are topologically complete [10]. B. J. Pettis noted [59] that Brown's latest version of the open mapping theorem could be extended to a closed graph version.

THEOREM (P). If $f:X \to Y$ is a closed graph homomorphism on a topologically complete group X to a Hausdorff group Y, then f is open if and only if f is almost open and f(X) is closed.

Note that if f is almost open and f(X) is closed then f(X) is also open for $f(X) \subseteq \underline{Int}\ \underline{Cl}\ f(X) = \underline{Int}\ f(X)$. Thus

such an open homomorphism has a clopen subgroup of Y as its image and must be surjective if Y is a connected group. Evidently, even when applied to topological groups and homomorphisms Theorem 41 is not subsumed by Theorem (P).

ALMOST CONTINUITY AND BAIRE SPACES

Generalized Continuity and Space Characteristics

Until recently, continuity was the primary mapping instrument for establishing certain characteristics of topological spaces. However, recently conditions weaker than continuity have become increasingly useful particularly for the purpose of characterizing topological spaces. Such a mapping characterization of Hausdorff k-spaces was obtained by Y.-F. Lin and Leonard Soniat [47] using their notion of weak continuity. Two more recent theorems from the literature illustrate the usefulness of a weaker condition than continuity as compared with continuity. Some preliminary definitions are required. Recall that a function $f:X\to Y$ has a strongly-closed graph $G(f) \subseteq X\times Y$, in the sense of Herrington and Long [29], if $G(f)$ is strongly closed with respect to the second coordinate space Y (Definition 4, Chapter I). A Hausdorff space (X,T) is a minimal Hausdorff space if T is contained in every Hausdorff subtopology of T on X. A Hausdorff space (X,T) is an H-closed space if it is a closed subspace of every Hausdorff space in which it is homeomorphically embedded. Mapping characterizations have been obtained for H-closed spaces and for minimal Hausdorff

spaces which are identical with the exception that continuity is used for one and weak continuity is used to obtain the other. Both of the following theorems are due to Larry L. Herrington and Paul E. Long ([29], [31]) and they are given in the order in which they were discovered. Let S denote a class of topological spaces containing the class of Hausdorff completely normal and fully normal spaces.

THEOREM (H-L)1. A Hausdorff space Y is H-closed if and only if for every topological space X belonging to S, each function $f:X\to Y$ with a strongly-closed graph is weakly continuous.

THEOREM (H-L)2. A Hausdorff space Y is minimal Hausdorff if and only if for every topological space X belonging to S, each function $f:X\to Y$ with strongly-closed graph is continuous.

A shorter and different proof of Theorem (H-L) than that given by Herrington and Long [29] is given by James E. Joseph [34].

Apparently, every minimal Hausdorff space is H-closed. This was known, for M. Katětov showed [36] that a Hausdorff space is minimal Hausdorff if and only if it is H-closed and semiregular. But H-closed spaces need not be minimal Hausdorff.

EXAMPLE 1. Let $X = [0,1]$ be the unit interval of real numbers with the smallest topology T containing the usual subspace open sets and containing all complements of countable sets. Then T is an extension of the usual subspace topology and hence is Urysohn. A basic open set in X is of the form $(A \cap X) - B$ where A is an open interval of real numbers and B is a countable subset of X. If $B_k = \{\frac{1}{n}: n \geq k,\ n \text{ an integer}\}$ and $A_k = X - B_k$ for each positive integer k, then $\cup\{A_k\} = X$ so that $\{A_k\}$ is a countable open cover of X which has no finite subcover so X is not countably compact. But X is Lindelöf for if $U = \{A_a - B_a: a \in I\}$ is a cover of X by basic open sets where each A_a is the intersection of X with an open interval of real numbers and B_a is countable, then there is a finite subcollection of U, $\{A_1-B_1,\ldots,A_n-B_n\}$ such that $A_1 \cup \ldots \cup A_n = X$. Since $B = B_1 \cup \ldots \cup B_n$ is countable and since $X - B \subseteq (A_1-B_1) \cup \ldots \cup (A_n-B_n)$, a countable subcover of U exists for X. Further, $\underline{Cl}\ (A-B) = \underline{Cl}\ A$ for each basic open set $A - B$ in X. And $\underline{Int}\ \underline{Cl}(A-B) = \underline{Int}\ \underline{Cl}\ A = A$ is a subspace open interval of X. Thus X is not semiregular. In fact the maximal semiregular subtopology of T is the usual subspace topology on X. But X is nearly compact. For if U is an open cover of X by basic open sets, then as before, there is a finite subcollection $\{A_1-B_1,\ldots,A_n-B_n\}$ of U so that $X = A_1 \cup \ldots \cup A_n = \underline{Int}\ \underline{Cl}(A_1-B_1) \cup \ldots \cup \underline{Int}\ \underline{Cl}(A_n-B_n)$.

Clearly, X is H-closed being both Hausdorff and almost compact [34], for every nearly compact space is almost compact. Yet X is not a minimal Hausdorff space since the usual subspace topology for X is a proper Hausdorff subtopology of T.

Evidently a generalized continuity condition may be the appropriate tool for characterizing a useful class of spaces larger than the class similarly characterized using continuity.

Using almost continuity, Shwu-Yeng T. Lin and Y.-F. Lin obtained the following characterization of Baire spaces (Theorem 3, [45]).

THEOREM (L-L)1. Let Y be an arbitrary second countable, regular, Hausdorff space that contains infinitely many points. Then a topological space X is a Baire space if and only if every mapping $f:X\to Y$ is almost continuous at each point of a dense subset D(f) of X.

The development of Theorem (L-L)1 can be traced to its origin where the notion of almost continuity was conceived. Continuity conditions were investigated while almost continuity laid dormant. Recently Husain awakened almost continuity from its slumber and the Lins' have shown that almost continuity accomplishes what continuity failed to achieve, a mapping characterization of Baire spaces.

Background and Blumberg Spaces

An especially interesting application of Theorem (L-L)1 is found by replacing the space Y by the Euclidean space of real numbers R (Theorem 4, [45]).

THEOREM (L-L)2. A topological space X is a Baire space if and only if every real-valued function on X has a dense set of points of almost continuity in X.

This theorem has its origin in the paper "New properties of all real functions" by Henry Blumberg [5] in which it was proved that every real-valued function on Euclidean n-space R^n has a comeager set of points of almost continuity. In fact, Blumberg invented the notion of pointwise almost continuity for real-valued functions. For a real-valued function $f:X\to R$ on a topological space X, Blumberg defined f to be densely approached at the point x or equivalently, the graph of f, G(f), to be densely approached at (x,f(x)), if for any positive ε, x has a neighborhood N so that $f^{-1}[B(f(x);\varepsilon)]$ is dense in N, where $B(f(x);\varepsilon)$ is the ε-ball about f(x). Thus f is densely approached at x if and only if $x \in \underline{\text{Int}}\ \underline{\text{Cl}}\ f^{-1}[B(f(x);\varepsilon)]$ for each positive ε. When R is replaced by an arbitrary topological space Y and $B(f(x);\varepsilon)$ is replaced by an

arbitrary open set V containing f(x), then the condition $x \in \underline{\text{Int}}\ \underline{\text{Cl}}\ f^{-1}(V)$ for each open set V containing f(x), is the condition of almost continuity of f at x as defined by T. Husain [32] (Definition 11, Chapter I).

If D(f) is the set of points of almost continuity of a function f, then Blumberg's first theorem was essentially that for any function $f:R^n \to R$, D(f) is comeager in R^n and conversely, if $D \subseteq R^n$ is a comeager set then there is a function $g:R^n \to R$ such that D = D(f). Blumberg's proof of the first part of his theorem does not exploit any special properties of R^n except indirectly in that he defined a relation between sets and points to be closed if each convergent sequence of points related to a fixed set had its limit point related to the same set. If this definition were modified to have limit points of each convergent net of points related to a fixed set to be related to that set, then the proof remains unchanged and yields a seemingly more general result.

THEOREM (B)1. If $f:X \to R$ is any real-valued function on an arbitrary topological space X, then D(f) is comeager in X.

Since a space is a Baire space if and only if each first category (meager) set has empty interior, each comeager subset of a Baire space X is dense in X. Hence an immediate corollary to Theorem (B)1 follows.

COROLLARY TO THEOREM (B)1. If X is a Baire space and $f:X\to R$ is any real-valued function on X, then D(f) is dense in X.

This is half of Theorem (L-L)2 which itself was a special case of Theorem (L-L)1.

Blumberg continued to seek the common properties of all real-valued functions on R^n and proved a second theorem which at the time, seemed more prominent than his first theorem. He showed that if $f:R^n\to R$ is any function, then there is a (non-unique) dense subset $D \subseteq R^n$, depending on f, such that f is continuous relative to D. In the same paper, Blumberg generalized his second theorem as follows (Theorem IV, [5]).

THEOREM (B)2. If X is a complete separable metric space without isolated points and if $f:X\to R$ is any real-valued function on X, then there is a dense subset D of X so that $f|D:D\to R$ is continuous.

A generalization of this theorem obtained by dismissing the restrictions on X has earned the name "Blumberg's Theorem" ([7], [42], [69]).

BLUMBERG'S THEOREM. If $f:X\to R$ is a real-valued function defined on a topological space X, then there is a dense subset D of X such that $f|D:D\to R$ is continuous.

A space X is called Blumberg [42] if Blumberg's Theorem is true for the space X. Thus Blumberg established that R^n and more generally any complete separable metric space without isolated points is Blumberg. A present avenue of research is to characterize or classify those spaces which are Blumberg. A fundamental result in this direction was obtained by J. C. Bradford and C. Goffman [7]. They proved that the metric Blumberg spaces are precisely the metric spaces of homogeneous second category. By a metric space X of homogeneous second category is meant a metric space each non-empty open sphere of which contains a second category subset of X. It is known that such a space is a metric Baire space. In "A note on Baire spaces" [46], Y.-F. Lin finds the following intrinsic characterization of a Baire space (Corollary, [46]).

THEOREM. A topological space Y is a Baire space if and only if every subset of the first category has no non-void interior.

An easy consequence of this theorem is the following characterization (p.75, [6]; p. 179, [70]).

THEOREM 1. A topological space X is a Baire space if and only if every non-empty open subset of X is of second category.

Easily then, every non-empty Baire space is of second category and every open subspace of a Baire space is a Baire space (Proposition 7.2 iii, [11]). The condition that every non-empty open set is of second category is clearly the pure topological form of the homogeneous second category condition for metric spaces. Thus the result of Bradford and Goffman can be restated as follows.

THEOREM (B-G)1. A metric space X is a Baire space if and only if it is Blumberg.

This theorem improves Theorem (B)2 since it shows that every complete metric space is Blumberg.

Bradford and Goffman were concerned only with metric spaces and yet as noted by H. E. White, Jr. [68], their proof that every metric Blumberg space is a Baire space did not depend on the properties of the metric. Thus, the following result holds.

THEOREM (B-G)2. Every topological Blumberg space is a Baire space.

Evidently, the class of all Blumberg spaces is a subclass of the class of Baire spaces. Thus a result of H. Bennett [4] that every semimetrizable Baire space is Blumberg can be restated as follows.

THEOREM. (Bennett) Every semimetrizable space X is Baire if and only if it is Blumberg.

Bennett's Theorem generalizes Theorem (B-G)1 since a semimetric on a space X is a distance function which does not necessarily satisfy the triangle inequality. In particular a semimetric on X is a real-valued function d on X×X such that $d(x,y) = d(y,x) \geq 0$ for all $x,y \in X$ and such that for each subset M of X, if $d(x,M) = \underline{\text{Inf}}\{ d(x,y):y\in M\}$ then $\underline{\text{Cl}}\ M = \{x:d(x,M)=0\}$. One might wonder if Bennett's Theorem can be strengthened or if the converse to Theorem (B-G)2 holds. Is every Baire space necessarily Blumberg? Counterexamples answering this question in the negative were discovered independently by Ronnie F. Levy ([41], [42]) and by H. E. White, Jr. ([68], [69]). White found completely regular, first countable, Hausdorff, Baire spaces which fail to be Blumberg assuming the Continuum Hypothesis. Levy gave an example of a totally ordered set with the interval topology which is a Baire space failing to be Blumberg without requiring the Continuum Hypothesis [42]. Levy's totally ordered non-Blumberg Baire space will now be constructed.

EXAMPLE 2. (Levy) If A and B are subsets of an ordered set (X,<), then define A<B if and only if a<b for all $(a,b) \in A\times B$. Define a<B if {a}<B and A<b if A<{b}. Following Leonard Gillman and Meyer Jerison [23], an ordered set X is called an η_1-set if whenever A and B are countable subsets of X, A<B implies that $A < x < B$ for

some $x \in X$. It is known that card(X) $\geq$ aleph-one for each η_1-set X but Gillman and Jerison proved without invoking the Continuum Hypothesis that in fact the card(X) $\geq$ c where c is the cardinality of R. They also constructed a minimal η_1-set Q (pp. 185-189, [23]) with card(Q) = c. Let X = Q be supplied with the interval topology. That is, $\{(a,b):a,b \in X\}$ is a basis for the topology on X where $(a,b) = \{x:x \in X \text{ and } a < x < b\}$. If $\{G_n:n = 1,2,\ldots\}$ is a collection of open dense subsets of X and if (a,b) is any non-empty basic open set then to show that $G = \bigcap\{G_n\}$ is dense in X it suffices to show $(a,b) \bigcap G \neq \emptyset$. To this end there is a non-empty open interval $(a_1,b_1) \subseteq (a,b) \bigcap G_1$. Inductively, there is a sequence of non-empty open intervals $\{(a_n,b_n)\}$ such that $(a_{n+1},b_{n+1}) \subseteq (a_n,b_n) \bigcap G_{n+1}$. Then $\{a_n\} < \{b_n\}$ so that there is an $x \in X$ with $\{a_n\} < x < \{b_n\}$. So $x \in (a,b) \bigcap G$ and G is dense in X. Thus X is a Baire space. A completely regular space Y is a P-space if all prime ideals are maximal in the commutative ring C(Y) of all continuous real-valued functions on Y (p. 62, [23]). If $y \in Y$ then $M_y = \{f:f \in C(Y) \text{ and } f(y) = 0\}$ is a maximal ideal containing the ideal $O_y = \{f:f \in C(Y) \text{ and } y \in \underline{\text{Int}}\ f^{-1}(0)\}$. If $O_y \neq M_y$ then there exists a non-maximal prime ideal P with $O_y \subseteq P$. If $M_y = O_y$ then y is called a P-point of Y. So Y is a P-space if and only if each point $y \in Y$ is a P-point of Y. The term "P-space" is a shortening of

"pseudo-discrete space" [22] since each point of a discrete space is a P-point. If a function $f:Y\to R$ is continuous at a P-point $y \in Y$, then f is constant on some neighborhood of y (Problem 4L3, [23]). Now X is totally ordered with the interval topology so that X is completely normal and T (pp. 66-67, [64]) and hence completely regular. Further, X is a P-space without isolated points. (Problem 13P, [64]). Since a dense subspace of a space without isolated points can have no isolated points and since P-points of a space are P-points in any subspace containing them, then every dense subspace of X is a P-space without isolated points. Since card(X) = card(R), let $f:X\to R$ be a bijection. If D is a dense subset of X and if $f|D$ is continuous at a point $x \in D$, then $f|D$ is constant on some non-singleton subset of D. This contradicts the fact that f is a bijection. Evidently $f|D$ is not continuous at any point of D, for any dense subset D of X. Thus X is not Blumberg.

The questions recently posed have been: "Which Baire spaces are Blumberg?" and "Which Baire spaces are not Blumberg?". In the study of Baire spaces certain classes of Baire spaces have been of special interest. These classes have become the first candidates for being in or for containing the class of Blumberg spaces.

It is well known that all complete (pseudo-)metric spaces and all locally compact (regular) Hausdorff spaces are Baire spaces (Theorem 6.34, [37]). Either of these

results is referred to as a Baire category theorem. Recently, unified Baire category theorems have been found by showing that a class of topological spaces which includes the complete metric spaces and the locally compact Hausdorff spaces is a subclass of the class of Baire spaces. Much attention has been given to such subclasses of the class of Baire spaces by J. M. Aarts and D. J. Lutzer in their paper "Completeness properties designed for recognizing Baire spaces" [2]. One of the prominent methods for obtaining a broad class of spaces is to include in the class all spaces for which a certain kind of subtopology called a cotopology exists and such that the cotopology has certain properties. In particular, the class of cocompact spaces is the class of topological spaces which have at least one compact cotopology. The notions of cotopology and in particular cocompactness, though introduced by Aarts, de Groot, and McDowell [1], were inspired by de Groot's earlier work [17] in which a unified Baire category theorem was found.

DEFINITION 1. For a topological space $T = (X,O)$, the space $*T = (X,*O)$ is a cospace of T and $*O$ is a cotopology of O if and only if 1) $*O \subseteq O$ and 2) for every $x \in X$ and for every $V \in O$ with $x \in V$, there is a (closed) neighborhood U of x in T such that $U \subseteq \underline{Cl}_T V$ and U is closed in *T.

Note that arbitrarily small (closed) neighborhoods of a point in a space T = (X,O) are complements of open sets in a cospace *T if a cotopology *O of O exists.

Another important subclass of Baire spaces for which a unified Baire category theorem holds is the class of α-favorable spaces introduced by G. Choquet (p. 116, [11]). It has been shown that the class of α-favorable spaces includes the complete metric spaces and the locally compact Hausdorff spaces. Further, open subspaces of (strongly) α-favorable spaces are (strongly) α-favorable and arbitrary products of (strongly) α-favorable spaces are (strongly) α-favorable. Finally, each strongly α-favorable space is α-favorable and each α-favorable space is a Baire space. Thus not only was a unified Baire category theorem obtained but more, arbitrary products of complete metric spaces and/or locally compact Hausdorff spaces are Baire. For example R^X is Baire for any set X if R is the usual space of real numbers. Thus real Hilbert space l_2 is humorously seen to be Baire since its field of scalars is locally compact and Hausdorff and because l_2 is homeomorphic to R^N where N is a countably infinite set [3]. More easily, l_2 is a Baire space being a complete metric space. On the other hand if X is an uncountable set, the Baire space R^X is neither metrizable (no σ-locally finite base exists) (p. 127, [37]) nor locally compact (p. 239, [19]). The non-metrizability of R^X could also be realized since A. H. Stone showed that R^X is not normal [65].

An α-favorable space is defined below.

DEFINITION 2. Let (X,T) be a topological space and let $T^* = T - \{\phi\}$. The space X is α-favorable if and only if there is a function (called the winning tactic or winning strategy) $f:T^* \to T^*$; such that $f(U) \subseteq U$ and such that for any sequence $V_1, V_3, V_5, \ldots$ (indexed by the odd natural numbers) of members of T^* defined inductively so that $V_1 \supseteq V_2 = f(V_1) \supseteq V_3 \supseteq V_4 = f(V_3) \supseteq V_5 \supseteq \ldots$ it follows that $\bigcap \{V_n\} \neq \phi$. Similarly, (X,T) is called strongly α-favorable if and only if there exists a function $f:P \to T^*$ where $P = \{(U,x): U \in T^* \text{ and } x \in U\}$ such that $x \in f((U,x)) \subseteq U$ for all $(U,x) \in P$ and such that for any sequence of pairs $(V_1,x_1), (V_3,x_3), (V_5,x_5), \ldots$ of P defined inductively such that $V_1 \supseteq V_2 = f((V_1,x_1)) \supseteq V_3 \supseteq V_4 = f((V_3,x_3)) \supseteq V_5 \supseteq \ldots$ it follows that $\bigcap \{V_n\} \neq \phi$.

The α-favorable (and strongly α-favorable) spaces are those spaces where the two player game (strong version of the game) played by α and β can always be won by α. The game is played by the players alternately selecting non-empty open subsets of the opponents immediately previous choice. The player β opens the game and wins if the game terminates. In the strong version of the game, each time β plays, an open set and an interior point of the set are specified. Then α must select an open subset containing the specified point but contained in β's set. Clearly every strongly α-favorable space is α-favorable. That

every α-favorable space is a Baire space will be proven next (Theorem 7.13, [11]).

THEOREM (C). If X is an α-favorable space then X is a Baire space.

Proof. Suppose that (X,T) is not a Baire space but that $f:T^*\to T^*$ is a winning tactic. Let $\{F_n\}$ be a sequence of closed nowhere dense subsets of X such that for some $V\in T^*$, $V \subseteq \bigcup\{F_n\}$. Let $V_1 = V$ and in general let $V_{2n+1} = V_{2n} \cap (X - F_n)$. Then $\bigcap\{V_n\} = V \cap (X - \bigcup\{F_n\}) = \emptyset$ so that f is not a winning tactic. The contradiction shows that (X,T) is a Baire space if it is α-favorable.

A space is defined to be β-favorable if player β has a winning tactic, and M. R. Krom proved [38] that a topological space X is β-favorable if and only if X is not a Baire space. This leads to a positive game theoretic characterization of Baire spaces (Theorem 2, [38]):

THEOREM (K). A topological space (X,T) is a Baire space if and only if for some basis B of T such that $\emptyset \notin B$, and for all $U \in B$ and for all decreasing functions $f:B\to B$ there exists a decreasing function $g:B\to B$ so that for all sequences $\{V_n : n \geq 0\}$ with $V_0 = U$ and $V_{n+1} = f(V_n)$ for n odd and $V_{n+1} = g(V_n)$ for n even, then $\bigcap\{V_n\} \neq \emptyset$.

Notice that the function $g:B\to B$ depends upon the starting position U for player β and a fixed strategy f for β.

This is in contrast to the α-favorable space where α has a winning strategy g regardless of the strategy or starting position of β. Thus the non-α-favorable Baire spaces are those for which neither player has a winning strategy [38].

A unified Baire category theorem also holds for the class of topologically complete spaces (in the sense of Čech)[10]. A space X is said to be topologically complete if and only if X is homeomorphic to a dense G_δ-set in a compact Hausdorff space Y. Clearly, every locally compact Hausdorff space X is topologically complete since X is an open dense subset of its Hausdorff one-point (Alexandroff) compactification $Y = X \cup \{\infty\}$ (p. 150, [37]). A metric space X is metrically complete if and only if X is homeomorphic to a complete metric space (Y,d). It is known that a metric space X is topologically complete if and only if X is metrically complete [10]. Furthermore, E. Čech showed that each topologically complete space is a Baire space [10]. Thus a unified Baire category theorem also holds for the class of topologically complete spaces. Other classes of Baire spaces have been investigated ([2], [56]) for which unified Baire category theorems hold. One such class investigated by J. C. Oxtoby contains the class of topologically complete spaces and has the additional advantage of being closed under arbitrary products. Thus a result obtained by Oxtoby [56] is that an arbitrary product of topologically complete spaces must be a Baire space.

In the case of metrizable spaces, the cocompact spaces are precisely the topologically complete spaces [1] which in turn are precisely the strongly α-favorable spaces (Theorem 8.7, [11]).

R. Sikorski raised the question of whether a product of metric Baire spaces must be a Baire space [60]. Using the Continuum Hypothesis a partial result in this direction was obtained by Oxtoby [56] who produced a completely regular Baire space X for which $X \times X$ is of first category and hence $X \times X$ is not a Baire space. Clearly, such a Baire space X cannot be topologically complete or α-favorable. Using the example of Oxtoby, Krom [38] answered Sikorski's question in the negative by constructing a metric Baire space Z such that $Z \times Z$ is not a Baire space. This metric Baire space is Blumberg by Theorem (B-G)1 and is clearly not topologically complete (metrically complete) and hence also not cocompact. Neither can it be α-favorable. Also it is evident that a product of Blumberg spaces need not be Blumberg. For $Z \times Z$ is a non-Baire space and so is not Blumberg by Theorem (B-G)2.

Since the space of Krom is Blumberg but not cocompact, topologically complete, or α-favorable, the class of Blumberg spaces is not contained in any of these special classes of Baire spaces. H. E. White reports [68] that in a letter Dr. B. J. Pettis suggested that perhaps the class of Blumberg spaces would include the cocompact spaces, or the α-favorable spaces, or the paracompact Baire spaces.

White showed that none of these conjectures is true assuming the Continuum Hypothesis. In fact, though it was known that a P-space need not be a Baire space, Levy's example shows that a strongly α-favorable P-space may not be Blumberg. Yet in contrast to Levy's example, assuming the Continuum Hypothesis, White showed [69] that every cocompact P-space is Blumberg.

The classification of Blumberg spaces remains an open problem. Certainly Levy's example in conjunction with Theorem (B-G)2 shows that the class of Blumberg spaces is a proper subclass of the class of Baire spaces even though these classes coincide in the class of metric spaces. Thus Blumberg's continuity condition fails to provide a mapping characterization for the class of Baire spaces.

Meanwhile, Husain [32] defined almost continuity of a function at a point with arbitrary range space generalizing Blumberg's notion of a real-valued function being densely approached at a point. Then Husain proved the following theorem (Theorem 4, [32]).

THEOREM (H). If X is a metric Baire space and $f:X\to R$ is a real-valued function on X, then D(f) is dense in X where D(f) is the set of points of almost continuity of f.

Though this theorem may appear to generalize Blumberg's initial results for real-valued functions defined on R^n, Husain's theorem actually follows from Blumberg's work, and in fact is a consequence of the Corollary to Theorem (B)1.

However, the similarity between Theorem (H) and Theorem (B-G)1 is striking and suggests some possible connection between the condition of having a continuous restriction to a dense subset of X and having D(f) dense in X for a real-valued function on X, at least when X is a metric Baire space. The existance of such a connection is given more credence in light of Theorem (B-G)2 and the Lins' characterization of Baire spaces, Theorem (L-L)2. The following result was known by You-Feng Lin and by Paul E. Long and Earl E. McGehee, Jr. (Theorem 3, [51]).

THEOREM 2. (Lin, Long and McGehee) If $f:X\to Y$ is a function between topological spaces such that for some dense subset D of X, $f|D:D\to Y$ is continuous, then $D \subseteq D(f)$ so that D(f) is dense in X.

Proof. Let D be a dense subset of X and let $f:X\to Y$ be a function so that $f|D:D\to Y$ is continuous. If $x\in D$ and V is an open set containing f(x), $(f|D)^{-1}(V) = f^{-1}(V) \cap D$ is an open set relative to D containing x. Thus for some open set U in X, $f^{-1}(V) \cap D = U \cap D$. So, $\underline{Cl}\ U = \underline{Cl}(U \cap D)$ since D is dense in X (p. 5, [23]), and $x \in U \subseteq \underline{Cl}\ U \subseteq \underline{Cl}\ f^{-1}(V)$. Therefore, $x \in \underline{Int}\ \underline{Cl}\ f^{-1}(V)$ and $D \subseteq D(f)$.

An immediate corollary to Theorem 2 follows.

COROLLARY 1 TO THEOREM 2. If X is a topological Blumberg space then for each real-valued function $f:X\to R$, D(f) is dense in X.

Because of Theorem (L-L)2 and Levy's example, the converse of this corollary is not true. Now Corollary 1 to Theorem 2 implies Theorem (B-G)2 in light of Theorem (L-L)2.

COROLLARY 2 TO THEOREM 2. Every Blumberg space is a Baire space.

Seeking to remove some of the inessential hypotheses of Theorem (H) and place the result in a more natural context, Shwu-Yeng T. Lin found a generalization of the Corollary to Theorem (B)1 and hence a generalization of Theorem (H). Her direct proof actually yields the following generalization of Theorem (B)1.

THEOREM (L). If $f:X\to Y$ is a function on a topological space X into a space Y satisfying the second axiom of countability, then D(f) is a comeager subset of X.

Proof. Let $\{V_n\}$ be a countable basis for the topology on Y. Define $E_n = (\underline{Cl}\ f^{-1}(V_n)) - \underline{Int}\ \underline{Cl}\ f^{-1}(V_n)$ for each n. Then each E_n is nowhere dense and $E = \bigcup\{E_n\}$ is a first category set in X. If $x \in X - D(f)$ then for some n, $f(x) \in V_n$ and $x \notin \underline{Int}\ \underline{Cl}\ f^{-1}(V_n)$. Hence $x \in E_n \subseteq E$. So $X - D(f)$ is a first category set being contained in E and D(f) is a comeager subset of X.

COROLLARY TO THEOREM (L). (Theorem 1, [44]) If $f:X\to Y$ is a function from a Baire space X into a space Y satisfying the second axiom of countability, then D(f) is a dense subset of X.

Theorem (L-L)1 is the result of this corollary and a converse to this corollary which the Lins discovered. Their converse of the Corollary to Theorem (L) (Theorem 2, [45]) follows.

THEOREM (L-L)3. Let Y be an arbitrary regular Hausdorff space that contains infinitely many points. If X is a topological space such that for each function $f:X\to Y$, D(f) is dense in X, then X is a Baire space.

The proof of Theorem (L-L)3 depended on a Lemma which will be stated following some general properties for a topological space X and a definition.

PROPERTY C1. There is a sequence $\{V_n:n = 1,2,3,\ldots\}$ of pairwise disjoint nonempty open subsets of X.

PROPERTY C2. There is a sequence $\{V_n:n = 1,2,3,\ldots\}$ of nonempty open subsets of X whose closures are pairwise disjoint.

DEFINITION 3. A topological space X is a C1 (C2) space if and only if Property C1 (Property C2) is satisfied for the space X.

An immediate observation is that every C2 space is a C1 space.

The following Lemma (p. 3, [45]) is the basis for half of the proof of Theorem (L-L)3.

LEMMA. Every regular Hausdorff space that contains infinitely many points is a C1 space.

The Lins' direct proof of this Lemma yields the following apparently stronger result.

THEOREM (L-L)4. Every regular Hausdorff space that contains infinitely many points is a C2 space.

This result is well-known and follows from the more general result now to be given (2.4, Chapter VII, [19]).

THEOREM (D)2. If Y is a regular Hausdorff space and $A \subseteq Y$ is any infinite subset, then there exists a sequence $\{U_n : n \geq 0\}$ of open sets whose closures are pairwise disjoint and such that $A \cap U_n \neq \emptyset$ for each $n \geq 1$.

If Y is a regular Hausdorff space that contains infinitely many points, then by taking A = Y, it is evident that Y is a C2 space.

Phillip Alan Hartman [26] used the conclusion of Theorem (D)2 to define a new separation axiom T_n for Hausdorff topological spaces which is strictly intermediate to the Hausdorff (T_2) and the regular Hausdorff (T_3) axioms.

AXIOM T_n. (Hartman) Given any infinite subset A of a Hausdorff space X, there is a sequence $\{U_n : n \geq 0\}$ of open subsets of X whose closures are pairwise disjoint and such that $U_n \cap A \neq \emptyset$ for each $n \geq 1$.

That the T_n separation axiom is intermediate to the Hausdorff and regular Hausdorff axioms is clear. Hartman showed that the Hausdorff axiom does not imply Axiom T_n and that Axiom T_n does not imply the regular Hausdorff axiom. Examples 3 and 5 below will verify these facts. Further, Hartman has shown that the T_n axiom does not imply either the Urysohn ($T_{2\frac{1}{2}}$) axiom or the Hausdorff semiregular axiom. However, in some respects, Axiom T_n behaves similarly to the Hausdorff, Urysohn, and regular Hausdorff axioms. For example, the T_n axiom is hereditary.

THEOREM 3. If A is a subspace of a T_n-space X, then A is a T_n-space.

Proof. If A is a subspace of a T_n-space X then A is Hausdorff. If B is an infinite subset of A, there is a sequence $\{U_n : n \geq 0\}$ of open sets in X whose closures are pairwise disjoint and such that $U_n \cap B \neq \emptyset$ for $n \geq 1$. Define $V_n = U_n \cap A$ for $n \geq 0$. Then $\{V_n : n \geq 0\}$ is a sequence of open sets in A such that $V_n \cap B \neq \emptyset$ for $n \geq 1$. Further, for each n, the subspace closure of V_n is contained in $\underline{Cl}\ U_n$, so that the subspace closures of the members of $\{V_n : n \geq 0\}$ are pairwise disjoint. Hence A is a T_n-space.

Like the Hausdorff and Urysohn axioms (but unlike the regular Hausdorff axiom), Axiom T_n is preserved under topology extensions.

THEOREM 4. If T_1 and T_2 are topologies on a set X with $T_1 \subseteq T_2$ and if $X_1 = (X,T_1)$ is a T_n-space, then $X_2 = (X,T_2)$ is a T_n-space.

Proof. Certainly X_2 is a Hausdorff space. If $A \subseteq X_2$ is an infinite set then there exists a sequence $\{U_n : n \geq 0\} \subseteq T_1$ whose closures are pairwise disjoint and such that $U_n \cap A \neq \emptyset$ for $n \geq 1$. Since $T_1 \subseteq T_2$, $\{U_n : n \geq 0\} \subseteq T_2$ and for each $n \geq 0$, the closure in X_2 of U_n is contained in the closure in X_1 of U_n so that the closures in X_2 of the members of $\{U_n : n \geq 0\}$ are pairwise disjoint. Thus X_2 is a T_n-space.

Note that the above method of proof also shows that the Urysohn axiom is preserved under topology extensions.

Theorem (L-L)4 and Theorem (D)2 suggest a connection between Axiom T_n and Property C2. Evidently every finite Hausdorff (discrete) space is a T_n-space whereas all C1 and C2 spaces must have infinitely many points. The following relationship does hold.

THEOREM 5. Every infinite T_n-space is a C2 space.

Proof. Let X be an infinite T_n-space. Choose a sequence $\{U_n : n \geq 0\}$ of open subsets of X whose closures

are pairwise disjoint and such that $U_n = U_n \cap X \neq \phi$ for each $n \geq 1$. Then $\{U_n : n \geq 1\}$ is such a sequence showing that X is a C2 space.

Theorem 5 is actually stronger than Theorem (L-L)4 as will be shown by the following example of an infinite T_n-space X which is not a regular Hausdorff space. Hartman [26] gave a similar example showing the existence of a non-regular T_n-space.

EXAMPLE 3. Let T_1 be the usual topology for the set R of real numbers. Let Z be the set of rational numbers and let $X = (R, T_2)$ be a space with topology T_2 being the smallest extension of T_1 such that $Q \in T_2$. Since (R, T_1) is a regular Hausdorff space by Theorem (D)2 it is a T_n-space. By Theorem 4, X is a T_n-space. But X is not regular since the open set Q contains no closed neighborhoods (Example 2, p. 141, [19]).

The converse of Theorem 3 may fail for two different reasons. Firstly, a C2 space may not be a Hausdorff space and secondly, the T_n axiom is a uniform property whereas Property C2 is a non-uniform property. The following two examples illustrate these two ways in which a C2 space may fail to satisfy Axiom T_n.

EXAMPLE 4. Any non-discrete infinite topological sum of indiscrete spaces must fail to be Hausdorff in some summand and hence must fail to be Hausdorff since the

Hausdorff axiom is hereditary. Yet such a space must be a C2 space since the summands are pairwise disjoint clopen (closed and open) subsets of the sum.

The next example shows that a Hausdorff C2 space need not be a T_n-space. Hartman gave a similar example to show that Axiom T_n is stronger than the Hausdorff axiom.

EXAMPLE 5. Let $X = (R, T_2)$ be the non-regular T_n-space of Example 3. Let N be the set of natural numbers and for each $k \in N$, define $A_k = \{n + \frac{1}{k\pi} : n \in N\}$. Then $P = \{A_k : k \in N\} \cup \{\{x\} : x \in X - \bigcup\{A_k : k \in N\}\}$ is a partition of X which induces an equivalence relation S on X. Let $Y = X/S$ be the quotient space with quotient topology T induced by the projection $p: X \to Y$. If $k \in N$, let $V_{k,\delta} =$

$$p\left(\bigcup\left\{\left(n + \frac{1}{k\pi} - \delta, n + \frac{1}{k\pi} + \delta\right) : n \in N\right\}\right) \text{ for } 0 < \delta < \frac{1}{2k(k+1)\pi}.$$

Then $V_{k,\delta}$ is an open set in Y containing A_k and if $i \neq k$, $V_{i,\delta} \cap V_{k,\delta} = \emptyset$. If $n \in N$ and $k \in N$ then $U_{n,k} =$

$p\left(\left(n - \frac{1}{(k+1)\pi}, n + \frac{1}{(k+1)\pi}\right) \cap Q\right)$ is an open set containing $p(n)$ and $U_{n,k} \cap V_{k,\delta} = \emptyset$. In general, Y is a Hausdorff space. If $x \in X - \bigcup\{A_k : k \in N\}$ and if $x \notin N$, then for each $\delta > 0$ such that $N \cap (x-\delta, x+\delta) = \emptyset$ and such that $(x-\delta, x+\delta) \subseteq X - \bigcup\{A_k : k \in N\}$, $W_{x,\delta} = p((x-\delta, x+\delta))$ is a basic open set containing x and if further, $x \in Q$, then $U_{x,\delta} =$ $p((x-\delta, x+\delta) \cap Q)$ is also a basic open set containing x. If $x \in N$ then for each positive $\delta < \frac{1}{2}$, $U_{x,\delta} =$ $p((x-\delta, x+\delta) \cap Q)$ is a basic open set containing x.

If $p(x) \neq p(y)$, $p(x)$ and $p(y)$ can be separated by disjoint open subsets of Y by choosing sufficiently small values for δ. By Theorem (D)2, since $(-\infty,0)$ is an infinite subset of the regular Hausdorff space of real numbers with the usual topology, there is a sequence $\{U_n : n \geq 1\}$ of non-empty open subsets of $(-\infty,0)$ whose closures are pairwise disjoint subsets of $(-\infty,0]$. Thus, $\{p(U_n) : n \geq 1\}$ is a sequence of non-empty open subsets of Y whose closures are pairwise disjoint so that Y is a C2 space. But Y fails to satisfy Axiom T_n. For $p(N)$ is an infinite subset of Y and if U and V are open subsets of Y containing $p(n)$ and $p(m)$ respectively with $n \neq m$, then for some $\delta > 0$ and $\delta' > 0$, $U_{n,\delta} \subseteq U$ and $U_{m,\delta'} \subseteq V$ and by choosing $k \in N$ with $\frac{1}{k} < \delta$ and $\frac{1}{k} < \delta'$ then $A_k \in \underline{Cl}\ U \cap \underline{Cl}\ V$.

The space (Y,T) of Example 5 is a connected Hausdorff C2 space without isolated points which fails to satisfy Axiom T_n. Using Theorem 3, a disconnected Hausdorff C2 space which is not a T_n-space can be constructed easily once the existence of a non-T_n Hausdorff space is known.

EXAMPLE 6. Let X be a regular Hausdorff space that contains infinitely many points. Then X is an infinite T_n-space by Theorem (D)2 and by Theorem 3, X is a C2 space. Let Y be any Hausdorff non-T_n-space. Then if $Z = X + Y$ is the topological sum of X and Y, then Z is a Hausdorff space and fails to be a T_n-space since the subspace Y is not a T_n-space. Since X is a clopen C2 subspace

of Z, Z is a C2 space.

The second half of the Lins' proof of Theorem (L-L)3 establishes the following fundamental result.

THEOREM (L-L)5. Let Y be a C1 space. If X is a topological space such that for each function $f:X\to Y$, D(f) is dense in X, then X is a Baire space.

Proof. Suppose that the topological space X is not a Baire space. Let $\{F_n:n \geq 1\}$ be a countable family of nowhere dense subsets of X such that $\underline{\text{Int}} \bigcup\{F_n:n \geq 1\} = U \neq \emptyset$. It may be assumed that $F_n \cap F_m = \emptyset$ for $n \neq m$. For otherwise let $E_1 = F_1$ and $E_n = F_n - \bigcup\{F_k:1 \leq k < n\}$ for each $n > 1$. Then $\bigcup\{E_n:n \geq 1\} = \bigcup\{F_n:n \geq 1\}$. Let $\{V_n:n \geq 0\}$ be a sequence of pairwise disjoint non-empty open subsets of Y. By the Axiom of Choice select $y_n \in V_n$ for each $n \geq 0$ and define a function $f:X\to Y$ by $f(x) = y_0$ if $x \in X-U$ and $f(x) = y_n$ if $x \in U \cap F_n$ for $n \geq 1$. To show that D(f) is not dense in X it is sufficient to show that $U \cap D(f) = \emptyset$. To this end let $x \in U$. Then for some $n \geq 1$, $x \in U \cap F_n$ and $f(x) \in V_n$. But $\underline{\text{Int}}\ \underline{\text{Cl}}\ f^{-1}(V_n) = \underline{\text{Int}}\ \underline{\text{Cl}}(U \cap F_n) = \emptyset$ since any subset of a nowhere dense set is nowhere dense. Thus $x \notin \underline{\text{Int}}\ \underline{\text{Cl}}\ f^{-1}(V_n)$ and $x \notin D(f)$. Therefore $U \cap D(f) = \emptyset$.

Improvements of Theorem (L-L)3 and Theorem (L-L)1 follow as corollaries.

THEOREM 6. Let Y be an infinite T_n-space. If X is a topological space for which every function $f:X\to Y$ has D(f) dense in X, then X is a Baire space.

Proof. Every infinite T_n-space is a C1 space by Theorem 5 and the fact that all C2 spaces are C1 spaces.

THEOREM 7. Let Y be an arbitrary second countable T_n-space containing infinitely many points. Then a topological space X is a Baire space if and only if for every function $f:X\to Y$, D(f) is a dense subset of X.

Proof. The Corollary to Theorem (L) proves the necessity. Theorem 6 proves the sufficiency.

Since the class of T_n-spaces properly contains the class of regular Hausdorff spaces, Theorem 6 is actually stronger than Theorem (L-L)3 and hence Theorem 7 is a strengthened version of Theorem (L-L)1.

C1 and C2 Spaces

The class of C1 spaces contains the class of C2 spaces which in turn contains the class of infinite T_n-spaces. The space Y of Theorem (L-L)5 is only required to have Property C1. Theorem 6 and also Theorem (L-L)1 might be strengthened further if a class of C1 spaces could be identified which would properly contain the class of infinite T_n-spaces. For this reason, Property C1 and Property C2 will be investigated more fully and characterizations of these properties for a topological space X will be sought.

Some immediate observations can be made. No space with a finite topology can have Property C1 or Property C2. In particular no indiscrete space has Property C1 or Property C2. Yet every infinite discrete space has Property C2 and hence both properties. More generally, if a space X has infinitely many isolated points, then X is a C1 space. Example 8 below will show that in the absence of the T_1 axiom such a space X need not be a C2 space. But every T_1-space with infinitely many isolated points is a C2 space for every isolated point x of a T_1-space must be a clopen (closed and open) point. Such a T_1-space is given in the next example.

EXAMPLE 7. Let $N = \{1,2,3,\ldots\}$ be the set of natural numbers and let $X = \{0\} \cup \{\frac{1}{n}: n \in N\}$ have the subspace topology induced by the usual topology on the set R of all real numbers. If $V_n = \{\frac{1}{n}\}$ for each $n \in N$, then $\{V_n: n \in N\}$ is a sequence of non-empty pairwise disjoint clopen subsets of X. Thus X is a C2 space.

EXAMPLE 8. Let $Y = \{0,1,2,\ldots\}$ be the set of whole numbers with topology T defined as follows. A proper subset V of Y is open in (Y,T) if and only if $0 \notin V$. Thus Y is the only neighborhood of the point 0. Further, 0 is the only closed point of Y so that Y is not a T_1-space. For any subset E of Y, either $E = \emptyset$ or $0 \in \underline{Cl}\ E$. Thus Y is not a C2 space. Yet the set of natural numbers N is an infinite collection of isolated points of Y so that Y is a C1 space. In particular, $\{\{n\}: n \in N\}$ is a sequence of non-empty pairwise disjoint open subsets of (Y,T).

It is known that every Hausdorff space X contains a copy of the integers (Introduction, [23]) in the sense that X has a countably infinite discrete subspace Z. As in the examples just given, every space having infinitely many isolated points must contain a copy of the integers. The following example will show that having a countably infinite discrete subspace is not a sufficient condition for a space to be a C1 space.

EXAMPLE 9. Note that the topology T for the space Y of Example 8 is closed under arbitrary intersections. Thus

the family of closed sets in (Y,T) is a topology S for Y. Unlike (Y,T), (Y,S) is neither Hausdorff nor compact and 0 is the only isolated point in (Y,S). The smallest open set in (Y,S) containing the arbitrary natural number n is {0,n}. Thus (Y,S) is not a C1 space. For just as (Y,T) could have no disjoint pair of nonempty closed subsets, (Y,S) can have no disjoint pair of nonempty open sets. Yet the subspace N of natural numbers in (Y,S) is a countably infinite discrete subspace.

The space (Y,S) of Example 9 could be called the one-point closed extension of a countably infinite discrete space (p. 46, [64]) for the proper closed subsets of (Y,S) are precisely the closed sets in the countably infinite discrete subspace Y - {0}. Similarly, the space (Y,T) of Example 8 is the one-point open extension of a countably infinite discrete space (p. 48, [64]) for the proper open subsets of (Y,T) are precisely the open sets in the countably infinite discrete subspace Y - {0}. An alternate characterization of the space (Y,T) of Example 8 will be explored in order to facilitate a mapping characterization of C1 spaces as those topological spaces on which a continuous surjection exists onto the space (Y,T).

A space Z is discrete if and only if Z has no non-isolated points. Frank Siwiec [62] has classified uniquely (up to homeomorphism) six countably infinite T_1-spaces having exactly one non-isolated point according to various inherent topological properties of each space.

For example, the space X of Example 7 is the unique example of a countable, compact T_1-space with exactly one non-isolated point. Notice that a countable T_1-space with exactly one non-isolated point is necessarily infinite. If the bijection $f:X\to(Y,T)$ is defined by $f(0) = 0$ and $f(\frac{1}{n}) = n$ for each $n > 0$ then f is continuous. If $f^{-1}(T) = \{f^{-1}(V):V \in T\}$, then $f^{-1}(T)$ is a compact subtopology for X which fails to make X a T_1-space. The space (Y,T) is the unique minimal countably infinite space with exactly one non-isolated point. This space is minimal in the sense that the topology T has no proper subtopology for which Y has exactly one non-isolated point. Thus if (X,T_1) is any countably infinite space with exactly one non-isolated point then T_1 has a subtopology T_0 for which (X,T_0) is homeomorphic to (Y,T). In particular there is a continuous bijection $f:(X,T_1)\to(Y,T)$. Thus (Y,T) is the continuous image of every countably infinite space having exactly one non-isolated point. Every countably infinite space X having exactly one non-isolated point, though not necessarily a C2 space as Example 8 shows, must be a C1 space. For more generally, if X is any topological space for which there exists a continuous surjection $f:X\to(Y,T)$, then $\{f^{-1}(n):n \in N\}$ is a sequence of non-empty pairwise disjoint open subsets of X. Conversely, it is not difficult to see that (Y,T) is the continuous image of every C1 space X. This mapping characterization of C1 spaces will follow two characterizations of C1 spaces in terms of quotient spaces.

THEOREM 8. A topological space X is a C1 space if and only if there exists an open subspace Y of X and an equivalence relation R on Y such that Y/R is a countably infinite discrete space.

Proof. If Y is an open subspace of X and R is an equivalence relation on Y so that Y/R is a countably infinite discrete space, then the quotient map $p:Y\to Y/R$ is a continuous surjection and Y is the countably infinite union of the open point inverses of p. Since each open subset of Y is open in X, $\{p^{-1}([y]):[y] \in Y/R\}$ is a sequence of non-empty pairwise disjoint open subsets of X, so that X is a C1 space.

Conversely, if X is a C1 space and $\{V_n:n \geq 1\}$ is a sequence of non-empty pairwise disjoint open subsets of X, then $Y = \bigcup\{V_n:n \geq 1\}$ is an open subspace of X and $\{V_n:n \geq 1\}$ is a partition of Y. Let R be the equivalence relation on Y induced by this partition. If $p:Y\to Y/R$ is the quotient map then $p^{-1}(V_n) = V_n$ is open in Y so that each point of Y/R is isolated and Y/R is a countably infinite discrete space.

THEOREM 9. A topological space X is a C1 space if and only if there is a proper open subspace Y of X and an equivalence relation R on X so that X - Y is an equivalence class and X/R - {X-Y} is a countably infinite discrete subspace of X/R.

Proof. If Y is a proper open subset of X and if R is an equivalence relation on X so that X-Y is an equivalence class then X/R - {X-Y} is an open subspace of X/R. For if $p:X\to X/R$ is the quotient map then $p^{-1}(X/R - \{X-Y\}) = Y$ is open in X. Further, if X/R - {X-Y} is a countably infinite discrete open subspace of X/R then each point of X/R - {X-Y} is open in X/R. Thus $U = \{p^{-1}(c):c \in X/R - \{X-Y\}\}$ is a countably infinite family of non-empty pairwise disjoint open subsets of X. By choosing a bijection $f:N\to U$ from the set of natural numbers onto U, the desired sequence is obtained.

Conversely, if X is a C1 space and $\{V_n:n \geq 0\}$ is a sequence of non-empty pairwise disjoint open subsets of X then define $Y = \bigcup\{V_n:n \geq 1\}$. Since $V_0 \subseteq X-Y$, Y is a proper open subspace of X. Furthermore $\{X-Y\} \cup \{V_n:n \geq 1\}$ is a partition of X which induces an equivalence relation R on X. For $n \geq 1$, V_n is open in X so that the point V_n is open in X/R and X/R - {X-Y} is a countably infinite discrete subspace of X/R.

Theorem 9 leads to the following aforementioned mapping characterization for C1 spaces.

THEOREM 10. Let Y be the one-point open extension of a countably infinite discrete space. A topological space X is a C1 space if and only if X admits a continuous surjection $f:X\to Y$.

Proof. Let Y be the space (Y,T) of Example 8. It has already been shown that if $f:X\to Y$ is a continuous surjection then X is a C1 space.

Conversely, let X be a C1 space. By Theorem 9 there is an open subset W of X and an equivalence relation R on X so that X - W is an equivalence class and $X/R - \{X - W\}$ is a countably infinite discrete subspace of X/R. Let $g:X/R\to Y$ be a bijection with $g(X-W) = 0$ and let $p:X\to X/R$ be the quotient map. Then g is continuous since $g^{-1}(V)$ is open for each open set V in Y. Also p is a continuous surjection so that $gp = f:X\to Y$ is a continuous surjection.

The proof of Theorem 10 shows that if X is a C1 space on which a continuous surjection $f:X\to Y$ is defined, then the kernel of f, $K(f) = \{(x_1,x_2):f(x_1) = f(x_2)\}$ is the equivalence relation on X guaranteed by Theorem 9. Furthermore, f can be factored through X/K(f). That is $f = gp$ where $p:X\to X/K(f)$ is the quotient map and $g:X/K(f)\to Y$ is a continuous bijection. Note that g may not be open and thus not a homeomorphism. Let X and Y be the spaces of Example 7 and Example 8 respectively. If $f:X\to Y$ is defined by $f(0) = 0$ and $f(\frac{1}{n}) = n$, then f is a continuous bijection so that K(f) is the relation of equality of numbers on X and $p:X\to X/K(f)$ is a homeomorphism. So $g:X/K(f)\to Y$ is open if and only if $f:X\to Y$ is open. Since X is a T_1-space and Y is not a T_1-space, evidently f is not a homeomorphism and hence not open. In particular $X - \{1\}$ is open in X

and yet $f(X - \{1\}) = Y - \{1\}$ is not open in Y. Hence h is not open. If the points of X were "doubled" by taking $X' = X \times \{0,1\}$ with indiscrete topology on $\{0,1\}$, then $p: X' \to X'/K(f')$ would still be open and continuous though not a homeomorphism if $f: X \to Y$ were extended to $f': X' \to Y$ by embedding X as $X \times \{0\} \subseteq X'$ and extending f to X' by setting $f((x,0)) = f((x,1))$ for each $x \in X$. But then f' is not open as f was not open so that neither is g and g and p may each fail to be a homeomorphism.

The investigation of the C1 or C2 property for topological spaces satisfying certain separation axioms may reveal important relationships or differences between the C1 and C2 properties. But first some general topological considerations of these properties can be made. From the definition of C1 and C2 spaces it is clear that the C1 and C2 properties are topological invariants. Note that Property C2 is invariate under any one-to-one open and closed multifunction f. Recall that a multifunction f is one-to-one if and only if f^{-1} is a function. Property C1 is invariant under any one-to-one open multifunction. Thus these properties are certainly invariant under homeomorphisms. More generally, since the inverse of a continuous surjection is a one-to-one open and closed multifunction, the following theorem holds.

THEOREM 11. Let $f:X\to Y$ be a continuous surjection from a topological space X onto a topological space Y. If Y is a C1 (C2) space then X is a C1 (C2) space.

Yet, since every constant mapping onto a singleton point space is continuous, open, and closed, the C1 and C2 properties are in general not invariant under continuous, open, or closed mappings or under quotient space formations. Also, in general open or closed subspaces of a C1 or C2 space may fail to be either a C1 or C2 space for each topological space X can be homeomorphically embedded as a clopen subspace of a C2 space Y where Y is a countably infinite topological sum of spaces each homeomorphic to X. Thus neither the C1 or C2 property is hereditary, weakly hereditary (p. 4, [64]), or inherited by open subspaces. However, though the C1 and C2 properties are not divisible, they are productive (p. 133, [37]). In fact a stronger statement can be made.

COROLLARY 1 TO THEOREM 11. Let $\{X_a:a \in A\}$ be a family of topological spaces and let $X = P\{X_a:a \in A\}$ be the product space. Then X is a C1 (C2) space if X_a is a C1 (C2) space for some $a \in A$.

Proof. Let $a \in A$ such that X_a is a C1 (C2) space. If $p_a:X\to X_a$ is the projection onto the coordinate space X_a, then p_a is a continuous surjection so that X is a C1 (C2) space by Theorem 11.

However, the converse of Corollary 1 to Theorem 11 does not hold. The following example shows that a product of finite spaces may be a C2 space.

EXAMPLE 10. Let $X_n = \{0,1\}$ be a discrete two-point space for each $n \in N$, the set of natural numbers, and let $X = P\{X_n : n \in N\}$ be the product space. For each $n \in N$ and $k \in N$ define $V_n(k)$ to be a subspace of X_k as follows. If $k < n$, $V_n(k) = \{1\}$. If $k = n$, $V_n(k) = \{0\}$ and if $k > n$ then $V_n(k) = X_k$. Let $V_n = P\{V_n(k) : k \in N\}$ be the product subspace of X. Then V_n is a clopen subset of X for each $n \in N$ and the sequence $\{V_n : n \in N\}$ consists of non-empty clopen pairwise disjoint subsets of X. For if $n < m$, then $V_n(n) \cap V_m(n) = \{0\} \cap \{1\} = \phi$ so that $V_n \cap V_m = \phi$. Thus X is a C2 product space and each factor space X_n fails to be a C1 space, being finite.

The method of Example 10 shows that an infinite product of Hausdorff spaces must be a C1 space if infinitely many of the factor spaces has cardinality greater than one. Likewise an infinite product of Urysohn spaces must be a C2 space if infinitely many of the factor spaces have cardinality greater than one. Also the construction of Example 10 proves that an infinite product of disconnected spaces is a C2 space.

THEOREM 12. Let $\{X_a : a \in A\}$ be an infinite family of disconnected topological spaces and let $X = P\{X_a : a \in A\}$ be the product space. Then X is a C2 space.

COROLLARY TO THEOREM 12. Let $\{X_a : a \in A\}$ be a family of topological spaces and let $X = P\{X_a : a \in A\}$ be the product space. If $B = \{a : a \in A \text{ and } X_a \text{ is disconnected}\}$ is an infinite subset of A then X is a C2 space.

Proof. Let $Y = P\{X_a : a \in B\}$ and let $Z = P\{X_a : a \in A-B\}$. Then if B is an infinite set, Y is a C2 space by Theorem 12. By Corollary 1 to Theorem 11 $Y \times Z$ is a C2 space and hence X is a C2 space since X is homeomorphic to $Y \times Z$ (p. 103, [37]).

Note that the Corollary to Theorem 12 guarantees the existence of non-Hausdorff (or non-T_0) C2 spaces by "slipping in" an extra indiscrete factor space with cardinality greater than one into a product of infinitely many disconnected spaces. Furthermore, every non-empty open subspace in such a product will have Property C2. Thus non-Hausdorff spaces exist which have Property C2 both uniformly and locally in some sense.

EXAMPLE 11. Let $X_0 = \{0,1\}$ have the indiscrete topology and let $X_n = \{0,1\}$ have the discrete topology for n an integer and $n \geq 1$. Let $X = P\{X_n : n \geq 0\}$ be the product space. By the Corollary to Theorem 12, X is a non-T_0, C2 space since the factor space X_0 is a non-T_0-space.

If U is a non-empty open subset of X and $W = P\{W_n : n \geq 0\}$ is a non-empty basic open set contained in U with each W_n open in X_n, then for sufficiently large n, say $n \geq N \geq 1$, $W_n = X_n$. Thus $Y = P\{X_n : n \geq N\}$ is a C2 space and hence W is a C2 space by the Corollary to Theorem 12. If $Z = P\{X_n : 0 \leq n < N\}$ then X is homeomorphic to $Z \times Y$. Select a sequence $\{V_n : n \geq 1\}$ of non-empty open subsets of Y whose closures in Y are pairwise disjoint. Then $\{Z \times V_n : n \geq 1\}$ is a sequence of non-empty open sets in $Z \times Y$ whose closures are pairwise disjoint. Consider the sequence $\{h(U) \quad (Z \times V_n) : n \geq 1\}$ of non-empty open sets in the subspace h(U), where $h : X \to Z \times Y$ is the natural hoemomorphism. To show that the closures in h(U) of these sets are pairwise disjoint it is sufficient to show that their closures relative to $Z \times Y$ are pairwise disjoint. But this holds for in fact the members of the sequence $\{Z \times V_n : n \geq 1\}$ have pairwise disjoint closures relative to $Z \times Y$ since the members of $\{V_n : n \geq 1\}$ have pairwise disjoint closures in Y. Thus U as an arbitrary non-empty open subspace of X is a C2 space being homeomorphic to h(U).

If T is a topology on an infinite set X then there is a subtopology $T_0 \subseteq T$ for which (X, T_0) is not a C1 space and there is an extension topology $T_1 \supseteq T$ for which (X, T_1) is a C2 space. For example let T_0 be the indiscrete topology and let T_1 be the discrete topology. Theorem 4 stated that Axiom T_n is preserved under topology extensions.

That Property C1 and Property C2 are also preserved under topology extensions is a result of Theorem 11.

COROLLARY 2 TO THEOREM 11. Let T_1 and T_2 be topologies on a set X with $T_1 \subseteq T_2$. Let $X_1 = (X,T_1)$ and let $X_2 = (X,T_2)$. Then X_2 is a C1 (C2) space if X_1 is a C1 (C2) space.

Proof. Let $f:X_2 \to X_1$ be the identity function on the set X. Then f is a continuous surjection and the result follows from Theorem 11.

The conclusion of Corollary 2 to Theorem 11 may be strengthened in the context of certain specific subtopologies or extensions of a topology. If T is a topology on the set X let $T_\theta \subseteq T$ be the subtopology of θ-open subsets of (X,T). Recall that a subset U of (X,T) is θ-open if and only if U is a union of open subsets of (X,T) whose closures in (X,T) are contained in U. Hence if (X,T) is a regular space $T = T_\theta$ since every open set in (X,T) is θ-open. For if $U \in T$ and $x \in U$ then for some $V(x) \in T$, $x \in V(x) \subseteq \underline{Cl}\ V(x) \subseteq U$ so that $U = \bigcup\{V(x):x \in U\} = \bigcup\{\underline{Cl}\ V(x):x \in U\}$ and $U \in T_\theta$. This proves half of the following characterization of regular spaces.

THEOREM 13. A topological space (X,T) is regular if and only if $T = T_\theta$.

Proof. It remains only to show that if $T = T_\theta$ then (X,T) is regular so assume that $T = T_\theta$. If $x \in U \in T$ then $U = \bigcup\{V_a : a \in A\}$ where each $V_a \in T$ and $\underline{Cl}\ V_a \subseteq U$. Thus for some $b \in A$, $x \in V_b \subseteq \underline{Cl}\ V_b \subseteq U$ so that (X,T) is a regular space.

For notational purposes let $X_\theta = (X,T_\theta)$ if (X,T) is a topological space. Corollary 2 to Theorem 11 shows that if X_θ is a C1 (C2) space then X is a C1 (C2) space. In fact a stronger statement can be made.

THEOREM 14. If X_θ is a C1 space then X is a C2 space.

Proof. Let $f:X_\theta \to Y$ be a continuous surjection where Y is the one-point open extension of the countably infinite discrete space of natural numbers N. Such a function exists by Theorem 10. By the Axiom of Choice select $x_k \in f^{-1}(k)$ for each $k \in N$. Since $f^{-1}(k) \in T_\theta$ for each $k \in N$, there is a $V_k \in T$ with $x_k \in V_k \subseteq \underline{Cl}\ V_k \subseteq f^{-1}(k)$. Hence $\{V_k : k \in N\}$ is a sequence of non-empty open sets in X whose closures are pairwise disjoint. Thus X is a C2 space if X_θ is a C1 space.

COROLLARY TO THEOREM 14. Property C1 is equivalent to Property C2 in the class of regular spaces.

Proof. Since Property C2 implies Property C1, it suffices to prove that every regular C1 space is a C2 space. If (X,T) is a regular C1 space then $X = X_\theta$ is a C1 space by Theorem 13 and hence X is a C2 space by Theorem 14.

Additional corollaries to Theorem 14 could be obtained by applying Theorems 8, 9, 10 to the class of regular topological spaces and replacing Property C1 with Property C2. The Corollary to Theorem 14 also shows the equivalence of Theorem (L-L)4 with the Lemma (p. 3, [45]) which stated that every infinite regular Hausdorff space is a C1 space. Thus a new proof of Theorem (L-L)4 can now be given using the following alternate characterizations of a C1 space.

THEOREM 15. A space X is a C1 space if and only if there is an open subspace Y of X and a filterbase of closed somewhere dense sets in Y having a nowhere dense total intersection in Y.

Proof. If Y is an open subspace of X and $B = \{F_a : a \in A\}$ is a filterbase of closed somewhere dense subsets of Y with $\underline{\text{Int}} \bigcap \{F_a : a \in A\} = \emptyset$ and such that the indexing function $F:A \to B$ defined by $F(a) = F_a$ is injective, then A is directed upward by containment for the members of B. That is $a \leq b$ if and only if $F(b) \subseteq F(a)$. Let $G = \bigcap \{F_a : a \in A\}$ and for each $b \in A$ let $A(b) = \{a : a \in A \text{ and } a > b\}$. Let a_1 be an arbitrary but fixed

element of A. If a_1 is maximal in A then $F(a_1) \subseteq F(a)$ for all $a \in A$ so that $F(a_1) = G$. But then $G \in B$ and G is both nowhere dense and somewhere dense. Thus A has no maximal elements and $A(b) \neq \emptyset$ for each $b \in A$. Since $\bigcup\{F(a_1) - F(a) : a \in A(a_1)\} = F(a_1) - G$, and since $(\underline{\text{Int}}\ F(a_1)) \cap (F(a_1) - G) \neq \emptyset$, then for some $a_2 \in A(a_1)$, $(\underline{\text{Int}}\ F(a_1)) \cap (F(a_1) - F(a_2)) \neq \emptyset$. Let $V_1 = (\underline{\text{Int}}\ F(a_1)) \cap (Y - F(a_2))$. Then V_1 is a non-empty open subset of Y and $V_1 \subseteq F(a_1) - F(a_2)$. Since Y is open in X, V_1 is an open subset of X also. Continuing this process yields an increasing subsequence $\{a_n : n \in N\}$ of A so that for each $n \in N = \{1,2,3,\ldots\}$, a non-empty open subset V_n in X exists such that $V_n \subseteq F(a_n) - F(a_{n+1})$. Clearly $\{V_n : n \in N\}$ is a sequence of pairwise disjoint non-empty open subsets of X so that X is a C1 space.

Conversely, if X is a C1 space and $\{V_n : n \in N\}$ is a sequence of pairwise disjoint non-empty open subsets of X, then $Y = \bigcup\{V_n : n \in N\}$ is an open subspace of X. For each $k \in N$ let $N_k = \{n : n \in N \text{ and } n \leq k\}$, and let $F_k = Y - \bigcup\{V_n : n \in N_k\}$. Then $\{F_k : k \in N\}$ is a filterbase of closed somewhere dense sets in the subspace Y and $\bigcap\{F_k : k \in N\} = \emptyset$ is nowhere dense in Y.

Theorem 15 can be strengthened by showing that the open subspace Y of X can always be chosen to be X.

COROLLARY 1 TO THEOREM 15. A topological space X is a C1 space if and only if there is a filterbase of closed somewhere dense subsets of X having a nowhere dense total intersection.

Proof. By Theorem 15 the sufficiency is proven by noting that X is an open subspace of itself.

For the necessity let X be a C1 space with open subspace Y and filterbase $\{F_a : a \in A\}$ of closed somewhere dense sets in Y such that $\underline{\text{Int}} \bigcap \{F_a : a \in A\} = \emptyset$. Let $E_a = F_a \cup \text{Fr}(Y)$ for each $a \in A$. Then since $\text{Fr}(Y)$ is nowhere dense in X, $\{E_a : a \in A\}$ is the desired filterbase in X.

COROLLARY 2 TO THEOREM 15. A topological space X has Property C1 if and only if some open subspace Y of X has Property C1.

Proof. An open subspace Y of X has Property C1 if and only if Y has a filterbase of closed somewhere dense sets whose total intersection is nowhere dense by Corollary 1 to Theorem 15. But an open subspace Y of X has such a filterbase if and only if X is a C1 space by Theorem 15.

A space X is regular if and only if the family of closed neighborhoods of each point is a local base for that point (p. 113, [37]). Such a local base of closed neighborhoods is a filterbase. If furthermore the regular space is a T_1-space (and hence also Hausdorff) then the total intersection of the closed neighborhoods of a point x

must be {x} since points are closed in a T_1-space. Thus if x is a non-isolated point of a regular Hausdorff space then {x} is not open so that Int {x} = ϕ and the closed neighborhood base at x has a nowhere dense total intersection. But for regular spaces Property C1 is equivalent to Property C2 so that Theorem (L-L)4 follows.

COROLLARY 3 TO THEOREM 15. (Theorem (L-L)4) Every infinite regular Hausdorff space is a C2 space.

Proof. Let X be an infinite regular Hausdorff space. If X has no non-isolated points then X is a discrete space and has Property C2. If X has a non-isolated point x then X has Property C1 since the filterbase of closed neighborhoods of x has a nowhere dense total intersection. By the Corollary to Theorem 14, X is a C2 space.

Further characterizations of C1 and non-C1 spaces follow from Corollary 1 to Theorem 15 and its proof using De Morgan's laws of complementation.

COROLLARY 4 TO THEOREM 15. A topological space X is a C1 space if and only if there is a family of non-dense open sets in X directed upward by inclusion whose total union is dense in X.

COROLLARY 5 TO THEOREM 15. A topological space X fails to be a C1 space if and only if either 1) every decreasing sequence of closed sets having nowhere dense

total intersection has a nowhere dense member, or 2) every increasing sequence of open sets having a dense union has a dense member.

By Theorem 5 and Example 3 or by Example 5, regularity is not essential for a Hausdorff space to be a C2 space and further a Hausdorff C2 space need not be a T_n-space. In fact the non-T_n, non-Urysohn space of Example 5 is a connected Hausdorff C2 space. Thus this space has no regular Hausdorff subtopology, for Axiom T_n and the Urysohn axiom are preserved under topology extensions and every regular Hausdorff space is a Urysohn T_n-space. This disproves the possible conjecture that a Hausdorff C2 space must have a regular Hausdorff subtopology. Note that having a regular Hausdorff subtopology is sufficient for an infinite space to have Property C2 since every infinite regular Hausdorff space has Property C2 which is preserved under topology extensions. Indeed the space of Example 3 is such a non-regular infinite Hausdorff space which has a regular Hausdorff subtopology.

The non-Hausdorff space of Example 8 was a C1, non-C2 space. The following example shows that Property C1 is not equivalent to Property C2 even in the class of Hausdorff spaces.

EXAMPLE 12. Let Q be the set of rational real numbers and let a be a fixed irrational real number. Let $X = \{(x,y): x,y \in Q \text{ and } y \geq 0\}$ be the upper rational half-plane

and let T be the a-slope topology on X (p. 93, [64]). The a-slope topology has for a basis the collection of all sets of the form $N_\varepsilon((x,y)) = \{(x,y)\} \cup (B_\varepsilon(x + \frac{y}{a}) \cap X) \cup (B_\varepsilon(x - \frac{y}{a}) \cap X)$ where $(x,y) \in X$, ε is a positive real number, and for a real number z, $B_\varepsilon(z) = \{(x,0): |x-z| < \varepsilon$ and x is a real number$\}$. Geometrically, $N_\varepsilon((x,y))$ consists of $\{(x,y)\}$ plus two intervals on the rational x-axis centered at the two irrational points $x + \frac{y}{a}$ and $x - \frac{y}{a}$; the lines joining these points to (x,y) have slopes -a and a respectively. Clearly, (X,T) has no isolated points. If (x_1,y_1) and (x_2,y_2) are distinct points in X then $\left(x_1 \pm \frac{y_1}{a}\right) \neq \left(x_2 \pm \frac{y_2}{a}\right)$ so that for sufficiently small $\varepsilon > 0$, $N_\varepsilon((x_1,y_1)) \cap N_\varepsilon((x_2,y_2)) = \emptyset$. Thus (X,T) is a Hausdorff space. Note that each open interval on the rational x-axis is open in (X,T) for $N_\varepsilon((x,0)) = B_\varepsilon(x) \cap X$ for each $x \in Q$. Further, if $U \in T$ then $B_\varepsilon(z) \cap X \subseteq U$ for some real number z and for some $\varepsilon > 0$. Now $\underline{Cl}(B_\varepsilon(z) \cap X)$ contains all points of X on lines of slope $\pm a$ passing through a point of $B_\varepsilon(z)$. Thus for any two real numbers z_1 and z_2 and for any $\varepsilon > 0$, $\underline{Cl}(B_\varepsilon(z_1) \cap X) \cap \underline{Cl}(B_\varepsilon(z_2) \cap X)$ contains an open ball of positive radius with respect to the usual Euclidean metric on X. Hence, X is not a Urysohn space. Further X is not a C2 space. But $B_\varepsilon(z) \cap X$ is an open regular Hausdorff subspace of X containing infinitely many points so that $B_\varepsilon(z) \cap X$ has Property C2 and hence also Property C1. By Corollary 2 to Theorem 15, X is a C1 space since X has an open C1 subspace.

Property C1 and Property C2 do very little toward insuring general topological properties for a space. One of the few characteristics enjoyed by every C1 or C2 space is that it must have an infinite topology and hence infinitely many points. The lack of space characteristics implied by either Property C1 or Property C2 is largely due to the fact that a topological sum of two spaces is C1 (C2) if either summand is C1 (C2). To help avoid this problem the following local and uniform versions of Property C1 (C2) are introduced for a topological space X.

PROPERTY LC1 (LC2). There is a basis for the topology on X each member of which is a C1 (C2) subspace of X.

PROPERTY UC1 (UC2). Each non-empty open subspace Y of X is a C1 (C2) space.

Property UC1 (UC2) implies Property LC1 (LC2) and Property UC1 (UC2) always implies Property C1 (C2) if $X \neq \emptyset$. If X has Property LC1 (LC2) then X has no isolated points.

DEFINITION 4. A topological space X is an LC1 (LC2) space if and only if X has Property LC1 (LC2). Also a topological space X is a UC1 (UC2) space if and only if X has Property UC1 (UC2).

THEOREM 16. Property LC1 is equivalent to Property UC1 and each of these implies Property C1.

Proof. By Corollary 2 to Theorem 15 Property LC1 implies Property UC1 which in turn implies Property LC1 and Property C1.

That in general Property C1 does not imply Property UC1 is shown in the following example.

EXAMPLE 13. Let $X = Y + P$ be the topological sum of a regular Hausdorff space Y without isolated points and of a singleton point space P. Then X is a C1 space since Y is a C1 space but X is not a UC1 space since P is a finite non-empty open subspace of X.

Neither does Property C2 imply either Property UC2 or Property LC2. The space X of Example 13 is a C2 space since Y is a clopen C2 subspace of X. But X is not a UC2 space since X fails to be a UC1 space. Since Property UC1 is equivalent to Property LC1, neither is X an LC2 space. Finally, Property LC2 does not imply Property C2. The space X of Example 12 is an LC2 space since it was shown that every basic open neighborhood $N_\varepsilon((x,y))$ for $\varepsilon > 0$ is a C2 subspace of X. But it was also shown that this space is not a C2 space. So Property LC2 cannot imply Property UC2 since Property UC2 implies Property C2.

The Hausdorff, Urysohn, and T_n axioms and regularity are hereditary properties. Furthermore, a T_1-space without isolated points contains no finite non-empty open sets.

For in a T_1-space, every point of a finite open set must be an isolated point. This proves the following theorem and corollary.

THEOREM 17. A T_n-space X is a UC2 space if and only if X has no isolated points.

COROLLARY TO THEOREM 17. A regular Hausdorff space X is a UC2 space if and only if X has no isolated points.

Non-Hausdorff UC2 spaces also exist. The space X of Example 10 was shown to be a non-T_0 UC2 space. The non-T_n, non-Urysohn, Hausdorff space X of Example 12 is an LC2 and hence LC1 and UC1 space. The following question arises. Does there exist a Hausdorff space without isolated points which fails to be a UC1 space? Such a space must necessarily have a non-empty open subspace failing to have Property C1. Thus the above question reduces to the following equivalent question. Does there exist a non-empty Hausdorff space without isolated points which fails to be a C1 space? The class of infinite Hausdorff spaces properly contains the class of non-empty Hausdorff spaces without isolated points and the immediately preceding question can be equivalently applied to the larger class of infinite Hausdorff spaces. For if X is an infinite Hausdorff space failing to have Property C1 and if Y is the set of isolated points of X, then Y is an open discrete subspace of X failing to have Property C1

so that Y is finite. Thus Y is a clopen subspace of X and X = (X-Y) + Y is a topological sum of a Hausdorff space X-Y without isolated points and a finite discrete space Y. Since X-Y must fail to have Property C1, the above questions are equivalent to the following. Does there exist an infinite Hausdorff space which fails to have Property C1? An affirmative answer would be of interest in classifying the C1 spaces and a negative answer would say that the class of C1 spaces properly contains the class of infinite Hausdorff spaces. The following theorem answers the question in the negative.

THEOREM 18. Every infinite Hausdorff topological space X has Property C1.

Proof. It suffices to show that every non-empty Hausdorff space without isolated points has Property C1. Let X be such a space and let x and x_1 be distinct elements of X. Choose disjoint open sets Y_1 and V_1 containing the points x and x_1 respectively. Since X has no isolated points, Y_1 is infinite. Moreover, Y_1 is a Hausdorff space without isolated points. Choose a point $x_2 \neq x$ with $x_2 \in Y_1$ and choose disjoint open sets Y_2 and V_2 in Y_1 with $x \in Y_2$ and $x_2 \in V_2$. Since Y_1 is open in X, Y_2 and V_2 are also open in X. Furthermore, the sets V_1, V_2, and Y_2 are pairwise disjoint. Now Y_2 is an infinite open subspace of X containing x so that a point distinct from x may be selected from Y_2. Proceeding in this manner, if for some

positive integer n, pairwise disjoint non-empty open sets $V_1, V_2, \ldots, V_n, Y_n$ have been chosen, with $x \in Y_n$, then Y_n is an infinite set and a point $x_{n+1} \neq x$ may be chosen from Y_n. Then disjoint open subsets of Y_n, Y_{n+1} and V_{n+1} can be chosen with $x \in Y_{n+1}$ and $x_{n+1} \in V_{n+1}$. Finally, the sets $V_1, V_2, \ldots, V_{n+1}, Y_{n+1}$ are non-empty pairwise disjoint open subsets of X. By Mathematical Induction the space X contains a sequence $\{V_n : n \geq 1\}$ of pairwise disjoint non-empty open subsets so that X is a C1 space.

Professor You-Feng Lin has remarked that the method of proof used for Theorem 18 can be used to obtain the following result similarly.

THEOREM 19. Every infinite Urysohn space has Property C2.

Theorem (L-L)4 follows as a corollary to Theorem 19 since every regular Hausdorff space is Urysohn. However, Theorem 5 does not follow from Theorem 19 since Hartman [26] has given an example of a non-Urysohn T_n-space.

The importance of Theorem 18 is that it combines with Theorem (L-L)5 to allow strengthened versions of Theorem (L-L)1 and Theorem 7. But Theorem 18 also allows Blumberg's first theorem [5] to be generalized in its entirety. If D(f) is the set of points of almost continuity in X of a function $f:X \to Y$ then Blumberg's first theorem stated that if R is the usual space of real numbers, then D is a

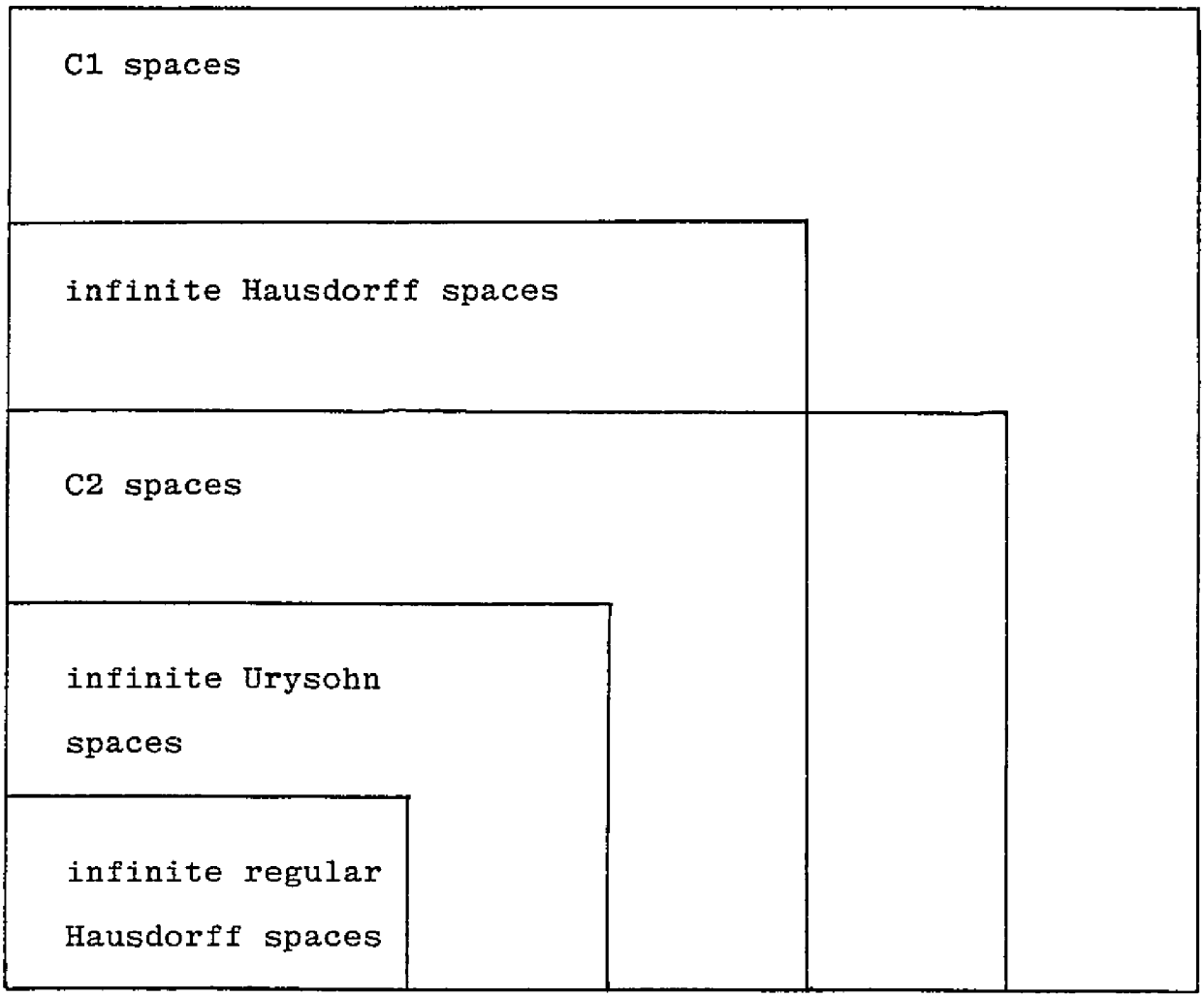

Figure 1. Class inclusion of topological spaces

comeager subset of R^n, Euclidean n-space, if and only if there is a function $f:R^n \to R$ such that $D = D(f)$. Half of this theorem was generalized by Theorem (L). The generalization of the other half yields the following combination of results.

THEOREM 20. If X is any space and Y is any second countable C1 space then a subset D of X is a comeager set in X if and only if $D = D(f)$ for some function $f:X \to Y$.

Proof. Theorem (L) establishes the sufficiency. For the necessity let D be a comeager subset of X. Then $X - D = \bigcup\{F_n : n \geq 1\}$ is a countable union of pairwise disjoint nowhere dense sets in X. Since Y is a C1 space, let $g:Y \to Z$ be a continuous surjection onto the one-point open extension $Z = \{0\} \cup N$ of the countably infinite discrete space of natural numbers N. By the Axiom of Choice, choose $y_0 \in g^{-1}(0)$ and choose $y_n \in g^{-1}(n)$ for each $n \in N$. Define $f:X \to Y$ by the $f(x) = y_0$ if $x \in D$ and $f(x) = y_n$ if $x \in F_n$ for $n \geq 1$. If $x \in D$ and V is open in Y containing $f(x) = y_0$ then $D \subseteq f^{-1}(V)$ so that $D \subseteq \underline{\text{Int}}\ \underline{\text{Cl}}\ f^{-1}(V) = X$ since D is dense in X. Thus f is almost continuous at each point $x \in D$ so that $D \subseteq D(f)$. But if $x \in F_n$ for some $n \geq 1$ then $g^{-1}(n) = V_n$ is an open set in Y containing $f(x) = y_n$ and $\underline{\text{Int}}\ \underline{\text{Cl}}\ f^{-1}(V_n) = \emptyset$ since $f^{-1}(V_n) = F_n$ is nowhere dense. Thus f is not almost continuous at x if $x \in X-D$. Thus $D = D(f)$.

COROLLARY TO THEOREM 20. If X is any space and Y is any second countable infinite Hausdorff space, then a subset D of X is a comeager set in X if and only if $D = D(f)$ for some function $f:X \to Y$.

Theorem 18 combines with Theorem (L-L)5 to generalize Theorem (L-L)3 which in turn can be combined with Theorem (L) to yield the following strengthening of Theorem (L-L)1 and Theorem 7.

THEOREM 21. Let Y be a second countable infinite Hausdorff space. The space X is a Baire space if and only if D(f) is dense in X for each function $f:X \to Y$.

In attempting to improve Theorem (L) and Theorem 21 by weakening the second countability condition resulted the following theorem.

THEOREM 22. Let X be a second category space and let Y be a Lindelöf space. If X fails to have Property C1 and $f:X \to Y$ is a function with closed graph G(f), then $D(f) \neq \emptyset$.

Proof. Let $y \in Y$ and $\{V_b : b \in B\}$ be a local open base at y where the indexing function $V:B \to \{V_b : b \in B\}$ defined by $V(b) = V_b$ is bijective. If $\underline{\text{Int}}\ \underline{\text{Cl}}\ f^{-1}(V_b) \neq \emptyset$ for each $b \in B$ then $\{\underline{\text{Cl}}\ f^{-1}(V_b) : b \in B\}$ is a filterbase of closed somewhere dense sets in X. If $x \in \underline{\text{Int}} \bigcap \{\underline{\text{Cl}}\ f^{-1}(V_b) : b \in B\}$ then $x \in \bigcap \{\underline{\text{Int}}\ \underline{\text{Cl}}\ f^{-1}(V_b) : b \in B\}$. Let $\{U_a : a \in A\}$ be a local base at x with indexing function $U:A \to \{U_a : a \in A\}$

defined by $U(a) = U_a$, being bijective. Let A and B be directed so that $a_1 < a_2$ if and only if $U(a_2) \subseteq U(a_1)$ and $b_1 < b_2$ if and only if $V(b_2) \subseteq V(b_1)$. Then let $A \times B$ be directed by the lexicographic partial ordering induced by the partial orderings for A and B. By the Axiom of Choice, for each $(a,b) \in A \times B$, choose $x(a,b) \in U_a \cap f^{-1}(V_b)$. Then $\{x(a,b)\}$ is a net converging to x and $\{f(x(a,b))\}$ is a net in Y which accumulates to y. Thus $\{(x(a,b), f(x(a,b)))\}$ accumulates to (x,y) in $X \times Y$ so that $(x,y) \in \underline{Cl}\ G(f)$ and $y = f(x)$. Suppose that $D(f) = \emptyset$. Then $x \notin D(f)$ so that for some V_b, $x \notin \underline{Int}\ \underline{Cl}\ f^{-1}(V_b)$. Thus $x \notin \bigcap \{\underline{Int}\ \underline{Cl}\ f^{-1}(V_b) : b \in B\}$. This contradiction proves that $\underline{Int} \bigcap \{\underline{Cl}\ f^{-1}(V_b) : b \in B\} = \emptyset$. If $\underline{Int}\ \underline{Cl}\ f^{-1}(V_b) \neq \emptyset$ for each $b \in B$ then $\{\underline{Cl}\ f^{-1}(V_b) : b \in B\}$ is a filterbase of closed somewhere dense subsets of X with nowhere dense total intersection. This would show X to be a Cl space by Corollary 1 to Theorem 15. But since X is not a Cl space $\underline{Int}\ \underline{Cl}\ f^{-1}(V_b) = \emptyset$ for some $b \in B$. For each $y \in Y$, select an open neighborhood V_y of y so that $f^{-1}(V_y)$ is nowhere dense in X. Since Y is a Lindelöf space a countable open cover $\{V_n : n \geq 1\}$ of Y exists so that each $f^{-1}(V_n)$ is nowhere dense. Thus $X = \bigcup \{f^{-1}(V_n) : n \geq 1\}$ is a first category space. This contradicts the hypothesis that X is a second category space. Thus $D(f) \neq \emptyset$.

Note that every non-empty open subspace of a Baire space X is a second category subspace by Theorem 1. Further, if X fails to be a C1 space, then every open subspace of X fails to have Property C1 by Corollary 2 to Theorem 15. By Theorem 1 of Chapter I, every weakly continuous function into a Hausdorff space has a closed graph. Also a weakly continuous function restricted to a subspace is weakly continuous on the subspace. Thus these facts and Theorem 22 support the following result.

THEOREM 23. Let X be a non-C1 Baire space and let Y be a Hausdorff Lindelöf space. If $f:X\to Y$ is a weakly continuous function then D(f) is a dense subset of X.

Proof. By Theorem 22, $D(f) \neq \emptyset$. Suppose that D(f) is not dense in X. Then there exists an open set U contained in X-D(f). But $f|U:U\to Y$ is weakly continuous and U is a second category non-C1 space so that f|U is almost continuous at a point $x \in U$. Thus if V is open in Y containing f(x), then $x \in \underline{\text{Int}}_U \underline{\text{Cl}}_U(f^{-1}(V) \cap U) \subseteq \underline{\text{Int}}\ \underline{\text{Cl}}\ f^{-1}(V)$ so that $x \in D(f)$ and $D(f) \cap U \neq \emptyset$. This contradiction proves that D(f) is a dense subset of X.

The space X of Theorem 23 cannot be an infinite Hausdorff Baire space. The following example of a non-C1 Baire space shows that Theorem 23 is not true vacuously.

EXAMPLE 14. Let X be an uncountable set with the countable-complement topology. Then a non-empty subset U of X is open if and only if its complement X-U is countable. Since no two non-empty open sets are disjoint, X is not a C1 space. Further, for this reason non-empty open sets are dense in X. Also a countable intersection of non-empty sets is open having a countable complement. Thus X is a Baire space.

SUMMARY

Throughout this paper credit is given to whom it is due for each of the previously published theorems, corollaries, and definitions. Important theorems are either identified by numbers or by initials of the authors responsible for the theorems. If more than one theorem from the same set of authors is introduced then a combination of initials and numbers is used to identify the theorem. The numbered theorems with the exception of Theorems 2 and 19 of Chapter II were obtained independently by the author. The idea of Theorem 2 of Chapter II was discovered independently but help was received from Professor Y.-F. Lin on the proof. Theorem 19 of Chapter II was obtained by Professor Y.-F. Lin using the method of proof of Theorem 18 discovered by the author. The corollaries are labeled clearly referring to the theorems from which they follow. Some of the numbered theorems or corollaries are known results in which case clear acknowledgement is made of the true authors. Furthermore, these known results with the exception of Theorem 2 of Chapter II are obtained by different methods than those of the original authors. For the most part the initialed theorems are not building blocks for this paper but are introduced

for perspective in comparison to the new results obtained. There are ninety-seven theorems and thirty-nine corollaries included in this paper. Of the ninety-seven theorems, thirty-three are lettered and eight of the sixty-four numbered theorems were known results. Seven of the thirty-nine corollaries were known results.

In Chapter I, eleven of the twenty-five definitions are original in this work and allow a new approach to known results and improvements of these results. Noiri showed (Theorem 1, Chapter I) that the graph of a weakly continuous function into a Hausdorff range space has a closed graph but did not connect this result to any existing results. Rather than using standard interior point methods the author finds this result by viewing the graph of the function as the inverse range of the diagonal of the range under a product function. To effect this proof the notion of a strongly closed subset of a product space with respect to certain coordinates was introduced generalizing the strongly closed graphs of Herrington and Long and also generalizing the θ-closed sets of Veličko. Five of the six corollaries of Theorem 1 are new or improvements of known results. Theorems concerning products of weakly continuous functions are obtained more easily than the analogous known results for almost continuous (S and S) functions. Inspired by Theorem 1, the closure of the graph of a multifunction is studied and more generally, properties of closed relations are investigated. Multifunctions with closed graphs

are upper semi-c-continuous generalizing the single-valued version of this result by Long and Hendrix. Also the set of fixed points for a multifunction from an arbitrary topological space into itself is closed if the graph is closed. Thus a closed graph multifunction retraction is closed and as a consequence the result of Noiri that every weak retract of a Hausdorff space is closed is obtained. Translating an investigation of Weston concerning topologies on a set into facts about bijective functions resulted in the discovery of new characterizations for weak continuity and almost continuity. The new characterization of weak continuity reveals superfluous hypotheses in several of the theorems in the literature due to Noiri and Long and Carnahan. In particular it destroys all previous attempts to deduce weak continuity from almost continuity plus conditions weaker than weak continuity. However a condition weaker than weak continuity is found in this work which allows weak continuity to follow from almost continuity. Similarly, almost continuity follows from weak continuity in the presence of almost openness, improving the result obtainable from known theorems that a weakly continuous open function is almost continuous. In an attempt to produce a continuity complement for almost continuity, semi-continuity is investigated leading to an improvement of a theorem by Hamlett. Strong semi-continuity is introduced and shown to be independent from continuity but stronger than semi-continuity. Almost continuity plus strong

semi-continuity implies continuity and this result can be used to obtain the result of Long and McGehee that an almost continuous function of a real variable into a locally connected space is continuous if inverse images of connected sets are connected. The decomposition of continuity into weak continuity plus weak* continuity of Levine is improved by introducing local weak* continuity. Local weak* continuity allows improvement of known results of Long and Herrington and Long and McGehee since in particular every function into a rim-compact space is locally weak* continuous. Thus a closed graph theorem is obtained slightly improving a known theorem of Long and McGehee. Dual versions of some of the results are obtained and in particular, an open mapping theorem dual to the closed graph theorem with a slight sacrifice in generality.

In Chapter II, the Lins' characterization of Baire spaces is traced to its origin and thus placed in its proper setting. Stemming from the same origin, the attempt to classify Blumberg spaces and Baire non-Blumberg spaces has been a concern of Levy, White, and others. A theorem of Long and McGehee independently discovered by Y.-F. Lin together with the Lins' characterization of Baire spaces provides a new proof that all Blumberg spaces are Baire spaces. An improvement of the Lins' characterization of Baire space is obtained using the T_n separation axiom of Hartman. A rather complete investigation of C1 and C2 spaces introduced by the author is for the purpose of

improving further the Lins' theorem on Baire spaces. A generalization of the converse portion of Blumberg's Theorem I is noted, an improved characterization of Baire spaces is obtained and finally it is shown that a closed graph function from a non-C1 Baire space into a Lindelöf space has a dense set of points of almost continuity.

An open question of Shwu-Yeng T. Lin and Y.-F. Lin asked whether an almost continuous closed graph function from a Baire space into a second countable space is necessarily continuous. The author answers this question in the negative but raises some related questions. The closed graph theorem of Chapter I shows that continuity is derivable if the domain is an arbitrary topological space and the range is strongly locally compact. It is unknown what more general conditions on the domain and range spaces would yield continuity of the almost continuous closed graph function.

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