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A Potential Method for Body and Surface Wave Propagation in Transversely Isotropic Half- and Full-Spaces

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ABSTRACT: The problem of propagation of plane wave including body and surface waves propagating in a transversely isotropic half-space with a depth-wise axis of material symmetry is investigated in details. Using the advantage of representation of displacement fields in terms of two complete scalar potential functions, the coupled equations of motion are uncoupled and reduced to two independent equations for potential functions. In this paper, the secular equations for determination of body and surface wave velocities are derived in terms of both elasticity coefficients and the direction of propagation. In particular, the longitudinal, transverse and Rayleigh wave velocities are determined in explicit forms. It is also shown that in transversely isotropic materials, a Rayleigh wave may propagate in different manner from that of isotropic materials. Some numerical results for synthetic transversely isotropic materials are also illustrated to show the behavior of wave motion due to anisotropic nature of the problem.

Keywords: *P*-Wave; Rayleigh Wave; Scalar Potential Function; *SH*-Wave; *SV*-Wave; Transverse Isotropy.

INTRODUCTION

The study of elastic wave propagation has its origin in the age-old research. Cauchy in 1822 discovered most of the elements of the pure theory of elasticity, including the notion of stress and the displacement equations of motion. Poisson was the first, who recognized that an elastic disturbance is in general composed of both types of fundamental displacement waves, the dilatational (longitudinal) and the equivoluminal (transverse) waves (see Miklowitz, 1978).

Cauchy in 1830 and Green in 1839, investigated the propagation of plane waves through a crystalline medium and obtained equations for the velocity of propagation in terms of the direction of the normal to the wave front. In the case of isotropic medium, in addition to a dilatational wave, also called as P - wave, two other waves correspond to transverse plane waves known as SV - and SH -waves (vertically and horizontally polarized shear waves) found to be existed (see Miklowitz, 1978).

Lord Rayleigh in 1887, made the very important finding of his well-known surface

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wave. This wave is generated by a pair of plane harmonic waves, dilatational and equivoluminal waves (*P*- and *SV*-waves), propagating over the free surface of an elastic half-space. In an isotropic half-space, it travels parallel to the surface with a wave speed that is slightly less than that of the equivoluminal body wave (see Miklowitz, 1978).

Recently, the wave propagation in anisotropic materials is of major concern because of their high performance in technological applications (see Raoofian Naeeni and Eskandari-Ghadi, 2016; Ardeshire-Behrestaghi et al., 2013; Eskandari-Ghadi et al., 2014). Most innovative materials such as composites, piezo-composites, and magnetics are anisotropic, and in application need to be modeled as either transversely isotropic or orthotropic materials. The early work of Stoneley (1949) revealed that wave propagation in a transversely isotropic medium gives rise to phenomena, which greatly differ from that of isotropic one. Later, some researches focused on the study of the elastodynamic problems pertinent to the transversely isotropic half-space under surface or buried loadings (see Eskandari-Ghadi and Sattar, 2009; Eskandari-Ghadi et al., 2011; Eskandari-Ghadi and Ardeshir-Behrestaghi, 2010; Raoofian Naeeni and Eskandari-Ghadi, 2016).

The potential method is a powerful tool for solving the coupled both equilibrium equations and equations of motion (see Eskandari-Ghadi and Ardeshir-Behrestaghi, 2011; Raoofian Naeeni et al., 2015a). Lame in 1852, showed that every sufficiently smooth solution for the displacement equations of motion in isotropic materials can be represented by the sum of two components for the displacement, the first is represented as the gradient of a scalar potential function, which is curl-free and the second is shown as a solenoidal vector field, which is divergence-free, where both satisfy wave equations having the dilatational and equivoluminal wave speeds, respectively (see Miklowitz, 1978). A rigorous completeness proof of this solution was given by Sternberg and Gurtin (1962). On the other hand, Lekhnitskii in 1940, considered elastostatics problems of transversely isotropic material characterized by torsionless axisymmetry and found the solution of the displacement equations of equilibrium in terms of a single stress function satisfying a fourth-order partial differential equation (see Lekhnitskii, 1981). Hu (1953) studied the general case of elastostatic problem in transversely isotropic media and generalized Lekhnitskii’s solution, now a day called Lekhnitskii–Hu–Nowacki solution. Eskandari-Ghadi (2005) introduced a complete solution for the general elastodynamics boundary value problems in a linear Green elasticity transversely isotropic mono-axial-convex domain in terms of two scalar potential functions, one of which describes *SH*-wave and the other gives both *SV*- and *P*-waves in any plane containing the axis of material symmetry (see Amiri-Hazaveh et al., 2013). His solution is reduced to only one potential function for torsionless axisymmetric problems.

Three different body waves can propagate in each direction in anisotropic elastic materials; however, while the associated displacement vectors are mutually perpendicular, the waves cannot in general be classified into dilatational and rotational types. For transversely isotropic materials one of the body waves is always purely transverse, and since this wave is polarized in planes perpendicular to the direction of symmetry, it may appropriately be referred to as an *SH*-wave. The other two body waves are called quasi-longitudinal (QL) and quasi-transverse (QT), the former being the wave for which the inclination of the displacement vector to the wave normal is least (see Raoofian Naeeni et al., 2015b; Slawinski, 2010).

In this study, for the first time, with the aid of a general and complete solution presented by Eskandari-Ghadi (2005), the propagation of harmonic waves in transversely isotropic media is investigated, particularly the explicit equations for the body-wave is presented from which the velocities of longitudinal and transverse waves are deduced. It is shown that the velocities of these waves are dependent on both compressive/tensile and shear moduli of the material. It is shown that the longitudinal wave velocity in some special direction is independent from shear moduli, and in the same way, the transverse wave velocity in some direction is independent from the compressive/tensile moduli. The Rayleigh-wave velocity is studied in detail, where two kinds of Rayleigh-waves are discovered and the conditions for the existence of any kind of Rayleigh-wave are addressed.

GOVERNING EQUATIONS AND POTENTIAL FUNCTION

The displacement equations of motion for a linear transversely isotropic Green elastic material in Cartesian coordinate system ($o: x_1, x_2, x_3$) and in the absence of body forces are expressed as (see Eskandari-Ghadi, 2005).

$$\begin{aligned}
&C_{66} u_{i,jj} + \frac{1}{2}\left(C_{11} + C_{12}\right) u_{j,ij} \\
&+ C_{44} u_{i,33} + \left(C_{13} + C_{44}\right) u_{3,i3} \\
&= \rho \ddot{u}_i, \quad i = 1,2 \\
&C_{44} u_{3,jj} + C_{33} u_{3,33} + \\
&\left(C_{13} + C_{44}\right) u_{j,3j} = \rho \ddot{u}_3
\end{aligned} \tag{1}$$

where C_{11}, C_{13}, C_{33}, C_{12}, C_{44} and $C_{66} = \left(C_{11} - C_{12}\right)/2$ are elasticity coefficients, u_1, u_2 and u_3 are components of displacement vector and ρ is the density of the medium. The summation convention is applied over repeated indices. Due to positive definiteness of strain energy function, the following inequalities must be hold:

$$\begin{aligned}
&C_{11} > 0, C_{33} > 0, \\
&C_{44} > 0, C_{66} > 0, \\
&C_{11} C_{33} - C_{13}^2 - C_{33} C_{66} > 0
\end{aligned} \tag{2}$$

The general solution of Eq. (1) in an x_3-convex medium may be expressed in terms of two scalar potential functions F and χ as (see Eskandari-Ghadi, 2005).

$$\begin{aligned}
&u_1(x_1, x_2, x_3, t) = -\alpha_3 \frac{\partial^2 F}{\partial x_1 \partial x_3} - \frac{\partial \chi}{\partial x_2}, \\
&u_2(x_1, x_2, x_3, t) = -\alpha_3 \frac{\partial^2 F}{\partial x_2 \partial x_3} + \frac{\partial \chi}{\partial x_1}, \\
&u_3(x_1, x_2, x_3, t) = (1 + \alpha_1) \\
&\left[\nabla^2 + \frac{\alpha_2}{1 + \alpha_1} \frac{\partial^2}{\partial x_3^2} - \frac{\rho_0}{1 + \alpha_1} \frac{\partial^2}{\partial t^2}\right] F
\end{aligned} \tag{3}$$

where $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ and

$$\begin{aligned}
&\alpha_1 = \frac{C_{66} + C_{12}}{C_{66}} > 0, \\
&\alpha_2 = \frac{C_{44}}{C_{66}} > 0, \;\; \alpha_3 = \frac{C_{13} + C_{44}}{C_{66}}, \\
&\alpha_4 = \frac{C_{33}}{C_{66}} > 0, \;\; \rho_0 = \frac{\rho}{C_{66}},
\end{aligned} \tag{4}$$

By replacing Eq. (3) into Eq. (1), the following partial differential equations for

F an χ may be derived (see Eskandari-Ghadi, 2005).

$$\left[\Box_1^2\Box_2^2 - \delta\frac{\partial^4}{\partial x_3^2\partial t^2}\right]F = 0,\ \ \Box_0^2\chi = 0 \qquad (5)$$

in which

$$\Box_i^2 = \nabla^2 + \frac{1}{s_i^2}\frac{\partial^2}{\partial x_3^2} - \frac{1}{c_i^2}\frac{\partial^2}{\partial t^2} \quad i = 0,1,2$$

and

$$\begin{aligned} &c_0^2 = \frac{C_{66}}{\rho},\ \ c_1^2 = \frac{C_{11}}{\rho},\ c_2^2 = \frac{C_{44}}{\rho}, \\ &s_0^2 = \frac{1}{\alpha_2},\ \delta = \left(1 - \frac{1}{s_2^2}\right)\left(\frac{1}{c_1^2} - \frac{1}{c_2^2 s_1^2}\right) \end{aligned} \qquad (6)$$

Moreover s_1^2 and s_2^2 are the roots of the following polynomial equation and they are not zero or negative.

$$\begin{aligned} &C_{33}C_{44}s^4 + \\ &\left(C_{13}^2 + 2C_{13}C_{44} - C_{11}C_{33}\right)s^2 + \\ &C_{11}C_{44} = 0 \end{aligned} \qquad (7)$$

s_1^2 and s_2^2 can be real and distinct, coalescent, or conjugate complex. It is easy to show, that if the roots are real, then they are positive (see Eskandari-Ghadi, 2005).

BODY WAVES

In this section, we give a brief account of the behavior of plane harmonic body waves propagating in an infinite media and governed by Eq. (5). To do so, we attach a Cartesian coordinate system $(o: x_1, x_2, x_3)$ and for more understanding a cylindrical coordinate system $(o: r, \theta, z)$ as shown in Figure 1 at an arbitrary point of the domain in such a way that the $x_3 -$axis is positive downward and is parallel to the axis of material symmetry at any point. We seek the solution of Eq. (5) in the form of plane wave as:

$$\begin{aligned} &F = Ae^{iw(Sx_p n_p - t)}, \\ &\chi = Be^{iw(Sx_p n_p - t)} \end{aligned} \qquad (8)$$

where ω is the circular frequency of the wave, *S* its slowness defined as the inverse of wave velocity, and $\mathbf{n} = (n_1, n_2, n_3 = \cos\varphi)$ (see Figure 1) is the direction of propagation of the plane wave defined above. It is also called the wave normal, which is an axis normal to the plane that has a constant displacement at any time. Shear-wave (*S*-wave) is defined as a displacement vector propagating in the direction of **n**, however, does not have any component in the propagating direction. On the other hand, *P*-wave (either compressive or tensile wave) is defined as a displacement propagating in the direction of **n**, and has just one component in the propagating direction. Both *S*- and *P*-waves are functions of position, say (x_1, x_2, x_3) in Cartesian coordinate system or (r, θ, z) in cylindrical coordinate system. *SH*-wave is called as the shear-wave, which is propagating in the direction shown by $\mathbf{n}$ in $r - z$ plane, however, it does not have any component in this plane. We denote *SV*-wave to be the shear-wave propagating in the direction given by **n** in $r - z$ plane, which has just one component normal to **n** in $r - z$ plane.

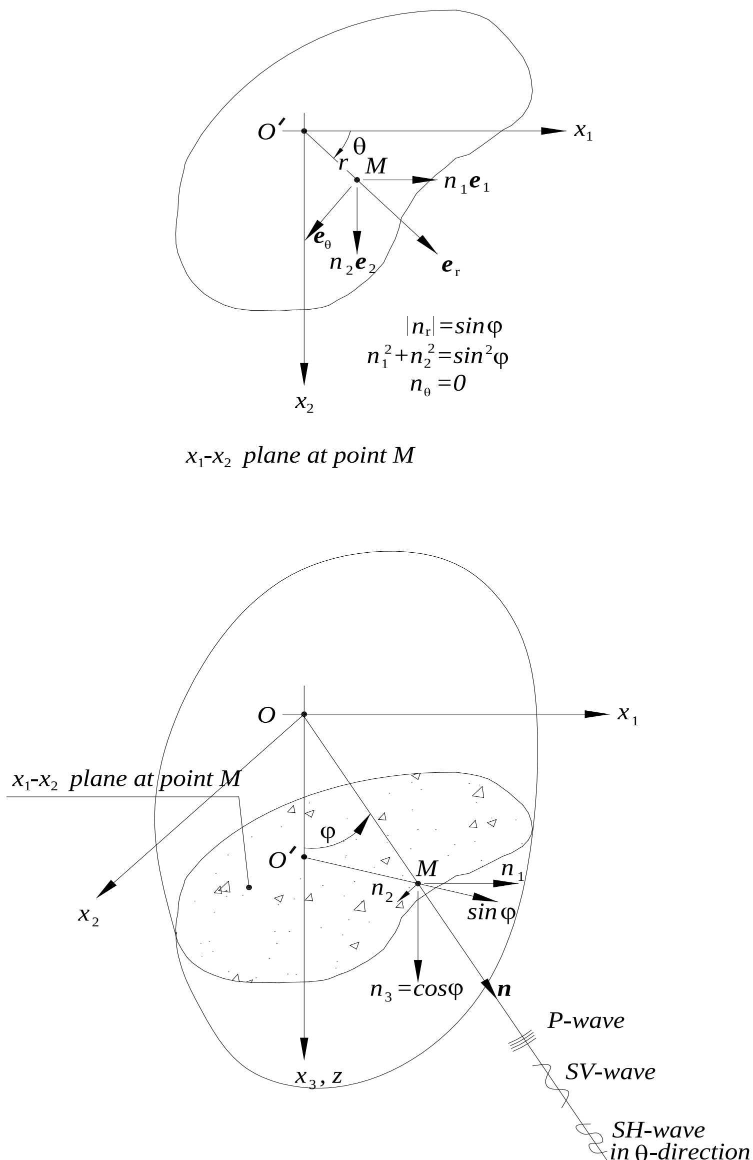

Fig. 1. A schematic illustration for showing *P*-, *SV*- and *SH*-waves traveling in the direction of **n** at a point *M* of a full-space occupied by transversely isotropic material

Substituting Eq. (8) into Eq. (5), results in two independent equations for the amplitude of different waves induced by *F* and *c*, which may be expressed as:

$$\left[\left(s_0^2\sin^2\varphi + \cos^2\varphi\right)S^2 - \frac{s_0^2}{c_0^2}\right]B = 0, \qquad (9)$$

$$\left[b_1S^4 + b_2S^2 + 1\right]A = 0, \qquad (10)$$

in which the coefficients b_1 and b_2 are:

$$b_1 = c_1^2c_2^2\left(\sin^2\varphi + \frac{\cos^2\varphi}{s_1^2}\right)\left(\sin^2\varphi + \frac{\cos^2\varphi}{s_2^2}\right), \qquad (11)$$

$$b_2 = c_2^2 + c_1^2\left(\sin^2\varphi + \frac{\cos^2\varphi}{s_1^2s_2^2}\right)$$

Let's consider Eq. (9) first. One may, from this equation, write $\left(s_0^2\sin^2\varphi+\cos^2\varphi\right)S^2-s_0^2\big/c_0^2=0$, which results in;

$$v_{SH}=S^{-1}=\sqrt{\left(C_{66}\sin^2\varphi+C_{44}\cos^2\varphi\right)\big/\rho} \tag{12}$$

v_{SH} is called *SH*-wave velocity. As observed, it is a function of the direction of propagation in any plane containing x_3-axis. Because of symmetry exists in x_3-plane (x_1-x_2-plane), v_{SH} is independent of the orientation of propagation direction in x_1-x_2-plane. The special cases of this velocity are discussed in the next sub-section. However, here it is proved that the displacements due to χ-function is related to *SH*-wave. To do so, it is observed from the linearity of the Eq. (1) with respect to u_i and Eq. (3) with respect to *F* and *c*, that the displacement field that merely corresponds to the function χ could be expressed as:

$$\mathbf{u}=\left(-\frac{\partial\chi}{\partial x_2},\frac{\partial\chi}{\partial x_1},0\right) \tag{13}$$

from which it is clear that there does not have any component in the x_3- direction. In the other words, it lies in the plane of isotropy $(o:x_1-x_2)$. Moreover, the inner product of the displacement vector and the direction of propagation is zero.

$$\mathbf{u}.\mathbf{n}=\left(-i\,\omega\,s\,n_2\,\chi,\,i\,\omega\,s\,n_1\,\chi,0\right).\left(n_1,\,n_2,\,n_3\right)=0 \tag{14}$$

which shows that it is perpendicular to the wave normal, namely it is polarized in the plane of isotropy. Eqs. (13) and (14) prove that the function χ makes solely *SH*-waves.

Now, we are going to investigate the waves due to the function F. To do so, we consider Eq. (10), which results in the following equation for the slowness:

$$b_1S^4+b_2S^2+1=0 \tag{15}$$

where b_1 and b_2 are given in Eq. (11). Using the formula for the roots of the quadratic equation, one may obtain the following velocities, as the inverse of slowness, for the waves induced by the function F.

$$v_L=\sqrt{\frac{C_{11}\sin^2\varphi+C_{33}\cos^2\varphi+C_{44}+\sqrt{\Omega}}{2\rho}},$$
$$v_T=\sqrt{\frac{C_{11}\sin^2\varphi+C_{33}\cos^2\varphi+C_{44}-\sqrt{\Omega}}{2\rho}}$$
$$\Omega=\begin{bmatrix}\left(C_{11}-C_{44}\right)\sin^2\varphi\\-\left(C_{33}-C_{44}\right)\cos^2\varphi\end{bmatrix}^2+\left(C_{13}+C_{44}\right)^2\sin^2 2\varphi \tag{16}$$

where v_L and v_T are the velocities for longitudinal and transverse waves, respectively. By the definition given earlier, v_L and v_T are respectively the velocities of *P*- and *SV*-waves. These waves are polarized in a plane containing x_3-axis. As seen, these velocities also depend on the direction of propagation in any vertical plane containing x_3-axis. Although v_L is the longitudinal wave velocity, it is a function of C_{44}, which is a shear modulus. Moreover, v_T is transverse wave velocity, however, it is a function of C_{11} and C_{33}, which are axial moduli of elasticity. Because of this, we call

v_L and v_T as quasi-longitudinal and quasi-transverse velocity, respectively.

Special Cases

In this section, we investigate the velocities obtained in previous section for some special directions, and special materials degenerated from transversely isotropic material. By this target, the following cases are considered:

Case (1)

The incident wave along the axis of symmetry $\varphi = 0$ (Figure 1): In this case, the wave travels in vertical direction and the velocities of *SH*-, *SV*- and *P*-waves are given by:

$$v_{SH} = \sqrt{C_{44}/\rho}, \quad v_L = \sqrt{C_{33}/\rho}, \quad v_T = \sqrt{C_{44}/\rho} \tag{17}$$

As seen, the velocity v_L is written in terms of longitudinal stiffness, and v_T depends on shear modulus. Thus, these two waves are respectively pure longitudinal and pure transverse waves.

Case (2)

The incident wave normal to the axis of symmetry $\varphi = 90$ (Figure 1): In this case, the wave propagates in a horizontal direction. The velocities of different waves are written as:

$$v_{SH} = \sqrt{C_{66}/\rho}, \quad v_L = \sqrt{C_{11}/\rho}, \quad v_T = \sqrt{C_{44}/\rho} \tag{18}$$

which shows that v_L is again a function of solely longitudinal stiffness and v_T is a function of shear stiffness only. As observed, the wave is either pure longitudinal or pure transverse wave.

Case (3)

The case where the parameter δ is equal to zero: This case is a constrained transversely isotropic material subjected to the equation:

$$\delta = \left(1 - \frac{1}{s_2^2}\right)\left(\frac{1}{c_1^2} - \frac{1}{c_2^2 s_1^2}\right) = 0 \tag{19}$$

One must reformulate the coefficient b_2 to derive the velocities for this case. Based on Eq. (11), b_2 may be written as:

$$b_2 = \left(c_1^2 + c_2^2\right)\sin^2\varphi + \left(\frac{c_1^2}{s_1^2} + \frac{c_2^2}{s_2^2} + \delta c_1^2 c_2^2\right)\cos^2\varphi \tag{20}$$

which is reduced to:

$$b_2 = \left(c_1^2 + c_2^2\right)\sin^2\varphi + \left(\frac{c_1^2}{s_1^2} + \frac{c_2^2}{s_2^2}\right)\cos^2\varphi \tag{21}$$

for $d = 0$. In this case, W is also changed as:

$$\sqrt{\Omega} = \left(c_1^2 - c_2^2\right)\sin^2\varphi + \left(\frac{c_1^2}{s_1^2} - \frac{c_2^2}{s_2^2}\right)\cos^2\varphi \tag{22}$$

which leads to:

$$v_L^2 = c_1^2\left(\sin^2\varphi + \frac{\cos^2\varphi}{s_1^2}\right), \quad v_T^2 = c_2^2\left(\sin^2\varphi + \frac{\cos^2\varphi}{s_2^2}\right). \tag{23}$$

As it is observed, in this case, the velocities of both longitudinal and transverse waves depend on both shear and compressive/tensile moduli. This matter is inherent in s^2_1 and s_2^2, which exist in the expression of v_L^2 and v_T^2 given in Eq. (23). To go more to the details of these velocities, we, from Eq. (19) , see that d = 0 results in either $s_2^2 = 1$ or $s_1^2 = c_1^2 \big/ c_2^2$ or both of these are valid. Referring to Eq. (7), one may deduce that:

$$\begin{aligned} & s_1^2 s_2^2 = C_{11} \big/ C_{33}\,, \\ & s_1^2 + s_2^2 \\ & = \left(C_{11} C_{33} - C_{13}^2 - 2C_{13} C_{44}\right) \big/ \left(C_{33} C_{44}\right). \end{aligned} \tag{24}$$

Therefore, if $s_2^2 = 1$ then:

$$\begin{aligned} & s_1^2 = C_{11} \big/ C_{33}\,, \\ & (C_{13} + C_{44})^2 = \\ & \left(C_{33} - C_{44}\right)\left(C_{11} - C_{44}\right) \end{aligned} \tag{25}$$

This is the solely case considered by Kirkner (1982), when he formulated forced vibration of a rigid disc attached on the surface of a homogeneous constrained transversely isotropic half-space. It should be mentioned that the forced vertical, rocking and horizontal vibrations of rigid disc attached on the surface of a transversely isotropic half-space or buried in a full-space or placed at an arbitrary depth of a half-space have been rigorously investigated in (see Eskandari-Ghhadi et al., 2010, 2012) with the use of the potential functions given in Eq. (3) and (5), for the general transversely isotropic materials.

If $s_2^2 = 1$ and Eq. (26) holds, then

$$\begin{aligned} & v_L = \sqrt{\left(C_{11} \sin^2 \varphi + C_{33} \cos^2 \varphi\right) \big/ \rho}, \\ & v_T = \sqrt{C_{44} \big/ \rho}. \end{aligned} \tag{26}$$

As seen, the longitudinal wave velocity is independent of shear moduli and in the same way, the transverse wave velocity does not depend on compressive/tensile elasticity coefficients. If $s_1^2 = c_1^2 \big/ c_2^2$ then

$$s_2^2 = C_{44} \big/ C_{33}\,, C_{13} + C_{44} = 0 \tag{27}$$

and the velocities are determined as:

$$\begin{aligned} & v_L = \sqrt{\left(C_{11} \sin^2 \varphi + C_{44} \cos^2 \varphi\right) \big/ \rho}, \\ & v_T = \sqrt{\left(C_{44} \sin^2 \varphi + C_{33} \cos^2 \varphi\right) \big/ \rho} \end{aligned} \tag{28}$$

Here, we see that both the longitudinal and transverse wave velocities depend on a combination of shear and axial moduli. Eventually, when both $s_2^2 = 1$ and $s_1^2 = c_1^2 \big/ c_2^2$ are valid, then

$$C_{13} + C_{44} = 0,\; C_{13} + C_{33} = 0 \tag{29}$$

and

$$\begin{aligned} & v_L = \sqrt{\left(C_{11} \sin^2 \varphi + C_{44} \cos^2 \varphi\right) \big/ \rho}, \\ & v_T = \sqrt{C_{44} \big/ \rho}. \end{aligned} \tag{30}$$

The velocities given in Eq. (30) shows that in this constrained transversely isotropic material, only the longitudinal wave velocity depends on both shear and axial moduli, while the transverse wave velocity does not depend on any axial elasticity coefficient.

Case (4)

The isotropic material: In this case the elasticity coefficients are expressed as:

$$C_{11} = C_{33} = \lambda + 2\mu,\ C_{12} = C_{13} = \lambda, C_{44} = \mu \tag{31}$$

where *1* and *m* are Lame's constants. Substituting Eq. (31) in Eq. (7) one may deduce that $s_1^2 = s_2^2 = 1$ and *d* = 0. From Eq. (26) for the case of *d* = 0 and considering Eqs. (31) and (12), Eq. (32) is obtained.

$$v_{SH} = \sqrt{\mu/\rho}, v_L = \sqrt{(\lambda + 2\mu)/\rho}, v_T = \sqrt{\mu/\rho} \tag{32}$$

RAYLEIGH WAVE

To study the Rayleigh wave in the domains containing transversely isotropic material in detail, we consider, as usual, a homogeneous transversely isotropic half-space in such a way that the planes of isotropy are perpendicular to the x_3-axis. We take the origin of the coordinate system at an arbitrary place on the free surface $(x_3 = 0)$ and the x_3-axis is considered as a pointing vertically downward into the half-space, so that the half-space is represented by $x_3 > 0$. As mentioned earlier, we also assume that the surface $x_3 = 0$ to be stress free, so that the Cauchy stress component S_{13}, S_{23} and S_{33} are zero at $x_3 = 0$. We choose x_1-axis in the direction of wave propagation so that all particles on a line parallel to x_2-axis are equally displaced. Therefore, all the field quantities will be independent from x_2-coordinate.

Following Chadwick (1989), we seek the solution of Eq. (5) in the form of (see Figure 2):

$$F = he^{i\omega\left(S x_1 + S\eta x_3 - t\right)} \tag{33}$$

where *h* is an unknown parameter that must be determined by substituting Eq. (34) into the Eq. (5) and also satisfying the regularity condition:

$$F \to 0 \quad \text{as} \quad x_3 \to \infty \tag{34}$$

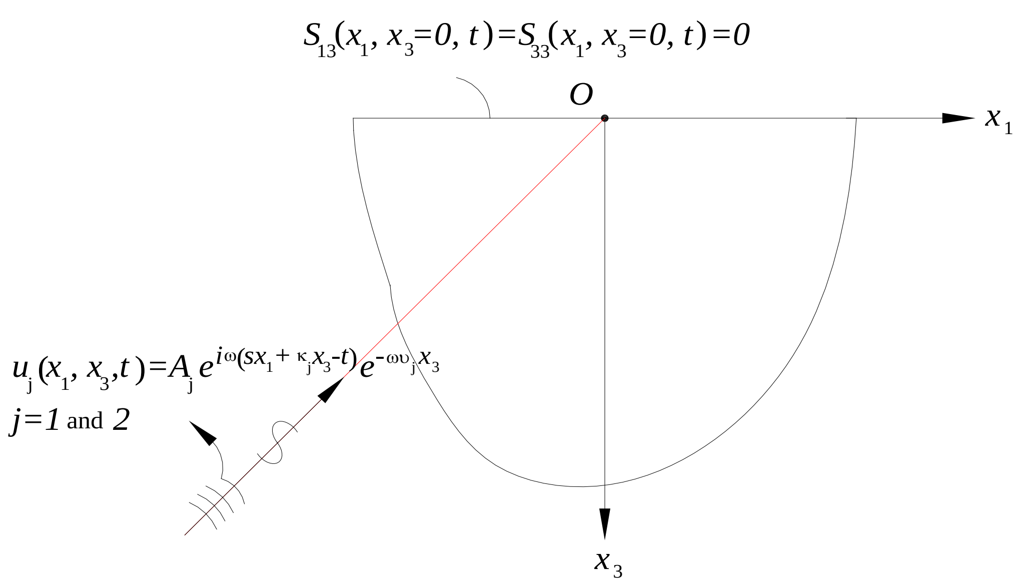

Fig. 2. A homogenous transversely isotropic half-space with an incident wave in $x_1 - x_3$ plane

Substituting Eq. (34) into Eq. (4), it is seen that there does not exist any component for the displacement in x_2 – direction. Replacing Eq. (33) into Eq. (5) results in the following equation for determination of η:

$$J_1\eta^4 + J_2\eta^2 + J_3 = 0 \tag{35}$$

where

$$J_1 = S^4,\ J_2 = \left(s_1^2 + s_2^2\right)S^4 - \left(\frac{s_1^2 s_2^2}{c_1^2} + \frac{1}{c_2^2}\right)S^2,$$
$$J_3 = \left[\left(\frac{1}{c_1^2 c_2^2} + S^4\right) - \left(\frac{1}{c_1^2} + \frac{1}{c_2^2}\right)S^2\right]s_1^2 s_2^2 \tag{36}$$

The solution of Eq. (35) may be determined as:

$$\eta_1 = \frac{1}{S}\eta_1' = \pm\frac{1}{S\sqrt{2}}\sqrt{a_1 - a_2S^2 + \sqrt{a_3S^4 - 2a_4S^2 + a_5}}$$
$$\eta_2 = \frac{1}{S}\eta_2' = \pm\frac{1}{S\sqrt{2}}\sqrt{a_1 - a_2S^2 - \sqrt{a_3S^4 - 2a_4S^2 + a_5}} \tag{37}$$

in which

$$a_1 = \frac{s_1^2 s_2^2}{c_1^2} + \frac{1}{c_2^2},\ a_2 = s_1^2 + s_2^2,$$
$$a_3 = \left(s_1^2 - s_2^2\right)^2$$
$$a_4 = \left(s_1^2 + s_2^2\right)\left(\frac{s_1^2 s_2^2}{c_1^2} + \frac{1}{c_2^2}\right) -$$
$$2s_1^2 s_2^2\left(\frac{1}{c_1^2} + \frac{1}{c_2^2}\right),\ a_5 = \left(\frac{s_1^2 s_2^2}{c_1^2} - \frac{1}{c_2^2}\right)^2 \tag{38}$$

Now, by defining η_j' for $j = 1, 2$ as:

$$\eta_j' = \pm\left(\kappa_j + i\vartheta_j\right),\ \vartheta_j > 0,\quad i = 1,2 \tag{39}$$

Since, we encounter the wave approaching the free surface $(x_3 = 0)$, those *hj* with negative imaginary part must be discarded. As a result, the function *F* is expressed as:

$$F = h_1 e^{-\omega\vartheta_1 x_3} e^{i\omega\left(Sx_1 + \kappa_1 x_3 - t\right)}$$
$$+ h_2 e^{-\omega\vartheta_2 x_3} e^{i\omega\left(Sx_1 + \kappa_2 x_3 - t\right)}. \tag{40}$$

With the use of Eq. (3) and the strain-displacement and the stress-strain relationships, the following relations for the stresses S_{33}, S_{31} are obtained:

$$S_{13} = C_{44}\left\{\begin{array}{l}\left(\alpha_2 - \alpha_3\right)\dfrac{\partial^3 F}{\partial x_1 \partial x_3^2} + \left(1 + \alpha_1\right)\\ \left(\dfrac{\partial^2}{\partial x_1^2} - \dfrac{\rho_0}{1 + \alpha_1}\dfrac{\partial^2}{\partial t^2}\right)\dfrac{\partial F}{\partial x_1}\end{array}\right\},$$
$$S_{33} = \left(C_{33}\left(1 + \alpha_1\right) - C_{13}\alpha_3\right)\frac{\partial^3 F}{\partial x_1^2 \partial x_3}$$
$$+ \alpha_2 C_{33}\frac{\partial^3 F}{\partial x_3^3} - \rho_0 C_{33}\frac{\partial^3 F}{\partial x_3 \partial t^2}, \tag{41}$$

and the stress S_{23} would be zero. From the above relations, the stress free conditions at the surface $x_3 = 0$ results in:

$$\beta_1 h_1 + \beta_2 h_2 = 0,\ \theta_1 h_1 + \theta_2 h_2 = 0 \tag{42}$$

where

$$\beta_j(S) = (\alpha_2 - \alpha_3)\eta_i'^2(S) + (1 + \alpha_1)S^2 - \rho_0.$$
$$\theta_j(S) = \left(\begin{array}{l}\left[C_{33}(1 + \alpha_1) - C_{13}\alpha_3\right]S^2 + \\ C_{33}\alpha_2\eta_j'^2(S) - C_{33}\rho_0\end{array}\right)\eta_i'(S), \tag{43}$$

for *j*=1,2. The determinant of the coefficient matrix must be zero to have a nontrivial solution for Eq. (42). This, results in the following equation for the Rayleigh wave slowness:

$$\theta_1(S)\beta_2(S)-\theta_2(S)\beta_1(S)=0. \tag{44}$$

Substitution of Eq. (43) into Eq. (44) and rationalizing the result, one may arrive at any of the following equations governing the Rayleigh wave slowness:

$$N=a_3S^4-2a_4S^2+a_5=0 \tag{45}$$

or

$$R=\left(f_2g_1-f_1g_2\right)M+\left(f_1g_1-f_2g_2N\right)=0 \tag{46}$$

in which a_3, a_4 and a_5 have been given in Eq. (39) and

$$\begin{aligned}
&M=a_1-a_2S^2\\
&f_1=\left(2\left[C_{33}(1+\alpha_1)-C_{13}\alpha_3\right]S^2+C_{33}\alpha_2M\right)^2+C_{33}^2\alpha_2^2N,\\
&f_2=2C_{33}\alpha_2\left(2\left[C_{33}(1+\alpha_1)-C_{13}\alpha_3\right]+C_{33}\alpha_2M\right)\\
&g_1=\left(2\left[(1+\alpha_1)S^2-\rho_0\right]+(\alpha_2-\alpha_3)M\right)^2+(\alpha_2-\alpha_3)^2N,\\
&g_2=2(\alpha_2-\alpha_3)\left(2\left[(1+\alpha_1)S^2-\rho_0\right]+(\alpha_2-\alpha_3)M\right)
\end{aligned} \tag{47}$$

Eq. (45) is a bi-quadratic equation in terms of S and its roots are $\pm\bar{S}_1$ and $\pm\bar{S}_2$, which are defined later in Eq. (58). However, Eq. (46) is a bi-quartic equation in S^2 and its roots are:

$$S_{1,2}^2=-\frac{O_4}{4O_5}-L^*\pm\sqrt{-4L^2-2k_1+\frac{k_2}{L^*}},\quad S_{3,4}^2=-\frac{O_4}{4O_5}+L^*\pm\sqrt{-4L^2-2k_1+\frac{k_2}{L^*}}. \tag{48}$$

The parameters O_4, O_5, k_1, k_2, L^* and the details of determining these roots are given in Appendix A.

Rayleigh Wave Slowness in Isotropic Material

In this section, we investigate the solution of Eq. (44) for isotropic materials to prove the validity of this equation. For simplicity of computations, and to avoid the emergence of auxiliary roots as a result of rationalizing of Eq. (44) for general transversely isotropic materials (see Eqs. (45) and (46)), Eq. (44) is directly evaluated for isotropic materials, first. It is emphasized that in the case of isotropic material with λ and μ as its Lame's constants, we have $N=\frac{1}{\hat{c}_1^2}-\frac{1}{\hat{c}_2^2}$ which is always non-zero with $\hat{c}_1=(\lambda+2\mu)/\rho$ and $\hat{c}_2=\mu/\rho$. Considering Eq. (31), one may deduce the following relations for isotropic materials

$$\begin{aligned}
&\eta_j'=\sqrt{\frac{1}{\hat{c}_j^2}-S^2},\ j=1,2\\
&\theta_1=\left[2\left(\lambda+\mu\right)S^2-\frac{\lambda+\mu}{\mu}\rho\right]\eta_1',\\
&\theta_2=2\left(\lambda+\mu\right)S^2,\\
&\beta_1=\frac{2\left(\lambda+\mu\right)}{\mu}S^2-\frac{2\left(\lambda+\mu\right)\rho}{\mu\left(\lambda+2\mu\right)},\\
&\beta_2=\frac{2\left(\lambda+\mu\right)}{\mu}S^2-\frac{\lambda+\mu}{\mu}\frac{\rho}{\mu},
\end{aligned} \tag{49}$$

and thus Eq. (44) changes to

$$\left(S^2-\frac{1}{2c_2^2}\right)^3-S^4\left(S^2-\frac{1}{\hat{c}_2^2}\right)\left(S^2-\frac{1}{\hat{c}_1^2}\right)=0, \tag{50}$$

which results in a bi-cubic equation for S and its solution may be given as

$$S_k^2 = -\frac{1}{3a_1'}\left(a_2' + \tau_k D + \frac{\Lambda_0}{\tau_k D}\right), \quad k = 1,2,3 \tag{51}$$

where

$$\begin{aligned}
&\tau_1 = 1, \quad \tau_2 = \frac{-1+\sqrt{3}i}{3}, \\
&\tau_3 = \frac{-1-\sqrt{3}i}{3}, a_2' = \frac{1}{\hat{c}_2^2}\left(\frac{3}{2\hat{c}_2^2} - \frac{1}{\hat{c}_1^2}\right), \\
&D = \sqrt[3]{\frac{\Lambda_1 + \sqrt{\Lambda_1^2 - 4\Lambda_0^3}}{2}}, \\
&\Lambda_0 = a_2'^2 - 3a_1'a_3', \\
&\Lambda_1 = 2a_2'^3 - 9a_1'a_2'a_3' + 27a_1'^2 a_4', \\
&a_1' = \left(\frac{1}{\hat{c}_1^2} - \frac{1}{\hat{c}_2^2}\right), \quad a_3' = -\frac{1}{2\hat{c}_2^6}, \quad a_4' = \frac{1}{16\hat{c}_2^8},
\end{aligned} \tag{52}$$

Considering the relation in Eq. (49) and noticing that $\hat{c}_1 > \hat{c}_2$, it is seen that $\eta_j'(j = 1,2)$ can be either real or pure imaginary number provided that $S < 1/\hat{c}_1$ or $S > 1/\hat{c}_2$, respectively. However, the radiation condition in Eq. (34) implies that $S > 1/\hat{c}_2$ must be hold. It should be mentioned that if $1/\hat{c}_1 < S < 1/\hat{c}_2$ then η_2' will be real number, which violates the radiation condition.

For the case of Poisson material, where $\lambda = \mu$, we have $\hat{c}_1^2 = 3\hat{c}_2^2$ and Eq. (51) accompanied with Eq. (52) results in

$$v_R = \frac{1}{S} = \begin{cases} 2\hat{c}_2, \quad \sqrt{2+2/\sqrt{3}}\ \hat{c}_2, \\ \sqrt{2-2/\sqrt{3}}\ \hat{c}_2 \end{cases} \tag{53}$$

where v_R denotes the Rayleigh wave velocity. The two first roots violate the radiation condition and as a result for Poisson material we have $v_R = \sqrt{2-2/\sqrt{3}}\hat{c}_2$ which is exactly the same as that reported Rayleigh wave velocity in Graff (1991).

Rayleigh Wave Slowness in Transversely Isotropic Materials

After proving the validity of Eq. (44) for isotropic materials, we, in this section, investigate the nature of the roots of Eq. (44) for general transversely isotropic materials. As it has been shown, the solutions of Eq. (44), must satisfy either Eq. (45) or Eq. (46), which have been derived in the course of rationalizing Eq. (44). First, we consider the nature of the roots of Eq. (46), when $N \neq 0$. To do so, we first describe the behavior of $\eta_j'(j = 1,2)$ in Eq. (37), which defines the critical bounds for the values of *S* derived from Eqs. (45) and (46). First, we should mention that $\eta_j'(j = 1,2)$ must be either complex conjugate numbers or pure imaginary numbers to satisfy the radiation condition. The formulation for $\eta_j'(j = 1,2)$ may be concisely written as

$$\begin{aligned}
\eta_1' &= \frac{1}{\sqrt{2}}\sqrt{M + \sqrt{N}}, \\
\eta_2' &= \frac{1}{\sqrt{2}}\sqrt{M - \sqrt{N}}
\end{aligned} \tag{54}$$

where *M* and *N* has been defined in Eqs. (45) and (47), respectively. Since *S* is the reciprocal of velocity, from the physical point of view it must be real and positive. Thus, it, from Eqs. (38) and (24), is clear that both *M* and *N* are real. To have η_1' and η_2' to be complex conjugates, *N* must necessarily be negative, otherwise both η_1' and η_2' become either real numbers or pure imaginary ones. To discuss about the sign of *N* as a function of *S*, one must first solve the equation $N(S) = a_3 S^4 - 2a_4 S^2 + a_5 = 0$ for *S*.

As seen, this equation is a bi-quadratic equation in terms of S and its discriminate may be expressed as

$$\Delta = 4\left(a_4^2 - a_3 a_5\right) = \frac{16 s_1^2 s_2^2 \left(1 - s_1^2\right)\left(1 - s_2^2\right)\left(c_1^2 - c_2^2 s_1^2\right)\left(c_1^2 - c_2^2 s_2^2\right)}{c_1^4 c_2^4}. \tag{55}$$

D, based on Eq. (55), is a real number. Here, assuming that all coefficients in Eq. (38) are nonzero, we are going to investigate the nature of the roots of *N* which depends on the combination of the signs of a_3, a_4 and *D*.

If $\Delta < 0$, then the roots of *N* are all complex conjugates and the sign of *N* is the same as the sign of a_3. From Eq. (38), and the nature of s_1^2 and s_2^2, it is clear that the coefficients a_1 and a_5 are always positive and the other coefficients could be either positive or negative numbers. If s_1^2 and s_2^2 are real, then a_3 would always be positive, and if they become complex conjugates, then a_3 would always be negative. However, when a_3 is negative, then Eq. (55) shows that Δ is necessarily positive, which is not the case. Therefore, if $\Delta < 0$, then *N* would always be positive.

If $\Delta = 0$, then *N* has duplicated roots, which may be either real or pure imaginary depends on the sign of a_4. The sign of *N* is the same as the sign of a_3. It is worth mentioning that $a_3 > 0$ is necessary condition for $\Delta = 0$ (see Eq. (55)). Consequently, similar to previous case, if $\Delta = 0$ then *N* is always positive.

If $\Delta > 0$ then the following cases may occur:

Case (i) $0 < \sqrt{\Delta} < a_4$, which shows that both a_3 and a_4 must be positive, then all four roots are real.

Case (ii) either $0 < a_4 < \sqrt{\Delta}$ or $-\sqrt{\Delta} < a_4 < 0$, which shows that a_4 may be either positive or negative and a_3 is negative, then two roots ($\overline{S}_1$ and $-\overline{S}_1$ in Eq. (58)) are real and the other two ($\overline{S}_2$ and $-\overline{S}_2$ in Eq.(58)) are pure imaginary.

Case (iii) $a_4 < -\sqrt{\Delta}$, which shows that a_4 is negative and a_3 is positive, then all four roots are pure imaginary.

In the Case (i), N is negative provided that

$$-\overline{S}_2 < S < -\overline{S}_1 \quad or \quad \overline{S}_1 < S < \overline{S}_2 \tag{56}$$

and it is positive if

$$\begin{aligned} &S < -\overline{S}_2 < 0 \quad or \\ &-\overline{S}_1 < S < \overline{S}_1 \quad or \\ &0 < \overline{S}_2 < S \end{aligned} \tag{57}$$

where

$$\begin{aligned} \overline{S}_1 &= \sqrt{\left(a_4 - \sqrt{\Delta}\right)\Big/ a_3}, \\ \overline{S}_2 &= \sqrt{\left(a_4 + \sqrt{\Delta}\right)\Big/ a_3}. \end{aligned} \tag{58}$$

However, S, based on its definition, must be positive. Thus, the conditions of Eqs. (56) and (57) could be modified as:

$$\overline{S}_1 < S < \overline{S}_2 \tag{59}$$

for N to be always negative, and

$$0 < S < \overline{S}_1 \quad or \quad 0 < \overline{S}_2 < S \tag{60}$$

for N to be always positive.

Based on the conditions given in Case (ii) and Eq. (58), it is seen that when Case (ii) occurs, then $\overline{S}_2$ is always a pure imaginary number and $\overline{S}_1$ is a real number. Therefore, N would be positive provided that

$$0 < S < \overline{S}_1 \tag{61}$$

and it is negative if

$$S > \overline{S}_1 \tag{62}$$

In the Case (iii), the sign of *N* is the same as a_3 which is always positive.

Since *M* is a real number, if *N* become positive then both η_1' and η_2' must be pure imaginary numbers in order that the radiation condition to be satisfied. The necessary and sufficient condition to have both η_1' and η_2' to be pure imaginary numbers is that

$$M < -\sqrt{N} \tag{63}$$

which guarantees that $M_1 < 0$. However, *M* is a negative number if and only if

$$s_1^2 + s_2^2 > 0 \quad \text{and} \quad S > S^* \tag{64}$$

where

$$S^* = \sqrt{\left(\left(s_1^2 s_2^2\right)\big/c_1^2 + 1\big/c_2^2\right)\Big/\left(s_1^2 + s_2^2\right)} \tag{65}$$

If $s_1^2 + s_2^2 < 0$, then *M* has imaginary roots and it is always positive. To have $M < -\sqrt{N}$ it is necessary that

$$0 < S < \min\left(\frac{1}{c_1}, \frac{1}{c_2}\right) \quad \text{or} \quad S > \max\left(\frac{1}{c_1}, \frac{1}{c_2}\right) \tag{66}$$

which depends on the condition that $c_1 > c_2$ or $c_1 < c_2$. Contrary to isotropic materials, where $\hat{c}_1 > \hat{c}_2$, in transversely isotropic media, the order of wave speeds depend on some combinations of elasticity coefficients (see Chadwick, 1989). As a result it can be said that two kinds of Rayleigh waves may propagate in transversely isotropic half-space. If both η_1' and η_2' are pure imaginary numbers then there exists a kind of Rayleigh wave, which attenuates continuously in exponential manner with depth. We call this kind of Rayleigh wave as the first kind. If η_1' and η_2' are complex conjugate numbers with positive imaginary parts, then there exists a wave, which propagates periodically in *z*-direction, however it is also attenuated exponentially with depth. We call this as the second kind of Rayleigh wave. It is clear that the first kind of Rayleigh wave is same as the one that propagates in isotropic half-spaces.

According to the above discussion, the nature of Rayleigh wave based on the sign of Δ, a_3 and a_4 could be concisely categorized as:

Case (a): The Rayleigh wave is of the first kind provided that $s_1^2 + s_2^2 > 0$ and any of the conditions

$$\Delta > 0 \quad \text{and} \quad \left\{\begin{array}{l} 0 < \sqrt{\Delta} < a_4 \\ 0 < S < \overline{S}_1 \quad or \quad S > \overline{S}_2 \\ S > S^* \\ 0 < S < \min\left(1/c_1, 1/c_2\right) \ \ or \ S > \\ \max\left(1/c_1, 1/c_2\right) \end{array}\right. \tag{67}$$

$$\Delta > 0 \quad \text{and} \quad \begin{cases} \sqrt{\Delta} > a_4 > 0 \quad or \quad -\sqrt{\Delta} < a_4 < 0 \\ 0 < S < \overline{S}_1 \\ S > S^* \\ 0 < S < \min\left(1/c_1, 1/c_2\right) \quad or \quad S > \\ \max\left(1/c_1, 1/c_2\right) \end{cases} \tag{68}$$

$$\Delta > 0 \quad \text{and} \quad \begin{cases} a_4 < -\sqrt{\Delta} \\ S > S^* \\ 0 < S < \min\left(1/c_1, 1/c_2\right) \quad or \quad S > \\ \max\left(1/c_1, 1/c_2\right) \end{cases} \tag{69}$$

$$\left(\Delta < 0 \quad \text{or} \quad \Delta = 0\right) \quad \text{and} \begin{cases} S > S^* \\ 0 < S < \min\left(1/c_1, 1/c_2\right) \quad or \\ S > \max\left(1/c_1, 1/c_2\right) \end{cases} \tag{70}$$

with *S* being the root of Eq. (46) is hold.

Case (b): The Rayleigh wave is of the second kind provided that either of the following two conditions is held.

$$\Delta > 0 \quad \text{and} \quad \begin{cases} 0 < \sqrt{\Delta} < a_4 \\ \overline{S}_1 < S < \overline{S}_2 \end{cases} \tag{71}$$

$$\Delta > 0 \quad \text{and} \quad \begin{cases} 0 < a_4 < \sqrt{\Delta} \quad \text{or} \\ -\sqrt{\Delta} < a_4 < 0 \\ S > \overline{S}_1 \end{cases} \tag{72}$$

According to Eq. (54), the equation (45), which itself has been derived from rationalizing of the Eq. (44), results in

$$\eta_1' = \eta_2' = \eta' = \frac{1}{2}\sqrt{M} \tag{73}$$

and therefore Eq. (40) degenerates to

$$F = he^{i\omega\left(Sx_1 + \eta' x_3 - t\right)} \tag{74}$$

Substituting Eq. (74) into Eq. (41) results in

$$\beta h = 0, \quad \theta h = 0 \tag{75}$$

where

$$\begin{aligned} &\beta(S) = (\alpha_2 - \alpha_3)\eta'^2 + \\ &(1 + \alpha_1)S^2 - \rho_0. \\ &\theta(S) = \\ &\left(\begin{matrix} \left[C_{33}(1 + \alpha_1) - C_{13}\alpha_3\right]S^2 \\ +C_{33}\alpha_2\eta'^2 - C_{33}\rho_0 \end{matrix}\right)\eta', \end{aligned} \tag{76}$$

One possible solution for Eq. (75) is $h = 0$, which shows that the Rayleigh wave is vanished. If $h \neq 0$, then the following two equations must simultaneously be satisfied:

$$\beta = 0, \quad \theta = 0 \tag{77}$$

Eq. (77) results in two different values for S, which means that there is not a unique solution to satisfy both of Eq. (77), simultaneously. The above terminologies demonstrate that $N = 0$ is physically absurd.

The Nature of Rayleigh Wave in Terms of Elasticity Coefficients

Having determined the possible nature of the Rayleigh wave in transversely isotropic half-space in terms of D, a_3 and a_4 (see Eqs. (67) to (72)), in this section, the sign of D, a_3 and a_4 are investigated based on the combinations of elasticity coefficients and determine the conditions for occurrence of two possible Rayleigh waves in terms of elastic moduli of transversely isotropic materials. At first, the situation of $\Delta > \mathbf{0}$ is considered, which includes Cases (i), (ii) and (iii).

When $\Delta > 0$, one of the following two conditions must be held

$$C_{13} > \sqrt{(C_{44} - C_{11})(C_{44} - C_{33})} - C_{44} \ or \ C_{13} < -\sqrt{(C_{44} - C_{11})(C_{44} - C_{33})} - C_{44}. \tag{78}$$

In Eq. (78), either $C_{44} > \max(C_{11}, C_{33})$ or $C_{44} < \min(C_{11}, C_{33})$ in order that the right hand sides of these inequalities do not contain a complex number. In the Case (i), both a_3 and a_4 are positive. However, a_3 is positive if one of the following conditions is held.

$$\begin{aligned} &C_{13} > \sqrt{C_{11}C_{33}}, \\ &C_{13} < -\sqrt{C_{11}C_{33}} - 2C_{44} \\ &\sqrt{C_{11}C_{33}} - 2C_{44} < C_{13} < -\sqrt{C_{11}C_{33}}, \\ &\text{and} \quad C_{44}^2 > C_{11}C_{33} \\ &-\sqrt{C_{11}C_{33}} < C_{13} < \sqrt{C_{11}C_{33}} - 2C_{44}, \\ &\text{and} \quad C_{44}^2 < C_{11}C_{33} \end{aligned} \tag{79}$$

and finally $a_4 > 0$ if

$$-\sqrt{\frac{(C_{44} - C_{33})(C_{44}^2 - C_{11}C_{33})}{C_{44} + C_{33}}} - C_{44} < C_{13} < \sqrt{\frac{(C_{44} - C_{33})(C_{44}^2 - C_{11}C_{33})}{C_{44} + C_{33}}} - C_{44} \tag{80}$$

In inequalities of Eq. (80), either $C_{44} > \max(C_{11}, C_{33})$ or $C_{44} < \min(C_{11}, C_{33})$ to have real numbers in the whole inequalities. By the combination of all conditions expressed in Eqs. (78) to (80) and considering the relations, one may derive the necessary and sufficient condition for Case (i) to be held as:

$$\begin{aligned} &C_{12} > 2C_{13}^2 \big/ C_{33} - C_{11} \quad \text{and} \\ &C_{44} < C_{11} < C_{33} \end{aligned} \tag{81}$$

in which C_{13} must satisfy one of the following inequalities.

$$\begin{aligned} &\sqrt{(C_{44} - C_{11})(C_{44} - C_{33})} - C_{44} < C_{13} < \sqrt{C_{11}C_{33}} - 2C_{44} \\ &-\sqrt{C_{11}C_{33}} < C_{13} < -C_{44} - \sqrt{(C_{44} - C_{11})(C_{44} - C_{33})} \end{aligned} \tag{82}$$

For the Case (ii), a_3 must be a negative number, which leads to one of the following combined conditions.

$$\begin{aligned} &C_{44} > \max\left(C_{11}, C_{33}\right) \quad \text{and} \\ &\begin{cases} -\sqrt{C_{11}C_{33}} - 2C_{44} < C_{13} < \sqrt{C_{11}C_{33}} - 2C_{44} \\ \qquad\qquad \text{or} \\ \qquad -\sqrt{C_{11}C_{33}} < C_{13} < \sqrt{C_{11}C_{33}} \end{cases} \\ &C_{44} < \ \min\left(C_{11}, C_{33}\right) \quad \text{and} \\ &\begin{cases} -\sqrt{C_{11}C_{33}} - 2C_{44} < C_{13} < -\sqrt{C_{11}C_{33}} \\ \qquad\qquad \text{or} \\ \sqrt{C_{11}C_{33}} - 2C_{44} < C_{13} < \sqrt{C_{11}C_{33}} \end{cases} \end{aligned} \tag{83}$$

On the other hand, in this case a_4 could be either positive or negative, which means that it can be any real number. Considering inequalities of Eqs. (83), (78) and (2), one may derive the necessary and sufficient condition for Case (ii) to be held as:

$$C_{12} > 2\frac{C_{13}^2}{C_{33}} - C_{11} \tag{84}$$

in which C_{13} must satisfy the following conditions.

$$-\sqrt{C_{11}C_{33}} < C_{13} < \sqrt{C_{11}C_{33}}, \tag{85}$$

if $C_{44} > \max\left(C_{11}, C_{33}\right)$, and

$$\sqrt{C_{11}C_{33}} - 2C_{44} < C_{13} < \sqrt{C_{11}C_{33}} \tag{86}$$

if $C_{44} < \min\left(C_{11}, C_{33}\right)$. In the Case (iii), a_4 is negative while a_3 is positive. From Eq. (80), it is seen that a_4 is negative if

$$\begin{gathered} C_{13} < -\sqrt{\frac{\left(C_{44}-C_{33}\right)(C_{44}^2 - C_{11}C_{33})}{C_{44}+C_{33}}} - C_{44} \quad \text{or} \\ C_{13} > \sqrt{\frac{\left(C_{44}-C_{33}\right)(C_{44}^2 - C_{11}C_{33})}{C_{44}+C_{33}}} - C_{44} \end{gathered} \tag{87}$$

Considering inequalities of Eqs. (87), (79), (78) and (2), one may derive the necessary and sufficient conditions for this case as:

$$C_{12} > 2\frac{C_{13}^2}{C_{33}} - C_{11} \quad \text{and} \quad C_{44} < C_{33} < C_{11} \tag{88}$$

in which C_{13} must satisfy either of the following inequalities.

$$\begin{gathered} \sqrt{(C_{44}-C_{11})(C_{44}-C_{33})} - \\ C_{44} < C_{13} < \sqrt{C_{11}C_{33}} - 2C_{44} \\ -\sqrt{C_{11}C_{33}} < C_{13} < -C_{44} - \\ \sqrt{(C_{44}-C_{11})(C_{44}-C_{33})} \end{gathered} \tag{89}$$

Based on the inequalities (89) and (82), it is seen that the conditions for the Case (i) and (iii) are similar, except that in the Case (i), $C_{11} < C_{33}$ and in the Case (iii) $C_{11} > C_{33}$.

Now, we consider the case of $\Delta < 0$ which results in

$$\begin{gathered} -\sqrt{(C_{44}-C_{11})(C_{44}-C_{33})} - \\ C_{44} < C_{13} < \\ \sqrt{(C_{44}-C_{11})(C_{44}-C_{33})} - C_{44} \end{gathered} \tag{90}$$

where C_{44} is either larger than $\max(C_{11}, C_{33})$ or smaller than $\min(C_{11}, C_{33})$ to have only real numbers in the inequalities of Eq. (95). Moreover, as it has been stated earlier, in this case a_3 is positive. Considering inequalities in Eqs. (90), (79) and (2), one may derive the following necessary and sufficient conditions for this case.

$$\begin{gathered} C_{12} > 2\frac{C_{13}^2}{C_{33}} - C_{11} \quad \text{and} \\ C_{44} < \min(C_{33}, C_{11}) \end{gathered} \tag{91}$$

in which C_{13} must satisfies inequalities of Eq. (90).

Eventually, we go through the case of $\Delta = 0$. This situation results in

$$\begin{gathered} C_{13} = -\sqrt{(C_{44}-C_{11})(C_{44}-C_{33})} - C_{44} \text{ or} \\ C_{13} = \sqrt{(C_{44}-C_{11})(C_{44}-C_{33})} - C_{44} \end{gathered} \tag{92}$$

while as before the condition for C_{44} is either $C_{44} > \max(C_{11}, C_{33})$ or $C_{44} < \min(C_{11}, C_{33})$. Similar to the previous case, here a_3 is also positive. Considering inequalities in Eqs. (92) and (79) and (2), we have

$$C_{12} > 2C_{13}^2 \big/ C_{33} - C_{11} \text{ and } \\ C_{44} < \min(C_{11}, C_{33}) \tag{93}$$

in which C_{13} must satisfy Eq. (92). From the previous discussions, we see that in the situations of $N > \mathbf{0}$, the Rayleigh wave is of the first kind. In this case, in addition to the constrains for D, a_3 and a_4, another auxiliary conditions should also be satisfied, which are (see Eqs. (67) to (70)).

$$\begin{cases} -C_{44} - \sqrt{C_{11}C_{33} + C_{44}^2} < C_{13} \\ < -C_{44} + \sqrt{C_{11}C_{33} + C_{44}^2} \\ \qquad S > S^* \\ 0 < S < \min\left(1/c_1, 1/c_2\right) \ or \\ S > \max\left(1/c_1, 1/c_2\right) \end{cases} \tag{94}$$

The first relation in inequalities of Eq. (94) arises from the condition $s_1^2 + s_2^2 > 0$.

Collecting the above results, the conditions for <u>the Rayleigh wave of the first kind</u> may be expressed as:

If inequalities in Eq. (67) are held, then by considering the inequalities in Eqs. (79) and (78), one may verify that:

$$\min\left(1/c_1, 1/c_2\right) < S^* < \bar{S}_2 \text{ and }, \\ \bar{S}_{1,2} > \max\left(1/c_1, 1/c_2\right) \tag{95}$$

from which the inequalities of Eq. (67) may be modified as:

$$C_{44} < C_{11} < C_{33} \qquad \text{and} \\ \begin{cases} \sqrt{(C_{44} - C_{11})(C_{44} - C_{33})} - \\ C_{44} < C_{13} < \sqrt{C_{11}C_{33}} - 2C_{44} \\ \qquad \text{or} \\ -\sqrt{C_{11}C_{33}} < C_{13} < -C_{44} - \\ \sqrt{(C_{44} - C_{11})(C_{44} - C_{33})} \end{cases} \tag{96}$$

where S^* is given in Eq. (70), and S must satisfy

$$\begin{cases} \max\left[S^*, 1/c_2\right] < S < \bar{S}_1 \quad or \quad S > \bar{S}_2 & if \ S^* < \bar{S}_1 \\ \qquad S > \bar{S}_2 & if \ S^* > \bar{S}_1 \end{cases} \tag{97}$$

If Eq. (68) is held, then

$$\bar{S}_1 > \max(1/c_1, 1/c_2), \\ S^* > \min(1/c_1, 1/c_2) \tag{98}$$

and the Eq. (68) may be modified as:

$$\begin{cases} -\sqrt{C_{11}C_{33}} \\ < C_{13} < -C_{44} + & C_{44} > \max(C_{11}, C_{33}) \\ \sqrt{C_{11}C_{33} + C_{44}^2} \\ \sqrt{C_{11}C_{33}} \\ -2C_{44} < C_{13} < -C_{44} + & C_{44} < \max(C_{11}, C_{33}) \\ \sqrt{C_{11}C_{33} + C_{44}^2} \end{cases} \tag{99}$$

in which S should satisfy

$$\begin{cases} \max\left[S^*, \max\left(1/c_1, 1/c_2\right)\right] & if \ \ S^* < \bar{S}_1 \\ < S < \bar{S}_1 \\ \ S > \max \\ \ \left[S^*, \max\left(1/c_1, 1/c_2\right)\right] & if \ S^* > \bar{S}_1 \end{cases} \tag{100}$$

If Eq. (69) is held, then

$$\min\left(1/c_1, 1/c_2\right) < S^* < \max\left(1/c_1, 1/c_2\right) \tag{101}$$

which modifies Eq. (69) as:

$$\begin{cases} \sqrt{(C_{44}-C_{11})(C_{44}-C_{33})}- \\ C_{44} < C_{13} < \sqrt{C_{11}C_{33}} - 2C_{44} \\ \qquad \text{or} \\ -\sqrt{C_{11}C_{33}} < C_{13} < -C_{44} - \\ \sqrt{(C_{44}-C_{11})(C_{44}-C_{33})} \end{cases} \text{and} \quad C_{44} < C_{33} < C_{11} \tag{102}$$

where

$$S > \max\left(1/c_1, 1/c_2\right) \tag{103}$$

If Eq. (70) is held, then S^* lays in the range given in Eq. (101), and in this case we have:

$$-\sqrt{(C_{44}-C_{11})(C_{44}-C_{33})}- \\ C_{44} \le C_{13} \le \\ \sqrt{(C_{44}-C_{11})(C_{44}-C_{33})} - C_{44} \tag{104}$$

in which S should be larger than the maximum of $\left(1/c_1, 1/c_2\right)$ as indicated in (103). Moreover, in this case if $\Delta < 0$ then $C_{44} < \min(C_{33}, C_{11})$ and if $\Delta = 0$ then $C_{11} \neq C_{33}$ and the condition for C_{44} is $C_{44} < \min(C_{11}, C_{33})$.

Finally, the conditions for the Rayleigh wave of the second kind may be expressed as either

$$C_{44} < C_{11} < C_{33} \quad \text{and} \\ \begin{cases} \sqrt{(C_{44}-C_{11})(C_{44}-C_{33})}- \\ C_{44} < C_{13} < \sqrt{C_{11}C_{33}} - 2C_{44} \\ \qquad \text{or} \\ -\sqrt{C_{11}C_{33}} < C_{13} < -C_{44} - \\ \sqrt{(C_{44}-C_{11})(C_{44}-C_{33})} \end{cases} \tag{105}$$

in which *S* should lay in the range of

$$\bar{S}_1 < S < \bar{S}_2 \tag{106}$$

or

$$C_{44} < C_{33} < C_{11} \quad and \\ \begin{cases} -\sqrt{C_{11}C_{33}} < C_{13} < \sqrt{C_{11}C_{33}} \\ and \;\; C_{44} > \max\left(C_{11}, C_{33}\right) \\ \qquad or \\ \sqrt{C_{11}C_{33}} - 2C_{44} < C_{13} < \sqrt{C_{11}C_{33}} \\ and \;\; C_{44} < \min\left(C_{11}, C_{33}\right) \end{cases} \tag{107}$$

in which *S* is larger than $\bar{S}_1$

$$S > \bar{S}_1 \tag{108}$$

In all cases, the elasticity coefficients must also satisfy Eq. (2).

NUMERICAL RESULTS

In this section, we present the numerical evaluations of body and Rayleigh wave velocities for some synthetic transversely isotropic materials. The normalized elasticity coefficients of the selected materials are listed in Table 1. They are normalized according to the elasticity parameters of isotropic material expressed in Table 2 (the coefficients of transversely isotropic materials are divided by corresponding isotropic one). The elastic parameters of the synthetic materials have been selected in such way that the conditions for positive definiteness of strain energy function according to Eq. (2) or are satisfied. Moreover to grasp the two different kind of Rayleigh wave propagation, and to clarify the characteristic nature of the roots of Rayleigh wave slowness functions in terms of elastic parameters, the restrictions placed on which according to inequalities of Eqs. (81), (82), (84), (85), (86), (88), (89), (90), (96), (99), (102), (105) and (107) are also considered. Figures 3-5 show respectively a graphical representation for quasi-longitudinal, quasi-transverse and *SH*- waves in terms of incidence angle *j* for the synthetic materials listed in Table 1. Material 1 is isotropic; therefore, as it is expected and could be seen in the figures, its body wave velocities are independent from the direction of propagation. The maximum longitudinal wave velocity belongs to Material 9 and its minimum occurs for Material 2. As could be seen in the Figure 3, the longitudinal wave velocities of Materials 5 and 6 are nearly the same. This also happens for Materials 7, 8 and 10. The transverse wave velocities for Materials 5, 6, 7, 8 and 10 are approximately equal and as it is observed in figure 4, the transverse waves of these materials are very small depend on the angle of incidence. Table 3 shows the Rayleigh wave slowness for different synthetic materials listed in Table 2. In this table, the nature of s_1^2, s_2^2, Δ, a_3, a_4 and the type of Rayleigh wave has also been included. The Rayleigh wave slowness for each material is derived from zeros of the polynomial function, $R = 0$, given in Eq. (46) or equivalently in Eq. (A1) (see Appendix). Since the equation $R = 0$ has been derived by rationalizing of Eq. (44), some auxiliary roots may be produced which have no relevance to Rayleigh wave slowness. The correct root that is related to Rayleigh wave slowness is determined with the use of the critical bounds on *S* presented in Eqs. (97), (100), (103), (106) and (108) depend on the conditions of Δ, a_3, and a_4 as stated in previous section.

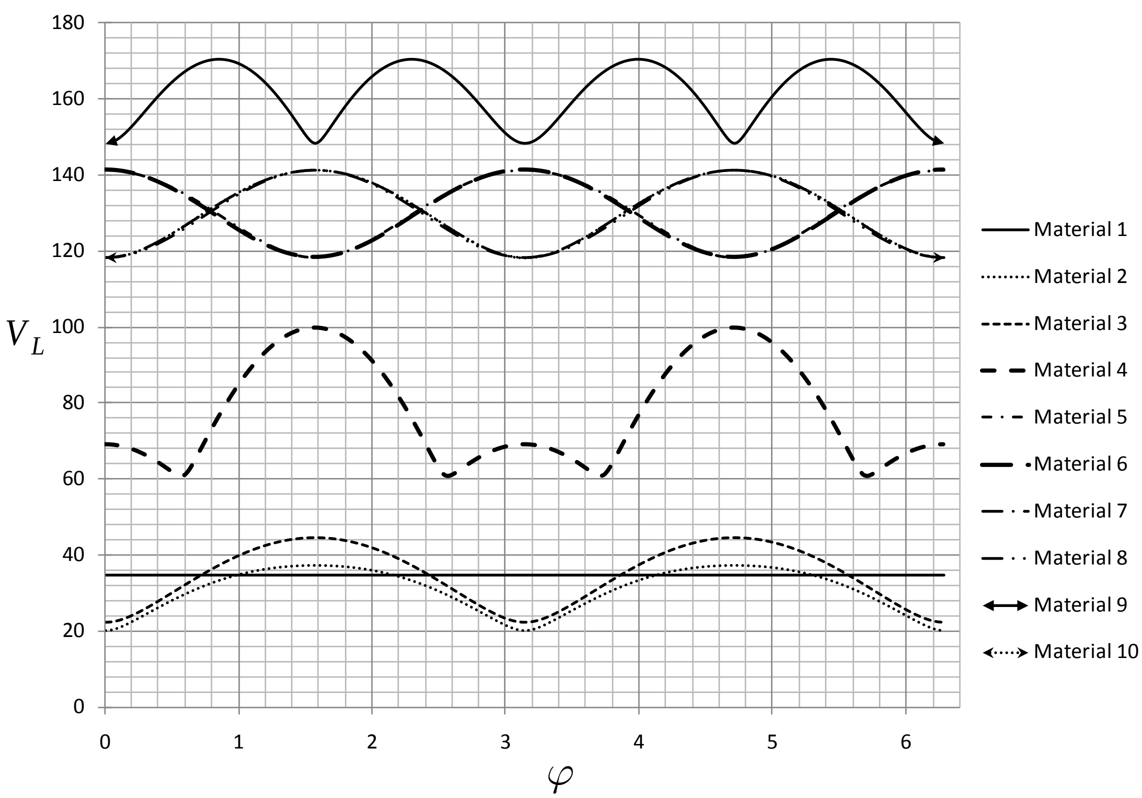

Fig. 3. Comparison of quasi-longitudinal wave velocity as a function of incidence angle

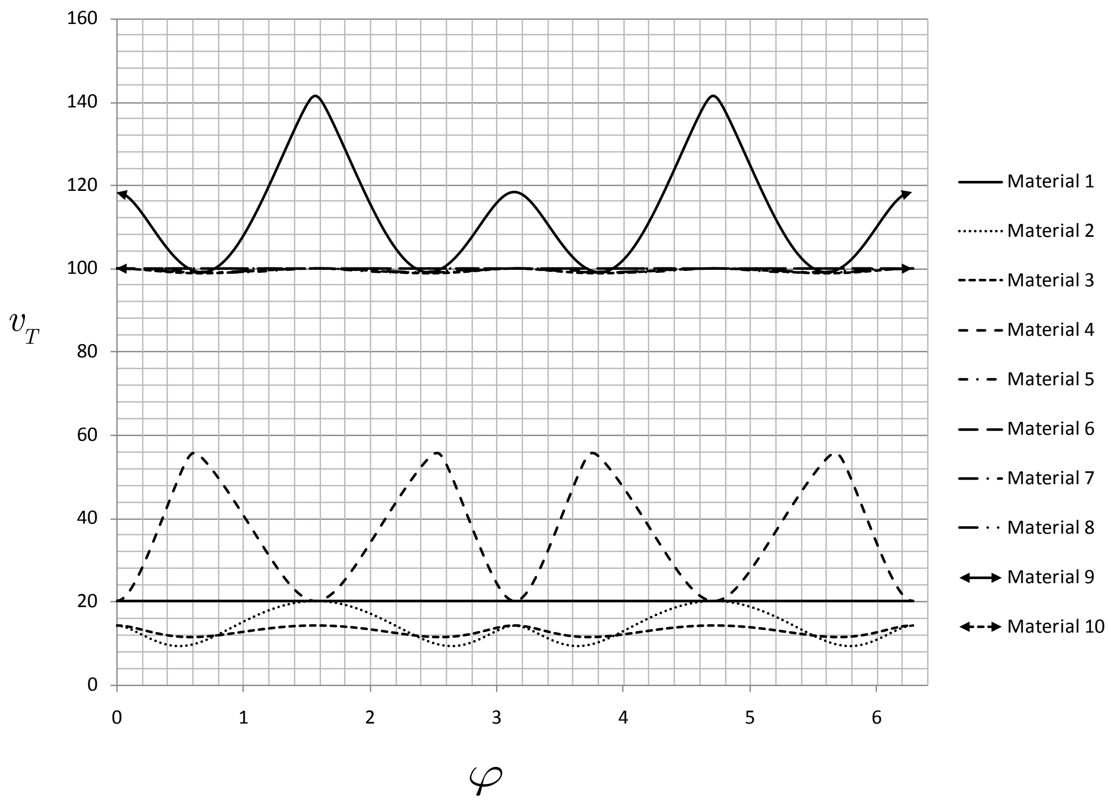

Fig. 4. Comparison of quasi-transverse wave velocity as a function of incident angle

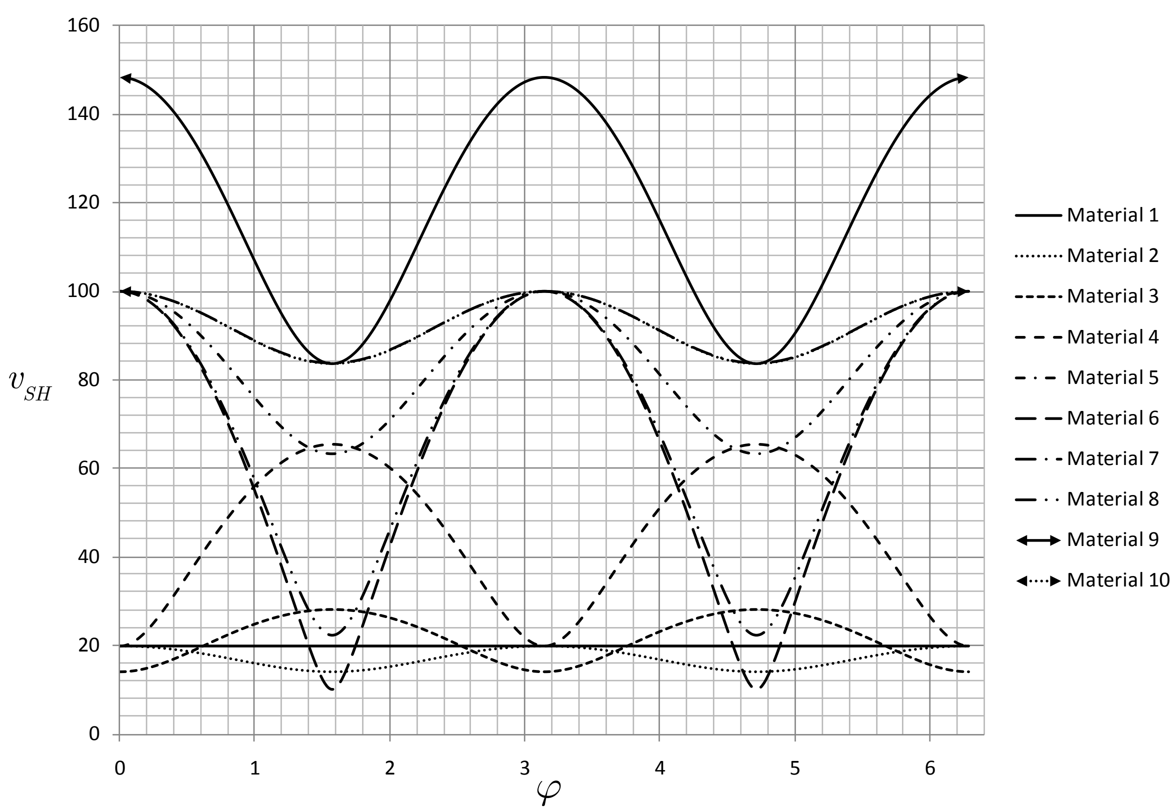

Fig. 5. Comparison of SH-wave velocity as a function of incident angle

CONCLUSIONS

With the use of potential function method as a powerful tool for solving a system of partial differential equations, the harmonic wave propagation in transversely isotropic media has been investigated in details. The velocities of the transverse and longitudinal waves have been derived explicitly and it has been shown that the shear wave velocity, in some directions of propagation, may depend on some elasticity constants other than the shear modulus. In the same way, the longitudinal wave velocity may depend on shear moduli for some propagation directions. Since, one of the potential functions used in this paper produces solely the displacements of horizontally polarized shear wave, the *SH*-wave velocity has been investigated in an elegant manner. The Rayleigh wave velocity has been rigorously investigated, where two different kinds for this wave has been recognized. Some transversely isotropic materials with different natures in wave propagation point of view have been produced and the velocities of different waves have been given.

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APPENDIX

The details for Rayleigh wave slowness:

Eq. (46) is expressed as a bi-quadratic polynomial in S^2:

$$R = O_5 S^8 + O_4 S^6 + O_3 S^4 + O_2 S^2 + O_1 = 0 \tag{A1}$$

where

$$\begin{aligned} O_1 &= \left(a_1^2 - a_5\right)\left[\left(a_1^2 - a_5\right)p_2^2 - 4p_3^2\right]q_1^2 + 8\left(a_1^2 - a_5\right)p_2\left(a_1 p_2 + 2p_3\right)q_1 q_3 \\ &\quad + 4\left[\left(3a_1^2 + a_5\right)p_2^2 + 8a_1 p_2 p_3 + 4p_3^2\right]q_3^2. \end{aligned} \tag{A2}$$

$$\begin{aligned} O_2 &= -4\Big\{\Big[\Big(\left(a_1 a_2 - a_4\right)\left(a_1^2 - a_5\right)p_2^2 + 2\left(a_1^2 - a_5\right)p_1 p_3 + 2\left(a_4 - a_1 a_2\right)p_3^2\Big)q_1 \\ &- \left(a_1^2 - a_5\right)p_2\left(a_1 p_2 + 2p_3\right)q_2\Big] - 2\Big[a_5 p_2\left(-2p_1 q_1 + a_2 p_2 q_1 + p_2 q_2\right) + 4p_3\left(a_4 p_2 q_1 + p_3 q_2\right) \\ &+ 2a_1 p_2\left(a_4 p_2 q_1 - 2a_2 p_3 q_1 + 4p_3 q_2\right) + a_1^2 p_2\left(2p_1 q_1 - 3p_2 a_2 q_1 + 3p_2 q_2\right)\Big]q_3 \\ &+ 2\left[a_4 p_2^2 - 4a_1 p_1 p_2 + 3a_1 a_2 p_2^2 - 4p_1 p_3 + 4a_2 p_2 p_3\right]q_3^3\Big\} \end{aligned} \tag{A3}$$

$$\begin{aligned} O_3 &= 2\Big\{a_5\left[\left(2p_1^2 + \left(a_3 - a_2^2\right)p_2^2\right)q_1^2 + 4p_2\left(-2p_1 + a_2 p_2\right)q_1 q_2 + 4p_2^2 q_2^2\right] \\ &+ a_1^2\left[-2p_1^2 q_1^2 + 8p_1 q_1 p_2 q_2 + p_2^2\left(\left(3a_2^2 - a_3\right)q_1^2 - 12a_2 q_1 q_2 + 6q_2^2\right)\right] \\ &+ 2\Big[a_4^2 p_2^2 q_1^2 - 4a_4 p_3 q_1\left(p_1 q_1 - 2p_2 q_2\right) + p_3^2\left(\left(a_3 - a_2^2\right)q_1^2 + 4q_2^2\right) \\ &- 4a_4 p_2\left(-2p_1 q_1 + a_2 p_2 q_1 + p_2 q_2\right)q_3 + 4p_3\left(a_2^2 p_2 q_1 - a_3 p_2 q_1 + 4p_1 q_2 - 4a_2 p_2 q_2\right)q_3 \\ &+ \left(4p_1^2 - 8a_2 p_1 p_2 + 3a_2^2 p_2^2 + a_3 p_2^2\right)q_3^2\Big] + 4a_1\Big[-a_2 q_1\left(a_4 p_2^2 q_1 - 2p_1 q_1 p_3 + 4p_2 q_2 p_3\right) + 3a_2^2 p_2^2 q_1 q_3 \\ &- 2a_2 p_2 q_3\left(2p_1 q_1 + 3p_2 q_2\right) + p_2\left(2a_4 p_2 q_1 q_2 + 4p_3 q_2^2 - a_3 q_3 p_2 q_1 + 8p_1 q_2 q_3\right)\Big]\Big\} \end{aligned} \tag{A4}$$

$$\begin{aligned} O_4 &= -4\Big\{a_4\left[\left(2p_1^2 + a_3 p_2^2 - a_2^2 p_2^2\right)q_1^2 + 4a_2 p_2^2 q_1 q_2 - 8p_1 p_2 q_1 q_2\right] \\ &+ 2p_3\left[a_2^2 p_1 q_1^2 - a_3 p_1 q_1^2 - 2a_2^2 p_2 q_1 q_2 + 2a_3 p_2 q_1 q_2 - 4p_1 q_2^2 + 4a_2 p_2 q_2^2\right] \\ &+ a_1\left[a_2 q_1^2\left(-2p_1^2 + a_2^2 p_2^2 - a_3 p_2^2\right) + 2p_2 q_1 q_2\left(4a_2 p_1 - 3a_2^2 p_2 + a_3 p_2\right) + 2p_2 q_2^2\left(-4p_1 + 3a_2 p_2\right)\right] \\ &+ 2q_3\left[\left(a_2^2 - a_3\right)\left(-2p_1 + a_2 p_2\right)p_2 q_1 - q_2\left(4p_1^2 - 8a_2 p_1 p_2 + 3a_2^2 p_2^2 + a_3 p_2^2\right)\right]\Big\} \end{aligned} \tag{A5}$$

$$\begin{aligned} O_5 &= \left(a_2^2 - a_3\right)\left(-4p_1^2 + a_2^2 p_2^2 - a_3 p_2^2\right)q_1^2 - 8\left(a_2^2 - a_3\right)\left(-2p_1 + a_2 p_2\right)p_2 q_1 q_2 \\ &+ 4q_2^2\left(4p_1^2 - 8a_2 p_1 p_2 + 3a_2^2 p_2^2 + a_3 p_2^2\right) \end{aligned} \tag{A6}$$

in which

$$\begin{aligned} &p_1 = C_{33}\left(1 + \alpha_1\right) - C_{13}\alpha_3, \quad p_2 = \alpha_2 C_{33}, \quad p_3 = -\rho_0 C_{33} \\ &q_1 = \alpha_2 - \alpha_3, \quad q_2 = 1 + \alpha_1, \quad q_3 = -\rho_0 \end{aligned} \tag{A7}$$

and a_1 to a_5 have been defined in Eq. (38). The solution of Eq. (A1), may be expressed as:

$$\begin{aligned} S_{1,2}^2 &= -\frac{O_4}{4O_5} - L^* \pm \sqrt{-4L^2 - 2k_1 + \frac{k_2}{L^*}} \\ S_{3,4}^2 &= -\frac{O_4}{4O_5} + L^* \pm \sqrt{-4L^2 - 2k_1 + \frac{k_2}{L^*}} \end{aligned} \tag{A8}$$

in which

$$\begin{aligned} k_1 &= \frac{8O_5O_3 - 3O_4^2}{8O_5^2}, \quad k_2 = \frac{O_4^3 - 4O_5O_4O_3 + 8O_5^2O_2}{8O_5^3} \\ L^* &= \frac{1}{2}\sqrt{-\frac{2}{3}k_1 + \frac{1}{3O_5}\left(\Theta + \frac{j_0}{\Theta}\right)}, \qquad j_0 = O_3^2 - 3O_4O_2 + 12O_5O_1 \\ \Theta &= \sqrt[3]{\frac{j_1 + \sqrt{j_1^2 - 4j_0^3}}{2}}, \quad j_1 = 2O_3^3 - 9O_4O_3O_2 + 27O_4^2O_1 + 27O_5O_2^2 - 72O_5O_3O_1 \end{aligned} \tag{A9}$$

Table A1. Normalized elasticity coefficients of synthetic transversely isotropic materials with respect to elastic parameters of isotropic material given in Table A2

Material No.	$\bar{C}_{11} = \frac{C_{11}}{C_{11iso}}$	$\bar{C}_{12} = \frac{C_{12}}{C_{12iso}}$	$\bar{C}_{13} = \frac{C_{13}}{C_{13iso}}$	$\bar{C}_{33} = \frac{C_{33}}{C_{33iso}}$	$\bar{C}_{44} = \frac{C_{44}}{C_{44iso}}$
1	1.000	1.000	1.000	1.000	1.000
2	1.16667	2.500	0.750	0.167	1.000
3	1.66667	1.000	1.750	0.417	0.500
4	8.333	3.500	0.750	4.000	1.000
5	11.667	15.00	-8.200	16.667	25
6	11.667	34.50	-40.85	16.667	25
7	16.667	15	-8.20	11.667	25
8	16.667	47.50	-40.85	11.667	25
9	16.667	15.00	-8.20	11.667	55
10	16.667	15.00	-8.00	11.6667	25

Table A2. Elasticity coefficients of isotropic materials used for normalization

C_{11iso}	C_{12iso}	C_{13iso}	C_{33iso}	C_{44iso}
60000	20000	20000	60000	20000

Table A3. Rayleigh wave velocity

Material No.	Nature of s_1^2, s_2^2	Sign of $s_1^2+s_2^2$	The Nature of Δ, a_4, a_3	Rayleigh Wave Slowness	The type of Rayleigh Wave
1	Real	Positive	-	0.054383	First Kind
2	Complex Conjugate	Negative	$\Delta>0,\ 0<a_4<\sqrt{\Delta}$ or $-\sqrt{\Delta}<a_4<0$	0.069093	Second Kind
3	Complex Conjugate	Positive	$\Delta>0,\ 0<a_4<\sqrt{\Delta}$ or $-\sqrt{\Delta}<a_4<0$	0.076581	Second Kind
4	Real	Positive	$\Delta<0$	0.050087	First Kind
5	Real	Positive	$\Delta>0,\ 0<\sqrt{\Delta}<a_4$	0.012317	Second kind
6	Real	Positive	$\Delta>0,\ 0<\sqrt{\Delta}<a_4$	0.053211	First Kind
7	Real	Positive	$\Delta>0,\ a_4<-\sqrt{\Delta}$	0.011969	First Kind
8	Reals	Positive	$\Delta>0,\ a_4<-\sqrt{\Delta}$	0.048758	First Kind
9	Complex Conjugate	Positive	$\Delta>0,\ 0<a_4<\sqrt{\Delta}$ or $-\sqrt{\Delta}<a_4<0$	0.010582	Second Kind
10	Complex Conjugate	Positive	$\Delta>0,\ 0<a_4<\sqrt{\Delta}$ or $-\sqrt{\Delta}<a_4<0$	0.011962	First Kind

Material No.	Corresponding Inequalities	Quai-Longitudinal Wave Velocity v_L, for Zero Incident Angle	Quai-Transverse Wave Velocity v_T for Zero Incident Angle	Horizontally Polarized Shear Wave Velocity v_{SH} for Normal Incident Angle	The Ratio of Rayleigh Wave Velocity to Quai-Transverse Wave Velocity
1	-	$20\sqrt{3}$	20	20	0.9194
2	Eq. (72)	$10\sqrt{2}$	20	$10\sqrt{2}$	0.7237
3	Eq. (72)	$10\sqrt{5}$	$10\sqrt{2}$	$20\sqrt{2}$	0.9233
4	Eq. (70)	$40\sqrt{3}$	20	$10\sqrt{43}$	0.9982
5	Eq. (71)	$100\sqrt{2}$	100	$20\sqrt{10}$	0.8118
6	Eq. (67)	$100\sqrt{2}$	100	10	0.1879
7	Eq. (69)	$20\sqrt{35}$	100	$10\sqrt{70}$	0.8355
8	Eq. (69)	$20\sqrt{35}$	100	$10\sqrt{5}$	0.2051
9	Eq. (72)	$20\sqrt{35}$	$20\sqrt{55}$	$10\sqrt{70}$	0.6371
10	Eq. (68)	$20\sqrt{35}$	100	$10\sqrt{70}$	