A SURVEY ON THE CYCLIC COLORING AND ITS RELAXATIONS

JÚLIUS CZAP

*Department of Applied Mathematics and Business Informatics
Faculty of Economics, Technical University of Košice
Němcovej 32, 04001 Košice, Slovakia*

e-mail: julius.czap@tuke.sk

MIRKO HORŇÁK

AND

STANISLAV JENDROL'

*Institute of Mathematics, P.J. Šafárik University
Jesenná 5, 04001 Košice, Slovakia*

e-mail: mirko.hornak@upjs.sk
stanislav.jendrol@upjs.sk

Abstract

A cyclic coloring of a plane graph is a vertex coloring such that any two vertices incident with the same face receive distinct colors. This type of coloring was introduced more than fifty years ago, and a lot of research in chromatic graph theory was sparked by it. This paper is a survey on the state of the art concerning the cyclic coloring and relaxations of this graph invariant.

Keywords: plane graph, edge coloring, vertex coloring.

2010 Mathematics Subject Classification: 05C15, 05C10.

REFERENCES

- [1] O. Amini, L. Esperet and J. van den Heuvel, *A unified approach to distance-two colouring of graphs on surfaces*, *Combinatorica* **33** (2013) 253–296.
doi:10.1007/s00493-013-2573-2
- [2] V. Andova, B. Lidický, B. Lužar and R. Škrekovski, *On facial unique-maximum (edge-) coloring*, *Discrete Appl. Math.* **237** (2018) 26–32.
doi:10.1016/j.dam.2017.11.024

- [3] K. Appel and W. Haken, *Every planar map is four colorable*, Bull. Amer. Math. Soc. **82** (1976) 711–712.
doi:10.1090/s0002-9904-1976-14122-5
- [4] K. Appel and W. Haken, *Every planar map is four colorable, Part I: Discharging*, Illinois J. Math. **21** (1977) 429–490.
doi:10.1215/ijm/1256049011
- [5] K. Appel, W. Haken and J. Koch, *Every planar map is four colorable, Part II: Reducibility*, Illinois J. Math. **21** (1977) 491–567.
doi:10.1215/ijm/1256049012
- [6] J. Azarija, R. Erman, D. Kráľ, M. Krnc and L. Stacho, *Cyclic colorings of plane graphs with independent faces*, European J. Combin. **33** (2012) 294–301.
doi:10.1016/j.ejc.2011.09.011
- [7] T. Bálint and J. Czap, *Facial parity 9-edge-coloring of outerplane graphs*, Graphs Combin. **31** (2015) 1177–1187.
doi:10.1007/s00373-014-1472-7
- [8] O.V. Borodin, *A new proof of the 6 color theorem*, J. Graph Theory **19** (1995) 507–521.
doi:10.1002/jgt.3190190406
- [9] O.V. Borodin, *Cyclic coloring of plane graphs*, Discrete Math. **100** (1992) 281–289.
doi:10.1016/0012-365x(92)90647-x
- [10] O.V. Borodin, *Cyclic degree and cyclic coloring of 3-polytopes*, J. Graph Theory **23** (1996) 225–231.
doi:10.1002/(sici)1097-0118(199611)23:3<225::aid-jgt2>3.0.co;2-u
- [11] O.V. Borodin, *Solution of Ringel’s problems on vertex-face coloring of plane graphs and coloring of 1-planar graphs*, Met. Diskret. Anal. **41** (1984) 12–26.
- [12] O.V. Borodin, H.J. Broersma, A. Glebov and J. van den Heuvel, *A new upper bound on the cyclic chromatic number*, J. Graph Theory **54** (2007) 58–72.
doi:10.1002/jgt.20193
- [13] O.V. Borodin and A.O. Ivanova, *List 2-facial 5-colorability of plane graphs with girth at least 12*, Discrete Math. **312** (2012) 306–314.
doi:10.1016/j.disc.2011.09.018
- [14] O.V. Borodin and D.R. Woodall, *Cyclic colorings of 3-polytopes with large maximum face size*, SIAM J. Discrete Math. **15** (2002) 143–154.
doi:10.1137/s089548019630248x
- [15] O.V. Borodin, D.P. Sanders and Y. Zhao, *On cyclic colorings and their generalizations*, Discrete Math. **203** (1999) 23–40.
doi:10.1016/s0012-365x(99)00018-7
- [16] Y. Bu and W. Wang, *Some sufficient conditions for a planar graph of maximum degree six to be class 1*, Discrete Math. **306** (2006) 1440–1445.
doi:10.1016/j.disc.2006.03.032

- [17] J. Czap, *Facial parity edge coloring of outerplane graphs*, Ars Math. Contemp. **5** (2012) 285–289.
doi:10.26493/1855-3974.228.ee8
- [18] J. Czap, *Facial rainbow edge-coloring of simple 3-connected plane graphs*, Opuscula Math. **40** (2020) 475–482.
doi:10.7494/OpMath.2020.40.4.475
- [19] J. Czap, *Parity vertex coloring of outerplane graphs*, Discrete Math. **311** (2011) 2570–2573.
doi:10.1016/j.disc.2011.06.009
- [20] J. Czap and S. Jendroľ, *Colouring vertices of plane graphs under restrictions given by faces*, Discuss. Math. Graph Theory **29** (2009) 521–543.
doi:10.7151/dmgt.1462
- [21] J. Czap, S. Jendroľ and F. Kardoš, *Facial parity edge colouring*, Ars Math. Contemp. **4** (2011) 255–269.
doi:10.26493/1855-3974.129.be3
- [22] J. Czap, S. Jendroľ and F. Kardoš, *On the strong parity chromatic number*, Discuss. Math. Graph Theory **31** (2011) 587–600.
doi:10.7151/dmgt.1567
- [23] J. Czap, S. Jendroľ, F. Kardoš and R. Soták, *Facial parity edge colouring of plane pseudographs*, Discrete Math. **312** (2012) 2735–2740.
doi:10.1016/j.disc.2012.03.036
- [24] J. Czap, S. Jendroľ and M. Voigt, *Parity vertex colouring of plane graphs*, Discrete Math. **311** (2011) 512–520.
doi:10.1016/j.disc.2010.12.008
- [25] J. Czap and Zs. Tuza, *Decompositions of plane graphs under parity constraints given by faces*, Discuss. Math. Graph Theory **33** (2013) 521–530.
doi:10.7151/dmgt.1690
- [26] Z. Dvořák, M. Hebdige, F. Hlásek, D. Kráľ and J.A. Noel: *Cyclic coloring of plane graphs with maximum face size 16 and 17*.
arXiv:1603.06722.
- [27] Z. Dvořák, R. Škrekovski and M. Tancer, *List-coloring squares of sparse subcubic graphs*, SIAM J. Discrete Math. **22** (2008) 139–159.
doi:10.1137/050634049
- [28] H. Enomoto and M. Horňák, *A general upper bound for the cyclic chromatic number of 3-connected plane graphs*, J. Graph Theory **62** (2009) 1–25.
doi:10.1002/jgt.20383
- [29] H. Enomoto, M. Horňák and S. Jendroľ, *Cyclic chromatic number of 3-connected plane graphs*, SIAM J. Discrete Math. **14** (2001) 121–137.
doi:10.1137/s0895480198346150

- [30] I. Fabrici and F. Göring, *Unique-maximum coloring of plane graphs*, Discuss. Math. Graph Theory **36** (2016) 95–102.
doi:10.7151/dmgt.1846
- [31] I. Fabrici, S. Jendroř and M. Vrbjarová, *Unique-maximum edge-colouring of plane graphs with respect to faces*, Discrete Appl. Math. **185** (2015) 239–243.
doi:10.1016/j.dam.2014.12.002
- [32] C. Godsil and G. Royle, *Algebraic Graph Theory* (Springer, 2001).
- [33] J.L. Gross and T.W. Tucker, *Topological Graph Theory* (Dover Publications, 2001).
- [34] H. Grötzsch, *Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel*, Wiss. Z. Martin-Luther-Universität, Halle-Wittenberg, Math.-Nat. Reihe **8** (1959) 109–120.
- [35] F. Havet, J.-S. Sereni and R. Škrekovski, *3-facial coloring of plane graphs*, SIAM J. Discrete Math. **22** (2008) 231–247.
doi:10.1137/060664124
- [36] F. Havet, D. Král, J.-S. Sereni and R. Škrekovski, *Facial colorings using Hall’s theorem*, European J. Combin. **31** (2010) 1001–1019.
doi:10.1016/j.ejc.2009.10.003
- [37] M. Hebdige and D. Král, *Third case of the cyclic coloring conjecture*, SIAM J. Discrete Math. **30** (2016) 525–548.
doi:10.1137/15m1019490
- [38] M. Horňák and S. Jendroř, *On a conjecture by Plummer and Toft*, J. Graph Theory **30** (1999) 177–189.
doi:10.1002/(sici)1097-0118(199903)30:3<177::aid-jgt3>3.0.co;2-k
- [39] M. Horňák and S. Jendroř, *On vertex types and cyclic colourings of 3-connected plane graphs*, Discrete Math. **212** (2000) 101–109.
doi:10.1016/s0012-365x(99)00212-5
- [40] M. Horňák and S. Jendroř, *Unavoidable sets of face types for planar maps*, Discuss. Math. Graph Theory **16** (1996) 123–141.
doi:10.7151/dmgt.1028
- [41] M. Horňák and J. Zlámalová, *Another step towards proving a conjecture by Plummer and Toft*, Discrete Math. **310** (2010) 442–452.
doi:10.1016/j.disc.2009.03.016
- [42] D. Huang and W. Wang, *Planar graphs of maximum degree six without 7-cycles are class one*, Electron. J. Combin. **19(3)** (2012) #P17.
doi:10.37236/2589
- [43] S. Jendroř, *Facial rainbow edge-coloring of plane graphs*, Graphs Combin. **34** (2018) 669–676.
doi:10.1007/s00373-018-1904-x
- [44] S. Jendroř and L. Kekenáková, *Facial rainbow coloring of plane graphs*, Discuss. Math. Graph Theory **39** (2019) 889–897.
doi:10.7151/dmgt.2047

- [45] S. Jendroľ and L. Kekeňáková, *Facial rainbow colorings of trees*, Australas. J. Combin. **69** (2017) 358–367.
- [46] S. Jendroľ and R. Soták, *On the cyclic coloring conjecture*, Discrete Math. (2020) accepted.
- [47] T. Kaiser, O. Rucký, M. Stehlík and R. Škrekovski, *Strong parity vertex coloring of plane graphs*, Discrete Math. Theor. Comput. Sci. **16** (2014) 143–158.
- [48] D. Kráľ, T. Madaras and R. Škrekovski, *Cyclic, diagonal and facial colorings*, European J. Combin. **26** (2005) 473–490.
doi:10.1016/j.ejc.2004.01.016
- [49] D. Kráľ, T. Madaras and R. Škrekovski, *Cyclic, diagonal and facial colorings—a missing case*, European J. Combin. **28** (2007) 1637–1639.
doi:10.1016/j.ejc.2006.07.005
- [50] M. Kriesell, *Contractions, cycle double covers, and cyclic colorings in locally connected graphs*, J. Combin. Theory Ser. B **96** (2006) 881–900.
doi:10.1016/j.jctb.2006.02.009
- [51] B. Lidický, K. Messerschmidt and R. Škrekovski, *A counterexample to a conjecture on facial unique-maximal colorings*, Discrete Appl. Math. **237** (2018) 123–125.
doi:10.1016/j.dam.2017.11.037
- [52] B. Lidický, K. Messerschmidt and R. Škrekovski, *Facial unique-maximum colorings of plane graphs with restriction on big vertices*, Discrete Appl. Math. **242** (2019) 2612–2617.
doi:10.1016/j.disc.2019.05.029
- [53] B. Lužar, M. Mockovčiaková, R. Soták, R. Škrekovski and P. Šugerek, *ℓ -facial edge colorings of graphs*, Discrete Appl. Math. **181** (2015) 193–200.
doi:10.1016/j.dam.2014.10.009
- [54] B. Lužar and R. Škrekovski, *Improved bound on facial parity edge coloring*, Discrete Math. **313** (2013) 2218–2222.
doi:10.1016/j.disc.2013.05.022
- [55] B. Mohar and C. Thomassen, *Graphs on Surfaces* (The Johns Hopkins University Press, 2001).
- [56] M. Montassier and A. Raspaud, *A note on 2-facial coloring of plane graphs*, Inform. Process. Lett. **98** (2006) 235–241.
doi:10.1016/j.ipl.2006.02.013
- [57] W.-P. Ni, *Edge colorings of planar graphs with $\Delta = 6$ without short cycles contain chords*, J. Nanjing Norm. Univ. Nat. Sci. Ed. **34** (2011) 19–24.
- [58] W.-P. Ni, *Edge colorings of planar graphs without adjacent special cycles*, Ars Combin. **105** (2012) 247–256.
- [59] W.-P. Ni and J. Wu, *Edge coloring of planar graphs which any two short cycles are adjacent at most once*, Theoret. Comput. Sci. **516** (2014) 133–138.
doi:10.1016/j.tcs.2013.11.011

- [60] O. Ore and M.D. Plummer, *Cyclic coloration of plane graphs*, in: Recent Progress in Combinatorics (Proceedings of the Third Waterloo Conference on Combinatorics), (Academic Press, 1969) 287–293.
- [61] M.D. Plummer and B. Toft, *Cyclic coloration of 3-polytopes*, J. Graph Theory **11** (1987) 507–515.
doi:10.1002/jgt.3190110407
- [62] N. Robertson, D.P. Sanders, P.D. Seymour and R. Thomas, *The four-colour theorem*, J. Combin. Theory Ser. B **70** (1999) 2–44.
doi:10.1006/jctb.1997.1750
- [63] D.P. Sanders and Y. Zhao, *A new bound on the cyclic chromatic number*, J. Combin. Theory Ser. B **83** (2001) 102–111.
doi:10.1006/jctb.2001.2046
- [64] D.P. Sanders and Y. Zhao, *Planar graphs of maximum degree seven are class 1*, J. Combin. Theory Ser. B **83** (2001) 201–212.
doi:10.1006/jctb.2001.2047
- [65] C.E. Shannon, *A theorem on coloring the lines of a network*, J. Math. Phys. **28** (1949) 148–152.
doi:10.1002/sapm1949281148
- [66] K. Štorgel, *Improved bounds for some facially constrained colorings*, Discuss. Math. Graph Theory, in press.
doi:10.7151/dmgt.2357
- [67] P.G. Tait, *Remarks on the colouring of maps*, Proc. Roy. Soc. Edinburgh **10** (1880) 729.
- [68] C. Thomassen, *The square of a planar cubic graph is 7-colorable*, J. Combin. Theory Ser. B **128** (2018) 192–218.
doi:10.1016/j.jctb.2017.08.010
- [69] V.G. Vizing, *Critical graphs with a given chromatic class*, Met. Diskret. Anal. **5** (1965) 9–17.
- [70] V.G. Vizing, *On an estimate of the chromatic class of a p -graph*, Diskret. Analiz **3** (1964) 25–30.
- [71] W. Wang and Y. Chen, *A sufficient condition for a planar graph to be class 1*, Theoret. Comput. Sci. **385** (2007) 71–77.
doi:10.1016/j.tcs.2007.05.032
- [72] W. Wang, S. Finbow and P. Wang, *An improved bound on parity vertex colourings of outerplane graphs*, Discrete Math. **312** (2012) 2782–2787.
doi:10.1016/j.disc.2012.04.009
- [73] Y. Wang and L. Xu, *A sufficient condition for a plane graph with maximum degree 6 to be class 1*, Discrete Appl. Math. **161** (2013) 307–310.
doi:10.1016/j.dam.2012.08.011

- [74] A. Wendland, *Coloring of plane graphs with unique maximal colors on faces*, J. Graph Theory **83** (2016) 359–371.
doi:10.1002/jgt.22002
- [75] J. Wu and L. Xue, *Edge colorings of planar graphs without 5-cycles with two chords*, Theoret. Comput. Sci. **518** (2014) 124–127.
doi:10.1016/j.tcs.2013.07.027
- [76] L. Xue and J. Wu, *Edge colorings of planar graphs without 6-cycles with two chords*, Open J. Discrete Math. **3** (2013) 83–85.
doi:10.4236/ojdm.2013.32016
- [77] W. Zhang and J. Wu, *A note on the edge colorings of planar graphs without 7-cycles with three chords*, Ars Combin. **138** (2018) 393–402.
- [78] W. Zhang and J. Wu, *Edge coloring of planar graphs without adjacent 7-cycles*, Theoret. Comput. Sci. **739** (2018) 59–64.
doi:10.1016/j.tcs.2018.05.006
- [79] W. Zhang and J. Wu, *Edge colorings of planar graphs without 6-cycles with three chords*, Bull. Malays. Math. Sci. Soc. **41** (2018) 1077–1084.
doi:10.1007/s40840-016-0376-5
- [80] G. Zhou, *A note on graphs of class I*, Discrete Math. **263** (2003) 339–345.
doi:10.1016/s0012-365x(02)00793-8
- [81] J. Zlámalová, *A note on cyclic chromatic number*, Discuss. Math. Graph Theory **30** (2010) 115–122.
doi:10.7151/dmgt.1481

Received 26 June 2020

Revised 21 October 2020

Accepted 21 October 2020