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DOUBLE TOTAL DOMINATOR CHROMATIC NUMBER OF GRAPHS

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Abstract

In this paper, we introduce and study a new coloring problem of graphs called the double total dominator coloring. A double total dominator coloring of a graph G with minimum degree at least 2 is a proper vertex coloring of G such that each vertex has to dominate at least two color classes. The minimum number of colors among all double total dominator coloring of G is called the double total dominator chromatic number, denoted by $\chi_{dd}^t(G)$. Therefore, we establish the close relationship between the double total dominator chromatic number $\chi_{dd}^t(G)$ and the double total domination number $\gamma_{\times 2, t}(G)$. We prove the NP-completeness of the problem. We also examine the effects on $\chi_{dd}^t(G)$ when G is modified by some operations. Finally, we discuss the $\chi_{dd}^t(G)$ number of square of trees by giving some bounds.

Keywords: coloring, domination, double total dominator coloring.

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1. INTRODUCTION

All graphs considered in this paper are finite, undirected and simple graphs. Let $G = (V(G), E(G))$ be a graph such that $V(G)$ is the *vertex* set and $E(G)$ is the *edge* set. Graph coloring and domination in graphs are two major concepts within graph theory which have been extensively studied [12, 14, 23].

A *proper vertex coloring* of a graph is a vertex coloring such that no two adjacent vertices have the same color. A proper vertex coloring $\mathcal{C}$ using k colors is called a (*proper*) *k-coloring*. A subset of vertices colored with the same color is called a *color class* and every such class forms an independent set. Thus, finding a proper *k-coloring* of a graph G is equivalent to the partitioning of its vertex set into k independent sets. The minimum number of colors among all proper colorings of G is the *chromatic number* of G , denoted by $\chi(G)$. A graph admitting a proper *k-coloring* is said to be *k-colorable*, and it is said to be *k-chromatic* if its chromatic number is exactly k . Finding the chromatic number is proved to be NP-complete in general case [9]. More variants of graph coloring can be found in [4, 5, 8, 24].

Graph coloring and domination problems are often correlated. The research of domination in graphs has a great importance in graph theory. Its basic concept is the dominating set and the domination number. First, a *dominating set* of a graph G is a set $S \subseteq V(G)$ such that every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S . The *domination number* of G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G . The literature on this subject has been surveyed and detailed in [15, 16]. Second, a *total dominating set*, abbreviated TD-set, of G is a set $S \subseteq V(G)$ such that every vertex in $V(G)$ is adjacent to at least one vertex in S . The *total domination number* of G , denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G . A TD-set of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$ -set. Total domination is well studied and detailed in [17, 20]. After that, many variants of the dominating set were introduced. Among the most famous, we have the *k-tuple total dominating set* [18], which is an extension of the total dominating set. A set S of vertices in G is a *k-tuple total dominating set*, abbreviated kTD-set, of G if every vertex of G is adjacent to least k vertices in S . The minimum cardinality of a kTD-set of G is the *k-tuple total domination number* of G , denoted by $\gamma_{\times k, t}(G)$. For a graph to have a kTD-set, its minimum degree is at least k . A kTD-set of cardinality $\gamma_{\times k, t}(G)$ is called a $\gamma_{\times k, t}(G)$ -set. The concept of kTD-set has been studied by several authors (see, for example, [19, 22] and elsewhere). When $k = 2$, a kTD-set is called a *double total dominating set*, abbreviated DTD-set, and the *k-tuple total domination number* is called the *double total domination number*, denoted by $\gamma_{\times 2, t}(G)$.

In [7], Chellali *et al.* showed some relations between the chromatic number and some domination parameters in graphs. Motivated by the relation between

the coloring and domination, the notion of dominator coloring was defined. In [12], Gera *et al.* introduced the *dominator coloring* as a proper coloring such that every vertex has to dominate at least one color class (possibly its own class). The minimum number of colors among all dominator colorings of G is the *dominator chromatic number* of G , denoted by $\chi_d(G)$. Zverovich, in [27], called this coloring parameter *strong coloring*. Gera studied further the problem in [10, 11] and she proved its NP-completeness. Chellali and Maffray gave a polynomial-time algorithm for computing χ_d on P_4 -free graphs [6]. Arumugam *et al.* showed that the dominator coloring problem is NP-hard on bipartite, planar and split graphs [2]. More results on this coloring could be found in [3, 21, 25, 26] and elsewhere.

In [14], Haddad and Kheddouci have introduced the *strict strong coloring* (SSColoring for short). In this coloring, each vertex has to dominate a color class different from its own class. They proved that computing the minimum cardinality of a strict strong coloring of an arbitrary graph is an NP-complete problem. In [23], Kazemi studied the same problem and called it *total dominator coloring*, abbreviated TD-coloring. The *total dominator chromatic number* of G , denoted by $\chi_d^t(G)$, is the minimum number of color classes in a TD-coloring of G . A $\chi_d^t(G)$ -coloring of G is any total dominator coloring with $\chi_d^t(G)$ colors. Kazemi studied this problem on several classes of graphs. He proved that every graph G of order n and without isolated vertices satisfies $\max\{\chi_d(G), \gamma_t(G)\} \leq \chi_d^t(G) \leq n$.

The TD-coloring problem is defined initially to embody the use of the dominance property in the dominator coloring. This latter allows the dominance of the empty color class which has not a sense in practice. To explain the assumption, Haddad *et al.* [13] have taken as an example the broadcasting application. They proposed a new approach to optimize the broadcast process in ad hoc networks. In particular, a broadcast should be defined such that all vertices are reachable (coverage property) with minimizing multiple message reception phenomena (redundancy property). Motivated by this application, in this paper, we generalize this approach and introduce the k -tuple total dominator coloring.

The k -tuple *total dominator coloring* of G (k TD-coloring for short) is a proper coloring of G in which each vertex of the graph is adjacent to every vertex of k color classes. We define the k -tuple *total dominator chromatic number*, denoted by $\chi_{k,d}^t$, as the minimum number of colors among all k TD-colorings. For a given graph G , G has a k TD-coloring, if its minimum degree is at least k . Since every $(k+1)$ -tuple total dominator coloring is also a k -tuple total dominator coloring, we note that $\chi_{k,d}^t(G) \leq \chi_{k+1,d}^t(G)$ for all graphs with minimum degree at least $k+1$. When $k=1$, a k -tuple total dominator chromatic number is the well-studied total dominator chromatic number. When $k=2$, a k TD-coloring is called a *double total dominator coloring*, abbreviated DTD-coloring, and the k -tuple total dominator chromatic number is called the *double total dominator chromatic number*, denoted by χ_{dd}^t . A χ_{dd}^t -coloring of G is any double total do-

minator coloring with χ_{dd}^t colors.

The goal of this paper is to study the double total dominator chromatic number. We give some introductory results in Section 2. The NP-completeness is proved in Section 3. Then in Section 4, we study the double total dominator chromatic number as well as finding general bounds. We also study its relation with chromatic number and some domination parameters. In Section 5, we examine the effects on $\chi_{dd}^t(G)$ when G is modified by some operations as the removal or contraction of edges or vertices. The effect of modifying the graph on the parameter is of practical importance. Finally, in Section 6, we characterize the trees T for which $\chi_{dd}^t(T^2) = \gamma_{\times 2, t}(T^2)$. We prove that if T is a non-star tree of order $n \geq 4$, then $\chi_{dd}^t(T^2) \leq n - 1$ and we characterize the trees achieving equality in this bound.

2. NOTATION AND PRELIMINARIES

Let $G = (V(G), E(G))$, simpler $G = (V, E)$, be a simple graph with vertex set V of order $n = |V(G)|$ and edge set $E(G)$ of size $m = |E(G)|$. The *distance* between two vertices in a graph is the length (number of edges) of the shortest path connecting them. The *diameter* of the graph is the greatest distance between any two vertices. We denoted the distance between u and v in G and the diameter of G by $dist_G(u, v)$ and $diam(G)$, respectively. The distance between a vertex v in the graph G and a proper subset X of $V(G)$ is $dist_G(v, X) = \min\{dist_G(v, u) : u \in X\}$. Let v be a vertex in $V(G)$, the *open neighborhood* of v is $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is $N_G[v] = \{v\} \cup N_G(v)$. The degree of a vertex v in G is $d_G(v) = |N_G(v)|$. The *minimum* and *maximum degree* of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. If the graph G is clear from the context, we simply write $d(v)$, $N(v)$, and $N[v]$ rather than $d_G(v)$, $N_G(v)$, and $N_G[v]$, respectively. For any subset $S \subseteq V(G)$, $G - S$ denotes the graph obtained from G by removing the set of vertices S and all edges incident with the set S from G . $G[S]$ denotes the *subgraph* of G induced by S .

We call a vertex of degree one a *leaf*, its adjacent vertex a *support* vertex and its incident edge a *pendant* edge. A *strong* support vertex is a support vertex with at least two leaf neighbors. A *path* of order n is denoted by P_n . Here it is assumed that the vertex set $V(P_n)$ of the path P_n is $\{v_1, v_2, \dots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n - 1\}$, we write simply $P_n = v_1 \dots v_n$. A *cycle* and a *complete* graph of n vertices are denoted by C_n and K_n , respectively. Let $G = (V, E)$ be a graph and p be a partition of V . The graph G is *k-partite* by $\mathcal{P}$ if for every $M \in p$, $G[M]$ is empty and $|p| = k$. It is *bipartite* when $|p| = 2$.

A *tree* is a connected graph without any cycle. A *rooted* tree T distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of $V(T)$, the *parent* of v is the neighbor of v on the unique path connecting r and v ((r, v) -path), while a *child* of

v is any other neighbor of v . A *descendant* of v is a vertex u such that the unique (r, u) -path contains v . Thus, every child of v is a descendant of v . The maximal subtree rooted at v is the subtree of T induced by v and its descendants, denoted by T_v . For instance, a *star*, denoted by $K_{1,n-1}$, is a tree of order $n \geq 2$ with exactly one non-leaf vertex, called the *center* of the star. A *double star*, denoted by $DS_{n,m}$, is a tree having $n + m + 2$ vertices with exactly two non-leaves, called *centers* of the double star. The *generalized friendship* graph, denoted by D_q^n , is a collection of n cycles (all of order q), meeting at a common vertex. The graph D_q^n has $n(q - 1) + 1$ vertices and qn edges. For two graphs G and H on disjoint sets of n and m vertices, respectively, the *corona* $G \circ H$ is the graph arising from the disjoint union of G with n copies of H , by adding edges between the i -th vertex of G and all vertices of i -th copy of H . Note that the corona $G \circ H$ has $n(m + 1)$ vertices and $|E(G)| + (n|E(H)| + m)$ edges.

Let $X \subseteq V$ and $v \in V \setminus X$. If the vertex v is adjacent to all vertices of X , we denote it by $v \sim X$. More formally, the *k-double total dominator coloring* of G is a proper k -coloring $C_1, C_2, \dots, C_k$ of G such that for every vertex $v \in V$, there exist two indices $i, j \in \{1, 2, \dots, k\}$, $i \neq j$, where v is adjacent to every vertex of C_i and C_j . We say that a color i is a *dominated* color by a vertex u , with $u \in V$ (or u *dominates* i) if and only if $u \sim C_i$. A color class C in a given DTD-coloring $\mathcal{C}$ of G is *free* if each vertex of G is adjacent to every vertex of some color class different from C . If $\mathcal{C}$ is a double total dominator coloring of G with the coloring classes $C_1, C_2, \dots, C_k$, we write simply $\mathcal{C} = (C_1, C_2, \dots, C_k)$.

We begin by the following straight forward observation.

Observation 1. *If v is an arbitrary vertex in G with $\delta(G) \geq 2$, then in every DTD-coloring of G the open neighborhood of v , $N(v)$, contains two color classes.*

Trivially, $\chi_{dd}^t(G) \geq \chi_d^t(G)$ holds for every graph G with minimum degree at least two. Moreover, as we shall show, the difference $\chi_{dd}^t(G) - \chi_d^t(G)$ can be arbitrarily large. To do this, we need to consider the total dominator chromatic number of a path. In [23], Kazemi proved the following theorem.

Theorem 2 [23]. *Let P_n be a path of order $n \geq 2$. Then*

$$\chi_d^t(P_n) = \begin{cases} 2 \lceil \frac{n}{3} \rceil - 1 & \text{if } n \equiv 1 \pmod{3}, \\ 2 \lceil \frac{n}{3} \rceil & \text{otherwise.} \end{cases}$$

Theorem 3. *There exists a connected graph G with $\delta(G) \geq 2$, such that $\chi_{dd}^t(G) - \chi_d^t(G)$ can be arbitrarily large.*

Proof. For $n \geq 3$, let us consider the graph G obtained from the path P_n , with $V(P_n) = \{v_1, v_2, \dots, v_n\}$, by adding the vertices $u_1, u_2, \dots, u_n$ such that $u_i v_i \in E(G)$ for all i ($1 \leq i \leq n$) and by adding the vertex u to the vertices u_i for all i ($1 \leq i \leq n$), as shown in Figure 1.

Figure 1. Graph G in the proof of Theorem 3.

Clearly, $G[\{v_1, v_2, \dots, v_n\}]$ is a path of order n . Let $\mathcal{C}$ be a TD-coloring of $G[\{v_1, v_2, \dots, v_n\}]$ with $\chi_d^t(P_n)$ color classes. We can extend $\mathcal{C}$ to a TD-coloring of G by coloring the vertices $u_1, u_2, \dots, u_n$ with a new color $\chi_d^t(P_n) + 1$ and coloring the vertex u with another new color $\chi_d^t(P_n) + 2$. Clearly, this is a TD-coloring of G with $\chi_d^t(P_n) + 2$ color classes. Then, we have

$$\chi_d^t(G) \leq \chi_d^t(P_n) + 2 = \begin{cases} 2 \lceil \frac{n}{3} \rceil + 1 & \text{if } n \equiv 1 \pmod{3}, \\ 2 \lceil \frac{n}{3} \rceil + 2 & \text{otherwise.} \end{cases}$$

Furthermore, for $1 \leq i \leq n$ with $n \geq 3$, we have $d(u_i) = 2$ and $\bigcap_{i=1}^n N(u_i) = \{u\}$. We color the vertices $v_1, v_2, \dots, v_n, u$ with $1, \dots, n+1$ colors, respectively. Since $N(v_1) = \{v_2, u_1\}$ and $N(v_n) = \{v_{n-1}, u_n\}$, then using Observation 1 we need two additional colors to color u_1 and u_n . This implies that an additional color is given to the vertices $u_2, u_3, \dots, u_{n-1}$. Thus, every DTD-coloring of G uses at least $n+4$ colors. Let $\mathcal{C}' = (C_1 = \{v_1\}, \dots, C_n = \{v_n\}, C_{n+1} = \{u_1\}, C_{n+2} = \{u_n\}, C_{n+3} = \{u\}, C_{n+4} = \{u_2, \dots, u_{n-1}\})$ be a proper coloring of G . Obviously, $\mathcal{C}'$ is a DTD-coloring of G with $n+4$ colors. Thus $\chi_{dd}^t(G) = n+4$. Therefore, we conclude that $\chi_{dd}^t(G) - \chi_d^t(G)$ can be arbitrarily large. ■

3. COMPLEXITY

In this section, we prove the NP-completeness of the decision problem of double total dominator coloring a graph G with k colors. We give the following formalization:

Instance: A graph $G = (V, E)$ with $\delta(G) \geq 2$ and a positive integer k .

Question: Is there a double total dominator coloring of G with k colors?

Theorem 4. *For $k \geq 5$, the k -double total dominator coloring problem of graphs is NP-complete.*

Proof. Since a non-deterministic algorithm can guess a solution in a polynomial time, the k -double total dominator coloring problem is in NP. Now, we give a polynomial time reduction from a k -coloring problem which is known to be NP-complete for $k \geq 3$. We construct a graph G' from G by adding two dominating vertices u and v to G (with u and v are adjacent and are adjacent to all vertices of G). We show that G admits a proper coloring with k colors if and only if G' admits a double total dominator coloring with $k + 2$ colors.

First, we prove the necessity. Let $C = (C_1, C_2, \dots, C_k)$ be a k -coloring of G . We construct a $(k + 2)$ -double total dominator coloring D of G' such that

$$D = (D_1 = C_1, D_2 = C_2, \dots, D_k = C_k, D_{k+1} = \{u\}, D_{k+2} = \{v\}).$$

It is easy to see that D is a double total dominator coloring of G' because:

- D is proper (obvious),
- for each vertex w in G , w dominates colors $k + 1$ and $k + 2$, u and v dominate colors i and j with $i \neq j \in \{1, \dots, k\}$.

Now, we prove the sufficiency. Let $C = (C_1, C_2, \dots, C_k, C_{k+1}, C_{k+2})$ be a $(k + 2)$ -double total dominator coloring of G' . We construct a k -coloring D of G . Since C is a double total dominator coloring, C is also proper. So, there exist two color classes C_i and C_j such that $C_i = \{u\}$ and $C_j = \{v\}$. Then D is the set of k other colors C_l such that $l \notin \{i, j\}$. ■

4. GENERAL BOUNDS

In this section, we present some sharp lower and upper bounds for the double total dominator chromatic number of a graph.

Consider an arbitrary $\chi_{dd}^t(G)$ -coloring of G such that $\delta(G) \geq 2$, and let D be a set consisting of one vertex from each of the $\chi_{dd}^t(G)$ color classes. Since every vertex in G dominates two color classes, then the set D is a DTD-set in G , implying that $\gamma_{\times 2, t}(G) \leq |D| = \chi_{dd}^t(G)$. Hence, we have the following result.

Observation 5. For every graph G with $\delta(G) \geq 2$, $\gamma_{\times 2, t}(G) \leq \chi_{dd}^t(G)$.

Moreover, the following result thereby establishes upper and lower bounds on the double total dominator chromatic number of an arbitrary graph in terms of its double total domination number and chromatic number.

Theorem 6. For every graph G with $\delta(G) \geq 2$ it holds

$$\max\{\gamma_{\times 2, t}(G), \chi(G)\} \leq \chi_{dd}^t(G) \leq \gamma_{\times 2, t}(G) + \chi(G).$$

Proof. To prove the upper bound, let $\mathcal{C}$ be a proper coloring of G with $\chi(G)$ colors. Now, we assign the colors $\chi(G) + 1, \chi(G) + 2, \dots, \chi(G) + \gamma_{\times 2, t}(G)$ to the

vertices of a $\gamma_{\times 2,t}(G)$ -set of G keeping the other vertices colored as before. The new coloring is a DTD-coloring of G since it is still a proper coloring and the double total dominating set provides the two color classes that every vertex has properly dominated. Therefore, $\chi_{dd}^t(G) \leq \gamma_{\times 2,t}(G) + \chi(G)$.

To prove the lower bound, we observe that every DTD-coloring of G is a proper coloring of G , and so $\chi(G) \leq \chi_{dd}^t(G)$. Hence, using Observation 5, we have $\max\{\gamma_{\times 2,t}(G), \chi(G)\} \leq \chi_{dd}^t(G)$. This completes the proof of the theorem. ■

As an immediate consequence of Theorem 6, we have the following result.

Corollary 7. *Let G be a bipartite graph with $\delta(G) \geq 2$. Then*

$$\gamma_{\times 2,t}(G) \leq \chi_{dd}^t(G) \leq \gamma_{\times 2,t}(G) + 2.$$

The next theorem presents the lower and upper bounds for the double total dominator chromatic number of a connected graph G of order n with minimum degree $\delta(G) \geq 2$.

Theorem 8. *Let G be a connected graph of order n , with $\delta(G) \geq 2$. Then $3 \leq \chi_{dd}^t(G) \leq n$. Furthermore, the following holds.*

1. $\chi_{dd}^t(G) = 3$ if and only if G is a complete 3-partite graph.
2. $\chi_{dd}^t(G) = n$ if and only if for every $x, y \in V(G)$ such that $xy \notin E(G)$, there exists $u \in N(x) \cup N(y)$ satisfying $d(u) = 2$.

Proof. Observation 5 implies that $\gamma_{\times 2,t}(G) \leq \chi_{dd}^t(G)$, and since the double total domination number of any graph is at least 3, we obtain $3 \leq \chi_{dd}^t(G) \leq n$. Now, we prove the following.

1. Assume that G is a complete 3-partite graph, obviously, $\chi_{dd}^t(G) = 3$. Conversely, assume that $\chi_{dd}^t(G) = 3$. Let $\mathcal{C} = (C_1, C_2, C_3)$ be a $\chi_{dd}^t(G)$ -coloring. Since $\mathcal{C}$ is a DTD-coloring, every vertex of C_1 dominates C_2 and C_3 . Similarly, any vertex of C_2 (respectively, C_3) dominates C_1 and C_3 (respectively, C_1 and C_2). Thus, G is a complete 3-partite graph with partitions C_1 , C_2 , and C_3 .

2. Assume that G is a connected graph with $\delta(G) \geq 2$ and let $x, y \in V(G)$ such that $xy \notin E(G)$. Suppose that for each $u \in N(x) \cup N(y)$, $d(u) \geq 3$. We show that $\chi_{dd}^t(G) < n$. We allow the color 1 to x, y and the colors $2, \dots, n-1$ to the remaining $n-2$ vertices. Clearly, this is a DTD-coloring of G . Thus, $\chi_{dd}^t(G) \leq n-1$, as desired. Conversely, assume that G is a graph of order n and for each $x, y \in V(G)$ such that $xy \notin E(G)$, there exists $u \in N(x) \cup N(y)$ with $d(u) = 2$. Clearly, if G is a complete graph of n vertices, then $\chi_{dd}^t(G) = n$. Now assume that $G \neq K_n$, this implies that there exist $x, y \in V(G)$ such that $xy \notin E(G)$. Hence there exists $u \in N(x) \cup N(y)$ such that $d(u) = 2$. By Observation 1, x or y (possibly x and y) used a unique color in every DTD-coloring of G . Apply the same logic for any two non-adjacent vertices. Obviously,

the remaining non-colored vertices form a complete subgraph, thus these vertices must be colored with a unique color. Therefore $\chi_{dd}^t(G) = n$. ■

As an immediate consequence of Theorem 8, we have the following results.

Corollary 9. *The following holds.*

1. For $n \geq 3$, $\chi_{dd}^t(K_n) = n$.
2. For $n \geq 3$, $\chi_{dd}^t(C_n) = n$.
3. For $n \geq 1$ and $q \geq 3$, we have $\chi_{dd}^t(D_q^n) = n(q - 1) + 1$.

Corollary 10. *For every connected graph G with $\delta(G) \geq 2$,*

$$\chi_{dd}^t(G \circ P_n) = (n + 1)|V(G)| \text{ if and only if } n = 2.$$

Proof. If $n = 2$, we can easily prove that for each $x, y \in V(G \circ P_2)$ such that $xy \notin E(G \circ P_2)$, there exists $u \in N(x) \cup N(y)$ with $d(u) = 2$. Using Theorem 8, we have $\chi_{dd}^t(G \circ P_2) = 3|V(G)|$, as desired. Conversely, assume that $\chi_{dd}^t(G \circ P_n) = (n + 1)|V(G)|$. By contradiction, suppose that $n \geq 3$. Let $v_1, \dots, v_n$ be the vertices of the i -th copy of P_n of the construction of $G \circ P_n$. We have $v_1 v_n \notin E(G \circ P_n)$ and for every $u \in N_{G \circ P_n}(v_1) \cup N_{G \circ P_n}(v_n)$, $d_{G \circ P_n}(u) \geq 3$. Then, by using Theorem 8, we have $\chi_{dd}^t(G \circ P_n) < |V(G)|(n + 1)$. This contradicts our assumption and hence $n = 2$. ■

The next theorem presents a sharp upper bound for the double total dominator chromatic number of a connected graph in terms of its double total domination number and the chromatic number of its induced subgraph.

Theorem 11. *Let G be a connected graph of order n with $\delta(G) \geq 2$ and let $\mathcal{D}_t(G)$ denote the set of all $\gamma_{\times 2, t}(G)$ -sets in G . Then*

$$\chi_{dd}^t(G) \leq \gamma_{\times 2, t}(G) + \min_{D \in \mathcal{D}_t(G)} \{\chi(G[V \setminus D])\}.$$

Proof. Let D be an arbitrary $\gamma_{\times 2, t}(G)$ -set and let $\mathcal{C}$ be a proper coloring of the graph $G[V \setminus D]$ using $\chi(G[V \setminus D])$ colors. We extend the coloring $\mathcal{C}$ to a coloring of the vertices of G by assigning to each vertex in D a new and distinct color. Let $\mathcal{C}'$ denote the resulting coloring of G . We note that $\mathcal{C}'$ uses $\gamma_{\times 2, t}(G) + \chi(G[V \setminus D])$ colors. Since D is a DTD-set of G , every vertex in G is adjacent to at least two vertices of D . Since each vertex of D belongs to one color class of $\mathcal{C}'$, then each vertex in G is therefore adjacent to every vertex of two color classes in the coloring $\mathcal{C}'$. Hence, $\mathcal{C}'$ is a DTD-coloring of G using $\gamma_{\times 2, t}(G) + \chi(G[V \setminus D])$ colors. Therefore, $\chi_{dd}^t(G) \leq \gamma_{\times 2, t}(G) + \chi(G[V \setminus D])$. This is true for every $\gamma_{\times 2, t}(G)$ -set D . The desired result follows by choosing D to be a $\gamma_{\times 2, t}(G)$ -set that minimizes $\chi(G[V \setminus D])$. ■

The next theorem gives some lower and upper bounds for the double total dominator chromatic number of a graph in terms of the double total dominator chromatic numbers of its connected components.

Theorem 12. *Let G be a graph with $\delta(G) \geq 2$. If $G_1, G_2, \dots, G_k$, $k \geq 2$, are all connected components of G , then*

$$\max_{1 \leq i \leq k} \{\chi_{dd}^t(G_i)\} + 3k - 3 \leq \chi_{dd}^t(G) \leq \sum_{i=1}^k \chi_{dd}^t(G_i)$$

and these bounds are sharp.

Proof. For each $1 \leq i \leq k$, the component G_i has color classes $C_{i1}, C_{i2}, \dots, C_{i_{i_i}}$. Clearly, $\mathcal{C} = (C_{i1}, C_{i2}, \dots, C_{i_{i_i}} : 1 \leq i \leq k)$ is a DTD-coloring of G . Thus, $\chi_{dd}^t(G) \leq \sum_{i=1}^k \chi_{dd}^t(G_i)$.

Next, we prove the lower bound. Let G_s be a component of G with maximum double total dominator chromatic number, then $\chi_{dd}^t(G_s) = \max_{1 \leq i \leq k} \{\chi_{dd}^t(G_i)\}$. Since we need to at least three new colors for coloring the vertices of every G_i , when $i \neq s$, we obtain

$$\chi_{dd}^t(G) \geq \max_{1 \leq i \leq k} \{\chi_{dd}^t(G_i)\} + 3k - 3.$$

Using Theorem 8, we trivially obtain that $\chi_{dd}^t(G) = \max_{1 \leq i \leq k} \{\chi_{dd}^t(G_i)\} + 3k - 3$ if and only if at most one connected component of G is not a complete 3-partite graph.

Consider the graph G constructed by k connected components, say $G_1, \dots, G_k$, such that each component is a complete 3-partite graph. Clearly, $\chi_{dd}^t(G) = \sum_{i=1}^k \chi_{dd}^t(G_i) = 3k$ and hence the upper bound of theorem is achieved. ■

5. DOUBLE TOTAL DOMINATOR CHROMATIC NUMBER OF SOME OPERATIONS ON GRAPHS

In this section, we examine the effects on $\chi_{dd}^t(G)$ when G is modified by deleting a vertex or deleting an edge. We also study the effects on $\chi_{dd}^t(G)$, when G is modified.

5.1. Vertex and edge removal

The graph $G - e$ is a graph obtained from G by simply removing the edge e and the graph $G - v$ is a graph made by deleting the vertex v and all edges that are incident with v . The following theorem gives an upper bound and a lower bound for $\chi_{dd}^t(G - e)$.

Theorem 13. *Let G be a connected graph such that $\delta(G) \geq 3$ and e be an edge of G . Then, we have $\chi_{dd}^t(G) - 1 \leq \chi_{dd}^t(G - e) \leq \chi_{dd}^t(G) + 2$.*

Proof. First, we prove that $\chi_{dd}^t(G) - 1 \leq \chi_{dd}^t(G - e)$. We shall present a DTD-coloring of $G - e$ such that $e = uv$. If we add the edge e to $G - e$, then we distinguish two cases. If the vertices u and v have the same color in the DTD-coloring of $G - e$, then we add a new color, to one of them. Since, every vertex dominates two classes in a DTD-coloring of $G - e$, then this is a DTD-coloring of G . Thus, we have $\chi_{dd}^t(G) \leq \chi_{dd}^t(G - e) + 1$. If the vertices u and v do not have the same color in the DTD-coloring of $G - e$, then the DTD-coloring of $G - e$ can be a DTD-coloring of G . So $\chi_{dd}^t(G) \leq \chi_{dd}^t(G - e)$ and therefore we have $\chi_{dd}^t(G) - 1 \leq \chi_{dd}^t(G - e)$. Now, we prove that $\chi_{dd}^t(G - e) \leq \chi_{dd}^t(G) + 2$. We shall present a DTD-coloring of G . Suppose that the vertex u has color i and v has color j . We denote by C_i and C_j the set of vertices colored with color i and j , respectively. We have the following cases.

Case 1. The vertex u do not dominate the color class j and v does not dominate the color class i in the DTD-coloring of G . Then, the DTD-coloring of G can be a DTD-coloring of $G - e$. Thus, we have $\chi_{dd}^t(G - e) \leq \chi_{dd}^t(G)$.

Case 2. The vertex u dominates the color class j but v does not dominate the color class i in the DTD-coloring of G . Since u dominates the color class j for the DTD-coloring, then we have two possibilities.

(i) If $|C_j| \geq 2$. This means that v and at least another vertex adjacent to u have color j . We give a new color l to all the adjacent vertices of u except v . This coloring is a DTD-coloring for $G - e$. Therefore, $\chi_{dd}^t(G - e) \leq \chi_{dd}^t(G) + 1$.

(ii) If $|C_j| = 1$. Then all adjacent vertices of u except v does not have color j . Since, $d_{G-e}(u) \geq 2$ and u dominates another color class, said C_k , in DTD-coloring of G , then there exists vertex w which is adjacent to u . If $|C_k| = 1$, we choose $w \notin C_k$; otherwise $w \in C_k$. We give to w a new color l and hence this coloring is a DTD-coloring for $G - e$. Thus, we have $\chi_{dd}^t(G - e) \leq \chi_{dd}^t(G) + 1$.

Case 3. The vertex u dominates the color class j and v dominates the color class i in the DTD-coloring of G . We have three possibilities.

(i) If $|C_i| \geq 2$ and $|C_j| \geq 2$, then there are some vertices that are adjacent to u and have color j . Hence we color all of them with color l . Besides, since there are some vertices that are adjacent to v having color i , then we color all of them with color k . So, this is a DTD-coloring for $G - e$.

(ii) If $|C_i| \geq 2$ and $|C_j| = 1$, then only vertex v has color j . We do the same as in (ii) of Case 2 by giving a new color k to an adjacent vertex of u (except v). In addition, there are some vertices which are adjacent to v and having color i . Then we do the same as in (i) of Case 2 by giving a new color l to all the adjacent vertices of v except u .

(iii) If $|C_i| = |C_j| = 1$, then only vertex u has color i and only vertex v has color j . We do the same as (ii) in of Case 2 by giving a new color l to one of the adjacent vertices of u and giving a new color k to one of the adjacent vertices of v .

Thus, we have $\chi_{dd}^t(G - e) \leq \chi_{dd}^t(G) + 2$. ■

Now, we present a lower bound and an upper bound for the double total dominator chromatic number of the graph $G - v$.

Theorem 14. *Let G be a connected graph with $\delta(G) \geq 3$ and v be a vertex of G . Then, we have $\chi_{dd}^t(G) - 3 \leq \chi_{dd}^t(G - v) \leq \chi_{dd}^t(G) + d_G(v) - 1$.*

Proof. First, we prove that $\chi_{dd}^t(G) - 3 \leq \chi_{dd}^t(G - v)$. We shall present a DTD-coloring of $G - v$. If we add vertex v and all its corresponding edges to $G - v$, then it suffices to give a new color i to vertex v and two new colors k and l only to two of the adjacent vertices of v . The renaming coloring remains the same. Since every vertex except v dominates two previous color classes in a DTD-coloring. Besides, the vertex v dominates the two color classes k and l . Thus, we have a DTD-coloring of G . Therefore, we have $\chi_{dd}^t(G) \leq \chi_{dd}^t(G - v) + 3$.

Now, we prove that $\chi_{dd}^t(G - v) \leq \chi_{dd}^t(G) + d_G(v) - 1$. We give a DTD-coloring of G . Suppose that the vertex v has color i . Then, we have the following cases.

Case 1. The adjacent vertices of v do not dominate the color class i in the DTD-coloring of G . In this case, every vertex dominates two color classes in a DTD-coloring of G and then this is a DTD-coloring of $G - v$. Thus, $\chi_{dd}^t(G - v) \leq \chi_{dd}^t(G)$.

Case 2. The adjacent vertices of v dominate the color class i in the DTD-coloring of G . Then, we have two possibilities.

(i) There exists another vertex with color i . In this case, every vertex dominates two classes in a DTD-coloring of G and then this is a DTD-coloring of $G - v$. So, $\chi_{dd}^t(G - v) \leq \chi_{dd}^t(G)$.

(ii) Only vertex v has color i . For every $u \in N(v)$, $d_{G-v}(u) \geq 2$ and u dominates one color class in a DTD-coloring of G (except i). Then for every $u \in N(v)$, we give to only one of the adjacent vertex of u a new color from $i, l_1, l_2, \dots, l_{d(v)-1}$. Obviously, this is a DTD-coloring for $G - v$.

Hence, for this case, we have $\chi_{dd}^t(G - v) \leq \chi_{dd}^t(G) + d_G(v) - 1$. ■

5.2. Vertex and edge contraction

Let v be a vertex in a graph G . The contraction of v in G , denoted by G/v , is the graph obtained by deleting v and adding a clique on the open neighborhood of v . Note that, if two neighbors of v are already adjacent, then they remain simply

adjacent. In a graph G , the contraction of an edge $e = uv$ is removing e and its two extremities u and v and replacing by a new vertex such that edges incident to the new vertex are the edges incident to either u or v . We denote this graph by G/e . In this subsection, we examine the effects on $\chi_{dd}^t(G)$ when G is modified by an edge or vertex contraction. First, we consider the edge contraction.

Theorem 15. *Let G be a connected graph such that $\delta(G) \geq 3$ and $e \in E(G)$. Then, we have $\chi_{dd}^t(G) - 4 \leq \chi_{dd}^t(G/e) \leq \chi_{dd}^t(G) + 1$.*

Proof. First, consider a DTD-coloring for G and construct a DTD-coloring of G/e . Suppose that $e = uv$ where $u, v \in V(G)$. The vertex u has the color i and the vertex v has the color j . We keep the same coloring for the vertices $V(G) \setminus \{u, v\}$. Now, we give the new color k to the new vertex $u = v$. Every vertex on $V(G) \setminus (N[u] \cup N[v])$ can dominate the previous color classes (or even k) in the new coloring. The new vertex $u = v$ dominates the previous color classes dominated by u or v except color classes i and j . Now, let $A = N(u) \cup N(v)$ and let $w \in A$. If w dominates a color class l different than i and j in a DTD-coloring of G , then w dominates l and k in the new coloring. However, if w dominates the color classes i and j in a DTD-coloring of G , then we have two possibilities. First, if there exists another vertex having color i (or j), then w dominates color classes k and i (or j). Second, if there is no vertex with colors i and j and since $d(w) \geq 3$, then it suffices to give color i to one of the adjacent vertices of w (except the vertex $u = v$). This is a DTD-coloring of G/e . Thus, we have $\chi_{dd}^t(G/e) \leq \chi_{dd}^t(G) + 1$.

To find the lower bound, we shall give a DTD-coloring of G/e . We add the removed vertices u and v and all the corresponding edges to G/e and keep the previous coloring for the new graph. We consider the edge $e = uv$ and remove the used color of the vertex $u = v$. Now, we give new colors i and j to vertices u and v . It suffices to give the new color l to one of the adjacent vertices of u and the new color k to one of the adjacent vertices of v . Clearly, all the vertices of the set $V(G) \setminus \{u, v\}$ dominate the two previous color classes. Moreover, the vertex u can dominate color classes j and l , and the vertex v can dominate color classes i and k . Thus, this is a DTD-coloring and hence we have $\chi_{dd}^t(G) \leq \chi_{dd}^t(G/e) + 4$. Therefore $\chi_{dd}^t(G) - 4 \leq \chi_{dd}^t(G/e)$. ■

The vertex contraction is studied in the following theorem.

Theorem 16. *Let G be a connected graph such that $\delta(G) \geq 3$ and $v \in V(G)$. Then, we have $\chi_{dd}^t(G) - d_G(v) - 3 \leq \chi_{dd}^t(G/v) \leq \chi_{dd}^t(G) + d_G(v) - 1$.*

Proof. First, we shall give a DTD-coloring of G . We remove the vertex v and construct the graph G/v . We consider one of the adjacent vertices of v , say u and do not change its color and give the new colors $i, i + 1, \dots, i + d(v) - 2$ to

the other adjacent vertices of v . Clearly, each vertex which is not adjacent to v can dominate two color classes and hence the new coloring is a DTD-coloring of G/v . Therefore, we have $\chi_{dd}^t(G/v) \leq \chi_{dd}^t(G) + d_G(v) - 1$.

To find the lower bound, at first we shall give a DTD-coloring of G/v . We add the vertex v , add all the removed edges and remove all the added edges. It suffices to give the new color i to the vertex v and give the colors l and k to only two of its adjacent vertices, say u and w . Besides, for each $x \in N_G(v)$ we give a new color to only one neighbor of x . We denote these colors by $a_1, a_2, \dots, a_{d(v)}$. All the vertices that are not adjacent to v can dominate two color classes. Clearly, all the adjacent vertices of v can dominate the color class i and one color from $a_1, a_2, \dots, a_{d(v)}$. The vertex v can dominate the color classes l and k . Obviously, this is a DTD-coloring for G . So we have $\chi_{dd}^t(G) \leq \chi_{dd}^t(G/v) + d_G(v) + 3$. This completes the proof. ■

6. DOUBLE TOTAL DOMINATOR CHROMATIC NUMBER OF THE SQUARE OF TREES

Let $T = (V, E)$ be a tree, the *square of tree* T , denoted by T^2 , is the graph with vertex set V such that two vertices u and v are adjacent in T^2 whenever $\text{dist}_T(u, v) \leq 2$.

Theorem 17. *Let T be a tree. Then $\gamma_{\times 2, t}(T^2) \leq \chi_{dd}^t(T^2) \leq \gamma_{\times 2, t}(T^2) + \Delta(T) + 1$. Moreover, all the $\Delta(T) + 2$ values can be achieved by χ_{dd}^t .*

Proof. By Observation 5 and Theorem 6, we have $\gamma_{\times 2, t}(T^2) \leq \chi_{dd}^t(T^2) \leq \gamma_{\times 2, t}(T^2) + \chi(T^2)$. Furthermore, since $\chi(T^2) = \Delta(T) + 1$, then the desired result follows.

Let $T_{p,q}$ be a tree constructed by taking a star, say $K_{1,k}$, and joining p leaves to p different paths of order 3 and joining q leaves to q different paths of order 4 where $k = p + q$ and $k \geq 2$ (see Figure 2).

Let v_0 be the center of the star and the set $\{v_1, \dots, v_k\}$ be the leaves. We denote by $P_{i,3}$ and $P_{i,4}$ the added paths to v_i where $1 \leq i \leq k$, such that $V(P_{i,3}) = \{v_1^i, v_2^i, v_3^i\}$ and $V(P_{i,4}) = \{v_1^i, v_2^i, v_3^i, v_4^i\}$. We note that for each $1 \leq i \leq p$, v_i is attached to $P_{i,3}$ and for each $p + 1 \leq i \leq k$, v_i is attached to $P_{i,4}$.

Let D be a DTD-set for $T_{p,q}^2$. This means that each vertex in $T_{p,q}^2$ is dominated by at least two vertices in D . Thus D contains the set $\{v_1^i, v_2^i : 1 \leq i \leq p\}$, the set $\{v_2^i, v_3^i : p + 1 \leq i \leq k\}$, one vertex from the set $\{v_i, v_3^i\}$ for each $1 \leq i \leq p$ and one vertex from the set $\{v_1^i, v_4^i\}$ for each $p + 1 \leq i \leq k$. Hence, $\gamma_{\times 2, t}(T_{p,q}^2) \geq 3p + 3q = 3k$. If $D = \{v_i, v_1^i, v_2^i : 1 \leq i \leq p\} \cup \{v_1^i, v_2^i, v_3^i : p + 1 \leq i \leq k\}$, then D is a DTD-set of $T_{p,q}^2$ with $|D| = 3k$. Therefore, $\gamma_{\times 2, t}(T_{p,q}^2) = 3k$. Moreover, since in every DTD-coloring of $T_{p,q}^2$, every vertex dominates two color classes, then the

Figure 2. A tree $T_{p,q}$.

sets $\{v_1^i\}$, $\{v_2^i\}$ where $1 \leq i \leq p$ and $\{v_2^i\}$, $\{v_3^i\}$ where $p+1 \leq i \leq k$ form a color classes in every DTD-coloring of $T_{p,q}^2$. In addition, for each $1 \leq i \leq p$ (respectively, $p+1 \leq i \leq k$), v_2^i (respectively, v_3^i) dominates two color classes, then the set $\{v_i, v_3^i\}$ (respectively, $\{v_1^i, v_4^i\}$) forms a color class or contains a color class as a proper subset. Besides, it is clear that the set $M = \{v_0, \dots, v_k\}$ is a maximum clique of $T_{p,q}^2$, then in every proper coloring of $T_{p,q}^2$ we need to color every vertex in M with an additional color. Hence, each DTD-coloring of $T_{p,q}^2$ uses at least $(\chi(T_{p,q}^2[M]) - p) + 3k = 3k + q + 1$ colors, implying that $3k + q + 1 \leq \chi_{dd}^t(T_{p,q}^2)$. Now, let $\mathcal{C}$ be a coloring of $T_{p,q}^2$ defined as follows.

1. $\mathcal{C}(v_i) = i$, for $1 \leq i \leq k$,
2. $\mathcal{C}(v_j^i) = jk + i$, for $1 \leq i \leq k$ and $j \in \{1, 2\}$,
3. $\mathcal{C}(v_3^i) = 3k - p + i$, for $p+1 \leq i \leq k$,
4. $\mathcal{C}(v_0) = \mathcal{C}(v_3^i) = \mathcal{C}(v_4^j) = 3k + q + 1$, for $1 \leq i \leq p$ and $p+1 \leq j \leq k$.

Clearly, $\mathcal{C}$ is a DTD-coloring of $T_{p,q}^2$ having $3k + q + 1$ colors. Thus, $\chi_{dd}^t(T_{p,q}^2) \leq 3k + q + 1$. Therefore, $\chi_{dd}^t(T_{p,q}^2) = 3k + q + 1$. Since $\Delta(T_{p,q}^2) = k$ and $k = p + q$, we have

$$\chi_{dd}^t(T_{p,q}^2) = \gamma_{\times 2, t}(T_{p,q}^2) + \Delta(T_{p,q}^2) + 1 - p \quad \text{where, } 0 \leq p \leq k.$$

If $p = 0$, then $q = k$, $\chi_{dd}^t(T_{0,k}^2) = \gamma_{\times 2, t}(T_{0,k}^2) + \Delta(T_{0,k}^2) + 1$, and hence the upper bound is achieved. Moreover, if $n \in \{3, 4, 8\}$, then $\chi_{dd}^t(P_n^2) = \gamma_{\times 2, t}(P_n^2)$ and hence the lower bound is achieved. Therefore, all the values are achieved by $\chi_{dd}^t(G)$. ■

6.1. Trees T Satisfying $\gamma_{\times 2,t}(T^2) = \chi_{dd}^t(T^2)$

In this part, we characterize trees T satisfying $\gamma_{\times 2,t}(T^2) = \chi_{dd}^t(T^2)$. Let $\mathcal{F}$ be the family of trees constructed as follows. Let $\mathcal{F}$ consists of the tree P_3 and all trees T that can be obtained from a disjoint union of $k \geq 1$ paths P_4 by adding $k - 1$ edges joining one leaf from each P_4 in such a way that the resulting graph is connected and each original P_4 contains a vertex that is a leaf in T . If $T \in \mathcal{F}$ and $T \neq P_3$, then we call each of the k original P_4 used to construct the tree T an underlying P_4 of T . We proceed further with the following property of trees in the family $\mathcal{F}$.

Lemma 18. *If $T \in \mathcal{F}$, then $\gamma_{\times 2,t}(T^2) = \chi_{dd}^t(T^2)$. Further, the color classes of a $\chi_{dd}^t(T^2)$ -coloring are unique, and consist of the support vertices and the leaf of every underlying P_4 of T .*

Proof. Let T be a tree of $\mathcal{F}$. If $T = P_3$, then the result is immediate. Hence we may assume that T has order at least 4. Thus, T can be obtained from a disjoint union of $k \geq 1$ paths of order 4 by adding $k - 1$ edges joining one leaf from each P_4 in such a way that the resulting graph is connected and each P_4 contains a vertex that is a leaf in T . Thus, T has $n = 4k$ vertices. Let $P_{1,4}, P_{2,4}, \dots, P_{k,4}$ be the k underlying P_4 of T . Assume that for each $i \in \{1, \dots, k\}$, $V(P_{i,4}) = \{v_1^i, v_2^i, v_3^i, v_4^i\}$ and $E(P_{i,4}) = \{v_j^i v_{j+1}^i : 1 \leq j \leq 3\}$. Consider the coloring $\mathcal{C}$ of T^2 defined as follows.

1. $\mathcal{C}(v_1^i) = \mathcal{C}(v_4^i) = i$, for $1 \leq i \leq k$,
2. $\mathcal{C}(v_j^i) = (j - 1)k + i$, for $1 \leq i \leq k$ and $j \in \{2, 3\}$.

Clearly, $\mathcal{C}$ is a DTD-coloring in T^2 using $3k$ color classes, then $\chi_{dd}^t(T^2) \leq 3k$. Further, since each DTD-set of a graph contains at least two vertices of the neighborhood of every vertex, every DTD-set of T^2 contains the set $\{v_2^i, v_3^i : 1 \leq i \leq k\}$ and one vertex from the vertices v_1^i and v_4^i . Hence, $3k \leq \gamma_{\times 2,t}(T^2)$. Therefore, by Observation 5, we see that $3k \leq \gamma_{\times 2,t}(T^2) \leq \chi_{dd}^t(T^2) \leq 3k$, implying that $\chi_{dd}^t(T^2) = \gamma_{\times 2,t}(T^2) = 3k = \frac{3n}{4}$.

Let $\mathcal{C}$ be a $\chi_{dd}^t(T^2)$ -coloring of T^2 . By Observation 1, each support vertex in every underlying P_4 of T forms a color class that consists only of that vertex. Further, for each support vertex v in T , the neighborhood $N_{T^2}(v)$ of v forms two color classes or contains two color classes as a proper subset (the vertex v must dominate two color classes). Hence, $\mathcal{C}$ contains at least $3k$ color classes. Besides, if v is the support vertex of T and the set $N_{T^2}(v)$ does not form two color classes in $\mathcal{C}$, then an additional color class is needed, contradicting the fact that $\chi_{dd}^t(T^2) = 3k$ and $\mathcal{C}$ is a $\chi_{dd}^t(T^2)$ -coloring. Thus, the color classes in $\mathcal{C}$ are uniquely determined, and consist of the support vertices and the leaf of every underlying P_4 of T . ■

Theorem 19. *Let T be a tree of order $n \geq 3$. Then, $\gamma_{\times 2, t}(T^2) = \chi_{dd}^t(T^2)$ if and only if $T \in \mathcal{F}$.*

Proof. The sufficiency follows from Lemma 18. To prove the necessity, we proceed by induction on the order $n \geq 3$ of a tree T that satisfies $\gamma_{\times 2, t}(T^2) = \chi_{dd}^t(T^2)$. If $n = 3$, then $T = P_3$, $\chi_{dd}^t(T^2) = \gamma_{\times 2, t}(T^2) = 3$, and $T \in \mathcal{F}$, as desired. This establishes the base case. Now, suppose that $n \geq 4$ and that if H is a tree of order n' , where $3 \leq n' < n$, satisfying $\gamma_{\times 2, t}(H^2) = \chi_{dd}^t(H^2)$, then $H \in \mathcal{F}$.

Let T be a tree of order n that satisfies $\gamma_{\times 2, t}(T^2) = \chi_{dd}^t(T^2)$. If $\text{diam}(T) = 2$, then T is a star. Since $\gamma_{\times 2, t}(T^2) = 3$ and $\chi_{dd}^t(T^2) = n$, then $n = 3$ which is impossible. If $\text{diam}(T) = 3$, then T is a double star, $\gamma_{\times 2, t}(T^2) = 3$, and $\chi_{dd}^t(T^2) = \Delta(T) + 1$. So we have $T = P_4$ and $T \in \mathcal{F}$. Hence, we can assume that $\text{diam}(T) \geq 4$. Let $\mathcal{C}$ be a $\chi_{dd}^t(T^2)$ -coloring in T^2 .

We proceed further with the following series of claims that we may assume are satisfied by the tree T .

Claim 20. *We may assume that the tree T has no strong support vertex.*

Proof. Suppose that T has a strong support vertex w that is adjacent to at least two leaves, u and v . Let $H = T - u$ and let $\mathcal{C}'$ be the restriction of the coloring $\mathcal{C}$ to the vertices of H^2 . Suppose that $\mathcal{C}'$ is not a DTD-coloring of H^2 . Then, since $\mathcal{C}$ is a DTD-coloring of T^2 , the vertices in H^2 that don't dominate two color classes in $\mathcal{C}'$ are the vertices of the set $N_{T^2}(u)$. This implies that in the coloring $\mathcal{C}$, the vertex u has a unique color. In addition, since u dominates two color classes, called C_i and C_j , and $N_{T^2}[u]$ is a complete graph, then $|C_i| = |C_j| = 1$. Without loss of generality, we may assume that $C_i = \{w\}$ and $C_j = \{x\}$ where x is a non-leaf neighbor of w . Therefore, considering the tree $H = T - v$, we see that the restriction of the coloring $\mathcal{C}$ to the vertices in H^2 is a DTD-coloring of H^2 . Hence, by renaming the vertices u and v , if necessary, we may assume that $\mathcal{C}'$ is a DTD-coloring of H^2 . Since the number of color classes in $\mathcal{C}'$ is at most the number of color classes in $\mathcal{C}$ and $\mathcal{C}$ has $\chi_{dd}^t(T^2)$ color classes, this implies that $\chi_{dd}^t(H^2) \leq \chi_{dd}^t(T^2)$. Further, since every DTD-set of H^2 contains at least two neighbors of the vertex v , and $N_{T^2}(u) \setminus \{v\} = N_{T^2}(v) \setminus \{u\}$, then every DTD-set of H^2 is a DTD-set of T^2 , implying that $\gamma_{\times 2, t}(T^2) \leq \gamma_{\times 2, t}(H^2)$. Therefore, by Observation 5, we see that

$$\chi_{dd}^t(T^2) = \gamma_{\times 2, t}(T^2) \leq \gamma_{\times 2, t}(H^2) \leq \chi_{dd}^t(H^2) \leq \chi_{dd}^t(T^2).$$

Consequently, we have $\gamma_{\times 2, t}(H^2) = \chi_{dd}^t(H^2)$. Applying the inductive hypothesis to the tree H , we see that $H \in \mathcal{F}$. Since $\text{diam}(T) \geq 4$, then H can be obtained from a k disjoint union of P_4 , where $k \geq 1$, by adding $k - 1$ edges joining one leaf from each P_4 in such a way that the resulting graph is connected and each original P_4 contains a vertex that is a leaf in H . Since the vertex w is a support vertex in H with v as a leaf neighbor, then w is a support vertex of one underlying

of H . We add the deleted vertex u back to the tree. By Lemma 18, $\{v, y\}$, $\{x\}$ and $\{w\}$ are color classes where y is an adjacent vertex of x and x is an adjacent vertex of w . Since $\mathcal{C}$ is a proper coloring, u has an additional color, and hence $\chi_{dd}^t(H^2) \leq \chi_{dd}^t(T^2) - 1$, a contradiction. Thus, we may assume that the tree T has no strong support vertex. $\square$

Let r and u be two vertices at maximum distance apart in T and root the tree at the vertex r . Recall that $\text{diam}(T) \geq 4$. Let v, w, x , and y be the parents of u, v, w , and x , respectively. As an immediate consequence of Claim 20, we see that $d_T(v) = 2$.

Claim 21. $d_T(w) = 2$

Proof. Suppose that $d_T(w) \geq 3$ and w has a child, v' , distinct from v , which is not a leaf. Analogously, as with the vertex v , we see that $d_T(v') = 2$. Let u' be the leaf neighbor of v' and consider the tree $H = T - \{u', v'\}$. We note that in the coloring $\mathcal{C}$, the vertices v, v' and w are assigned unique colors. Since $\mathcal{C}$ is a DTD-coloring in T^2 , every set consisting of exactly one, but an arbitrary, vertex from each color class of $\mathcal{C}$ is a DTD-set of T^2 . Since $\chi_{dd}^t(T^2) = \gamma_{\times 2, t}(T^2)$ such that a DTD-set is in fact a $\gamma_{\times 2, t}(T^2)$ -set. We choose a set D which consists of v, v' and w , and one vertex from every color class. The resulting set D contains one vertex from each color class of $\mathcal{C}$ and is therefore $\gamma_{\times 2, t}(T^2)$ -set. To the contrary, suppose that the vertices x and u have a different color. Hence, we may choose D so that $\{x, u\} \subset D$. But $D \setminus \{u\}$ is a DTD-set of T^2 , contradicting the minimality of the set D . Thus, the vertices x and u have the same color. Analogously, as with the vertex u , we see that u' and x have the same color. Hence, u, u' , and x have the same color. Thus, we may choose D such that $\{v, v', w, x\} \subseteq D$. But, now $D \setminus \{x\}$ is a DTD-set of T^2 , contradicting the minimality of the set D . Therefore, since T has no strong support vertex, we see that $d_T(w) = 3$ and the child, v' , of w distinct from v is a leaf.

In the coloring $\mathcal{C}$, the vertices v and w are assigned unique colors. We now consider the tree $H = T - v'$ and let $\mathcal{C}'$ be the restriction of the coloring $\mathcal{C}$ to the vertices of H^2 . Suppose to the contrary, that $\mathcal{C}'$ is not a DTD-coloring of H^2 , this implies that v' is assigned a unique color in $\mathcal{C}$. We choose now the set D which consists of one vertex from each color classes of $\mathcal{C}$. Hence, we may choose D such that $\{x, w, v, v'\} \subseteq D$. But $D \setminus \{v'\}$ is a DTD-set of T^2 , contradicting the minimality of the set D . Thus, $\mathcal{C}'$ is a DTD-coloring in H^2 , implying that $\chi_{dd}^t(H^2) \leq \chi_{dd}^t(T^2)$. Further, $\gamma_{\times 2, t}(T^2) \leq \gamma_{\times 2, t}(H^2)$ since as before we can choose a $\gamma_{\times 2, t}(H^2)$ -set which contains the vertices v, w and x . Therefore, by Observation 5, we see that $\chi_{dd}^t(T^2) = \gamma_{\times 2, t}(T^2) \leq \gamma_{\times 2, t}(H^2) \leq \chi_{dd}^t(H^2) \leq \chi_{dd}^t(T^2)$, implying that $\gamma_{\times 2, t}(H^2) = \chi_{dd}^t(H^2)$, $\chi_{dd}^t(H^2) = \chi_{dd}^t(T^2)$ and that $\mathcal{C}'$ is a $\chi_{dd}^t(H^2)$ -coloring. As before, we see that $H \in \mathcal{F}$ and using Lemma 18, $\{x, u\}$ is a color class in $\mathcal{C}'$.

But then in the coloring $\mathcal{C}$ the vertex v' has an additional color, this implies that $\chi_{dd}^t(H^2) \leq \chi_{dd}^t(T^2) - 1$, a contradiction. Therefore $d_T(w) = 2$. $\square$

Claim 22. $d_T(x) = 2$.

Proof. Suppose that $d_T(x) \geq 3$ and x has a child, w' , distinct from w , which is not a leaf and w' has a child v' and v' has a child u' . Analogously, as with the vertices v and w , we see that $d_T(v') = d_T(w') = 2$. Consider the tree $H = T - \{u', v', w'\}$ and let $\mathcal{C}'$ be the restriction of the coloring $\mathcal{C}$ to the vertices of H^2 . We note that in the coloring $\mathcal{C}$, w, w', v and v' are assigned unique colors. Since $\mathcal{C}$ is DTD-coloring in T^2 , then v dominates two color classes C_i and $C_j = \{w\}$. If $C_i = \{x\}$, (respectively, $C_i = \{u\}$; we interchange the colors of u and x), then $\mathcal{C}'$ is therefore a DTD-coloring, implying that $\chi_{dd}^t(H^2) \leq \chi_{dd}^t(T^2) - 2$. Now, if $C_i = \{u, x\}$, then u' has a unique color. We interchange the colors of u' and x and we obtain that $\mathcal{C}'$ is a DTD-coloring, implying that $\chi_{dd}^t(H^2) \leq \chi_{dd}^t(T^2) - 2$. Let D' be a $\gamma_{\times 2, t}(H^2)$ -set. Then $v, w \in D'$. If $u \in D'$, then we can simply replace the vertex u in D' with the vertex x . Hence, we may choose D' such that $x \in D'$. But then $D' \cup \{v', w'\}$ is a DTD-set of T^2 , implying that $\gamma_{\times 2, t}(T^2) \leq |D'| + 2 = \gamma_{\times 2, t}(H^2) + 2$. Therefore, by Observation 5, we see that

$$\chi_{dd}^t(T^2) = \gamma_{\times 2, t}(T^2) \leq \gamma_{\times 2, t}(H^2) + 2 \leq \chi_{dd}^t(H^2) + 2 \leq \chi_{dd}^t(T^2).$$

Consequently, we have $\gamma_{\times 2, t}(H^2) = \chi_{dd}^t(H^2)$. Further, $\chi_{dd}^t(H^2) = \chi_{dd}^t(T^2) - 2$, implying that $\mathcal{C}'$ is a $\chi_{dd}^t(H^2)$ -coloring. Applying the inductive hypothesis to the tree H , we see that $H \in \mathcal{F}$. Since the vertex v is a support vertex in H with u as a leaf neighbor, then the vertices v and w are the support vertices of one of the underlying P_4 of H . By Lemma 18, $\{u, x\}$ is a color class in $\mathcal{C}'$. In order to dominate two color classes by the vertex v' in the coloring $\mathcal{C}$, then the vertex u' is assigned a unique color in $\mathcal{C}$. But then the DTD-coloring $\mathcal{C}'$ of H^2 contains three less color classes than does $\mathcal{C}$, implying that $\chi_{dd}^t(H^2) \leq \chi_{dd}^t(T^2) - 3$, a contradiction. Therefore, we see that $d_T(x) \geq 3$ and the child, w' , of x distinct from w has a child v' , which is a leaf.

Now, we consider the tree $H = T - \{v', w'\}$. We note that in the coloring $\mathcal{C}$, the vertices w, w', v and x are assigned unique colors, then the restriction, $\mathcal{C}'$, of the coloring $\mathcal{C}$ to the vertices of H is therefore a DTD-coloring, implying that $\chi_{dd}^t(H^2) \leq \chi_{dd}^t(T^2) - 1$. Let S be a $\gamma_{\times 2, t}(H^2)$ -set. Then $\{v, w, x\} \subseteq S$. But then $S \cup \{w'\}$ is a DTD-set of T^2 , implying that $\gamma_{\times 2, t}(T^2) \leq |S| + 1 = \gamma_{\times 2, t}(H^2) + 1$. Therefore, by Observation 5, we see that

$$\chi_{dd}^t(T^2) = \gamma_{\times 2, t}(T^2) \leq \gamma_{\times 2, t}(H^2) + 1 \leq \chi_{dd}^t(H^2) + 1 \leq \chi_{dd}^t(T^2).$$

Hence, $\gamma_{\times 2, t}(H^2) = \chi_{dd}^t(H^2)$ and $\chi_{dd}^t(H^2) = \chi_{dd}^t(T^2) - 1$, implying that $\mathcal{C}'$ is a $\chi_{dd}^t(H^2)$ -coloring. Applying the inductive hypothesis to the tree H , we see that $H \in \mathcal{F}$. By Lemma 18, $\{u, x\}$ is a color class in $\mathcal{C}'$; which contradicts the fact

that x is assigned a unique color in $\mathcal{C}$. Therefore, since T has no strong support vertex, we see that $d_T(x) = 3$ and the child w' is a leaf.

In the coloring $\mathcal{C}$, the vertices v and w are assigned unique colors. We now consider the tree $H = T - w'$ and let $\mathcal{C}'$ be the restriction of the coloring $\mathcal{C}$ to the vertices of H^2 . Clearly, $\mathcal{C}'$ is a DTD-coloring in H^2 , implying that $\chi_{dd}^t(H^2) \leq \chi_{dd}^t(T^2)$. Since as before we can choose a $\gamma_{\times 2,t}(H^2)$ -set which contains the vertex x , then $\gamma_{\times 2,t}(T^2) \leq \gamma_{\times 2,t}(H^2)$. Therefore, by Observation 5, we see that

$$\chi_{dd}^t(T^2) = \gamma_{\times 2,t}(T^2) \leq \gamma_{\times 2,t}(H^2) \leq \chi_{dd}^t(H^2) \leq \chi_{dd}^t(T^2).$$

This implies that $\gamma_{\times 2,t}(H^2) = \chi_{dd}^t(H^2)$, $\chi_{dd}^t(H^2) = \chi_{dd}^t(T^2)$ and $\mathcal{C}'$ is a $\chi_{dd}^t(H^2)$ -coloring. As before, we see that $H \in \mathcal{F}$ and $\{u, x\}$ is a color class in $\mathcal{C}'$. But then in the coloring $\mathcal{C}$ the vertex w' is assigned an additional color and $\chi_{dd}^t(H^2) \leq \chi_{dd}^t(T^2) - 1$, a contradiction. Therefore, $d_T(x) = 2$. $\square$

By Claims 20, 21 and 22, we see that $d_T(x) = d_T(w) = d_T(v) = 2$. Recall that $\mathcal{C}$ is defined earlier to be a $\chi_{dd}^t(T^2)$ -coloring.

Claim 23. *The sets $\{u, x\}$, $\{v\}$ and $\{w\}$ form a color class in $\mathcal{C}$.*

Proof. Since $\mathcal{C}$ is a DTD-coloring in T^2 , every set consisting of exactly one, but an arbitrary, vertex from each color class of $\mathcal{C}$ is a DTD-set of T^2 . Suppose that the vertices u , v , w , and x are assigned different colors. In this case, we choose a set D to consist of u , v , w , x , and one vertex from each color class that does not contain u , v , w or x . The resulting set D contains one vertex from each color class of $\mathcal{C}$ and is therefore a $\gamma_{\times 2,t}(T^2)$ -set. Moreover, $D \setminus \{u\}$ is a DTD-set of T , contradicting the minimality of the set D . Hence, at most three colors are used to color the vertices u , v , w and x . Since $\mathcal{C}$ is a DTD-coloring of T^2 and $d_{T^2}(u) = 2$, the vertices v and w are assigned unique colors. Therefore, the vertices u and x are assigned the same color. Since the vertex v must dominate two color classes, then the set $\{u, x\}$ forms a color class. Thus, the sets $\{v\}$, $\{w\}$, and $\{u, x\}$ form a color class in $\mathcal{C}$. $\square$

We consider the tree $H = T - \{u, v, w, x\}$.

Claim 24. $\gamma_{\times 2,t}(T^2) = \gamma_{\times 2,t}(H^2) + 3$ and $\gamma_{\times 2,t}(H^2) = \chi_{dd}^t(H^2)$.

Proof. Every DTD-set in H^2 can be extended to a DTD-set in T^2 by adding the vertices x , w and v , implying that $\gamma_{\times 2,t}(T^2) \leq \gamma_{\times 2,t}(H^2) + 3$. Let $\mathcal{C}'$ be the restriction of the coloring $\mathcal{C}$ to the vertices in H^2 . By Claim 23, the sets $\{v\}$, $\{w\}$ and $\{u, x\}$ form a color classes in the coloring $\mathcal{C}$. Thus, the coloring $\mathcal{C}'$ has three fewer color classes than does the coloring $\mathcal{C}$. Suppose that $\mathcal{C}'$ is not a DTD-coloring in H^2 . The only possible vertex in H which does not dominate two color classes in $\mathcal{C}'$ is the vertex y , implying that $\chi_{dd}^t(H^2) \leq \chi_{dd}^t(T^2) - 2$. Therefore, by Observation 5, we see that

$$\chi_{dd}^t(T^2) = \gamma_{\times 2,t}(T^2) \leq \gamma_{\times 2,t}(H^2) + 3 \leq \chi_{dd}^t(H^2) + 3 \leq \chi_{dd}^t(T^2) + 1.$$

Consequently, $\gamma_{\times 2,t}(T^2) - 3 \leq \gamma_{\times 2,t}(H^2) \leq \gamma_{\times 2,t}(T^2) - 2$. Thus, we distinguish two cases as follow.

Case 1. If $\gamma_{\times 2,t}(H^2) = \gamma_{\times 2,t}(T^2) - 2$, then $\chi_{dd}^t(H^2) = \gamma_{\times 2,t}(H^2)$. Applying the inductive hypothesis to the tree H , we see that $H \in \mathcal{F}$. By Claim 23 and Lemma 18, the coloring $\mathcal{C}'$ is a DTD-coloring, a contradiction.

Case 2. If $\gamma_{\times 2,t}(H^2) = \gamma_{\times 2,t}(T^2) - 3$, every set consisting of exactly one vertex from each color class of $\mathcal{C}$ is a $\gamma_{\times 2,t}(T^2)$ -set. Let D be a $\gamma_{\times 2,t}(T^2)$ -set. Since $T^2[N_H[y]]$ is a clique, the vertices of $N_H[y]$ are assigned different colors in $\mathcal{C}$. This implies that we can choose D to consist of $x, v, w, N_H[y]$, and one vertex from every color class that does not contain x, v, w or $N_H[y]$. Clearly, $D' = D \setminus \{x, v, w\}$ is a DTD-set of H^2 so that $y \in D'$. Since $\gamma_{\times 2,t}(H^2) = \gamma_{\times 2,t}(T^2) - 3$, D' is a $\gamma_{\times 2,t}(H^2)$ -set. But, since $\mathcal{C}$ is a DTD-coloring of T^2 and r is a root in T and using Claims 20 and 21, we get that $D' \setminus \{y\}$ is a DTD-set of H^2 , contradicting the minimality of the set D' .

Thus, $\mathcal{C}'$ is DTD-coloring in H^2 and $\chi_{dd}^t(H^2) \leq \chi_{dd}^t(T^2) - 3$. Therefore, by Observation 5, we see that

$$\chi_{dd}^t(T^2) = \gamma_{\times 2,t}(T^2) \leq \gamma_{\times 2,t}(H^2) + 3 \leq \chi_{dd}^t(H^2) + 3 \leq \chi_{dd}^t(T^2).$$

In particular, $\gamma_{\times 2,t}(H^2) = \chi_{dd}^t(H^2)$ and $\gamma_{\times 2,t}(T^2) = \gamma_{\times 2,t}(H^2) + 3$. $\square$

We now continue with the proof of Theorem 19. By Claim 24, $\chi_{dd}^t(H^2) = \gamma_{\times 2,t}(H^2)$. Applying the inductive hypothesis to the tree H , we see that $H \in \mathcal{F}$. If $H = P_3$, then $T = P_7$, $\gamma_{\times 2,t}(T^2) = 5$, and $\gamma_{\times 2,t}(H^2) = 3$; which is impossible. Thus, H has order at least 4 and then it can be obtained from a k disjoint union of P_4 , where $k \geq 1$, by adding $k - 1$ edges joining one leaf from each P_4 in such a way that the resulting graph is connected and each original P_4 contains a vertex that is a leaf in H . We denote by $P_4(y)$ the underlying P_4 of the tree H that contains the vertex y .

Claim 25. *The following holds.*

- (i) *The vertex y is a leaf in the $P_4(y)$.*
- (ii) *The $P_4(y)$ contains a vertex that is a leaf in T .*

Proof. (i) Suppose to the contrary, that y is an internal vertex of $P_4(y)$. Let D be a $\gamma_{\times 2,t}(H^2)$ -set. As observed earlier, the set D contains support vertices and a leaf of each underlying P_4 of the tree H . We can choose D containing the k leaves in H . Let y' be the leaf of H in the underlying $P_4(y)$ that belongs to the set D . Then the set $(D \setminus \{y'\}) \cup \{x, w, v\}$ is a DTD-set of T^2 , implying that $\gamma_{\times 2,t}(T^2) \leq |D| + 2 = \gamma_{\times 2,t}(H^2) + 2$, contradicting Claim 24. Thus, the vertex y is a leaf in the $P_4(y)$.

(ii) Suppose to the contrary that the $P_4(y)$ contains no leaf of T . By (i), we have $k \geq 2$. Since $H \in \mathcal{F}$, we note that z is the parent of y and t is the parent of z in T . Let D be a $\gamma_{\times 2, t}(H^2)$ -set. We can choose the set D which consists of each support vertex of each underlying P_4 and the k leaves in H , implying that t, z and y belong to D . But $(D \setminus \{t\}) \cup \{x, w, v\}$ is a DTD-set of T^2 , implying that $\gamma_{\times 2, t}(T^2) \leq |D| + 2 = \gamma_{\times 2, t}(H^2) + 2$, once again contradicting Claim 24. Thus, the $P_4(y)$ contains a vertex that is a leaf in T . $\square$

By Claim 25, the vertex y is a leaf of $P_4(y)$. Further, the $P_4(y)$ contains a vertex that is leaf in T different of y . This implies that $T \in \mathcal{F}$, where the underlying P_4 in the tree T consists of the P_4 induced by $\{u, v, w, x\}$ and the k underlying P_4 of the tree H . This completes the proof of Theorem 19. $\blacksquare$

6.2. Square of trees with large double total dominator chromatic number

In this section, we establish an upper bound on the double total dominator chromatic number of the square of trees in terms of their order. In addition, we characterize the trees whose square has a large double total dominator chromatic number.

For $i \geq 1$, let $\mathcal{F}_i$ be the family of all trees T that can be obtained from a star, $K_{1,k}$ where $k \geq 2$, by attaching a path of order i to one leaf vertex, said v , of $V(K_{1,k})$. The leaf v is called *base leaf*. By setting $\mathcal{M} = \bigcup_{i=1}^4 \mathcal{F}_i$. Clearly, if $K_{1,k}$, where $k \geq 2$, is the star used to construct a tree $T \in \mathcal{M}$, then T has order $n = k + 1 + i$.

Further, if $T \in \mathcal{F}_4$ and uvw is the path added to the base leaf when constructing T , then in every DTD-coloring of T^2 the vertices of $K_{1,k}$, v , and w are assigned $k + 3$ distinct colors. Moreover, the set $\{u, x\}$ contains at least one color class, implying that every DTD-coloring of T^2 uses at least $k + 4$ color classes. So, $\chi_{dd}^t(T^2) \geq k + 4$. Now, assigning the same color to the vertices u and x , and assigning a new color to each remaining vertex of T . This is a DTD-coloring of T^2 with $k + 4$ color classes, and then $\chi_{dd}^t(T^2) \leq k + 4$. Consequently, for each $T \in \mathcal{F}_4$, $\chi_{dd}^t(T^2) = k + 4 = n - 1$. In the same manner, we have for each $T \in \mathcal{F}_i$ where $1 \leq i \leq 3$, $\chi_{dd}^t(T^2) = k + i = n - 1$. We state this formally as follows.

Observation 26. *If T is a tree of $\mathcal{M}$ with order $n \geq 4$, then $\chi_{dd}^t(T^2) = n - 1$.*

We are now in a position to present an upper bound on the double total dominator chromatic number of the square of a non-star tree in terms of its order.

Theorem 27. *Let T be a non-star tree of order $n \geq 4$. Then, $\chi_{dd}^t(T^2) \leq n - 1$ with equality if and only if $T \in \mathcal{M}$.*

Proof. We proceed by induction on the order $n \geq 4$ of a non-star tree T . If $n = 4$, then $T = P_4$, $T \in \mathcal{F}_1$, and $\chi_{dd}^t(T^2) = 3 = n - 1$. This establishes the base case. Suppose that $n \geq 5$ and that for every tree H of order n' , where $4 \leq n' < n$, we have $\chi_{dd}^t(H^2) \leq n' - 1$, with equality if and only if $H \in \mathcal{M}$.

Let T be a non-star tree of order $n \geq 5$. If T is a double star, $DS_{p,q}$, then $\Delta(T) \leq p + q = n - 2$ and hence $\chi_{dd}^t(T^2) = \Delta(T) + 1 \leq n - 1$. If $\chi_{dd}^t(T^2) = n - 1$, then $\Delta(T) = n - 2$, $T = DS_{1,n-3}$, and hence $T \in \mathcal{F}_1$. Now, we assume that $\text{diam}(T) \geq 4$.

Assume that T has at least two strong support vertices v_1 and v_2 . If u_1 and u_2 (respectively, w_1 and w_2) are leaf neighbors of v_1 (respectively, v_2), then assign the color 1 to the vertices u_1 and w_1 , the color 2 to the vertices u_2 and w_2 , and assign a new color to each remaining vertex of T . This is a DTD-coloring of T^2 with $n - 2$ colors, and hence $\chi_{dd}^t(T^2) \leq n - 2$.

Now, we may assume that T has at most one strong vertex, called s . Let r and u be two vertices at maximum distance apart in T and root the tree at the vertex r . Let v be the parent of u , let w be the parent of v and let x be the parent of w . Let H be the tree, of order n' , obtained from T by deleting the vertex w and all its descendants, $H = T - V(T_w)$. Since $\text{diam}(T) \geq 4$, we note that $n' \geq 2$. We proceed further with the following series of claims.

Claim 28. *If $n' = 2$ and $T \notin \mathcal{F}_2$, then $\chi_{dd}^t(T^2) < n - 1$.*

Proof. Suppose that $n' = 2$ and $T \notin \mathcal{F}_2$. We distinguish two cases. If w is the strong support vertex of T , it implies that $w = s$. Then, T can be obtained from a star $K_{1,k}$, where $k \geq 4$, by subdividing p edges exactly once where $2 \leq p \leq k - 2$ (see Figure 3(e)). Then, $n = k + p + 1$ and $\chi_{dd}^t(T^2) = k + 1 = n - p$. Since $p \geq 2$, $\chi_{dd}^t(T^2) \leq n - 2$. If w is not a strong support vertex of T . Then we have two possibilities as follows.

- T has not a strong support vertex. Then T is obtained from a star $K_{1,k}$, where $k \geq 3$, by subdividing at least $k - 1$ edges exactly once. Assume first that $k - 1$ edges of the star are subdivided, that is T is isomorphic to the tree in Figure 3(b). Then $n = 2k$ and $\chi_{dd}^t(T^2) = k + 1 = n - k + 1 \leq n - 2$. Assume now that k edges of the star are subdivided, that is T is isomorphic to the tree in Figure 3(a). Then $n = 2k + 1$ and $\chi_{dd}^t(T^2) = k + 1 = n - k \leq n - 3$.
- T has a strong support vertex s with $q \geq 2$ leaf neighbors. Then T is obtained from a double star $DS_{k,q}$ with support vertices w and s , where $k \geq 2$ leaves are attached at w and $q \geq 2$ leaves are attached at s , by subdividing at least $k - 1$ pendant edges incident with w exactly once. Assume first that exactly $k - 1$ pendant edges incident with w are subdivided, that is, T is isomorphic to the tree in Figure 3(c). Hence $n = 2k + q + 1$ and $\chi_{dd}^t(T^2) = k + q + 1 = n - k \leq n - 2$. Assume now that the k pendant edges incident with w are subdivided,

that is, T is isomorphic to the tree in Figure 3(d). Then $n = 2k + q + 2$ and $\chi_{dd}^t(T^2) = k + q + 2 = n - k \leq n - 2$. $\square$

We note that the tree $R_{5,1}$ is the tree obtained from a P_5 by adding a new vertex attached by an edge to one support vertex of P_5 .

Claim 29. *If $n' = 3$ and $T \notin \mathcal{F}_3 \cup \{R_{5,1}\}$, then $\chi_{dd}^t(T^2) < n - 1$.*

Proof. The proof of this claim is analogous to that of Claim 28. $\square$

Suppose that $n' = 2$. By Claim 28, if $T \notin \mathcal{F}_2$, then $\chi_{dd}^t(T^2) < n - 1$. By Observation 26, if $T \in \mathcal{F}_2$, then $\chi_{dd}^t(T^2) = n - 1$. Now, suppose that $n' = 3$. By Claim 29, if $T \notin \mathcal{F}_3 \cup \{R_{5,1}\}$, then $\chi_{dd}^t(T^2) < n - 1$. By Observation 26, if $T \in \mathcal{F}_3$, then $\chi_{dd}^t(T^2) = n - 1$. Now, if $T = R_{5,1}$, then $T \in \mathcal{F}_2$. Thus, $\chi_{dd}^t(T^2) = 5 = n - 1$. Therefore, we may assume that $n' \geq 4$.

Figure 3. The possible configurations of T_w .

Claim 30. *If $\chi_{dd}^t(H^2) = \gamma_{\times 2, t}(H^2)$, then $\chi_{dd}^t(T^2) \leq n - 1$. Further, if $\chi_{dd}^t(T^2) = n - 1$, then $T \in \mathcal{F}_4$.*

Proof. Assume that $\chi_{dd}^t(H^2) = \gamma_{\times 2, t}(H^2)$, then since $n' \geq 4$, we have by Theorem 19, $H \in \mathcal{F}$, implying that $\chi_{dd}^t(H^2) = \frac{3n'}{4} \leq n' - 1$.

If $d_T(w) \geq 3$, then the three possible configurations of T_w are as follows.

- T_w is obtained from a star $K_{1,k}$ where $k \geq 2$, by subdividing at least $k - 1$ edges exactly once. Assume first that $k - 1$ edges of the star are subdivided, that is T_w is isomorphic to the tree in Figure 3(b). Then $n' = n - 2k$ and every $\chi_{dd}^t(H^2)$ -coloring can be extended to a DTD-coloring of T^2 by coloring each of the k support vertices in T_w with a unique color and adding one additional color to color the k leaves of T_w . Therefore, in this case $\chi_{dd}^t(T^2) \leq \chi_{dd}^t(H^2) + k + 1 \leq n - 2k - 1 + k + 1 = n - k \leq n - 2$. Assume now that k edges of the star are subdivided, that is T_w is isomorphic to the tree in Figure 3(a). Then $n' = n - 2k - 1$ and every $\chi_{dd}^t(H^2)$ -coloring can be extended to a DTD-coloring of T^2 by coloring each of the k support vertices in T_w with a unique color, coloring the vertex

w with a unique color and adding one additional color to the k leaves of T_w . Therefore, in this case $\chi_{dd}^t(T^2) \leq \chi_{dd}^t(H^2) + k + 2 \leq n - k \leq n - 2$.

- T_w is obtained from a star $K_{1,k}$, where $k \geq 3$, by subdividing p edges exactly once where $1 \leq p \leq k - 2$ (see Figure 3(e)). Then $n' = n - k - p - 1$ and every $\chi_{dd}^t(H^2)$ -coloring can be extended to a DTD-coloring of T^2 by coloring each of the $p + 1$ support vertices in T_w with a unique color, coloring each leaf neighbor of w with a unique color ($k - p$ colors) and assigning one color from these $k - p$ colors to each remaining vertex of T_w . Therefore, in this case $\chi_{dd}^t(T^2) \leq \chi_{dd}^t(H^2) + k + 1 \leq n - p - 1 \leq n - 2$.

- T_w is obtained from a double star $DS_{k,q}$ with support vertices w and s , where $k \geq 1$ leaves are attached at w and $q \geq 2$ leaves are attached at s , by subdividing at least $k - 1$ pendant edges incident with w exactly once. Assume first that exactly $k - 1$ pendant edges incident with w are subdivided, that is, T_w is isomorphic to the tree in Figure 3(c). Then $n' = n - 2k - q - 1$ and every $\chi_{dd}^t(H^2)$ -coloring can be extended to a DTD-coloring of T^2 by coloring each of the $k + 1$ support vertices in T_w with a unique color, coloring each leaf neighbor of s with a unique color (q colors) and assigning one color from these q colors to each remaining vertex of T_w . In this case, $\chi_{dd}^t(T^2) \leq \chi_{dd}^t(H^2) + k + q + 1 \leq n - k - 1 \leq n - 2$.

Assume now that the k pendant edges incident with w are subdivided, that is, T_w is isomorphic to the tree in Figure 3(d). Then $n' = n - 2k - q - 2$ and every $\chi_{dd}^t(H^2)$ -coloring can be extended to a DTD-coloring of T^2 by coloring each of the $k + 1$ support vertices in T_w with a unique color, coloring the vertex w with a unique color, coloring each leaf neighbor of s with a unique color (q colors) and assigning one color from these q colors to each remaining vertex of T_w . Thus, in this case $\chi_{dd}^t(T^2) \leq \chi_{dd}^t(H^2) + k + q + 2 \leq n - k - 1 \leq n - 2$.

In the following, we can assume that $d_T(w) = 2$. Then $n' = n - d_T(w) - 1$ and every $\chi_{dd}^t(H^2)$ -coloring can be extended to a DTD-coloring of T^2 by coloring each vertex of $N_T[v]$ with a unique color. Thus, $\chi_{dd}^t(T^2) \leq \chi_{dd}^t(H^2) + d_T(w) + 1 \leq n - 1$. If further $\chi_{dd}^t(T^2) = n - 1$, then we must have equality throughout the previous inequality chain. In particular, $\chi_{dd}^t(H^2) = n' - 1$. Since $\chi_{dd}^t(H^2) = \frac{3n'}{4}$, we deduce that $n' = 4$ and thus H is a path P_4 that we label its vertices in order a_1, a_2, a_3, a_4 . Without loss of generality, we have $x \in \{a_1, a_2\}$. If $x = a_2$, then assigning a unique color to each vertex in $V(T) \setminus \{a_1, a_4\}$ and assigning the color used by u to a_1 and a_4 provides a DTD-coloring of T^2 using $d_T(w) + 3 = n - 2$ colors, contradicting $\chi_{dd}^t(T^2) = n - 1$. Hence $x = a_1$ and therefore $T \in \mathcal{F}_4$. $\square$

By Claim 30, we may assume that $\chi_{dd}^t(H^2) \neq \gamma_{\times 2, t}(H^2)$, otherwise the desired result follows. Recall that $H = T - V(T_w)$. First, we assume that H is a star $K_{1,k}$ centered at v_0 . Since $n' \geq 4$, we have $k \geq 3$ and hence $s = v_0$. Thus every child of w is a support vertex of degree two. If $d_T(w) \geq 3$, then one

can construct a DTD-coloring of T^2 using $n - 2$ colors, a contradiction. Hence $d_T(v) = 2$ and T_w consists of the path wvu . Recall that x is the parent of w in T . We distinguish two cases. If $x = v_0$, then $T \in \mathcal{F}_2$. Otherwise, if x is a leaf of H , then $T \in \mathcal{F}_3$. In both cases, by Observation 26, we have $\chi_{dd}^t(T^2) = n - 1$. Assume now that H is a non-star tree of order $n' \geq 4$. Applying the inductive hypothesis to the tree H , we see that $\chi_{dd}^t(H^2) \leq n' - 1$, with equality if and only if $H \in \mathcal{M}$.

Claim 31. *If $d_T(w) \geq 3$, then $\chi_{dd}^t(T^2) < n - 1$.*

Proof. The proof of this claim is analogous to that of Claim 30. $\square$

Now, we continue with the proof of Theorem 27. By Claim 31, we may assume that $d_T(w) = 2$, otherwise $\chi_{dd}^t(T^2) < n - 1$. Thus, T_w is obtained from a star $K_{1,k}$, where $k \geq 2$ centered at v . Then $n' = n - d_T(v) - 1$ and every $\chi_{dd}^t(H^2)$ -coloring can be extended to a DTD-coloring of T^2 by coloring each vertex of $N_T[v]$ with a unique color. Thus, $\chi_{dd}^t(T^2) \leq \chi_{dd}^t(H^2) + d_T(v) + 1 \leq n - 1$. If further $\chi_{dd}^t(T^2) = n - 1$, then we must have equality throughout the previous inequality chain. In particular, $\chi_{dd}^t(T^2) = \chi_{dd}^t(H^2) + d_T(v) + 1$ and $\chi_{dd}^t(H^2) = n' - 1$. Applying the inductive hypothesis to the tree H , we see that $H \in \mathcal{F}_i$ where $1 \leq i \leq 4$. Let $S_{1,q}$, where $q \geq 2$, be a star used to construct H . Let $V(S_{1,q}) = \{v_j : 0 \leq j \leq q\}$ such that v_0 is the center of $S_{1,q}$ and v_1 is the base leaf of H . For $1 \leq i \leq 4$, let P_i be the path used to construct the tree H such that $V(P_i) = \{u_j : 1 \leq j \leq i\}$, $P_i = u_1 \dots u_i$ and $u_i v_1 \in E(H)$. Let $\mathcal{C}'$ be the $\chi_{dd}^t(H^2)$ -coloring of H^2 that colors the vertices u_1 and v_2 with color α and colors all remaining vertices of H^2 with a unique color. Recall that x is the parent of w in T .

Claim 32. 1. *If $d_T(v) \geq 3$, then $x = u_1$ and H is a path P_4 .*

2. *If $d_T(v) = 2$ and $x \in V(S_{1,q})$, then $x = v_2$ and H is a path P_4 .*

3. *If $d_T(v) = 2$ and $x \notin V(S_{1,q})$, then $x = u_1$ and $H \in \mathcal{F}_1$.*

Proof. 1. Assume that $d_T(v) \geq 3$. If $x \in V(H) \setminus \{u_1\}$, then the coloring $\mathcal{C}'$ can be extended to a DTD-coloring of T^2 by coloring each vertex of $N_T[v] \setminus \{u\}$ with a unique color and assigning the color used by u_1 to u provides a DTD-coloring of T^2 using $\chi_{dd}^t(H^2) + d_T(v)$ colors, contradicting $\chi_{dd}^t(T^2) = \chi_{dd}^t(H^2) + d_T(v) + 1$. Hence $x = u_1$. Recall that $H \in \mathcal{F}_i$, where $1 \leq i \leq 4$. Assume first that $H \in \mathcal{F}_i$, where $2 \leq i \leq 4$, that is, the coloring $\mathcal{C}'$ can be extended to a DTD-coloring of T^2 by coloring each vertex of $N_T[v] \setminus \{u\}$ with a unique color and assigning the color used by u_1 to u provides a DTD-coloring of T^2 using $\chi_{dd}^t(H^2) + d_T(v)$ colors, contradicting $\chi_{dd}^t(T^2) = \chi_{dd}^t(H^2) + d_T(v) + 1$. Hence $H \in \mathcal{F}_1$. Recall that $S_{1,q}$, where $q \geq 2$, is a star used to construct H . Since T has at most one strong support vertex, then $q = 2$. Thus, H is a path P_4 .

2. Assume that $d_T(v) = 2$ and $x \in V(S_{1,q})$, where $q \geq 2$. Suppose first that $q \geq 3$. If $x \neq v_2$, then we consider the coloring $\mathcal{C}$ obtained from $\mathcal{C}'$ by assigning a unique color to each of w and v , and assigning the color used by u_1 to u . If $x = v_2$, then we consider the coloring $\mathcal{C}$ defined previously and we interchange the colors of v_2 and v_3 . In both cases, we have $\mathcal{C}$ is a DTD-coloring of T^2 , and then $\chi_{dd}^t(T^2) \leq \chi_{dd}^t(H^2) + 2$, contradicting $\chi_{dd}^t(T^2) = \chi_{dd}^t(H^2) + d_T(v) + 1 = \chi_{dd}^t(H^2) + 3$. Thus, $q = 2$ and $H \in \{P_4, P_5, P_6, P_7\}$. If $x \neq v_2$ or $x = v_2$ and $H \neq P_4$, then one can construct a DTD-coloring of T^2 using $n - 2$ colors, a contradiction. Hence $x = v_2$ and $H = P_4$.

3. Assume that $d_T(v) = 2$ and $x \notin V(S_{1,q})$, where $q \geq 2$. If $x \neq u_1$, then we consider the coloring $\mathcal{C}$ obtained from $\mathcal{C}'$ by assigning a unique color to each of w and v , and assigning the color used by u_1 to u . Thus, $\mathcal{C}$ is a DTD-coloring of T^2 , implying that $\chi_{dd}^t(T^2) \leq \chi_{dd}^t(H^2) + 2$, a contradiction. Hence $x = u_1$. Recall that P_i , where $1 \leq i \leq 4$, is the path used to construct the tree H . Assume that $i \geq 2$. If $q = 2$, then $T = P_{6+i}$ and clearly $\chi_{dd}^t(T^2) \leq n - 2$, a contradiction. Hence let $q \geq 3$. Then the coloring $\mathcal{C}$ obtained from $\mathcal{C}'$ by interchanging the colors of u_1 and u_2 , coloring each of w and v with a unique color and assigning the color used by v_2 to u , is a DTD-coloring of T^2 . This implies that $\chi_{dd}^t(T^2) \leq \chi_{dd}^t(H^2) + 2$, a contradiction. Hence $x = u_1$ and $H \in \mathcal{F}_1$. $\square$

We now continue with the proof of Theorem 27. By Claim 32(1), if $d_T(v) \geq 3$, then $x = u_1$ and H is a path P_4 . Thus, $T \in \mathcal{F}_4$. By Claim 32(2), if $d_T(v) = 2$ and $x \in V(S_{1,q})$, then $x = v_2$ and H is a path P_4 . Hence T is a path P_7 and therefore $T \in \mathcal{F}_4$. By Claim 32(3), if $d_T(v) = 2$ and $x \notin V(S_{1,q})$, then $x = u_1$ and $H \in \mathcal{F}_1$. Therefore $T \in \mathcal{F}_4$. This completes the proof of Theorem 27. $\blacksquare$

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