ON THE WEIGHT OF MINOR FACES IN TRIANGLE-FREE 3-POLYTOPES

OLEG V. BORODIN¹

Institute of Mathematics Siberian Branch
Russian Academy of Sciences, Novosibirsk, 630090, Russia
e-mail: brdnoleg@math.nsc.ru

AND

ANNA O. IVANOVA²

Ammosov North-Eastern Federal University
Yakutsk, 677000, Russia
e-mail: shmgnanna@mail.ru

Abstract

The weight $w(f)$ of a face f in a 3-polytope is the degree-sum of vertices incident with f . It follows from Lebesgue's results of 1940 that every triangle-free 3-polytope without 4-faces incident with at least three 3-vertices has a 4-face with $w \leq 21$ or a 5-face with $w \leq 17$. Here, the bound 17 is sharp, but it was still unknown whether 21 is sharp.

The purpose of this paper is to improve this 21 to 20, which is best possible.

Keywords: plane map, plane graph, 3-polytope, structural property, weight of face.

2010 Mathematics Subject Classification: 05C15.

REFERENCES

- [1] S.V. Avgustinovich and O.V. Borodin, *Neighborhoods of edges in normal maps*, Diskretn. Anal. Issled. Oper. **2** (1995) 3–9, in Russian.

¹This work was supported by grants 16-01-00499 and 15-01-05867 of the Russian Foundation for Basic Research and NSh-1939.2014.1 of President Grants for Government Support the Leading Scientific Schools of the Russian Federation.

²The author's work was performed as a part of government work "Organizing research" and supported by grant 15-01-05867 of the Russian Foundation for Basic Research.

- [2] O.V. Borodin, *Solution of Kotzig's and Grünbaum's problems on the separability of a cycle in a planar graph*, Mat. Zametki **46** (1989) 9–12, in Russian.
- [3] O.V. Borodin, *Joint generalization of the theorems of Lebesgue and Kotzig on the combinatorics of planar maps*, Diskret. Mat. **3** (1991) 24–27, in Russian.
- [4] O.V. Borodin, *Minimal weight of a face in plane triangulations without 4-vertices*, Mat. Zametki **51** (1992) 16–19, in Russian.
- [5] O.V. Borodin, *Triangulated 3-polytopes with restricted minimal weight of faces*, Discrete Math. **186** (1998) 281–285.
doi:10.1016/S0012-365X(97)00240-9
- [6] O.V. Borodin and D.V. Loparev, *The height of small faces in planar normal maps*, Diskretn. Anal. Issled. Oper. **5** (1998) 6–17, in Russian.
- [7] O.V. Borodin and D.R. Woodall, *The weight of faces in plane maps*, Mat. Zametki **64** (1998) 648–657, in Russian.
- [8] O.V. Borodin and D.R. Woodall, *Cyclic degrees of 3-polytopes*, Graphs Combin. **15** (1999) 267–277.
- [9] O.V. Borodin, *An improvement of Lebesgue's theorem on the structure of minor faces of 3-polytopes*, Diskretn. Anal. Issled. Oper. **9** (2002) 29–39, in Russian.
- [10] O.V. Borodin, *Colorings of plane graphs: a survey*, Discrete Math. **313** (2013) 517–539.
doi:10.1016/j.disc.2012.11.011
- [11] O.V. Borodin and A.O. Ivanova, *Describing 3-faces in normal plane maps with minimum degree 4*, Discrete Math. **313** (2013) 2841–2847.
doi:10.1016/j.disc.2013.08.028
- [12] O.V. Borodin, A.O. Ivanova and A.V. Kostochka, *Describing faces in plane triangulations*, Discrete Math. **319** (2014) 47–61.
doi:10.1016/j.disc.2013.11.021
- [13] O.V. Borodin, A.O. Ivanova and D.R. Woodall, *Light C_4 and C_5 in 3-polytopes with minimum degree 5*, Discrete Math. **334** (2014) 63–69.
doi:10.1016/j.disc.2014.06.024
- [14] O.V. Borodin, A.O. Ivanova and A.V. Kostochka, *Every 3-polytope with minimum degree 5 has a 6-cycle with maximum degree at most 11*, Discrete Math. **315–316** (2014) 128–134.
doi:10.1016/j.disc.2013.10.021
- [15] O.V. Borodin and A.O. Ivanova, *The vertex-face weight of edges in 3-polytopes*, Sibirsk. Mat. Zh. **56** (2015) 338–350, in Russian.
- [16] O.V. Borodin and A.O. Ivanova, *Every 3-polytope with minimum degree 5 has a 7-cycle with maximum degree at most 15*, Sibirsk. Mat. Zh. **56** (2015) 775–789, in Russian.
- [17] B. Ferencová and T. Madaras, *On the structure of polyhedral graphs with prescribed edge and dual edge weight*, Acta Univ. M. Belii Ser. Math. **12** (2005) 13–18.

- [18] B. Ferencová and T. Madaras, *Light graph in families of polyhedral graphs with prescribed minimum degree, face size, edge and dual edge weight*, Discrete Math. **310** (2010) 1661–1675.
doi:10.1016/j.disc.2009.11.027
- [19] B. Grünbaum, *Polytopal graphs*, in: Studies in Graph Theory, D.R. Fulkerson, Ed., MAA Studies in Mathematics **12** (1975) 201–224.
- [20] M. Horňák and S. Jendrol', *Unavoidable sets of face types for planar maps*, Discuss. Math. Graph Theory **16** (1996) 123–142.
doi:10.7151/dmgt.1028
- [21] S. Jendrol', *Triangles with restricted degrees of their boundary vertices in plane triangulations*, Discrete Math. **196** (1999) 177–196.
doi:10.1016/S0012-365X(98)00172-1
- [22] S. Jendrol' and H.-J. Voss, *Light subgraphs of graphs embedded in the plane and in the projective plane—a survey*, Discrete Math. **313** (2013) 406–421.
doi:10.1016/j.disc.2012.11.007
- [23] A. Kotzig, *Contribution to the theory of Eulerian polyhedra*, Mat.-Fyz. Čas. **5** (1955) 101–113, in Russian.
- [24] A. Kotzig, *From the theory of Eulerian polyhedra*, Mat. Čas. **13** (1963) 20–34.
- [25] A. Kotzig, *Extremal polyhedral graphs*, Ann. New York Acad. Sci. **319** (1979) 569–570.
- [26] H. Lebesgue, *Quelques conséquences simples de la formule d'Euler*, J. Math. Pures Appl. **19** (1940) 27–43.
- [27] T. Madaras and R. Škrekovski, *Heavy paths, light stars, and big melons*, Discrete Math. **286** (2004) 115–131.
doi:10.1016/j.disc.2013.11.052
- [28] T. Madaras and R. Soták, *The 10-cycle C_{10} is light in the family of all plane triangulations with minimum degree five*, Tatra Mt. Math. Publ. **18** (1999) 35–56.
- [29] T. Madaras, R. Škrekovski and H.-J. Voss, *The 7-cycle C_7 is light in the family of planar graphs with minimum degree 5*, Discrete Math. **307** (2007) 1430–1435.
doi:10.1016/j.disc.2005.11.080
- [30] B. Mohar, R. Škrekovski and H.-J. Voss, *Light subgraphs in planar graphs of minimum degree 4 and edge-degree 9*, J. Graph Theory **44** (2003) 261–295.
doi:10.1002/jgt.10144
- [31] O. Ore and M.D. Plummer, *Cyclic coloration of plane graphs*, Recent Progress in Combinatorics, W.T. Tutte, Ed. (Academic Press, New York, 1969) 287–293.
- [32] M.D. Plummer, *On the cyclic connectivity of planar graph*, Graph Theory and Application (Springer-Verlag, Berlin, 1972) 235–242.
- [33] M.D. Plummer and B. Toft, *Cyclic coloration of 3-polytopes*, J. Graph Theory **11** (1987) 507–515.
doi:10.1002/jgt.3190110407

- [34] E. Steinitz, *Polyheder und Raumeinteilungen*, Enzykl. Math. Wiss. (Geometrie) **3AB** (1922) 1–139.
- [35] P. Wernicke, *Über den Kartographischen Vierfarbensatz*, Math. Ann. **58** (1904) 413–426.

Received 27 November 2014
Revised 21 September 2015
Accepted 29 September 2015