ADJUSTMENT OF PARABOLIC AND LINEAR CURVES TO OBSERVATIONS TAKEN AT EQUAL INTERVALS OF THE INDEPENDENT VARIABLE

By Harry M. Roeser

Perhaps the two formulas which have the widest utility in the representation of the variation between two mutually dependent physical quantities X and Y are

$$Y = A + BX + CX^2 \quad (1)$$

and $Y = A + BX \quad (2)$

The best or most probable values of the constants A , B , and C of equation (1), or A and B of equation (2) can only be determined by the method of least squares, which involves considerable computation, apparently superficial, and often too excessive to warrant the application of that method. Recourse is often had to various graphical or empirical methods. These usually depend upon the skill of the experimenter in placing a line by eye, or the arbitrary selection of certain of the observations as having undue weight, and consequently the values obtained bear the defect of personal judgment. A reduction, either sufficiently approximate or absolutely correct, of the method of least squares to utilitarian convenience is therefore desirable.

The author purposes to develop a scheme which leaves the method of least squares without approximation as far as its application in many special cases to forms (1) and (2) is concerned. By following this scheme the major portion of the quantities involved in the mathematical expressions for the constants can be written down at once from inspection of the observations or consulting Table 5 accompanying the paper. The mathematical processes consist of the most fundamental arithmetic operations.

The proposed scheme depends upon the simple expedient of taking observations of Y at equal intervals of X . This is quite generally done in practice either involuntarily or to secure a desired continuity in a graphical representation. In such routine

practice where least-square solutions are necessary, the proposed scheme apparently has limited application on account of the requirement that observations be taken at equal intervals of the independent variable. However, the tedious arithmetical labor of making the regular solutions is conspicuously reduced with no ensuing loss in accuracy. Adapting the collection of routine data to the conditions to which the scheme applies will generally be found practicable and advantageous.

The method in its utmost simplicity applies to observations of equal weight, although by following the general plan the reduction of observations of unequal weight can be greatly expedited. The analytical development is unessential to the practical application. All necessary definitions and formulas are given in connection with Table 5.

1. SOLUTION FOR $Y=A+BX+CX^2$

Consider observations that fall along a curve of type (1).

The normal equations which are set up by the method of least squares to determine the most probable values of A , B , and C from more than three observations are—

$$\left. \begin{aligned} nA + B\Sigma X + C\Sigma X^2 &= \Sigma Y \\ A\Sigma X + B\Sigma X^2 + C\Sigma X^3 &= \Sigma XY \\ A\Sigma X^2 + B\Sigma X^3 + C\Sigma X^4 &= \Sigma XY^2 \end{aligned} \right\} \quad (3)$$

where n is the number of observations.

The general objection that the labor involved in a least-square solution is excessive is forced upon the mind by a consideration of equations (3), especially if n is a large number and the X 's contain three or more significant figures. Aside from the labor of setting up the normal equations that of merely solving them when the coefficients are large is considerable.

Let the center of coordinates be transferred to the point (X_m, Y_m) , and in the new system let equation (1) change to

$$y = a + bx + cx^2 \quad (4)$$

that is, let $y = Y - Y_m$, and $x = X - X_m$

Then, as becomes apparent from simple substitution of these values and reduction,

$$A = Y_m + a - X_m (b - cX_m) \quad (5)$$

$$B = b - 2cX_m \quad (6)$$

$$C = c \quad (7)$$

in which a , b , and c remain to be determined. Equations (3) reduce to—

$$\left. \begin{aligned} na + b\Sigma x + c\Sigma x^2 &= \Sigma y \\ a\Sigma x + b\Sigma x^2 + c\Sigma x^3 &= \Sigma xy \\ a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 &= \Sigma x^2 y \end{aligned} \right\} \quad (8)$$

Now it is a well-known property of the arithmetic mean of a set of values, that the summation of the odd powers of differences or “deviations” of the values from the arithmetic mean is zero. Therefore in equations (8), Σx , Σy , and Σx^3 are zero, since x and y are, respectively, measured from the mean values of X and Y .

After dividing each of equations (8) by n , they reduce to—

$$\left. \begin{aligned} a + \frac{c\Sigma x^2}{n} &= 0 \\ \frac{b\Sigma x^2}{n} &= \frac{\Sigma xy}{n} \\ a\frac{\Sigma x^2}{n} + \frac{c\Sigma x^4}{n} &= \frac{\Sigma x^2 y}{n} \end{aligned} \right\} \quad (9)$$

If the observations are taken at equal intervals of X , equations (9) may be further simplified by taking advantage of the fact that $\frac{\Sigma x^2}{n}$ and $\frac{\Sigma x^4}{n}$ may be made independent of x .

Let i be the interval between successive values of X . Then—

$$X_1 = X_m - \frac{(n-1)}{2} \cdot i; \text{ or, } x_1 = -\frac{(n-1)}{2} \cdot i$$

$$X_2 = X_m - \frac{(n-3)}{2} \cdot i; \quad x_2 = -\frac{(n-3)}{2} \cdot i$$

$$X_3 = X_m - \frac{(n-5)}{3} \cdot i; \quad x_3 = -\frac{(n-5)}{2} \cdot i$$

.....

$$X_{(n-1)} = X_m - \frac{n-(2n-3)}{2} \cdot i; \text{ or, } x_{(n-1)} = -\frac{n-(2n-3)}{2} \cdot i$$

$$X_n = X_m - \frac{n-(2n-1)}{2} \cdot i; \text{ or, } x_n = -\frac{n-(2n-1)}{2} \cdot i$$

Therefore

$$\frac{\Sigma x^2}{n} = \frac{i^2}{4n} \left[(n-1)^2 + (n-3)^2 + (n-5)^2 + \cdots + \{n-(2n-1)\}^2 \right] \quad (10)$$

and

$$\frac{\Sigma x^4}{n} = \frac{i^4}{16n} \left[(n-1)^4 + (n-3)^4 + (n-5)^4 + \cdots + \{n-(2n-1)\}^4 \right] \quad (11)$$

After summing the terms within the brackets, (10) and (11), reduce to—

$$\frac{\Sigma x^2}{n} = \frac{i^2}{12}(n-1)(n+1) \quad (12)$$

$$\frac{\Sigma x^4}{n} = \frac{i^4}{240}(n-1)(n+1)(3n^2-7) \quad (13)$$

The right-hand members of (12) and (13) are independent of x . Write now

$$\frac{i^2}{12}(n-1)(n+1) = i^2 u \quad (14)$$

$$\frac{i^4}{240}(n-1)(n+1)(3n^2-7) = i^4 v = \frac{i^2}{20}(3n^2-7)u \quad (15)$$

Equations (9) thus reduce to

$$\left. \begin{aligned} a + i^2 uc &= 0 \\ i^2 ub &= \frac{\Sigma xy}{n} \\ i^2 ua + i^4 vc &= \frac{\Sigma x^2 y}{n} \end{aligned} \right\} \quad (16)$$

These equations yield for the required values of a , b , and c ,

$$a = \frac{1}{i^2} \times \frac{\Sigma x^2 y}{n} \times \frac{u}{u^2 - v} \quad (17)$$

$$b = \frac{1}{i^2 u} \times \frac{\Sigma xy}{n} \quad (18)$$

$$c = -\frac{a}{i^2 u} \quad (19)$$

Let

$$S' = \frac{\Sigma xy}{n}; \quad T' = \frac{\Sigma x^2 y}{n}; \quad \frac{u}{v - u^2} = w.$$

Then

$$a = -\frac{T'w}{i^2} \quad (20)$$

$$b = \frac{S'}{i^2} \times \frac{1}{u} \quad (21)$$

$$c = -\frac{a}{i^2} \times \frac{1}{u} \quad (22)$$

For convenience in writing down the solutions values of w and $\frac{I}{u}$ are tabulated in Table 5 for values of n from 5 to 31.

The computation of S' and T' from the observations on X and Y will be greatly facilitated by use of the results obtained from the following considerations:

$$S' = \frac{\sum xy}{n} = \frac{\sum (X - X_m)(Y - Y_m)}{n}.$$

Arrange the constituent parts of the summation in pairs as follows: (1) First and last, (2) second and next to the last, (3) third and third from the last, etc., thus:

$$\left. \begin{aligned} (X_1 - X_m)(Y_1 - Y_m) &= -i \frac{(n-1)}{2} (Y_1 - Y_m) \\ (X_n - X_m)(Y_n - Y_m) &= -i \frac{n-(2n-1)}{2} (Y_n - Y_m) \end{aligned} \right\}$$

$$\left. \begin{aligned} (X_2 - X_m)(Y_2 - Y_m) &= -i \frac{(n-3)}{2} (Y_2 - Y_m) \\ (X_{n-1} - X_m)(Y_{n-1} - Y_m) &= -i \frac{n-(2n-3)}{2} (Y_{n-1} - Y_m) \end{aligned} \right\}$$

....., etc.

Now collect terms in the pairs of expressions connected by brackets; there results:

$$S' = \frac{\sum xy}{n} = \frac{i}{2n} [(n-1)(Y_n - Y_1) + (n-3)(Y_{n-1} - Y_2) + \dots] = iS \quad (23)$$

Similarly for $T' = \frac{\sum x^2 y}{n}.$

Arrange the observations in pairs:

$$\left. \begin{aligned} (X_1 - X_m)^2(Y_1 - Y_m) &= i^2 \left(\frac{n-1}{2} \right)^2 (Y_1 - Y_m) \\ (X_n - X_m)^2(Y_n - Y_m) &= i^2 \left[\frac{n-(2n-1)}{2} \right]^2 (Y_n - Y_m) \end{aligned} \right\}$$

$$\left. \begin{aligned} (X_2 - X_m)^2(Y_2 - Y_m) &= i^2 \left(\frac{n-3}{2} \right)^2 (Y_2 - Y_m) \\ (X_{n-1} - X_m)^2(Y_{n-1} - Y_m) &= i^2 \left[\frac{n-(2n-3)}{2} \right]^2 (Y_{n-1} - Y_m) \end{aligned} \right\}$$

....., etc.

Therefore,

$$\begin{aligned}
 T' &= \frac{\Sigma x^2 y}{n} - \frac{i^2}{4n} \left[(n-1)^2 (Y_1 + Y_{n-2} Y_m) + (n-3)^2 (Y_2 + Y_{n-1} - 2 Y_m) + \dots \right] \\
 &= \frac{i^2}{4n} \left[(n-1)^2 (Y_1 + Y_n) + (n-3)^2 (Y_2 + Y_{n-1}) + \dots \right] \\
 &\quad - \frac{i^2}{4n} 2 Y_m \left[(n-1)^2 + (n-3)^2 + \dots \right] \\
 &= \frac{i^2}{4n} \left[(n-1)^2 (Y_1 + Y_m) + (n-3)^2 (Y_2 + Y_{n-1}) + \dots \right] - i^2 u Y_m \\
 &= i^2 T
 \end{aligned} \tag{24}$$

Equations (20), (21), and (22) then reduce to—

$$a = -Tw \tag{25}$$

$$b = \frac{S}{i} \cdot \frac{I}{u} \tag{26}$$

$$c = -\frac{a}{i^2 u} \tag{27}$$

These last equations serve to determine completely the values of A , B , and C , according to equations (5), (6), and (7).

To save mental effort in the summations, values are furnished in Table 5 for the squares of the terms containing n . Thus if n is an even number, say 18, $(n-1)^2$ is given in the table opposite 18; $(n-3)^2$ is the next preceding tabular value in that column; $(n-5)^2$ is the second preceding value, and so forth. The arrangement is similar for n , an odd number.

A numerical example of the application of the above to a practical problem will now be given. Consider the problem of determining the depth-velocity curve discussed on page 131 of Merriam's "Method of Least Squares," eighth edition; 1913, Wiley & Sons. The original data give observations on the velocity of flow of the Mississippi River at different percentages of the total depth. The values of velocity Y , plotted against depth X , follow closely a smooth curve whose equation is assumed to be $Y = A + BX + CX^2$. It is required to determine the most probable values of A , B , and C from the observations. The observations follow:

TABLE 1.—Velocities at Different Depths Below the Surface of the Mississippi River

[See Merriman, Method of least squares, p. 131, 8th ed.; 1913, Wiley & Sons.]

Observation No.	Depths in decimals of total depth	Velocity (feet per second)	Observation No.	Depths in decimals of total depth	Velocity (feet per second)
1.....	a 0.0	3.1950	6.....	0.5	3.2282
2.....	.1	3.2299	7.....	.6	3.1807
3.....	.2	3.2532	8.....	.7	3.1266
4.....	.3	3.2611	9.....	.8	3.0594
5.....	.4	3.2516	10.....	.9	2.9759

a Surface.

For this set of observations, $X_m=0.45$; $Y_m=3.17616$; $n=10$, and $i=0.1$. Following is the detailed solution:

TABLE 2.—Solution for the Constants of $Y=A+BX+CX^2$ from the Observations Given in Table 1

Observation No.	Depth X	Velocity Y	I $Y_n - Y_1,$ $Y_{n-1} - Y_2,$ etc.	II $(n-1),$ $(n-3),$ etc.	III $Y_n + Y_1,$ $Y_{n-1} + Y_2,$ etc.	IV $(n-1)^2,$ $(n-3)^2,$ etc.	Products		Calculation for T
							I × II	III × IV	
1.....	0.0	3.1950	} -0.2191	9	6.1709	81	-1.9719	499.8429	
10.....	.9	2.9759							
2.....	.1	3.2299	} - .1705	7	6.2893	49	-1.1935	308.1757	
9.....	.8	3.0594							
3.....	.2	3.2532	} - .1266	5	6.3798	25	- .6330	159.4950	
8.....	.7	3.1266							
4.....	.3	3.2611	} - .0804	3	6.4418	9	- .2412	57.9762	
7.....	.6	3.1807							
5.....	.4	3.2516	} - .0234	1	6.4798	1	- .0234	6.4798	
6.....	.5	3.2282							
							2nS= -4.0630 S= -0.20315		1031.9696÷4n= 25.79924 -u Y _m = -26.20332
									T= -0.40408

From Table 5

$X_m=0.45$ $u=8.25$
 $Y_m=3.17616$ $I=0.121212$
 $\bar{u}$
 $w=0.15625$

$a = -Tw = +0.0631375$ $A = Y_m + a - X_m(b - cX_m) = 3.19513$
 $b = \frac{S}{i} \frac{I}{u} = -0.2462424$ $B = b - 2cX_m = 0.44253$
 $i \bar{u}$
 $c = \frac{-a}{i^2 u} = -0.7652970$ $C = c = 0.76530$

Therefore $Y = 3.19513 + 0.44523X - 0.76530X^2$.

This checks exactly with the equation given by Merriman.

2. SOLUTION FOR $Y = A + BX$.

The normal equations which yield the most probable values of A and B from n observations of equal weight of X and Y are,

$$\left. \begin{aligned} nA + B\sum X &= \sum Y \\ A.\sum X + B\sum X^2 &= \sum XY \end{aligned} \right\} \quad (28)$$

If the observations are taken at equal intervals of X and after plotting in a system of rectangular coordinates the center of coordinates be transferred to $X, Y = X_m, Y_m$, equations (28) reduce to,

$$\left. \begin{aligned} n.a. &= 0 \\ b\sum x^2 &= \sum xy \end{aligned} \right\} \quad (29)$$

in which $x = X - X_m$ and $Y = Y_m$

$$\text{Therefore,} \quad a = 0 \quad (30)$$

$$b = \frac{\sum xy}{\sum x^2} = \frac{S}{i \cdot u} \quad (31)$$

in which u is defined as in equation (14) and S as in equation (23).

In the old system of coordinates whose center is at $X, Y = 0, 0$

$$B = b \quad (32)$$

$$A = Y_m - bX_m \quad (33)$$

As an example of the practical application of the above, a solution is given of a problem adapted from page 74, Weld, *Theory of Errors and Least Squares*, 1916 (MacMillan).

To measure the total resistance of a Wheatstone bridge wire and to calibrate the wire at the same time are desired. The wire is 100 cm long. The resistance was measured of the first 10 cm, then of the first 20 cm, etc., and finally of the whole wire. The resistance of the connections entered each time as a constant term in the above resistances.

Let A be the resistance of the connections, B the unit resistance of the bridge wire, X the length of wire for which the resistance was observed, and Y the observed resistance; then,

$$Y = A + BX$$

The observations follow:

TABLE 3.—Observations of Length and Resistance of a Wheatstone Bridge Wire
[See Weld, Theory of Errors and Least Squares, 1916, p. 76]

Observation No.	Length measured X	Observed resistance Y	Observation No.	Length measured X	Observed resistance Y
	Cm	Ohm		Cm	Ohm
1.....	10	0.116	6.....	60	0.595
2.....	20	.205	7.....	70	.675
3.....	30	.295	8.....	80	.760
4.....	40	.388	9.....	90	.850
5.....	50	.503	10.....	100	.926

TABLE 4.—Solution for the Constants of $Y=A+BX$ from the Observations Given in Table 3

Observation No.	Length measured X	Observed resistances Y	I $Y_{10}-Y_1,$ $Y_9-Y_2,$ etc.	II $(n-1),$ $(n-3),$ etc.	Products $I \times II$
	Cm	Ohm			
1.....	10	0.116			
10.....	100	.926	0.810	9	7.290
2.....	20	.205			
9.....	90	.850	.645	7	4.515
3.....	30	.295			
8.....	80	.760	.465	5	2.325
4.....	40	.388			
7.....	70	.675	.287	3	.861
5.....	50	.503			
6.....	60	.595	.092	1	.092
$n=10$					$2nS=15.083$
$i=10$					$S=0.75415$

$X_m=55$

$Y_m=0.5313$

From Table 5, $\frac{I}{u}=0.12121$

Therefore,

$b=B=\frac{S}{i} \cdot \frac{I}{u}=0.009141$

$A=Y_m-BX_m=0.0285$

Therefore, finally,

$Y=0.0285+0.009141X$

This equation is not the same as that obtained by Weld. There is an error in his solution. The least-square solution gives values for the constants which check with the above.

3. SOLUTION WHEN A OR B OR BOTH A AND B ARE ZERO.

Should any of the constants A , B , and C be zero, that is, if the curve to be fitted to the observations is of one of the following forms—

$$Y = CX^2 \quad (34)$$

$$Y = BX + CX^2 \quad (35)$$

$$Y = A + CX^2 \quad (36)$$

$$Y = BX \quad (37)$$

the solutions given above for the constants do not hold. Such curves are constrained to fulfill certain geometric conditions independent of the observations themselves; namely, types (34), (35), and (37) are constrained to pass through $(X, Y) \equiv (0, 0)$; and (34) and (36) must be parallel to the X at $(X, Y) \equiv (0, 0)$. When an observer decides to fit any of these curves to a set of observations, the point $(X, Y) \equiv (0, 0)$ is automatically assigned an infinite weight, and therefore the development above can not apply.

The least-square solutions for curves of these types can be reduced so that Table 5 can be utilized for determining the constants. The mathematical work is in all respects similar to that above, except that the literal summations are made without first shifting the origin to the point $(X, Y) \equiv (X_m, Y_m)$. The final solutions are given in conjunction with Table 5.

4. SUMMARY

Least-square reductions of observations taken at equal intervals of the independent variable that follow approximately a parabolic or linear law frequently occur in physics and engineering practice. Makeshift devices are often employed to avoid the arithmetical work of determining the constants of curves which properly represent such data.

In this paper the ordinary least-square formulas are subjected to mathematical treatment and rigorous solutions are evolved which require an ultimate minimum of arithmetical work. A table is furnished from which a large portion of the solutions can be written down from a mere inspection of the observations.

Application of the solutions is made to typical problems and an absolute check with the ordinary least-square solutions is shown.

$$v = \frac{(n-1)(n+1)(3n^2-7)}{240} = \frac{u}{20}(3n^2-7)$$

$$w = \frac{u}{v-u^2}$$

$$S = \frac{1}{2n} \left[(n-1)(Y_n - Y_1) + (n-3)(Y_{n-1} - Y_2) \right. \\ \left. + (n-5)(Y_{n-3} - Y_3) + \dots \right]$$

$$T = \frac{1}{4n} \left[(n-1)^2(Y_n + Y_1) + (n-3)^2(Y_{n-1} + Y_2) \right. \\ \left. + (n-5)^2(Y_{n-3} + Y_3) + \dots \right] - uY_m$$

$$U = \frac{1}{4n} \left[(n-1)^2(Y_n + Y_1) + (n-3)^2(Y_{n-1} + Y_2) \right. \\ \left. + (n-5)^2(Y_{n-3} + Y_3) + \dots \right]$$

$$a = -Tw$$

$$b = \frac{S}{i} \cdot \frac{1}{u}$$

$$c = \frac{a}{i^2} \cdot \frac{1}{u}$$

5. DEFINITIONS AND FORMULAS

Let n = number of observations of equal weight entering into the computations.

i = interval between successive values of the independent variable.

X_m = arithmetic mean of the observed values of the independent variable.

Y_m = arithmetic mean of the observed values of the dependent variable.

$$u = \frac{(n-1)(n+1)}{12}$$

Then for $Y = A + BX + CX^2$

$$A = Y_m + a - X_m(b - cX_m)$$

$$B = b - 2cX_m$$

$$C = c$$

for $Y = A - BX$

$$A = Y_m - bX_m$$

$$B = b$$

for $Y = CX^2$

$$C = \frac{X_m Y_m + i^2 X_m S + i^2 U}{X_m^4 + i^2 6 X_m^2 u + i^4 v}$$

for $Y = BX + CX^2$

$$C = \frac{[X_m S + iU][X_m^2 - i^2 u] - i^2 u [X_m^2 Y_m - iU]}{i[X_m^2 + i^2 u][X_m^2 u + i^2 v] - i^3 4 X_m^2 u^2}$$

$$B = \frac{X_m Y_m + iS - C X_m [X_m^2 + i^2 3 u]}{X_m^2 + i^2 u}$$

for $Y = A + CX^2$

$$C = \frac{2 X_m S + iT}{i[4 X_m^2 u + i^2(v - u^2)]}$$

$$A = Y_m - C[X_m^2 + i^2 u]$$

for $Y = BX$

$$B = \frac{X_m Y_m + iS}{X_m^2 + i^2 u}$$

TABLE 5.—Values of Coefficients for Use in the Least Squares Solution of Parabolic and Linear Equations

<i>n</i>	<i>v</i>	<i>u</i>	log <i>u</i>	$\frac{1}{u}$	$\log \frac{1}{u}$	<i>w</i>	log <i>w</i>	$\frac{(n-1)^2}{n}$ an even number	$\frac{(n-1)^2}{n}$ an even number
5	6.8000	2.0000	0.3010300	0.50000	9.6989700-10	0.71429	9.8538720-10	-----	16
6	14.736	2.9167	.4648868	.34285	9.5351132	.46875	9.6709413	25	-----
7	28.000	4.0000	.6020600	.25000	9.3979400	.33333	9.5228787	-----	36
8	48.562	5.2500	.7201593	.19048	9.2798407	.25000	9.3979400	49	-----
9	78.667	6.6667	.8239087	.15000	9.1760913	.19481	9.2896006	-----	64
10	120.86	8.2500	.9164539	.12121	9.0835461	.15625	9.1938200	81	-----
11	178.00	10.000	1.0000000	.10000	9.0000000	.12821	9.1079054	-----	100
12	253.26	11.917	1.0761548	.083916	8.9238452	.10714	9.0299633	121	-----
13	350.00	14.000	1.1461280	.071429	8.8538720	.090909	8.9586074	-----	144
14	472.06	16.250	1.2108534	.061538	8.7891466	.078125	8.8927900	169	-----
15	623.47	18.667	1.2710667	.053571	8.7289333	.067873	8.8316990	-----	196
16	808.56	21.250	1.3273589	.047059	8.6726410	.059524	8.7746908	225	-----
17	1032.0	24.000	1.3802112	.041667	8.6197888	.052632	8.7212464	-----	256
18	1298.8	26.917	1.4300213	.037152	8.5699787	.046875	8.6709413	289	-----
19	1614.0	30.000	1.4771213	.033333	8.5228787	.042017	8.6234231	-----	324
20	1983.4	33.250	1.5217916	.030075	8.4782084	.037879	8.5783961	361	-----
21	2412.7	36.667	1.5642715	.027273	8.4357285	.034325	8.5356099	-----	400
22	2908.1	40.250	1.6047659	.024845	8.3952341	.031250	8.4948500	441	-----
23	3476.0	44.000	1.6434527	.022727	8.3565473	.028571	8.4559320	-----	484
24	4123.3	47.917	1.6804866	.020870	8.3195134	.026224	8.4186953	529	-----
25	4856.8	52.000	1.7160033	.019231	8.2839967	.024155	8.3829997	-----	576
26	5684.1	56.250	1.7501225	.017778	8.2498775	.022321	8.3487220	625	-----
27	6612.7	60.667	1.7829501	.016484	8.2170499	.020690	8.3057533	-----	676
28	7650.6	65.250	1.8145805	.015326	8.1854195	.019231	8.2839967	729	-----
29	8806.0	70.000	1.8450980	.014286	8.1549020	.017921	8.2533658	-----	784
30	10088	74.917	1.8745785	.013348	8.1254215	.016741	8.2237833	841	-----
31	11504	80.000	1.9090300	.012500	8.0909700	.015674	8.1951798	-----	900

WASHINGTON, February 20, 1920.