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ON FEJÉR SETS IN LINEAR AND SPHERICAL SPACES

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INTRODUCTION

1. Let M be a metric space of points $a, b, \cdots$, with the distance from a to b denoted by ab. Let A be a given subset of M. If p and p' are points of M such that

$$(1) \qquad px > p'x \, , \quad \text{for every point} \quad x \in A \, ,$$

we say that p' is point-wise closer than p to the set A. We say that p is a point-wise closest point to the set A if no point p' exists which is point-wise closer than p to A. For brevity we shall also refer to a point-wise closest point to the set A as a minimal point of the set A. The set of minimal points of A will be called the Fejér set of A and denoted by F(A). This set is never void since $F(A) \supseteq A$. F(A) is a monotone set function since

$$(2) \qquad A_1 \subseteq A_2 \quad \text{implies} \quad F(A_1) \subseteq F(A_2) \quad .$$

Let p, p' be distinct points of M. The set of points x such that

$$px \geq p'x$$

is called a half-space of M and denoted by the symbol $H = H(p,p')$; p,p' are its defining points or foci, p being exterior to H, p' interior. The convex hull K(A) of a given set A is defined as follows: If A is not contained in any half-space, we set $K(A) = M$. If there exist half-spaces containing A, then K(A) is defined as the intersection of all such half-spaces:

$$(3) \qquad K(A) = \bigcap_{H \supseteq A} H \quad .$$

Notice that $K(A)$ is always closed and $A \subseteq K(A)$. A further general inclusion is

(4) $$K(A) \subseteq F(A) \quad .$$

Indeed, if $p \bar{\in} F(A)$, then a point p' exists such that (1) holds. This implies $A \subseteq H = H(p,p')$ while evidently $p \bar{\in} H$. By (3) $p \bar{\in} K(A)$ and (4) is established.

2. In 1922 Fejér [4], noticed the interesting fact that in the euclidean plane $M = E_2$ we have

(5) $$F(A) = K(A) \quad .$$

One half of Fejér's own argument has just been used to establish (4) for any metric space. The other half of his proof in E_2 is as follows. Let us show that $F(A) \subseteq K(A)$. Indeed, if $p \bar{\in} K(A)$, then there exists a half-plane $H \subset E_2$ such that $p \bar{\in} H$, $A \subseteq H$. If p' is the point of H which is nearest to p, then (1) evidently holds showing that $p \bar{\in} F(A)$.

The last paragraph allows of a wider setting. Let M be a real inner-product space. By this we mean that M is a real linear space whose norm springs from an inner product (a,b) and that M is complete with respect to the metric just described. Under these assumptions the convex hull $K(A)$ is found to be identical with the least closed and linearly connected set containing A. Fejér's argument applies unchanged to prove the following

THEOREM 1. *If* M *is a real inner-product space and* $A \subseteq M$, *then*

(6) $$F(A) = K(A) \quad .$$

3. In Parts I and II we investigate the Fejer sets of finite sets of points in Banach spaces. The notion of the global distribution of points on spheres is introduced (Definition 1) and Fejér sets are described in terms of this concept (Theorem 5). Fejér's result is shown to hold in rather arbitrary 2-dimensional Banach spaces (Theorem 6). The situation is quite different for dimensions exceeding two. A very weak form (2.1) of Fejér's result is shown to imply that the Banach space is an inner-product space. In Part III we determine the Fejér set of a subset of spherical space.

I. ON FEJÉR SETS OF FINITE SETS OF POINTS IN BANACH SPACES

4. An application of Helly's theorem. Let B be a Banach space to be also denoted by B_n in case its dimension n happens to be finite. The Fejér set F(a,b) of two points is described by

THEOREM 2. *In any space B we have that*

(1.1) $$F(a,b) = E(x;\ \|a-x\| + \|x-b\| = \|a-b\|)\ .$$

Proof: Indeed, a point x of E is clearly minimal for the set $\{a,b\}$, because the existence of a point x' point-wise closer than x to $\{a,b\}$ would violate the triangle inequality. On the other hand, if $\|a-x\| + \|x-b\| > \|a-b\|$, we can find a point x' on the segment joining a to b such that $\|a-x\| > \|a-x'\|$, $\|b-x\| > \|b-x'\|$. This proves (1.1).

THEOREM 3. *Let A be a finite set of points of B_n (n finite) and let us assume that A contains at least n+1 points. Then*

(1.2) $$F(A) = \bigcup F(p_0, p_1, \ldots, p_n)\ , \qquad (p_i \in A),$$

<u>where the elements of the union are formed for every combination of n+1 distinct points of A.</u>

<u>Proof</u>: Let $G(A)$ denote the set on the right-hand side of (1.2). By (2) we have

$$G(A) \subseteq F(A) \quad . \tag{1.3}$$

Assume now that

$$p \overline{\in} G(A) \tag{1.4}$$

and let us show that $p \overline{\in} F(A)$, i.e., that p is not minimal. For every point $a \in A$ we consider the open sphere

$$S_a : \|x-a\| < \|p-a\| \quad . \tag{1.5}$$

Let $p_o, p_1, \cdots, p_n$ be any distinct points of A. By (1.4) $p \overline{\in} F(p_o, \cdots, p_n)$. This means that a point-wise closer point exists for the set $\{p_o, \cdots, p_n\}$, or

$$S_{p_o} \cap S_{p_1} \cap \cdots \cap S_{p_n} \neq 0 \quad . \tag{1.6}$$

The spheres S_a are convex sets, finite in number, every n+1 of which have a common point, in view of (1.6). By Helly's theorem (see [5])

$$\bigcap_{a \in A} S_a \neq 0 \quad .$$

If $p' \in \cap S_a$, then $p' \in S_a$, for every a, or $\|p-a\| > \|p'-a\|$ for every $a \in A$. Hence $p \overline{\in} F(A)$. Thus $G(A) \supseteq F(A)$. In view of (1.3), the identity (1.2) is established.

<u>An example</u>. Let $B_n = M_n$ be the Minkowski space of points $x = (x_1, \cdots, x_n)$ with the "cubical" norm $\|x\| = \max_i |x_i|$. In this particular case the result of Theorem 2 may be improved as follows:

If A is a finite set of points in M_n then

$$F(A) = \cup F(p_0, p_1) \quad , \tag{1.7}$$

where the union is formed for all pairs of distinct points of A.

Indeed, in this special case the spheres (1.5) are open cubes with edges parallel to the axes of coordinates. On repeating the argument used in proving Theorem 3 we find that every two among the cubes have a common point. If follows that the projections of the cubes on each coordinate axis have a common point and that therefore also all cubes have a common point.

Let $n = 3$ and let $A = \{a, b, c\}$, where

$$a = (2, 0, 0), \quad b = (0, 2, 0), \quad c = (0, 0, 2) \quad . \tag{1.8}$$

By (1.7) we have

$$F(a, b, c) = F(a, b) + F(a, c) + F(b, c) \quad .$$

The inspection of a diagram will show that the three right-hand side sets are lozenges: $F(a,b)$ is the lozenge of consecutive vertices (2, 0, 0), (1, 1, 1), (0, 2, 0), (1, 1, -1). Notice in particular the curious fact that the centroid (2/3, 2/3, 2/3) of the triangle $\Delta(a, b, c)$ is not a point of the Fejér set $F(a, b, c)$. However the point $p = (1, 1, 1)$ does belong to $F(a, b, c)$. We shall say that a,b,c are *globally distributed* on the sphere $\|x-p\|=1$ and investigate the general concept in our next section.

5. *On points globally distributed on spheres and their characterization.* DEFINITION 1. *Let B be a Banach space and let*

$$S : \|x-c\| = r \quad , \tag{1.9}$$

be the surface of the sphere of radius r *and center at* c. *Let* $p_1, p_2, \cdots, p_k$ *be* k *points of* S. *We shall say that the points* $p_1, \cdots, p_k$ *are* globally distributed *on* S *provided*

$$c \in F(p_1, p_2, \cdots, p_k) \quad . \tag{1.10}$$

In other words: There is no solid sphere $\|x-c'\| \leq r'$, *of a smaller radius* $r' < r$, *which covers all points* $p_1, \cdots, p_k$.

In what follows we denote by $[a,b)$ the half-open segment of points $a(1-t) + bt$ $(0 \leq t < 1)$, joining a with b, and use similar notations $[a,b]$, (a,b), for closed or open segments.

DEFINITION 2. *Let* $p_1, \cdots, p_k$ *be* k *points on the sphere* (1.9) *of Definition* 1. *We say that the points* $p_1, \cdots, p_k$ *are* well visible *provided there exists a point* q *outside* S, $\|q-c\| > r$, *such that the segments* $[q,p_i)$ *contain only points exterior to* S *and that there are points* s_i, *such that* $p_i \in (q,s_i)$ *and that the segments* $(p_i,s_i]$ *have only points interior to* S.

Example. In E_2 k points on a circle $\|x-c\| = r$ are well visible if and only if they are contained in an open half-circle. The general relation between Definitions 1 and 2 is shown by the following

THEOREM 4. *The points* $p_1, p_2, \cdots, p_k$ *on* S *are globally distributed on* S *if and only if they are not well visible*.

Proof: a. *The condition is necessary*. Indeed, suppose that our condition is not satisfied and, on the contrary, the points p_i are well visible from the outside point q. By Definition 2 we can extend the segment $[q,p_i]$ by a segment $(p_i,s_i]$ all points of which are interior to S. Let us now "shrink" S from the center of similitude q in the ratio $1 : \lambda$ $(0 < \lambda < 1)$, obtaining a new sphere

S' of radius $r' = r\lambda < r$. Let the segment (p_i, s_i) be transformed into (p_i', s_i') by this similitude transformation. Since (p_i, s_i) is interior to S, (p_i', s_i') will be interior to S'. It is clear that

$$p_i \in (p_i', s_i') , \qquad (i = 1, \cdots, k),$$

provided that λ is sufficiently close to unity. But then all p_i are inside S', showing that $p_1, \cdots, p_k$ are not globally distributed on S.

b. The condition is sufficient. Indeed, let us assume that $p_1, \cdots, p_k$ are not globally distributed on S and let us conclude that they are well visible. Accordingly, let p_i be covered by a sphere S_1 of radius $r_1 < r$. Inflate S_1 slightly from its center into a larger sphere of radius $r' < r$. Now all points p_i are interior to S. Let c' be the center of the sphere S'. Since the p_i are on S and inside S', we must have $c' \neq c$. The spheres S and S' are similar with respect to the (exterior) center of similitude

$$q = \frac{rc' - r'c}{r - r'} .$$

We denote by $p' = Tp$ the similitude transformation with center at q and ratio $r : r'$. Let $p_i' = Tp_i$. Since p_i is on S, p_i' is on S'. But p_i is by construction inside S'. It follows that all points of $(p_i', p_i]$ are inside S'. Let s_i be such that

$$p_i = Ts_i .$$

Now $(p_i, s_i]$ goes over into $(p_i', p_i]$ by our transformation. Since $(p_i', p_i]$ was shown to be inside S', it follows that $(p_i, s_i]$ has only points interior to S. We now claim that all points of $[q, p_i)$

are outside S. For if any point of this segment were inside or on S, the fact that s_i is inside S would imply that also p_i were inside S, which is not the case. We have just shown that $[q, p_i)$ is outside S, $(p_i, s_i]$ is inside S, $(i = 1,\cdots,k)$. But this is precisely what we mean when we say that the points p_i are well visible from q.

6. A description of Fejér sets in terms of global distribution.

THEOREM 5. *Let* $A = \{p_1, \cdots, p_k\}$ *be a finite set of points of* B. *Let* $p \bar{\in} A$. *Draw about* p *as center a sphere*

$$S : \|x-p\| = r$$

and project p_i *from* p *onto the surface* S *into* q_i. *Denote by* $F_1(A)$ *the set of those points* p *such that the points* $q_1, q_2, \cdots, q_k$ *are globally distributed on* S. *Then*

$$F(A) = A + F_1(A) \quad . \tag{1.11}$$

Proof: We have to show that $p \in F(A)$ if and only if the q_i are globally distributed on S or equivalently:

$$p \bar{\in} F(A) \tag{1.12}$$

if and only if

$$q_1, \cdots, q_k \text{ are not globally distributed on } S. \tag{1.13}$$

The size of the radius $r > 0$ is clearly immaterial.

a. Let us assume (1.13) and prove (1.12). Choose r such that

$$r < \min_i \|p-p_i\| \quad .$$

The assumption (1.13) means that $p \bar{\in} F(q_1, \cdots, q_k)$. Therefore there exists a point p' such that

$$\|q_i - p\| > \|q_i - p'\| , \qquad (i = 1, \cdots, k).$$

But then

$$\|p_i - p\| = \|p_i - q_i\| + \|q_i - p\| > \|p_i - q_i\| + \|q_i - p'\| \geqq \|p_i - p'\|$$

or

$$\|p_i - p\| > \|p_i - p'\| .$$

which proves (1.12).

b. We now assume that (1.12) holds and wish to prove (1.13). Choose r such that

$$r > \max_i \|p - p_i\| .$$

By (1.12) there is a point p' such that

$$\|p - p_i\| > \|p' - p_i\| , \quad \text{for all } i.$$

Hence

$$r = \|p - q_i\| = \|p - p_i\| + \|p_i - q_i\| > \|p' - p_i\| + \|p_i - q_i\| \geqq \|p' - q_i\|$$

or

$$\|p' - q_i\| < r , \quad \text{for all } i.$$

Setting

$$r' = \max_i \|p' - q_i\| < r$$

we see that the smaller sphere $\|x - p'\| \leqq r'$ will cover all points q_i, which proves (1.13).

7. <u>On Fejér sets in</u> B_2. As an application of Theorem 4 we wish to describe the Fejér sets of finite sets of points in a

Minkowski plane B_2. By Theorem 2 we conclude the following: The Fejer set $F(a,b)$ is identical with the segment $[a,b]$, for every pair of points a,b, if and only if the norm of B_2 has the property:

$$(1.14) \qquad \begin{gathered} \|p\| + \|q\| = \|p+q\| \ , \quad p \neq 0 \ , \quad q \neq 0 \\ \text{imply that } p = \alpha q \ , \quad (\alpha > 0) \ . \end{gathered}$$

It is known that (1.14) holds if and only if the gauge-curve $\|x\|=1$ contains no segment, in which case the curve $\|x\|=1$ may be described as being "round."

We assume (1.14) to hold and consider in B_2 a set $A = \{p_1,p_2,p_3\}$ of three points. Let $p \bar{\in} A$ and let q_1,q_2,q_3 be the points on the gauge-curve S as described in Theorem 4. By Theorem 5 $p \bar{\in} F(p_1,p_2,p_3)$ if and only if q_1,q_2,q_3 are not globally distributed on S. By Theorem 4 this is the case if and only if q_1,q_2,q_3 are well visible (from a point q outside S). S being "round," this is evidently the case if and only if p is outside the closed triangle $\Delta(q_1, q_2, q_3)$ or, equivalently, outside the triangle $\Delta(p_1, p_2, p_3)$. Thus

$$F(p_1, p_2, p_3) = \Delta(p_1, p_2, p_3) \ .$$

Applying Theorem 3 we obtain

THEOREM 6. Let B_2 be a Minkowski plane with the property (1.14). The Fejér set of a finite set of points is identical with the closed convex polygon spanned by the set.

This result is a special property of 2-dimensional Banach spaces as will be shown in Part II.

II. A CHARACTERIZATION OF INNER-PRODUCT SPACES

8. The main theorem. Let M be the inner-product space of Theorem 1 and let the set A consist of three points, p_1, p_2, p_3, distinct or not. Then K(A), being identical with the least closed and linearly connected set containing these points, is evidently the triangle $\Delta(p_1, p_2, p_3)$ having as vertices the points p_1, p_2, p_3. By Theorem 1 we have

$$F(p_1, p_2, p_3) = \Delta(p_1, p_2, p_3) \quad . \tag{2.1}$$

Let now M be an arbitrary Banach space. By Theorem 6 we know that (2.1) again holds provided that $B = B_2$ is 2-dimensional and that its metric has the property (1.14). Which higher-dimensional Banach spaces B enjoy the property (2.1)? An answer is given by the following

THEOREM 7. A space B of dimension ≥ 3 has the property (2.1) if and only if it is an inner-product space.

9. A few lemmas. For the proof of Theorem 7 we need a number of results concerning the 3-dimensional euclidean space E_3.

LEMMA 1. Let S_2 denote the surface of a sphere in E_3 with center at o. Let C be a simple closed curve on S_2 and let H_1 and and H_2 be the two open components of the complementary part: $S_2 = C + H_1 + H_2$. Let $K^*(H_1)$ and $K^*(H_2)$ denote the least convex sets in E_3 containing H_1 and H_2, respectively. If

$$o \,\bar{\epsilon}\, K^*(H_1) \cup K^*(H_2) \tag{2.2}$$

then C is necessarily a great circle.

<u>Proof</u>: By (2.2) $o \bar{\in} K^*(H_1)$. Therefore $\underline{o}$ is either outside or on the boundary of the closed convex set

$$\overline{K^*(H_1)} = K^*(H_1 + C) \quad .$$

We can therefore draw through $\underline{o}$ a plane Π which is a bounding plane or a plane of support of $K^*(H_1 + C)$. Let $\Gamma = S_2 \cap \Pi$ and let h_1 and h_2 denote the open half-spheres in which Π divides S_2: $S_2 = \Gamma + h_1 + h_2$. By our choice of Π and for a proper labelling of the h_i we have

$$h_1 + \Gamma \supseteqq H_1 \ .$$

Passing to complements on the sphere we see that $h_2 \subseteqq H_2 + C$, and since h_2 is open on the sphere, we have

$$h_2 \subseteqq H_2 \quad . \tag{2.3}$$

We now claim that

$$h_1 \cap H_2 = 0 \quad . \tag{2.4}$$

Indeed, if $p \in h_1 \cap H_2$, then $K^*(p, h_2)$ would contain the point $\underline{o}$ and by $p \in H_2$ and (2.3) a fortiori

$$o \in K^*(H_2) \quad ,$$

in contradiction to (2.2). By (2.3) no point of C is in h_2 and by (2.4) no point of C is in h_1. Thus $C \subseteqq \Gamma$ and therefore $C = \Gamma$ showing that C is a great circle.

A seemingly more general version of Lemma 1 to be used below is as follows.

LEMMA 2. <u>Let S be the surface of a convex body in E_3 having the point o as center of symmetry. Let C be a simple closed curve</u>

on S and let H_1, H_2 be the two open components of the complement: $S = C + H_1 + H_2$. Let $K^*(H_1), K^*(H_2)$ denote the least convex sets in E_3 containing H_1 and H_2, respectively. If

$$o \bar{\in} K^*(H_1) \cup K^*(H_2)$$

then C is a "great circle" of S, i.e., C is the intersection of S by a plane through o.

Proof: Draw a sphere S_2 with center at o. Central projection of S only S_2 with the center of projection at o reduces Lemma 2 to Lemma 1.

We turn to our last lemma which is due to Blaschke. Let S be the surface of a convex body in E_3. We assume that the surface S contains no segment, a fact which we describe by saying that S is "round." Let $\vec{v}$ be a fixed vector and let us illuminate S by rays of light all parallel to $\vec{v}$ in direction and sense. Let C denote the light-shade boundary on S, that is, the boundary between the illuminated part of S and the part of S which is shaded. We claim that C is a simple closed curve. Indeed, let π be a fixed plane normal to $\vec{v}$. The orthogonal projection of S onto π is a plane convex domain D whose boundary curve we denote by Γ. Let $p \in \Gamma$. The point p is the projection of a point q of S. Since S contains no segment this point q is unique and is readily shown to vary continuously as p varies continuously on Γ. As p describes Γ the point q describes our curve C which is clearly simple and closed.

LEMMA 3 (Blaschke). If S has the property that the light-

<u>shade boundary C is a plane curve for all possible directions of $\vec{v}$ then S is necessarily an ellipsoid</u>.

Blaschke proves this result ([1], pp. 157-159) by assuming that S is an "Eifläche," by which he means that S is analytic and regular at all its points and has everywhere non-vanishing curvature ([1], p. 147). However, the reader will have no difficulty in carrying through Blaschke's proof on the basis of our simplified assumption that S is "round."

10. <u>Proof of Theorem</u> 7. We are to show that the property (2.1) implies that B is an inner-product space. If $p_2 = p_3$ then (2.1) implies that

$$F(p_1, p_2) = [p_1, p_2] \quad .$$

The metric of B must therefore have the property (1.14): The gauge-surface

$$\Sigma : \|x\| = 1$$

of B is "round."

Let B_3 be an arbitrary but fixed 3-dimensional linear subspace of B. B_3 is a 3-dimensional Banach space whose gauge-surface is

$$S = B_3 \cap \Sigma \quad .$$

In terms of a coordinate system in B_3 we can also think of B_3 as being a Minkowski space whose points are those of an E_3 which is metrized by means of the convex gauge-surface S. By a theorem of Jordan and von Neumann [3], it suffices to show that S <u>is an ellipsoid</u>.

We know already that S is "round" because Σ has this property

($S \subset \Sigma$ and Σ contains no segments). Let us illuminate S from a direction parallel to $\vec{v}$. Let C be the light-shade boundary on S and let H_1 be the illuminated part of S, H_2 the shaded part, $S = C + H_1 + H_2$. <u>We claim that</u>

$$o \overline{\in} K^*(H_1) \quad . \tag{2.5}$$

Indeed, let us assume for the moment $\underline{o}$ to be a point of the convex set $K^*(H_1)$. H_1 being connected it follows, by a sharpened version of a theorem of Fenchel,[1] that $\underline{o}$ is the centroid of some three points of H_1: p_1, p_2, p_3, say. Thus

$$p_1, p_2, p_3 \in H_1 \quad , \tag{2.6}$$

$$o \in \Delta(p_1, p_2, p_3) \quad . \tag{2.7}$$

These conclusions, however, are contradictory with our previous assumptions, for on the one hand the points p_1, p_2, p_3 are well visible from a point at infinity in the direction $-\vec{v}$. From this it follows easily that they are well visible from a point q at finote distance and sufficiently far out in the direction of $-\vec{v}$. By Theorem 4 we conclude that the points p_1, p_2, p_3 <u>are not globally distributed on the sphere</u> S <u>of the space</u> B_3.

On the other hand by (2.7) and our basic assumption (2.1) we conclude that $o \in F(p_1, p_2, p_3)$. Thus $\underline{o}$ is a minimal point of the set $A = \{p_1, p_2, p_3\}$ in the space B. It follows a fortiori that

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[1]The sharpened version of the theorem of Fenchel is as follows: <u>Let</u> H <u>be a connected subset of</u> E_n <u>and let</u> $K^*(H)$ <u>be the least convex set containing</u> H. <u>Then every point of</u> $K^*(H)$ <u>is a centroid of some</u> n <u>points of</u> H. This result is due to L. N. H. Bunt. See [2] for Bunt's proof (pages 589-590) and for references.

that $\underline{o}$ is minimal point of A with respect to the subspace B_3. Thus p_1, p_2, p_3 are globally distributed on S in B_3, in direct contradiction to the conclusion of our previous paragraph.

This proves (2.5) and we may similarly show that

$$o \bar{\in} K^*(H_2) \quad .$$

By Lemma 2 we conclude that C is a plane curve and by Blaschke's Lemma 3 we learn that S is an ellipsoid.

III. ON FEJÉR SETS IN SPHERICAL SPACES

11. Let M be the real inner-product space of section 2. We are now confining our attention to the surface S of its unit sphere

$$\|x\| = 1 \quad .$$

By the distance xy of two points of S we mean the arc defined by

$$\cos xy = (x,y) \ , \quad o \leq xy \leq \pi \quad .$$

If p and p' are distinct points of S then the closed half-sphere H(p, p') may be defined by the inequality px $\geq$ p'x or equivalently by (p, x) $\leq$ (p', x) or (p'-p, x) $\geq$ 0. A being a given subset of S we may now define the convex hull K(A) as in section 1.

THEOREM 8. *Let A be a subset of S.* a. *If there is no open half-sphere containing A then*

(3.1) $$F(A) = S \quad .$$

b. *If there is an open half-sphere H_o such that $A \subseteq H_o$, then*

(3.2) $$F(A) = K(A).$$

Proof: a. In order to prove (3.1) we have to show that every point $p \in S$ is a minimal point of S. This is clear, for otherwise there would exist a point p' such that (1) holds. However (1) implies that A is in the open half-sphere $H_o(p, p')$ which contradicts our assumption.

b. Let us assume that

$$A \subseteqq H_o \quad , \tag{3.3}$$

where H_o is an open half-sphere defined by

$$H_o : (x,b) > 0 \quad , \tag{3.4}$$

and let us prove (3.2). However, the inclusion

$$K(A) \subseteqq F(A) \quad ,$$

or (4), has already been established in section 1 for any metric space. There remains to show that

$$F(A) \subseteqq K(A) \quad , \tag{3.5}$$

or that

$$p \bar{\in} K(A) \tag{3.6}$$

implies

$$p \bar{\in} F(A) \quad . \tag{3.7}$$

Assuming (3.6) means that there is a closed half-sphere H such that

$$p \bar{\in} H \; , \quad H \supseteqq A \quad . \tag{3.8}$$

Let the half-sphere H be defined by

$$H : (x,a) \geqq 0 \quad . \tag{3.9}$$

By the first relation (3.8) we know that $(p,a) < 0$. Choose $\epsilon > 0$

so small as to make sure that

$$(3.10) \qquad (p,a) + \epsilon(p,b) < 0 \ .$$

Consider the open half-sphere

$$(3.11) \qquad H_o' : (x,a) + \epsilon(x,b) > 0 \ .$$

If $x \in A$ then $x \in H$, by (3.8), and $x \in H_o$, by (3.3). The point x thus satisfies the inequalities (3.4), (3.9) and therefore also (3.11):

$$(3.12) \qquad A \subseteqq H_o' \ .$$

On the other hand (3.10) shows that

$$p \,\overline{\in}\, \overline{H_o'} \ .$$

But then clearly p is not a minimal point of A. Indeed let p' be the symmetric of p with respect to the hyperplane

$$(x,\ a+\epsilon b) = 0$$

which bounds H_o'. Clearly $H_o' = H_o(p,\ p')$. By (3.12) we see that p' is point-wise closer than p to A. This proves (3.7) and therefore also our theorem. Notice that the dimensionality of the space M, finite, denumerable or non-denumerable, does not affect our theorem.

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