Automatic Computing Methods for Special Functions

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(May 25, 1970)

Some of the pitfalls of automatic computation are described as well as their possible avoidance. A detailed description is given of the methods for computing the error and complementary error function. The implementation of the method is given in a double precision ASA FORTRAN program. The test ("driver") program and test results are also given.

Key words: Approximations; computer programs; computing pitfalls; continued fractions; error function; recurrence relations; series.

1. Introduction

AMS55[1]¹—the Handbook of Mathematical Functions—in addition to defining functions in terms of their integrals, power series, asymptotic expansions, etc., also supplies tables and approximating polynomials as well as methods of computing values both within and outside the tabular range. With the present trend to large scale computer systems, questions frequently arise concerning methods of computing the tabular values themselves. In many, if not most cases, the method used for computing the table is certainly not the best method for computing discrete values. While many specialized methods and programs exist [3, 6, 7, 8], as a supplement to the Handbook and to satisfy general needs, a systematic effort is being made to supply methods of computing special functions, accurate within machine precision, valid throughout the entire range of the function, efficient, and readily programmable and adaptable for varying precision and computers. Part I of the present paper is devoted to the error, probability, and related functions.

2. General Methods

Methods used in computing functions vary from expansion in series—power, Taylor, asymptotic, etc.; integration of the differential equation; numerical quadrature; evaluation of continued fractions; polynomial or rational approximations; recurrence techniques; iteration methods; stored tables with the use of interpolation; addition or duplication formulas, etc. Numerical quadrature and integration of the differential equation, while ideal for preparing a table, are generally inefficient for obtaining discrete values. Since the coefficients in approximating polynomials vary with precision, this method will be avoided if possible. The methods chosen for computing the functions will be those which avoid some of the common pitfalls or those pitfalls not easily remedied.

3. Common Pitfalls and Their Avoidance

Computation with alternating series leads to a loss of significant figures due to either or both subtraction and accumulated round-off errors. The use of Taylor series leads to difficulties of premature truncation if any of the derivatives possess zeros. The use of slowly convergent series leads to the difficulty of a heavy accumulation of round-off errors as well as of devising a satisfactory cut-off procedure. In computing with such series one might impose a condition on

¹ Figures in brackets indicate the literature references on p. 212.

the number of terms to be used in the computation. However, this number varies with precision and the method would impose a timing penalty in certain ranges. After the n th term has satisfied the convergence criteria, the sum of the n th to the $(n+p)$ th term should be checked for its contribution. The loss of significant figures away from or at the zeros of the functions will be noted by comparing the largest term with the sum and the convergent term. This method of curtailing the series will lead to time savings and frequently avoid overflow and underflow conditions. In using asymptotic expansions the test for convergence should precede the test for divergence.

In computing with continued fractions [2] to avoid premature truncation and or possible overflow or underflow difficulties, the convergents should only be obtained after the numerator and denominator of the ratio are tested for zeros. In this circumstance, the test for convergence should likewise be omitted. In continued fractions where the partial numerators and denominators are always positive, no zeros arise and the value of the continued fraction always lies between two consecutive convergents, except possibly for round-off errors in the calculation. Use of the transformed continued fraction containing only the even or odd convergents of the original fraction, while more efficient generally, may lead in certain cases to a loss of significant figures and to overflow conditions more quickly.

In recurrence techniques, there may be a loss of significant figures or a heavy buildup of round-off errors. Forward and backward techniques or a combination of both may be useful [4, 5]. Attention must be paid to the proper choice of normalization factor, avoiding the use of values at or near zeros.

In using iterative methods, common pitfalls are the imposition of too stringent a convergence criteria and/or a failure to impose a limit on the maximum number of iterations.

4. General Procedures

In addition to detailing the methods of computing, the ASA FORTRAN programs will be presented, prepared in such a way that only a minimal number of changes need be made to accommodate the change in precision. Initially the test ("driver") program will also be given as well as test results and timing in the hope of detecting software and hardware errors.

The authors, hoping to satisfy the needs of all users, earnestly solicit suggestions for improvements and comments relative to testing on other computers. In this way it is hoped that an excellent library of special functions will be established.

5. References

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Automatic Computing Methods for Special Functions

Part I. Error, Probability, and Related Functions

1. Definitions

Error Function

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad \operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$
$$= 1 - \operatorname{erf} x$$

$$\operatorname{erf}(-x) = -\operatorname{erf} x \quad \operatorname{erfc}(-x) = 2 - \operatorname{erfc} x$$

Normal or Gaussian Probability Function

$$Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad A(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^x e^{-t^2/2} dt = \int_{-x}^x Z(t) dt$$
$$= \operatorname{erf}(x/\sqrt{2})$$

$$P(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt \quad Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2} dt$$
$$= \int_{-\infty}^x Z(t) dt \quad = \int_x^\infty Z(t) dt$$
$$= \frac{1}{2} [1 + \operatorname{erf}(x/\sqrt{2})] \quad = 1 - P(x) = \frac{1}{2} \operatorname{erfc}(x/\sqrt{2})$$

Confluent Hypergeometric Function

$$M(\frac{1}{2}, \frac{3}{2}, -x^2) = (\sqrt{\pi}/2x) \operatorname{erf} x$$

$$M(1, \frac{3}{2}, x^2) = (\sqrt{\pi}/2x) e^{x^2} \operatorname{erf} x$$

$$U(\frac{1}{2}, \frac{1}{2}, x^2) = \sqrt{\pi} e^{x^2} \operatorname{erfc} x$$

Incomplete Gamma Function

$$\gamma(\frac{1}{2}, x^2) = 2 \int_0^x e^{-t^2} dt = \sqrt{\pi} \operatorname{erf} x$$

$$\Gamma(\frac{1}{2}, x^2) = 2 \int_x^\infty e^{-t^2} dt = \sqrt{\pi} \operatorname{erfc} x$$

2. Method

x , a real variable

NBC = the number of binary digits in the characteristic of a floating point number

NBM = accuracy desired or the maximum number of binary digits in the mantissa of a floating point number

For $x=0$, $\operatorname{erf} x=0$; $\operatorname{erfc} x=1.0$.

The power series is used for $|x| \leq \text{ULPS}(=1)$, the upper limit for the power series

$$\operatorname{erf} x = \frac{2x}{\sqrt{\pi}} e^{-x^2} \sum_{N=0}^{\infty} \frac{(2x^2)^N}{1 \cdot 3 \cdot \dots \cdot (2N+1)} = \frac{2x}{\sqrt{\pi}} e^{-x^2} \left[\sum_{N=1}^M T_N + 1 \right]$$

$$\text{where } T_0 = 1, D_0 = 1, T_N = \frac{2x^2}{2N+1} T_{N-1} = \frac{\text{PWR}}{D_N} T_{N-1}; D_N = D_{N-1} + 2; N \geq 1.$$

Since $\min \sum_0^{\infty} T_N = 1$, it is valid to obtain M for the final term by testing for

$$T_M \leq \text{TOLER} (=2^{-\text{NBM}}).$$

NOTE: In the program, in order to minimize round-off errors, summing starts at $N=1$.

One then obtains $\operatorname{erfc} x = 1 - \operatorname{erf} x$, $\operatorname{erf}(-x) = -\operatorname{erf} x$, $\operatorname{erfc}(-x) = 2 - \operatorname{erfc} x$.

The continued fraction

$$\operatorname{erfc} x = \frac{e^{-x^2}}{\sqrt{\pi}} \left[\frac{2x}{2x^2+1} - \frac{1 \cdot 2}{2x^2+5} - \frac{3 \cdot 4}{2x^2+9} - \dots \right] = \frac{e^{-x^2}}{\sqrt{\pi}} F$$

is used for $1 < |x| \leq \text{ULCF} (=0.83 [2^{(\text{NBC}-1)/2}])$, the upper limit (machine bound) for the continued fraction. The above continued fraction yields only even convergents, forming a monotonically increasing sequence, of the following continued fraction

$$\operatorname{erfc} x = \frac{e^{-x^2}}{\sqrt{\pi}} \left[\frac{1}{x+x} \frac{1/2}{x+x} \frac{1}{x+x} \frac{3/2}{x+x} \dots \right].$$

Setting

$F_{-1} = 0, F_0 = 2x, G_{-1} = 1, G_0 = 2x^2 + 1, W_1 = 1, B_1 = G_0 + 4$ with $W_N = W_{N-1} + 2, B_N = B_{N-1} + 4, N > 1$, one obtains

$$A_N = -W_N(W_N + 1), F_N = A_N F_{N-2} + B_N F_{N-1}, G_N = A_N G_{N-2} + B_N G_{N-1}, N \geq 1.$$

If $\left| 1 - \frac{F_N/G_N}{F_{N-1}/G_{N-1}} \right| \leq \text{TOLER} (=2^{-\text{NBM}})$, then $F_N/G_N = F$. If F_{N-1}/G_{N-1} denoted by $\text{PREV} \leq F_N/G_N$ the sequence is permitted to continue after first checking for boundedness ($G_N < \text{scaling factor}$, $\text{SCF} = 2^{C2-\text{NBC}}$ where $C2 = 2^{\text{NBC}-1}$). If $\text{PREV} > F_N/G_N$, $\text{PREV} = F$. One then obtains the quantities

$$\operatorname{erf} x = 1 - \operatorname{erfc} x, \operatorname{erf}(-x) = -\operatorname{erf} x, \operatorname{erfc}(-x) = 2 - \operatorname{erfc} x.$$

For $|x| > 0.83 [2^{(\text{NBC}-1)/2}]$, $\operatorname{erf} |x| = 1$, $\operatorname{erfc} |x| = 0$, the limiting values of the functions.

Beyond $|x| = 1$, a loss of significant figures will occur in subtraction in obtaining $\operatorname{erfc} x$, and in addition round-off errors begin to accumulate more rapidly. If only $\operatorname{erf} x$ is needed, a time saving is achieved by extending the power series to $|x| = 2$. The continued fraction should then be used until the appropriate critical value is reached.

While the continued fraction is convergent for $x > 0$, the number of convergents needed increases rapidly and without scaling underflow or overflow occur for x nearing zero. For $\text{NBC} = 8$, $\text{NBM} = 27$, underflow occurred near $x = 0.91$. The partial numerators and denominators F_{N-1} ,

G_{N-1} , F_{N-2} and G_{N-2} are scaled but greater time saving is achieved by using the power series. If one uses the continued fraction for $|x| < 0.61$ both F_N and G_N must be tested for boundedness.

The maximum precision attainable with the use of the asymptotic expansion

$$\operatorname{erfc} x = \frac{e^{-x^2}}{\sqrt{\pi x}} \left(1 + \sum_{N=1}^{\infty} \frac{(-1)^N 1 \cdot 3 \cdot \dots \cdot (2N-1)}{(2x^2)^N} \right)$$

is for

$$x = 4.6, |\epsilon| \leq 1 \text{ unit in the 9th significant figure,}$$

$$x = 6.7, |\epsilon| \leq 1 \text{ unit in the 19th significant figure.}$$

The following table gives an indication of the number of terms needed for particular values of NBM

	Maximum number of terms	
	NBM = 27	NBM = 60
Power series—erf 1.....	13	21
Continued fraction—erfc 1.....	24	124
Asymptotic expansion—erfc 6.7	9	47
erfc 9.3	6	17
Continued fraction—erfc 6.7.....	3	8
erfc 9.3.....	3	6

3. Range

The range covered is limited by the machine bound for the exponential function.

For $|x| \leq 0.83[2^{(NBC-1)/2}]$, the computations are performed with $|x| \leq 9.3$ for NBC = 8, $|x| \leq 26.5$ for NBC = 11.

For $|x| > .83[2^{(NBC-1)/2}]$, $\operatorname{erf} |x| = 1$, $\operatorname{erfc} |x| = 0$.

4. Accuracy

At least (NBM-I-3) significant binary digits where I is the number of binary digits representing the integer part of x^2 . (This is essentially the accuracy of the exponential routine, with round-off error allowance.)

5. Precision

Variable—by setting the desired NBM.

6. TIMING (Seconds—Univac 1108)

For the range		NBM = 27	NBM = 60	Method
$x = 0(.005)1$	(201 values).....	0.08	0.22	Power series Continued fraction
$x = 1(.05)9.4$	(169 values).....	.16	.53	
$x = 9.2(.2)26.6$	(88 values).....		.13	
Maximum Time/Evaluation (Seconds).....		.003	.015	

7. Testing

To avoid errors due to incorrect arguments, the beginning argument, the interval, and end argument were read in as scaled integers. The generation was performed with integer values, double precision conversion made, then single precision values obtained.

The double precision results were checked, when possible, against those obtained by using the polynomial or rational approximations of Clenshaw [1],¹ Cody [2], Hart [3], and Strecok [6]. In addition, special values were checked by the asymptotic expansion and numerical integration. The single precision results were then checked against the double precision results. In all cases the results obtained agreed within the reported accuracy.

For those desiring both functions accurately with limited precision throughout the range, the approximations of Cody were found the most efficient and accurate.

8. Critical Values

erfc 9.3=0.16532 44184 030157 (-38) (NBC=8)	
erfc 26.5=0.22109 07664 26373 (-306) (NBC=11)	
erf 4.33	$ \epsilon < 1(-9)$ NBM=27
erf 4.82	$ \epsilon < 1(-11)$ NBM=36
erf 5.68	$= 1 - \epsilon$ $ \epsilon < 1(-15)$ NBM=48
erf 6.43	$ \epsilon < 1(-19)$ NBM=60

9. Special Constants

$2(\pi)^{-\frac{1}{2}} = 1.12837$	91670	95512	57389	61589	03121
$(\pi)^{\frac{1}{2}}/2 = 0.88622$	69254	52758	01364	90837	41670
$(\pi)^{-\frac{1}{2}} = 0.56418$	95835	47756	28694	80794	51560
$\pi^{\frac{1}{2}} = 1.77245$	38509	05516	02729	81674	83341
$(2\pi)^{-\frac{1}{2}} = 0.39894$	22804	01432	67793	99460	59934
$(2\pi)^{\frac{1}{2}} = 2.50662$	82746	31000	50241	57652	84811
$\pi = 3.14159$	26535	89793	23846	26433	83279
$(2)^{-\frac{1}{2}} = 0.70710$	67811	86547	52440	08443	62104
$2^{\frac{1}{2}} = 1.41421$	35623	73095	04880	16887	24209

$$\text{erf } 1 = 0.84270 \quad 07929 \quad 49714 \quad 86934 \quad 12206$$

$$\text{erf } \rho = 0.5$$

$$\rho = 0.47693 \quad 62762 \quad 04469 \quad 87338$$

¹ Figures in brackets indicate the literature references on p. 217.

$$2^{-27} = 0.74505 \quad 80596 \quad 92382 \quad 8125 \quad (-8)$$

$$2^{-36} = 0.14551 \quad 91522 \quad 83668 \quad 51806 \quad 64062 \quad 5 \quad (-10)$$

$$2^{-48} = 0.35527 \quad 13678 \quad 80050 \quad 09293 \quad 55621 \quad 33789 \quad 0625 \quad (-14)$$

$$2^{-60} = 0.86736 \quad 17379 \quad 88403 \quad 54720 \quad 59622 \quad 40695 \quad 95336 \quad 91406 \quad 25 \quad (-18)$$

10. References

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1*          SUBROUTINE ERRINT (X, ERF, ERFC)
2*          C          (ASA FORTRAN)
3*          C  USAGE.   CALL ERRINT (X, ERF, ERFC)
4*          C          ENTERING WITH X(A REAL VARIABLE), ONE OBTAINS BOTH ERF X
5*          C          AND ERFC X TO THE MAXIMUM MACHINE ACCURACY POSSIBLE.
6*          C
7*          C          THE PROGRAM, AS SET UP BELOW
8*          C          (1) WITH THE DOUBLE PRECISION TYPE STATEMENT,
9*          C          (2) NBC=11, NBM=60 ON THE FIRST DATA CARD AND
10*         C          (3) THE REMAINING CONSTANTS ON THE SECOND AND THIRD DATA
11*         C          CARDS GIVEN AS DOUBLE PRECISION CONSTANTS, YIELDS DOUBLE
12*         C          PRECISION RESULTS ON THE UNIVAC 1108 SYSTEM.
13*         C
14*         C          SINGLE PRECISION RESULTS ARE OBTAINED BY
15*         C          (1) DELETING THE TYPE STATEMENT
16*         C          (2) SETTING NBC=8, NBM=27 ON THE FIRST DATA CARD
17*         C          (3) CHANGING THE D'S TO E'S ON THE SECOND AND THIRD DATA
18*         C          CARDS.
19*         C          (4) CHANGING FUNCTION TYPE-DABS TO ABS, DEXP TO EXP
20*         C          FOR OTHER VALUES OF NBM, THE CONSTANT ON THE THIRD DATA
21*         C          CARD SHOULD BE GIVEN TO THE CORRESPONDING PRECISION.
22*         C
23*         C          CAUTION.      THE SUBROUTINE CANNOT READILY BE ADAPTED TO
24*         C          COMPUTE THE ERROR FUNCTION FOR A COMPLEX
25*         C          ARGUMENT.
26*         C           $ERF(X) = (2/\sqrt{PI}) * \text{INTEGRAL FROM 0 TO X OF } EXP(-T**2) DT$ 
27*         C           $ERFC(X) = (2/\sqrt{PI}) * \text{INTEGRAL FROM X TO INFINITY OF}$ 
28*         C           $EXP(-T**2) DT = 1 - ERF(X)$ 
29*         C
30*         C           $ERF(-X) = -ERF(X)$ 
31*         C           $ERFC(-X) = 2 - ERFC(X)$ 
32*         C
33*         C          NBC=NO. OF BINARY DIGITS IN THE CHARACTERISTIC OF A FLOATING POINT
34*         C          NUMBER.
35*         C          NBM=ACCURACY DESIRED OR MAX. NO. OF BINARY DIGITS IN THE MANTISSA OF
36*         C          A FLOATING POINT NUMBER.
37*         C
38*         C  METHOD.      POWER SERIES FOR ABS(X) .LE. 1,ULPS, UPPER LIMIT FOR
39*         C          SERIES, CONTINUED FRACTION FOR ABS(X) .GT. 1 AND .LE.
40*         C          ULCF (UPPER LIMIT OF CONTINUED FRACTION)
41*         C
42*         C  RANGE.     ABS(X) .LE. ULCF          ULCF=.83*(2.**((NBC-1)/2))
43*         C          ABS(ULCF) APPROX. 9.3, NBC=8
44*         C          26.5, NBC=11
45*         C          BEYOND THIS RANGE THE LIMITING VALUES OF THE FUNCTIONS
46*         C          ARE SUPPLIED  ERF(INF)=1.0,      ERFC(INF)=0.
47*         C
48*         C  ACCURACY.   ROUTINE WILL RETURN (NBM-I-3) SIGNIFICANT BINARY DIGITS
49*         C          WHERE I IS THE NUMBER OF BINARY DIGITS REPRESENTING THE

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50*   C           INTEGER PART OF X**2. (THIS IS ESSENTIALLY THE ACCURACY
51*   C           OF THE EXPONENTIAL ROUTINE.)
52*   C
53*   C   PRECISION.      VARIABLE-BY SETTING DESIRED NBM.
54*   C
55*   C   MAXIMUM TIME.      UNIVAC 1108-EXEC II
56*   C           (SECONDS)      NBM=27      NBM=60
57*   C           .003           .015
58*   C
59*   C
60*   C
61*   C
62*   C           DOUBLE PRECISION AN, BN, CONS, C1, DN, ERF, ERFC, F, FN, FNM1,
63*   C   1           FNM2, FOUR, GN,GNM1 GNM2, ONE, PREV, PWR, RNBC, SCF, SUM,
64*   C   2           TN, TOLER, TRRTPI, TWO, ULCF, ULPS, WN, X, Y, YSQ
65*   C
66*   C           DATA NBC, NBM/11, 60/
67*   C           DATA ONE, TWO, FOUR, ULPS, CONS / 1.DO,2.DO,4.DO,1.DO, .83DO/
68*   C           DATA TRRTPI/1. 128379167095512574DO/
69*   C
70*   C           RNBC=NBC
71*   C           TOLER=TWO ** (-NBM)
72*   C
73*   C           TEST ON ZERO
74*   C
75*   C           IF (X) 2,1,2
76*   C
77*   C   1           ERF=0
78*   C           ERFC=ONE
79*   C
80*   C           RETURN
81*   C
82*   C   2           Y=DABS(X)
83*   C           YSQ=Y ** 2
84*   C
85*   C           IF (Y-ULPS) 3,3,4
86*   C
87*   C           MAXIMUM ARGUMENT
88*   C
89*   C   4           C1=TWO ** ((RNBC-ONE)/TWO)
90*   C           ULCF=CONS * C1
91*   C
92*   C           SCALE FACTOR
93*   C
94*   C           SCF=TWO ** (C1 ** 2—RNBC)
95*   C
96*   C           LIMITING VALUE
97*   C
98*   C           IF (Y-ULCF) 10,10,11
99*   C
100*   C   11          ERF=ONE
101*   C           ERFC=0

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102*      GO TO 7
103*      C
104*      C      METHOD——POWER SERIES
105*      C
106*      3      SUM=0
107*      DN=ONE
108*      TN=ONE
109*      PWR=TWO*YSQ
110*      6      DN=DN + TWO
111*      TN=PWR*TN/DN
112*      SUM=TN + SUM
113*      C
114*      C      TOLERANCE CHECK
115*      C
116*      IF (TN-TOLER) 5,6,6
117*      C
118*      5      ERF=(SUM + ONE) *TRRTPI*Y*DEXP(—YSQ)
119*      ERFC=ONE—ERF
120*      C
121*      C      NEGATIVE ARGUMENT
122*      C
123*      7      IF (X) 8, 9, 9
124*      C
125*      8      ERF=—ERF
126*      ERFC=TWO—ERFC
127*      C
128*      9      RETURN
129*      C
130*      C      METHOD——CONTINUED FRACTION
131*      C
132*      10     FNM2=0
133*      GNM2=ONE
134*      FNM1=TWO * Y
135*      GNM1=TWO * YSQ + ONE
136*      C
137*      PREV=FNM1/GNM1
138*      C
139*      WN=ONE
140*      BN=GNM1 + FOUR
141*      C
142*      14     AN=—WN * (WN + ONE)
143*      C
144*      FN=BN * FNM1 + AN * FNM2
145*      GN=BN * GNM1 + AN * GNM2
146*      C
147*      F=FN/GN
148*      C
149*      C      TOLERANCE CHECK
150*      C
151*      IF (DABS (ONE— (F/PREV))—TOLER) 12, 12, 13
152*      13     IF (PREV—F) 17, 17, 18
153*      C      BOTH FN AND GN MUST BE TESTED IF ABS (X) .LT. .61

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154*      17      IF (GN .LT. SCF) GO TO 16
155*      C
156*      C              SCALING
157*      C
158*      15      FN = FN/SCF
159*      GN = GN/SCF
160*      FNM1 = FNM1/SCF
161*      GNM1 = GNM1/SCF
162*      C
163*      16      FNM2 = FNM1
164*      GNM2 = GNM1
165*      FNM1 = FN
166*      GNM1 = GN
167*      C
168*      WN = WN + TWO
169*      BN = BN + FOUR
170*      PREV = F
171*      GO TO 14
172*      C
173*      18      F = PREV
174*      12      ERFC = F*DEXP (-YSQ)*TRRTPI/TWO
175*      ERF = ONE - ERFC
176*      C
177*      GO TO 7
178*      C
179*      END

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1*   C
2*   C
3*   C   TEST PROGRAM FOR DOUBLE PRECISION ERROR FUNCTION AND COMPLI-
4*   C   MENTARY ERROR FUNCTION      (UNIVAC 1108)
5*   C   CALL ERRINT (X, ERF, ERFC)
6*   C
7*   C
8*   C   DOUBLE PRECISION DN, DSCALE, ERF, ERFC, X
9*   C
10*  C   DIMENSION M(12)
11*  C   DIMENSION X (500), ERF (500), ERFC (500)
12*  C
13*  C   DATA M(1), M(2), M(3), M(4), M(5), M(6), M(7), M(8), M(9),
14*  C   1 M(10), M(11), M(12)/0, 5, 100, 100, 10, 5, 95, 10,
15*  C   2      95, 5, 270, 10/
16*  C   J=1
17*  C
18*  C   1   N=M(J)
19*  C   NINCR=M(J+1)
20*  C   NENDX=M(J+2)
21*  C   DSCALE=M(J+3)
22*  C
23*  C   I=1
24*  C   5   DN=N
25*  C   X(I)=DN/DSCALE
26*  C
27*  C   IF (N-NENDX) 2, 3, 3
28*  C
29*  C   2   I=I+1
30*  C   N=N+NINCR
31*  C   GO TO 5
32*  C
33*  C   3   DO 4 L=1, I
34*  C   4   CALL ERRINT (X (L), ERF (L), ERFC (L))
35*  C
36*  C   WRITE (6, 1002)
37*  C   DO 6 L=1, I
38*  C   6   WRITE (6, 1001) X (L), ERF (L), ERFC (L)
39*  C
40*  C   IF (J .EQ. 9) STOP
41*  C   J=J+4
42*  C   GO TO 1
43*  C
44*  C   1001  FORMAT (3D26. 18)
45*  C   1002  FORMAT (1H1, 11X, 1HX, 23X, 6HERF(X), 20X, 7HERFC(X) ./)
46*  C
47*  C   END

```

X	ERF(X)	ERFC(X)
.0000000000000000	.0000000000000000	.1000000000000000 + 001
.5000000000000000 — 001	.563719777970166236 — 001	.943628022202983377 + 000
.1000000000000000 + 000	.112462916018284892 + 000	.887537083981715109 + 000
.1500000000000000 + 000	.167995971427363495 + 000	.832004028572636506 + 000
.2000000000000000 + 000	.222702589210478454 + 000	.777297410789521546 + 000
.2500000000000000 + 000	.276326390168236935 + 000	.723673609831763065 + 000
.3000000000000000 + 000	.328626759459127429 + 000	.671373240540872572 + 000
.3500000000000000 + 000	.379382053562310322 + 000	.620617946437689678 + 000
.4000000000000000 + 000	.428392355046668455 + 000	.571607644953331545 + 000
.4500000000000000 + 000	.475481719786923677 + 000	.524518280213076324 + 000
.5000000000000000 + 000	.520499877813046541 + 000	.479500122186953459 + 000
.5499999999999999 + 000	.563323366325108957 + 000	.436676633674891044 + 000
.5999999999999999 + 000	.603856090847925924 + 000	.396143909152074076 + 000
.6500000000000000 + 000	.642029327355671843 + 000	.357970672644328157 + 000
.7000000000000000 + 000	.677801193837418476 + 000	.322198806162581524 + 000
.7500000000000000 + 000	.711155633653515133 + 000	.288844366346484867 + 000
.7999999999999999 + 000	.742100964707660485 + 000	.257899035292339516 + 000
.8499999999999999 + 000	.770668057608352530 + 000	.229331942391647470 + 000
.9000000000000000 + 000	.796908212422832131 + 000	.203091787577167870 + 000
.9500000000000000 + 000	.820890807273277944 + 000	.179109192726722056 + 000
.1000000000000000 + 001	.842700792949714871 + 000	.157299207050285130 + 000

X	ERF(X)	ERFC(X)
.1000000000000000 + 001	.842700792949714871 + 000	.157299207050285130 + 000
.1500000000000000 + 001	.966105146475310728 + 000	.338948535246892735 — 001
.2000000000000000 + 001	.995322265018952735 + 000	.467773498104726582 — 002
.2500000000000000 + 001	.999593047982555043 + 000	.406952017444958938 — 003
.3000000000000000 + 001	.999977909503001415 + 000	.220904969985854420 — 004
.3500000000000000 + 001	.99999256901627659 + 000	.743098372341412774 — 006
.4000000000000000 + 001	.99999984582742101 + 000	.154172579002800195 — 007
.4500000000000000 + 001	.9999999803383957 + 000	.196616044154288756 — 009
.5000000000000000 + 001	.9999999998462541 + 000	.153745979442803493 — 011
.5499999999999999 + 001	.999999999992643 + 000	.735784791797439867 — 014
.5999999999999999 + 001	.999999999999979 + 000	.215197367124989151 — 016
.6500000000000000 + 001	.1000000000000000 + 001	.384214832712064783 — 019
.7000000000000000 + 001	.1000000000000000 + 001	.418382560777941505 — 022
.7500000000000000 + 001	.1000000000000000 + 001	.277664938603056939 — 025
.7999999999999999 + 001	.1000000000000000 + 001	.112242971729829286 — 028
.8499999999999999 + 001	.1000000000000000 + 001	.276232407133377193 — 032
.9000000000000000 + 001	.1000000000000000 + 001	.413703174651381081 — 036
.9500000000000000 + 001	.1000000000000000 + 001	.376921448565488099 — 040

X	ERF(X)	ERFC(X)
.9500000000000000 + 001	.1000000000000000 + 001	.376921448565488099 -- 040
.1000000000000000 + 002	.1000000000000000 + 001	.208848758376254514 -- 044
.1050000000000000 + 002	.1000000000000000 + 001	.703592809017752444 -- 049
.1100000000000000 + 002	.1000000000000000 + 001	.144086613794369515 -- 053
.1150000000000000 + 002	.1000000000000000 + 001	.179330964357678251 -- 058
.1200000000000000 + 002	.1000000000000000 + 001	.135626116920590467 -- 063
.1250000000000000 + 002	.1000000000000000 + 001	.623194278197991275 -- 069
.1300000000000000 + 002	.1000000000000000 + 001	.173955731546672518 -- 074
.1350000000000000 + 002	.1000000000000000 + 001	.294943311325798927 -- 080
.1400000000000000 + 002	.1000000000000000 + 001	.303722984775031350 -- 086
.1450000000000000 + 002	.1000000000000000 + 001	.189939594197950386 -- 092
.1500000000000000 + 002	.1000000000000000 + 001	.721299417245120960 -- 099
.1550000000000000 + 002	.1000000000000000 + 001	.166320164004887325 -- 105
.1600000000000000 + 002	.1000000000000000 + 001	.232848575157153193 -- 112
.1650000000000000 + 002	.1000000000000000 + 001	.197913057555326932 -- 119
.1700000000000000 + 002	.1000000000000000 + 001	.102122801509426159 -- 126
.1750000000000000 + 002	.1000000000000000 + 001	.319886381234348273 -- 134
.1800000000000000 + 002	.1000000000000000 + 001	.608236923181640276 -- 142
.1850000000000000 + 002	.1000000000000000 + 001	.701996157498568456 -- 150
.1900000000000000 + 002	.1000000000000000 + 001	.491772283925648080 -- 158
.1950000000000000 + 002	.1000000000000000 + 001	.209095414792273150 -- 166
.2000000000000000 + 002	.1000000000000000 + 001	.539586561160790480 -- 175
.2050000000000000 + 002	.1000000000000000 + 001	.845084236957246622 -- 184
.2100000000000000 + 002	.1000000000000000 + 001	.803245387102246359 -- 193
.2150000000000000 + 002	.1000000000000000 + 001	.463333653965844984 -- 202
.2200000000000000 + 002	.1000000000000000 + 001	.162190586093347461 -- 211
.2250000000000000 + 002	.1000000000000000 + 001	.344534886046460488 -- 221
.2300000000000000 + 002	.1000000000000000 + 001	.444126594808806166 -- 231
.2350000000000000 + 002	.1000000000000000 + 001	.347405949564997476 -- 241
.2400000000000000 + 002	.1000000000000000 + 001	.164898258315193574 -- 251
.2450000000000000 + 002	.1000000000000000 + 001	.474936126406738376 -- 262
.2500000000000000 + 002	.1000000000000000 + 001	.830017257119653210 -- 273
.2550000000000000 + 002	.1000000000000000 + 001	.880166269072796021 -- 284
.2600000000000000 + 002	.1000000000000000 + 001	.566319240885615157 -- 295
.2649999999999999 + 002	.1000000000000000 + 001	.221090766426373696 -- 306
.2700000000000000 + 002	.1000000000000000 + 001	.0000000000000000

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