LUMINOSITY AND TEMPERATURE.

By P. G. Nutting.

With light expressed in terms of radiation and radiation a given function of temperature and wave length, light or luminosity is expressible directly in terms of temperature. These relations of light to radiation and of radiation to temperature have recently received considerable attention. Assuming them to be known, let us see what may be done toward establishing the direct relation of luminosity to temperature. The numerical values of some of the constants involved are not known with sufficient precision to permit of very precise numerical deductions, but the general relations developed show the interrelations of the various quantities involved. The relation of greatest practical interest is that of luminous efficiency to temperature.

Previous work has been limited to portions of the general problem. There are some good experimental determinations of the amount of light (monochromatic and total) emitted by bodies at known temperatures, but no one appears to have attacked the general problem directly.

Paschen and Wanner¹ in 1899 used a photometric method for determining the second constant of the Wien-Paschen radiation function

$$J = c_1 \lambda^{-\alpha} e^{-c_2/\lambda T}$$

c_1 , c_2 , and α being constants, λ wave length, and T absolute temperature. Working at constant wave length, they wrote this in the form

$$\log J = r_1 - \frac{r_2}{T}$$

¹ Paschen and Wanner, Sitzber. d. Berliner Akad., **2**, pp. 5-11; 1899.

or for two different temperatures T_1 and T_2

$$\log \frac{J_1}{J_2} = r_2 \left(\frac{1}{T_2} - \frac{1}{T_1} \right),$$

r_1 and r_2 being constants. Assuming light proportional to radiation, they computed r_2 for four different wave lengths from their data. The limitations of this assumption we shall discuss later on.

Lummer and Kurlbaum² studied the photometric brightness of a hole in a uniformly heated opaque envelope as a function of temperature. They compared the total light with that from a constant source. They expressed their results in the form

$$\frac{H_1}{H_2} = \left(\frac{T_1}{T_2} \right)^x$$

H being brightness, T temperature, and x a variable having the following values:

$T = 900$	1000	1100	1200	1400	1600	1900
$x = 30$	25	21	19	18	15	14

They state that x approaches the value 12, but this is not apparent from the plotted curve, and later Rasch showed (see below) that their equation was essentially defective.

Lummer and Pringsheim³ then took up the absolute measurement of the total light in Hefners emitted by a square millimeter of a perforation in such an opaque envelope.

They found

Temp. C.	Light/mm ² .
1175°	0.0042 H
1325	0.0220
1435	0.0635

and extrapolated

1500° C	0.1 H (about)
1700	0.5
1800	1.0

² Lummer and Kurlbaum, *Verh. d. Phys. Gesell.*, **2**, pp. 89-90; 1900.

³ Lummer and Pringsheim, *Phys. Zs.*, **3**, pp. 97-100; 1901.

but did not attempt any formulation of their results. This work was later greatly extended by Nernst,⁴ using an iridium furnace cross-checked by an auxiliary heated rod of oxide. He obtained for the light (Hefners) per sq. mm.

Temp. absolute.	Light H/mm^2 .	Temp. absolute.	Light H/mm^2 .
1464°	0.005	2357°	4.0
1524	0.01	2446	6.0
1685	0.05	2516	8.0
1764	0.1	2571	10.
1982	0.5	2619	12.
2092	1.0	2680	15.
2217	2.0	2763	20.

His measurements extended to about 2290° absolute. The above values were obtained from the relation

$$\log H = B - \frac{A}{T}$$

where H is Hefners per mm^2 , T is absolute temperature, $B = 5.367$ and $A = 11230$, the constants being determined from observations between 1400° and 2200°. Temperature measurements were made with a Wanner pyrometer, an instrument based on the monochromatic Paschen-Wanner relation mentioned above. His observations then indicate that the monochromatic and total light emission of a complete radiator between 1400° and 2200° differ only by a constant factor.

Rasch⁵ appears to have been the first to attempt a rational foundation for the relation of luminosity to temperature. Since retinal action is essentially a chemical process, he reasoned that the thermal equation of chemical equilibrium should apply, the luminous sensation being related to "retinal temperature" and this again to the rate of absorption of energy. In van't Hoff's reaction isochore

$$\frac{dK}{K} = - \frac{q}{R} \frac{dT}{T^2}$$

⁴ W. Nernst, *Phys. Zs.*, **7**, pp. 380-383; June 1, 1906.

⁵ E. Rasch, *Ann. d. Phys.*, **14**, pp. 193-203; 1904.

he replaces the equilibrium constant K by brightness H and the constants $-q/R$ by κ . Hence by integration

$$\log H = c - \frac{\kappa}{T}$$

$$\log \frac{H_2}{H_1} = \kappa \left(\frac{1}{T_1} - \frac{1}{T_2} \right)$$

This indicates that the quantity x in Lummer and Kurlbaum's equation (see above) is $x = \kappa/T$. Hence it is not a constant nor does it approach a constant value. It will be noticed that Rasch's equations are the same in form and but slightly different in interpretation from those of Paschen and Wanner.

Rasch computed a mean value of $\kappa = 26750$ from Lummer and Kurlbaum's data. Lucas,⁶ from the relation $\kappa = c_2/\lambda$ and c_2 (the second constant of the Wien-Paschen equation) = 14500, finds for λ the value 0.542μ , an equivalent mean wave length in dealing with total light.

Goldhammer⁷ from the data of König on visual sensibility and the work of Rasch just mentioned, derived the function

$$V = \left(\frac{\lambda_0}{\lambda} e^{1 - \frac{\lambda_0}{\lambda}} \right)^m$$

to represent the ratio of light to radiation at each wave length, λ_0 being the wave length of maximum sensibility and m a constant varying from 100 to 300 for different individuals. Both λ_0 and m he found to vary somewhat with the intensity of the radiation. Goldhammer recognized that it was necessary to introduce this auxiliary function of radiation, the *visibility* (V), to express light in terms of radiation. He multiplied the above function by the Wien-Paschen function $J = c_1 \lambda^{-5} e^{-c^2/\lambda T}$ and thus obtained for the total light from a complete radiator

$$H = a \left(\frac{T}{T + \epsilon} \right)^n$$

a , ϵ and n being constants. He did not attempt to construct a visibility function to represent the average human eye. The

⁶ R. Lucas, Phys. Zs., **6**, 19-20; Jan. 1, 1906.

⁷ D. A. Goldhammer, Ann. Physik., **16**, 621-652; April, 1905.

one he did construct contains the large exponent m and is always steeper on the blue side of the maximum, while the experimental curves are steeper on either side in about equal numbers. It differs only in its constants from the Wien-Paschen function for non selective radiation.

The writer⁸ has developed a visibility function of the form

$$V = V_0 e^{-\kappa(\lambda - \lambda_m)^2}$$

to represent the sensibility of the average human eye. Here V_0 is a constant depending upon the units employed, κ is 4.6 at low intensities (<0.05 meter-candle) and 2.0 at high (>10 m. c.); while λ_m , the wave length of maximum sensibility, is similarly $.511\mu$ at low intensities (rod vision) and $.565\mu$ at high (cone vision) intensities.

If the light is expressed in meter-candles and the radiation in meter-watts per unit wave length, then V_0 is the number of candles per watt at the wave length of maximum sensibility, $\lambda = \lambda_m$, for here $L = EV$ becomes $L = EV_0$, E being the radiation. To find this ratio of light to radiation it is only necessary to measure the same monochromatic radiation as light and as power. This ratio the author (l. c.) found to be 13.5 candles-watt, while Drysdale⁹ found 16.7. König's data on visual sensibility indicate that this ratio is a constant except for intensities ranging from about 0.3 m.-c. up to 30 m.-c., which is the range during which the transition from rod to cone vision occurs.

Visibility then is the constant of proportionality between radiation and light. It varies enormously with the wave length, but at constant wave length is constant at very high and very low intensities, with an uncertainty slightly larger than the possible error in its determination through a small range of moderate intensities.

Let us now apply the visibility function to known types of radiation. The best known is the non-selective radiation from the interior of a nearly closed cavity having opaque walls at a uniform temperature. This is known to be related to wave

⁸ Electrical World, June 27; 1908.

This Bulletin, 5, pp. 261-305; 1908

⁹ C. V. Drysdale, Proc. Roy. Soc. 80, pp. 19-25; 1907.

length and the temperature of the radiator by the Planck function

$$E = \frac{c_1 \lambda^{-5}}{e^{c_2/\lambda T} - 1}$$

This is difficult to use, but if it is expanded in a power series the first term is the Wien-Paschen function

$$(1) \quad E = c_1 \lambda^{-n} e^{-c_2/\lambda T}$$

which is sufficiently accurate in dealing with the short wave lengths of the visible spectrum. Multiplying this by the visibility function

$$(2) \quad V = V_0 e^{-\kappa (\lambda - \lambda_m)^2}$$

the product is the desired relation between luminosity and temperature

$$(3) \quad \begin{aligned} L &= E V \\ &= c_1 V_0 \lambda^{-n} e^{-c_2/\lambda T - \kappa (\lambda - \lambda_m)^2} \end{aligned}$$

Hence the isochromatic relation (λ constant) is of the form

$$(4) \quad \log L = A - \frac{B}{T}$$

where

$$A = \log (c_1 V_0 \lambda^{-n}) - \kappa (\lambda - \lambda_m)^2$$

and

$$B = c_2/\lambda$$

The value of the constant B is seen to be independent of the form of the visibility function while the constant A is not. Further, the form of (4) does not in any way depend upon the form of the visibility function, but only upon the form of the emission function $E(\lambda, T)$.

From (4) may be deduced the relative temperatures at which the various spectral colors appear. For $L = 1$ the temperature at which complete radiators will attain a specific luminosity of one light unit (the threshold value in this case) per unit wave length will be given by

$$T = \frac{B}{A}$$

from which T may be calculated for each λ .

Equation (4) assumes a constant wave length, hence it can apply to but a limited spectral region at a time. The reason that it has been found (by Nernst and others) experimentally to hold approximately for total light emission is that for a complete radiator, 90 per cent of the light is confined to a spectral region only 0.1μ broad, because of the narrow form of the visibility curve. The fact that both κ and λ_m vary considerably with intensity is of no consequence so long as the product $\kappa(\lambda - \lambda_m)^2$ remains constant.

The relation between total light emission and temperature is obtained by integrating $EVd\lambda$ from zero to infinity. In the form (3) above it is not readily integrable but using the alternative form¹⁰ of visibility function

$$(5) \quad V = V_0 \left(\frac{\lambda_m}{\lambda} \right)^\nu e^{\nu \left(1 - \frac{\lambda_m}{\lambda} \right)}$$

the product EV may be easily integrated in the form

$$(6) \quad L = \int_0^\infty EV d\lambda = c_1 V_0 \lambda_m^\nu e^\nu \Gamma(n + \nu - 1) \left(\frac{c_2}{T} + \nu \lambda_m \right)^{-(n + \nu - 1)}$$

which is of the simple form

$$(7) \quad L = A \left(\frac{B}{T} + 1 \right)^{-p}$$

where

$$A = c_1 V_0 \lambda_m^\nu e^\nu \Gamma(n + \nu - 1) (\nu \lambda_m)^{-p}$$

$$= c_1 V_0 \left(\frac{B}{c_2} \right)^{n-1} \frac{\Gamma(n + \nu - 1)}{\sqrt{2\pi\nu} \Gamma(\nu + 1)}$$

$$B = c_2 / \nu \lambda_m$$

$$p = n + \nu - 1$$

This expression for the light emitted applies to any body whose radiation is expressible by equation (1) viewed by any eye whose chromatic sensibility may be represented by equation (5); that is, it is of general application to nonselective radiation.

¹⁰This Bulletin, 5, p. 277; 1909.

Defining luminous efficiency F as the ratio of the total light emitted to the total energy radiated from the same area we have from (1) and (7)

$$(8) \quad F = \frac{\int_0^\infty EV d\lambda}{\int_0^\infty Ed\lambda} \\ = A \left(\frac{B}{T} + 1 \right)^{-\nu} : \alpha T^{n-1}$$

since

$$\int_0^\infty Ed\lambda = c_1 \left(\frac{T}{c_2} \right)^{n-1} \Gamma(n-1) = \alpha T^{n-1}$$

Equation (8) may be reduced to

$$F = V_0 \sqrt{2\pi\nu} B^{n-1} \frac{\Gamma(n+\nu-1)}{\Gamma(\nu+1) \Gamma(n-1)} \frac{T^\nu}{(B+T)^{n+\nu-1}}$$

a form more convenient for computation, by means of the relation

$$x! = x^x e^{-x} \sqrt{2\pi x} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \dots \right)$$

Since ν is greater than 100 the series in parenthesis has the value unity to within .001.

The luminous efficiency has a maximum value

$$(9) \quad F_m = V_0 \left(\frac{n-1}{n+\nu-1} \right)^{\frac{1}{2}}$$

at a temperature

$$(10) \quad T_m = B \frac{\nu}{n-1} = \frac{c_2}{(n-1)\lambda_m}$$

The numerical values of all the constants in (6) and (8) have been determined, but in no case with great precision. From König's data on visual sensibility,¹¹ ν may be calculated to be 120 ± 10 , a pure number with an uncertainty of about 10 units. The wave length of maximum visual sensibility is $0.565 \pm 0.005 \mu$. General average values of ν and λ_m for a large number of individuals might lie outside even these limits of uncertainty, but it is hardly probable. The values here quoted apply to cone vision. For low intensities (rod vision) ν is about 300 while $\lambda_m = 0.511 \mu$, but the value of ν_0 has not been determined, so that we are unable

¹¹ This Bulletin, 5, p. 279; 1908-9.

at present to compute the luminosity of very faint sources and weak reflected light.

For enclosed radiators of the "black body" type $C_2 = 14500 \pm 300$, when wave lengths are in microns (μ) and temperature in centigrade degrees absolute. $n = 5$, an integer so far as we know. For other bodies, n varies from 5.5 to 7. Coblentz¹² for an untreated carbon filament found $n = 5.2$ to 6.5 (decreasing with increase in temperature) for platinum $n = 6$ to 8, "metallized" carbon $n = 5.8$ to 6.0, silicon carbide $n = 6.4$. For lamps run at normal voltage he found for

Metallized carbon	$n = 6.1$
Tantalum	$n = 6.3$
* Tungsten	$n = 6.6$
Osmium	$n = 6.9$

For radiation from a cavity or "black body," then, the temperature of maximum luminous efficiency is

$$T_m = \frac{14500}{(5 - 1) \cdot 565} = 6420^\circ \pm 200^\circ \text{ abs}$$

the value of that efficiency being

$$F_m = 15 \left(\frac{5 - 1}{5 + 120 - 1} \right)^{1/2} = 2.7 \pm 0.2 \frac{\text{cand.}}{\text{watt.}}$$

The value $\nu_0 = 15.0$ is the mean of the values of Drysdale (16.5) and the author (13.5), and the uncertainty in the result is due chiefly to the uncertainty in this value.

This result indicates that such a radiator could give at best (6420°) only about 0.18 as much light as it would if all its radiation were confined to the yellow-green.

At a temperature of 2000° abs. (1727° C.) the computed luminous efficiency is

$T = 2000^\circ$	$C_2 = 14500$	$\nu = 120$	$V_0 = 15$
$n = 5.0$	6.0	7.0	
$F = .055$	.209	5.05 cand./watt.	

The first case ($n = 5$) corresponds to a perfect radiator or black body. This table is made on the assumption of C_2 constant. But

¹²This Bulletin, 5, p. 339; 1908-9.

little is known as to its variation with temperature and with the substance radiating. Lummer and Pringsheim's results indicate that for a given substance, C_2 varies in proportion to n , so that

$$\lambda_{max}T = C_2/n = \text{const.}$$

and they give for platinum $C_2 = 15600$. For platinum, then, at $2000^\circ \text{ abs.} = 1727^\circ \text{ C.}$, just below the melting point, $F = .121$ cand./watt.

The variation of luminous efficiency with temperature

$$\frac{\partial E}{\partial T} = F \left(\frac{\nu}{T} - \frac{n + \nu - 1}{B + T} \right)$$

is not very rapid at high temperatures, so that (8) could hardly be applied to advantage to the measurement of temperatures, nor could (8) be applied to the determination of the visual constant ν . The variation of F with ν is for

$T = 2000$	$n = 5$	$\nu_0 = 15.$
$\nu = 120$	110	100
$F = .051$	.058	.062 cand./watt.

Its most useful field appears to be in the determination of C_2 , with which F varies rapidly.

The agreement of the above calculated efficiencies with observed values are sufficiently close to give formula (8) considerable weight. The rapid increase in computed efficiency with the constant n , shown above, is quite in accord with experience with the new glow lamps, and Coblenz's values of this constant for such lamps.

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