JOURNAL OF RESEARCH of the National Bureau of Standards — B. Mathematical Sciences
Vol. 73B, No. 1, January–March 1969

The Cube of Every Connected Graph is 1-Hamiltonian *

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(November 15, 1968)

Let G be any connected graph on 4 or more points. The graph G^3 has as its point set that of G, and two distinct points u and v are adjacent in G^3 if and only if the distance between u and v in G is at most three. It is shown that not only is G^3 hamiltonian, but the removal of any point from G^3 still yields a hamiltonian graph.

Key Words: Cube of a graph; graph; hamiltonian.

Let G be a graph (finite, undirected, with no loops or multiple lines). A *walk* of G is a finite alternating sequence of points and lines of G, beginning and ending with a point and where each line is incident with the points immediately preceding and following it. A walk in which no point is repeated is called a *path*; the *length* of a path is the number of lines in it.

A graph G is *connected* if between every pair of distinct points there exists a path, and for such a graph, the *distance* between two points u and v is defined as the length of the shortest path if $u \neq v$ and zero if $u=v$.

A walk with at least three points in which the first and last points are the same but all other points are distinct is called a *cycle*. A cycle containing all points of a graph G is called a *hamiltonian cycle* of G, and G itself a *hamiltonian graph*.

Throughout the literature of graph theory there have been defined many graph-valued functions f on the class of graphs. In certain instances results have been obtained to show that if G is connected and has sufficiently many points, then the graph $f(G)$ (or its iterates $f^n(G)$) is a hamiltonian graph. Examples of such include the line-graph function $L(G)$ and the total graph function $T(G)$ (see [2, 1],[1] respectively).

The *line-graph* $L(G)$ of graph G is a graph whose point set can be put in one-to-one correspondence with the line set of G such that adjacency is preserved. The *total graph* $T(G)$ has its point set in one-to-one correspondence with the set of points and lines of G in such a way that two points of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent or incident.

Another example which always yields a hamiltonian graph is the cube function. In fact, if x is any line in a connected graph G with at least three points, then the cube of G has a hamiltonian cycle containing x. This follows from a result due to Karaganis [4] by which the cube of any connected graph G on $p(\geqslant 3)$ points turns out to be *hamiltonian-connected*, i.e., between any two points there exists a path containing all points of G. Now if x is any line joining points u and v in G, then the addition of x to the hamiltonian path between u and v in the cube of the graph produces a hamiltonian cycle of G containing x.

The *cube* G^3 of a connected graph G has as its point set that of G, and two distinct points u and v are adjacent in G^3 if and only if the distance between u and v in G is at most three. The purpose of this note is to prove that if G is a connected graph (with $p \geqslant 4$ points) then not only is G^3 hamiltonian, but the removal of any point from G^3 still yields a hamiltonian graph. Graphs enjoying this property have been referred to as 1-*hamiltonian* in [3].

THEOREM. *If* G *is a connected graph on* p $\geqslant$ 4 *points, then* G^3 *is 1-hamiltonian.*

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*An invited paper.

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[1] Figures in brackets indicate the literature references at the end of this paper.

Proof. That G^3 is hamiltonian is already known; indeed, there exists a hamiltonian cycle of G^3 which contains any specified line of G as we have already noted. We now show that the deletion of any point from G^3 also results in a hamiltonian graph. The proof is by induction on p, the result being true for $p=4$, since the cube of any connected graph on 4 points yields the complete graph, and the removal of any point from this leaves a cycle on three points. Clearly the theorem follows if the result is proved for a spanning tree T of G. Assume the statement to be true for all trees on n points, $4 \leqslant n < p$, and let T be any tree on p points. Consider the forest F obtained by deleting any point u from T. Denote the components of F by T_i, $1 \leqslant i \leqslant k$, where k is the degree of u in T. Furthermore, let T_i have p_i points so that $p=1+\sum_{i=1}^{k} p_i$, and let u_i be the point in T_i which is adjacent to u in T.

If $k=1$, then F is a tree. If $p_1 \geqslant 4$, then T_1^3 is 1-hamiltonian by hypothesis, while if $p_1=3$, T_1^3 is a triangle; in either case T_1^3 is hamiltonian.

Assume $k>1$. For all i such that $1 \leqslant i \leqslant k$ and $p_i \geqslant 3$, it follows from earlier remarks that a hamiltonian cycle C_i may be selected in T_1^3 to contain the line u_iv_i where v_i is adjacent to u_i in T_i. If $p_i=2$, T_i has only two points, namely u_i and v_i. We now construct a hamiltonian cycle C in T^3-u as follows.

For each C_i so defined for $p_i \geqslant 3$, remove the line u_iv_i to obtain a path P_i beginning at u_i and ending at v_i. If $p_i=2$, then T_i is itself a path P_i beginning at u_i and ending at v_i. For completeness, if $p_i=1$, we assign a second label v_i to the point u_i and speak of the trivial path P_i. Observe that for $r \neq s$, the distance from v_r to u_s is two if P_r is trivial, and is three otherwise. In either case the line v_ru_s is present in T^3-u. The desired cycle C is $P_1, v_1u_2, P_2, \ldots, P_j, v_ju_{j+1}, P_{j+1}, \ldots, P_k, v_ku_1$.

The previous result cannot be improved so that the removal of any two points from the cube of a connected graph with at least five points results in a hamiltonian graph. For example, if P is a path containing adjacent points u and v, neither an end-point, then $P^3-\{u, v\}$ is not hamiltonian.

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(Paper 73B1–287)