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FINITE GROUPS IN WHICH EVERY NON-NILPOTENT MAXIMAL INVARIANT SUBGROUP OF ORDER DIVISIBLE BY *p* IS A TI-SUBGROUP

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Abstract. Let A and G be finite groups such that A acts coprimely on G by automorphisms. We obtain a complete description of the structure of finite groups in which every non-nilpotent maximal A-invariant subgroup of order divisible by p is a TI-subgroup for any fixed prime divisor p of the order of groups.

Keywords: coprime action, non-nilpotent maximal *A*-invariant subgroup, TI-subgroup. 2020 *Mathematics Subject Classification:* 20D10.

1. INTRODUCTION

All groups are considered to be finite. Let *G* be a group and *H* a subgroup of *G*. If $H^g \cap H = 1$ or *H* for each $g \in G$, then *H* is called a TI-subgroup of *G*. In [10, Theorem 1.1] Shi and Zhang proved that every non-nilpotent subgroup of a group *G* is a TI-subgroup if and only if every non-nilpotent subgroup of *G* is normal. As a generalization, Shi and Li [7, Theorem 1] showed that every self-centralizing non-nilpotent subgroup of *G* is subnormal. Subgroup or a subnormal subgroup if and only if every non-nilpotent subgroup of *G* is subnormal. Consider the coprime action of groups, let *A* and *G* be groups such that *A* acts coprimely on *G* by automorphisms. Shao and Beltrán [5, Theorem B] obtained that every non-nilpotent *A*-invariant subgroup of *G* is normal. Moreover, Liu and Shi [2, Theorem 1.1] showed that every self-centralizing non-nilpotent *A*-invariant subgroup of *G* is a TI-subgroup if and only if every non-nilpotent *A*-invariant subgroup of *G* is normal.

Consider non-nilpotent maximal subgroups of a group, Lu, Pang and Zhong [3, Theorem 3.5] proved that every non-nilpotent maximal subgroup of a group G is a TI-subgroup if and only if every non-nilpotent maximal subgroup of G is normal. Combine the coprime action of groups, Shi and Liu [9, Theorem 1.8] obtained that every non-nilpotent maximal A-invariant subgroup of a group G is a TI-subgroup if and only if every non-nilpotent maximal non-nilpotent maximal subgroup of G is normal.

Let G be a group and p any fixed prime divisor of |G|. In [8, Theorems 1.3, 1.4 and 1.6] Shi, Li and Shen provided a complete description of group G in which every non-nilpotent maximal subgroup of order divisible by p is normal. Moreover, Beltrán and Shao [1, Theorem C] indicated that G is solvable if every non-nilpotent maximal A-invariant subgroup of G of order divisible by p is normal.

As a generalization of above research, Shi [6, Theorem 1.1] had the following result.

THEOREM 1.1. [6, Theorem 1.1] Let G be a group and p a fixed prime divisor of |G|. Then every maximal subgroup of G is nilpotent or a TI-subgroup or has p'-order if and only if one of the following statements holds:

(1) Every maximal subgroup of G is nilpotent or normal or has p'-order;

(2) $G = Z_q^m \rtimes (Z_p \rtimes H)$ is a Frobenius group with Z_q^m being its Frobenius kernel and $Z_p \rtimes H$ being its Frobenius complement such that $Z_p \rtimes H$ acts irreducibly on Z_q^m , where m > 1, $Z_p \in Syl_p(G)$, H is cyclic or a direct product of a generalized quaternion group and a cyclic group such that $[Z_p, H] \neq 1$.

In this paper, as a further generalization of [6, Theorem 1.1], consider the coprime of groups, we obtain the following result, the proof of which is given in Section 3.

THEOREM 1.2. Let A and G be groups such that A acts coprimely on G by automorphisms, let p be any fixed prime divisor of |G|. Then every non-nilpotent maximal A-invariant subgroup of G of order divisible by p is a TI-subgroup if and only if one of the following statements holds:

(1) every non-nilpotent maximal A-invariant subgroup of G of order divisible by p is normal;

(2) $G = Z_q^m \rtimes (Z_p \rtimes K)$ is a Frobenius group with kernel Z_q^m and complement $Z_p \rtimes K$, where Z_q^m is a minimal A-invariant normal subgroup of G and m > 1, Z_p is an A-invariant Sylow subgroup of G and K is an A-invariant cyclic group or an A-invariant direct product of a generalized quaternion group and a cyclic group of odd order with $[Z_p, K] \neq 1$.

The following result is a direct corollary of Theorem 1.2.

COROLLARY 1.3. Let A and G be groups such that A acts coprimely on G by automorphisms, let p be the smallest prime divisor of |G|. Then every non-nilpotent maximal A-invariant subgroup of G of order divisible by p is a TI-subgroup if and only if every non-nilpotent maximal A-invariant subgroup of G of order divisible by p is normal.

2. TWO NECESSARY LEMMAS

LEMMA 2.1. [5, Lemma 2.3] Suppose that a group A acts coprimely on a group G. If every maximal A-invariant subgroup of G is normal, then G is nilpotent.

LEMMA 2.2. [9, Lemma 2.3] Let A and G be groups such that A acts coprimely on G by automorphisms. Suppose that M is a maximal A-invariant subgroup of G, then M is either self-normalizing in G or normal in G.

3. PROOF OF THEOREM 1.2

Proof. For the necessity part.

When every non-nilpotent maximal A-invariant subgroup of G of order divisible by p is normal, it is obvious that every non-nilpotent maximal A-invariant subgroup of G of order divisible by p is a TI-subgroup.

When *G* has non-normal non-nilpotent maximal *A*-invariant subgroups of order divisible by *p*. Suppose that *M* is a non-normal non-nilpotent maximal *A*-invariant subgroup of *G* of order divisible by *p*. It is easily seen that $N_G(M)$ is also a non-nilpotent *A*-invariant subgroup of *G* of order divisible by *p* and $N_G(M) < G$. Therefore, $M = N_G(M)$ by the maximality of *M*. Since *M* is a TI-subgroup of *G* by the hypothesis, it follows that *G* is a Frobenius group with complement *M*. Let *N* be the kernel. Then $G = N \rtimes M$.

Note that both N and M are Hall-subgroups of G. One has that N is a characteristic subgroup of G, which implies that N is also an A-invariant subgroup of G. By the maximality of M, N is a minimal A-invariant normal subgroup of G. It follows that N is a characteristic simple group which is a direct product of some isomorphic simple groups. Since N is nilpotent by [4, Theorem 10.5.6], N is an elementary abelian group. Let $N = Z_q^m$, where $q \neq p, m > 1$ since M is non-nilpotent.

Let *P* be an *A*-invariant Sylow *p*-subgroup of *M*. We will show that *P* is normal in *M*. Assume that *P* is not normal in *M*. Then $N_M(P) < M$. Note that $N_M(P)$ is also an *A*-invariant subgroup of *M*. It follows that there is a maximal *A*-invariant subgroup M_0 of *M* such that $N_M(P) \le M_0$. Since $N \rtimes M_0$ is a non-nilpotent maximal *A*-invariant subgroup of *G* of order divisible by $p, N \rtimes M_0$ is a TI-subgroup of *G*. However, $(N \rtimes M_0)^x \cap (N \rtimes$ $M_0) = (N \rtimes M_0^x) \cap (N \rtimes M_0) \ge N \ne 1$ for each $x \in G$, which implies that $N \rtimes M_0$ is normal in $G = N \rtimes M$. It follows that M_0 is normal in M. By Frattini's argument, one has $M = N_M(P)M_0 = M_0$, a contradiction. Therefore, P is normal in M.

If every maximal *A*-invariant subgroup of *M* is normal, then *M* is nilpotent by Lemma 2.1, a contradiction. Thus *M* has non-normal maximal *A*-invariant subgroups. Let *K* be a non-normal maximal *A*-invariant subgroup of *M*. By above argument, one has that |K| is not divisible by *p*. Then $P \cap K = 1$. It follows that $M = P \rtimes K$. By the maximality of *K*, *P* is a minimal *A*-invariant normal subgroup of *M*, which implies that *P* is an elementary abelian subgroup. Then *P* is a cyclic group of order *p* by [4, Theorem 10.5.6]. Let $P = Z_p$. Then $M = Z_p \rtimes K$. Suppose that K_0 is any maximal *A*-invariant subgroup of *K*. Then $N \rtimes (Z_p \rtimes K_0)$ is a non-nilpotent maximal *A*-invariant subgroup of *G* of order divisible by *p*. Arguing as above, one has that K_0 is normal in *K*, which implies that *K* is nilpotent. Then *K* is cyclic or a direct product of a generalized quaternion group and a cyclic group of odd order by [4, Theorem 10.5.6] and $[Z_p, K] \neq 1$ since *M* is non-nilpotent.

For the sufficiency part.

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We only need to verify group G that belongs to case (2). Let L be any non-nilpotent maximal A-invariant subgroup of G of order divisible by p. Note that Z_q^m is a minimal A-invariant normal subgroup of G.

When the case that $Z_q^m \leq L$. Then $G = Z_q^m L$. It is easy to see that $Z_q^m \cap L$ is an A-invariant normal subgroup of G and $Z_q^m \cap L < Z_q^m$, by the minimality of Z_q^m , one has $Z_q^m \cap L = 1$. It follows that $G = Z_q^m \rtimes L$. Note that Z_q^m is a normal Hall-subgroup of G. By Schur-Zassenhaus theorem, one has that L and $Z_p \rtimes K$ are conjugate in G. It follows that L is also a Frobenius complement of G and then L is a TI-subgroup of G.

When the case that $Z_q^m < L$. Then $L = Z_q^m \rtimes (L \cap (Z_p \rtimes K)) = Z_q^m \rtimes (Z_p \rtimes (L \cap K))$ since |L| is divisible by *p*. Note that $L \cap K$ is a maximal *A*-invariant subgroup of *K* and *K* is nilpotent, one has that $L \cap K$ is normal in *K* by Lemma 2.2, which implies that $L = Z_q^m \rtimes (Z_p \rtimes (L \cap K))$ is normal in *G*.

ACKNOWLEDGEMENTS

We are very thankful to the referee who provides us valuable comments and suggestions. This research was supported in part by Shandong Provincial Natural Science Foundation, China (ZR2017MA022) and NSFC (11761079).

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Received December 20, 2023