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Spatial-temporal Model with Heterogeneous Random Effects

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Abstract: In this paper, we propose a novel spatial-temporal model with individual random effects characterized by a location-scale structure, which allows us to flexibly capture the pure influence of space-specific factors in the framework of quantile regression. A hybrid two-stage estimation procedure is introduced for this model, where the first stage proposes a Gaussian quasi-maximum likelihood estimator (QMLE) for the spatial-temporal effects while the second stage constructs a weighted conditional quantile estimator (WCQE) to study the conditional quantiles of the random effects related to space-specific attributes. The validity of the two-stage hybrid estimation is verified, and the asymptotic properties of our estimators are established. Our simulation study indicates that the proposed estimation procedure performs well in different scenarios with finite samples, and a real case study on the air quality of China is used to illustrate the application of the proposed method.

Keywords: Dynamic spatial autoregressive models; Hybrid estimation; Quantile regression; Quasi-maximum likelihood estimation; Random effects.

1. Introduction

With the rapid development of world economy and gradual deterioration of global ecological environment, countries around the world have taken measures such as establishing and improving environmental protection mechanisms to maintain the ecological equilibrium (Streck, 2004; Busch and Jörgens, 2005). Meanwhile, urban air qualities have attacked in-

tensive attentions (Wang and Hao, 2012; Gulia et al., 2015) since poor air qualities impose adverse impacts on human health and survival (Cakmak et al., 2011; Bowatte et al., 2015). Researchers have increasingly focused on the quantitative analysis of the impact of social and anthropogenic driving force on urban air qualities. For example, De Bruyn et al. (1998) and Chen and Xu (2017) studied the relationship between the air quality and the economic development; Cramer (1998) and Chen et al. (2020) investigated the influence of population density on the air quality; Li et al. (2018) and Yu and Liu (2020) considered the impact of industrial structure change on the air quality. However, as a complex and large system, the urban ambient air system is not only affected by human activities, but also closely related to spatial-temporal effects including historical patterns, geographical location, meteorological conditions and so on (Yang et al., 2007; Giri et al., 2008; Zhou et al., 2022). As a result, exploring the social driving force of urban air quality pressure after removing spatial-temporal effects becomes meaningful and necessary.

In literatures, spatial effects which depict the geographical correlation, are usually modeled in three forms: (i) spatial autoregressive (SAR) models with the spatial effect built at the response variable (Lee, 2004; Yu et al., 2008); (ii) spatial error models with the spatial effect modeled in the disturbance term (Kapoor et al., 2007; Kelejian and Prucha, 2010; Su and Yang, 2015); and (iii) spatial panel data models with the spatial effect simultaneously reflected in both responses and errors (Kelejian and Prucha, 1998, 1999). Some of previous models include temporal effects which capture autocorrelation in the form of lagged response by the same way as in time series models. As the practical need for analyzing spatial-temporal data increases, spatial panel data models including spatial and temporal effects simultaneously have been widely studied. For example, Yu et al. (2008) considered

the Gaussian quasi-maximum likelihood estimator (QMLE) for the spatial dynamic panel data (SDPD) model with fixed effects, when both the individual dimension N and time dimension T are large; Lee and Yu (2014) proposed the generalized method of moments estimators for the SDPD model with fixed effects, when N is large and T can be large but small relative to N ; and Su and Yang (2015) studied the QMLE for dynamic panel models with spatial errors and random/fixed effects when N is large but T is fixed. In this paper, we propose a dynamic spatial autoregressive (DSAR) model to capture the spatial-temporal effects and investigate its estimation.

For spatial panel data models, the individual effect is another important ingredient, which is widely investigated in form of fixed effect (Yu et al., 2008; Lee and Yu, 2010; Shi and Lee, 2017) or random effect (Baltagi et al., 2013; Su and Yang, 2015). Both fixed and random effects have merits and demerits: the random effect model avoids the “incidental parameter” problem (Neyman and Scott, 1948) in fixed effect models and has more efficient estimators, but it neglects possible correlations between individual effects and regressors; meanwhile, estimators of fixed effect models are robust in a sense that the fixed effect can be arbitrarily related to regressors (Mundlak (1978); Li and Yang (2021); Page 624 of Hansen (2021)). Note that the random effects in previous literatures are assumed to be with zero means and constant variances, which implies that the random effects are homogeneous for individuals. However, individual random effects can be influenced by space-specific covariates and this possibly makes the random effects heterogeneous. For example, the influence of socio-economic factors on air qualities possesses individual heterogeneity among cities. Generally heterogeneity across individuals can be described by fixed effects or heteroskedastic random effects (Baltagi et al., 2010). Both fixed and heteroskedastic random effects

only emphasize the difference among individuals, but cannot help to explore information from the heterogeneity. Motivated by this fact, this paper alternatively considers a space-specific-covariates-dependent location-scale structure for the individual effect. Specifically, the location of individual effect depicts the difference among individuals on the conditional mean and can be arbitrarily related to time-variant regressors, thus it can be viewed as a fixed effect; whereas the scale of individual effect is random and allows for conditional heteroscedasticity. Hence the location-scale structure combines advantages of fixed and random effects. Moreover, it can flexibly extract the influence of space-specific covariates on the whole distribution of the individual random effect.

It is well known that quantile regression (Koenker and Bassett, 1978) is a popular tool to capture heterogeneity and exploit the distributional information among data, which can provide a complete picture for studying influence of covariates on response. Reconsider the example of the air quality, we are interested in exploring the influence of socio-economic factors on the air quality after removing spatial-temporal effects, not only on the average level but also on the different quantile levels especially the lower and upper ones. This motivates us to adopt quantile regression for the proposed individual effects in location-scale structure. In summary, we propose a new framework to capture individual heterogeneity in the presence of spatial-temporal effects, and explore influence of space-specific covariates on response. The contributions of this paper are threefold.

- (a) We propose a DSAR model with possibly heterogeneous random effects. To our best knowledge it is the first time that the random effect of spatial models is assumed to have a location-scale structure that can be affected by space-specific covariates. As a result, this model can capture the pure influence of some space-specific factors on

different quantile levels of the response with spatial-temporal effects removed.

- (b) We introduce a two-stage hybrid estimation procedure for the proposed model. Specifically, a QMLE is proposed in the first stage to estimate spatial-temporal effects, and a weighted conditional quantile regression estimator (WCQE) is designed in the second stage by replacing unobserved random effects with observed pseudo ones based on the QMLE.
- (c) The theoretical contributions can be summarized as follows: (i) we derive the consistency and asymptotic normality for the QMLE; (ii) we establish the weak convergence of the quantile regression process and the asymptotic distribution of the WCQE; (iii) two estimators are proposed for the conditional scale coefficients of random effects and their asymptotic results are established.

The remainder of the paper is organized as follows. Section 2 introduces the model. Section 3 proposes a two-stage hybrid estimation procedure for this model, and establishes all related asymptotic results. A simulation study is conducted in Section 4 to assess the finite sample performance of the estimation procedure, and an empirical analysis on the air quality is provided in Section 5 to illustrate the usefulness of the proposed model and its inference tools. Conclusion and discussion are made in Section 6. A bootstrapping procedure for the QMLE as well as the technical details are relegated to the Supplementary Material. Throughout this paper, $\otimes$ denotes the Kronecker product of two matrices, $|\cdot|$ denotes the absolute value of a scalar/vector or the determinant of a matrix, $\|\cdot\|$ denotes the Euclidean norm of a vector, and $tr(\cdot)$ denotes the trace of a matrix. Let $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ be the smallest and largest eigenvalues of a matrix, respectively. Denote N and T as the total numbers of spatial units and time periods, respectively. Define $I(\cdot)$ as the indicator function.

For a positive integer m , denote I_m as the $m \times m$ identity matrix, $\mathbf{0}_m$ as the $m \times 1$ vector of zeros, $\mathbf{1}_m$ as the $m \times 1$ vector of ones, and let $J_m = \mathbf{1}_m \mathbf{1}_m'$. The operator E denotes the expectation with respect to the probability measure, $\mathbb{E}_n$ denotes the expectation with respect to the empirical measure, and $\mathbb{G}_n = \sqrt{n}(\mathbb{E}_n - E)$. $\ell^\infty(\mathcal{T})$ denotes the space of all uniformly bounded functions on $\mathcal{T}$. Moreover, $\rightarrow_p$ denotes the convergence in probability, $\rightarrow_d$ denotes the convergence in distribution, and $\rightsquigarrow$ denotes the weak convergence. In addition, an R package implementing the proposed two-stage hybrid estimation procedure is available at <https://github.com/wyLI2020/QuantileDSAR>, and the dataset analyzed in Section 5 is also included.

2. Model Specification

Consider the dynamic spatial autoregressive (DSAR) model with random effects below,

$$y_{it} = \theta_i + \alpha y_{i,t-1} + \lambda \sum_{j=1}^N w_{N,ij} y_{jt} + \sum_{\ell=1}^q \gamma_\ell z_{\ell it} + \varepsilon_{it}, \quad i = 1, \dots, N, t = 1, \dots, T, \quad (2.1)$$

where θ_i 's are the space-specific or individual random effects, the scalar parameters α and λ characterize the dynamic effect and spatial autoregressive effect, respectively, γ_ℓ 's are the coefficients of time-varying exogenous variables $\{z_{\ell it}\}$, $w_{N,ij}$'s are specified constant spatial weights that generate the spatial dependence, and ε_{it} 's are independent and identically distributed (*i.i.d.*) across i and t with mean zero and finite variance σ_ε^2 . To flexibly depict the possible influence of space-specific covariates on the individual random effect, this paper assumes a location-scale structure for θ_i 's, that is,

$$\theta_i = \psi_0 + \sum_{\ell=1}^p \psi_\ell x_{\ell i} + \left(\beta_0 + \sum_{\ell=1}^p \beta_\ell |x_{\ell i}| \right) \eta_i, \quad (2.2)$$

where $\{x_{\ell i}\}$ are time-invariant constant regressors, ψ_ℓ 's and β_ℓ 's are the coefficients of $\{x_{\ell i}\}$ and $\{|x_{\ell i}|\}$, respectively, $\beta_0 > 0, \beta_\ell \geq 0$ for $1 \leq \ell \leq p$, and η_i 's are *i.i.d.* disturbances with

mean zero and finite variance σ_η^2 ; see also Koenker and Zhao (1994) and Zhao and Xiao (2014). Let $\mu_i = \psi_0 + \sum_{\ell=1}^p \psi_\ell x_{\ell i}$ and $\vartheta_i = (\beta_0 + \sum_{\ell=1}^p \beta_\ell |x_{\ell i}|) \eta_i$, it then follows that

$$\theta_i = \mu_i + \vartheta_i.$$

The location-scale structure combines the advantages of fixed and random effects. Specifically, the location μ_i depicts the difference among individuals on the conditional mean and can be arbitrarily related to time-variant regressors $\{z_{\ell it}\}$, while the scale effect ϑ_i allows for conditional heteroscedasticity and reflects the influence of $\{x_{\ell i}\}$ on the fluctuation. Note that if $\psi_\ell = \beta_\ell = 0$ for all $\ell = 1, \dots, p$, then θ_i will reduce to the commonly used individual random effect. Moreover, if $\beta_0 = \beta_1 = \dots = \beta_p = 0$, then θ_i will be the individual fixed effect captured by space-specific covariates with $p + 1$ unknown parameters, which can avoid the “incidental parameters” problem due to N unknown parameters in commonly used individual fixed effect models.

Denote $\mathbf{Y} = (\mathbf{y}'_1, \dots, \mathbf{y}'_T)'$ and $\mathbf{Y}_{-1} = (\mathbf{y}'_0, \dots, \mathbf{y}'_{T-1})'$ with $\mathbf{y}_t = (y_{1t}, \dots, y_{Nt})'$, $Z = (Z'_1, \dots, Z'_T)'$ with $Z_t = (\mathbf{z}_{1t}, \dots, \mathbf{z}_{Nt})'$ and $\mathbf{z}_{it} = (z_{1it}, \dots, z_{qit})'$, $X = (\mathbf{x}_1, \dots, \mathbf{x}_N)'$ with $\mathbf{x}_i = (1, x_{1i}, \dots, x_{pi})'$, and $\boldsymbol{\varepsilon} = (\boldsymbol{\varepsilon}'_1, \dots, \boldsymbol{\varepsilon}'_T)'$ with $\boldsymbol{\varepsilon}_t = (\varepsilon_{1t}, \dots, \varepsilon_{Nt})'$. Let $S_{NT}(\lambda) = I_T \otimes B_N(\lambda)$ with $B_N(\lambda) = I_N - \lambda W_N$ and $W_N = \{w_{N,ij}, 1 \leq i, j \leq N\}$ being the $N \times N$ specified constant spatial weights matrix, and denote $\tilde{Z} = (\mathbf{Y}_{-1}, Z, (\boldsymbol{\iota}_T \otimes I_N)X)$. Then model (2.1) with (2.2) can be rewritten in the following matrix form:

$$S_{NT}(\lambda)\mathbf{Y} = \tilde{Z}\boldsymbol{\phi} + (\boldsymbol{\iota}_T \otimes I_N)\boldsymbol{\vartheta} + \boldsymbol{\varepsilon}, \quad (2.3)$$

where $\boldsymbol{\phi} = (\alpha, \boldsymbol{\gamma}', \boldsymbol{\psi}')'$ with $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_q)'$ and $\boldsymbol{\psi} = (\psi_0, \psi_1, \dots, \psi_p)'$, and $\boldsymbol{\vartheta} = (\vartheta_1, \dots, \vartheta_N)'$ with $\vartheta_i = \mathbf{x}'_{ai}\boldsymbol{\beta}\eta_i$, $\mathbf{x}_{ai} = (1, |x_{1i}|, \dots, |x_{pi}|)'$ and $\boldsymbol{\beta} = (\beta_0, \beta_1, \dots, \beta_p)'$. The following proposition provides a sufficient condition for the covariance stationarity of $\mathbf{y}_t$.

Proposition 1. *Suppose that η_i and ε_{it} are mutually independent. If $|\alpha B_N^{-1}(\lambda)| < 1$ and the process $\{\mathbf{z}_{it}\}$ is covariance stationary, then there exists a unique covariance stationary solution to models (2.1) and (2.2). The solution is in form of*

$$\mathbf{y}_t = \sum_{m=0}^{\infty} \alpha^m [B_N^{-1}(\lambda)]^{m+1} (\boldsymbol{\theta} + Z_{t-m}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}_{t-m}).$$

Proposition 1 indicates that the stationarity of $\{\mathbf{y}_t\}$ restricts the spatial and temporal effects to some extent by imposing conditions on the weights matrix W_N , spatial effect parameter λ and autoregressive coefficient α . Specifically, $|\alpha| + |\lambda| < 1$ implies $|\alpha B_N^{-1}(\lambda)| < 1$ if W_N is row-normalized. Moreover, the stationarity of $\{\mathbf{y}_t\}$ also depends on that of exogenous covariates $\{\mathbf{z}_{it}\}$. But assuming both $\{\mathbf{y}_t\}$ and $\{\mathbf{z}_{it}\}$ stationary makes the application of the proposed model very limited. This motivates us to consider an estimation method that is theoretically valid without any stationarity assumption on $\{\mathbf{y}_t\}$.

We are interested in the parameter estimation for $\boldsymbol{\phi}$, $\boldsymbol{\beta}$, λ , σ_ε^2 and σ_η^2 , and more importantly in the conditional quantile estimation of individual random effects θ_i 's. Let $Q_\eta(\tau)$ be the τ -th quantile of η_i . Denote $\boldsymbol{\theta} = (\theta_1, \dots, \theta_N)'$, $X_a = (\mathbf{x}_{a1}, \dots, \mathbf{x}_{aN})'$. The conditional quantile of $\boldsymbol{\theta}$, given the space-specific covariates X , has the form of

$$Q_{\boldsymbol{\theta}}(\tau|X) = X\boldsymbol{\psi} + X_a\boldsymbol{\varphi}(\tau), \quad (2.4)$$

where $\boldsymbol{\varphi}(\tau) = (\varphi_0(\tau), \varphi_1(\tau), \dots, \varphi_p(\tau))' = Q_\eta(\tau)\boldsymbol{\beta}$. Notice that the individual effects $\boldsymbol{\theta}$ are unobservable, to estimate $Q_{\boldsymbol{\theta}}(\tau|X)$ in the framework of model (2.3), we propose a two-stage hybrid estimation procedure in the following section.

3. Two-stage Hybrid Estimation

The space-specific random effect θ_i is latent, so it is natural for the estimation under model (2.4) to obtain its observable approximate by (2.1) based on reasonable estimates of λ , α

and γ , and then apply the quantile regression estimation method to model (2.4) with θ_i replaced by its observable approximate to obtain estimates of ψ and $\varphi(\tau)$. Note that ψ is independent of τ and it is not necessary to estimate it by conditional quantile estimation in our model setting, while $\varphi(\tau)$ is τ -dependent and can be estimated by the quantile estimation method. Moreover, λ and ϕ can be estimated directly and thus the observable approximate of ϑ_i is easily obtained by model (2.3). These together with the fact that $Q_{\theta}(\tau|X) = X\psi + Q_{\vartheta}(\tau|X)$, motivate us to alternatively obtain observable approximates of ϑ_i 's and then apply quantile regression to them as given by

$$Q_{\vartheta}(\tau|X) = X_a\varphi(\tau). \quad (3.1)$$

Specifically, this paper adopts the QMLE for model (2.3), denoted by $\hat{\zeta} = (\hat{\phi}', \hat{\sigma}_{\varepsilon}^2, \hat{\lambda}, \hat{\beta}^*)'$ for $\zeta = (\phi', \sigma_{\varepsilon}^2, \lambda, \beta^*)'$ with $\beta^* = \beta\sigma_{\eta}/\sigma_{\varepsilon}$. Then the observable approximates of ϑ_i 's called the pseudo scale effects, denoted by $\hat{\vartheta}_i$'s, can be constructed based on $\hat{\lambda}$ and $\hat{\phi}$; see (3.11). We next build a quantile regression model for $\hat{\vartheta} = (\hat{\vartheta}_1, \dots, \hat{\vartheta}_N)'$ using X_a as the regressor, and construct a WCQE $\hat{\varphi}(\tau)$ for $\varphi(\tau)$ and a weighted quantile average estimator (WQAE) $\hat{\beta}$ for β . In summary, a two-stage hybrid estimation procedure is proposed as follows:

Stage 1: Obtain the Gaussian QMLE $\hat{\zeta}$ based on model (2.3).

Stage 2: Define the pseudo scale effects $\{\hat{\vartheta}_i\}$ by (3.11) based on $\hat{\zeta}$. Then obtain the WCQE

$\hat{\varphi}(\tau)$ for $\varphi(\tau)$ based on model (3.1) with ϑ replaced by $\hat{\vartheta}$, and propose the WQAE $\hat{\beta}$ for β by combing the information among $\hat{\varphi}(\cdot)$ at multiple quantile levels τ 's.

As a result, if $S_{NT}(\hat{\lambda})$ is invertible, then the conditional mean of the data $\mathbf{Y}$ can be estimated by $\hat{E}(\mathbf{Y}|X) = S_{NT}^{-1}(\hat{\lambda})\tilde{Z}\hat{\phi}$ based on model (2.3) and $\hat{\zeta}$. Moreover, the conditional quantile of random effects θ can be estimated by $\hat{Q}_{\theta}(\tau|X) = X\hat{\psi} + X_a'\hat{\varphi}(\tau)$ based on model (2.4), $\hat{\psi}$

and $\hat{\varphi}(\tau)$. Statistical inference tools, such as significance tests for ϕ , $\varphi(\tau)$ and β can also be constructed based on $\hat{\phi}$, $\hat{\varphi}(\tau)$ and $\hat{\beta}$. More details of the proposed estimation procedure and its asymptotics are referred to Sections 3.1 and 3.2.

To establish the asymptotics for the proposed two-stage hybrid estimation procedure, the regularity conditions for N and T are assumed below.

Assumption 3.1. $T = cN^r$ for some constants $r \geq 0$ and $c > 0$.

Assumption 3.2. $T = cN^r$ for some constants $0 \leq r \leq 1$ and $c > 0$.

Assumption 3.3. $T = cN^r$ for some constants $0 < r \leq 1$ and $c > 0$.

Assumptions 3.1 and 3.2 allow T to be fixed or goes to infinity as N goes to infinity, and Assumption 3.2 restricts T at a rate not greater than N when it goes to infinity. Assumption 3.3 requires both T and N go to infinity, and it is a special case of Assumption 3.2. Specifically, the consistency of QMLE in the first stage is established under Assumption 3.1, while the asymptotic normality of QMLE is obtained under Assumption 3.2. For the asymptotic normality of QMLE, restricting the rate of T (i.e. $T/N \rightarrow c'$ for some constant $c' \geq 0$) is needed to prove the asymptotic normality of $(NT)^{-1/2} \partial \ln \mathcal{L}_{NT}(\zeta_0) / \partial \zeta$ using Theorem A.1 of Kelejian and Prucha (2001); see details in the proof of Theorem 2 in the Supplementary Material. Moreover, since $T \rightarrow \infty$ is necessary for the validity of replacing ϑ with $\hat{\vartheta}$ in quantile estimation, Assumption 3.3 is used to establish asymptotic results of WCQE and WQAE in the second stage; see details in Section 3.2.

3.1 The Quasi-maximum Likelihood Estimation

Recall that the parameter vector of model (2.3) is $\zeta = (\phi', \sigma_\varepsilon^2, \lambda, \beta^{*'})'$ with $\beta^* = \beta\sigma_\eta/\sigma_\varepsilon$, and let $\zeta_0 = (\phi'_0, \sigma_{\varepsilon 0}^2, \lambda_0, \beta_0^{*'})'$ be the true value of ζ . Denote $V_{NT}(\lambda, \phi) = S_{NT}(\lambda)Y -$

$\tilde{Z}\phi$, $\mathbf{V}_{NT} = (\iota_T \otimes I_N)\boldsymbol{\vartheta} + \boldsymbol{\varepsilon}$, $B_{0N} = B_N(\lambda_0)$ and $S_{0NT} = S_{NT}(\lambda_0)$, then it follows that $\mathbf{V}_{NT}(\lambda_0, \phi_0) = \mathbf{V}_{NT} = S_{0NT}\mathbf{Y} - \tilde{Z}\phi_0$. This paper considers the case that initial observations are generated exogenously, that is, $\mathbf{y}_0$ is exogenous such that it can be treated as a constant vector; see also Su and Yang (2015).

Assume that ε_{it} and η_i are distributed with mean zeros and variances σ_ε^2 and σ_η^2 , respectively, which implies that $\text{cov}\{\mathbf{V}_{NT}(\lambda, \phi)\} = \sigma_\varepsilon^2 \Omega_{NT}(\boldsymbol{\beta}^*)$, where $\Omega_{NT}(\boldsymbol{\beta}^*) = J_T \otimes A_N(\boldsymbol{\beta}^*) + I_{NT}$ with $A_N(\boldsymbol{\beta}^*) = \text{diag}\{(\mathbf{x}'_{a1}\boldsymbol{\beta}^*)^2, \dots, (\mathbf{x}'_{aN}\boldsymbol{\beta}^*)^2\}$. The Gaussian quasi-likelihood function of $\boldsymbol{\zeta}$ is then given by

$$\mathcal{L}_{NT}(\boldsymbol{\zeta}) = (2\pi\sigma_\varepsilon^2)^{-\frac{NT}{2}} |S'_{NT}(\lambda)\Omega_{NT}^{-1}(\boldsymbol{\beta}^*)S_{NT}(\lambda)|^{\frac{1}{2}} \exp\left\{-\frac{\mathbf{V}'_{NT}(\lambda, \phi)\Omega_{NT}^{-1}(\boldsymbol{\beta}^*)\mathbf{V}_{NT}(\lambda, \phi)}{2\sigma_\varepsilon^2}\right\},$$

and the corresponding quasi-log-likelihood function of $\boldsymbol{\zeta}$ is

$$\begin{aligned} \ln \mathcal{L}_{NT}(\boldsymbol{\zeta}) = & -\frac{NT}{2} \ln(2\pi) - \frac{NT}{2} \ln \sigma_\varepsilon^2 - \frac{1}{2} \ln |\Omega_{NT}(\boldsymbol{\beta}^*)| + \ln |S_{NT}(\lambda)| \\ & - \frac{1}{2\sigma_\varepsilon^2} \mathbf{V}'_{NT}(\lambda, \phi)\Omega_{NT}^{-1}(\boldsymbol{\beta}^*)\mathbf{V}_{NT}(\lambda, \phi). \end{aligned} \quad (3.2)$$

Maximizing (3.2) gives the Gaussian QMLE of $\boldsymbol{\zeta}$, denoted by $\hat{\boldsymbol{\zeta}}$. If ε_{it} and η_i are normally distributed, then $\hat{\boldsymbol{\zeta}}$ is the MLE.

For easy computation, we consider the concentrated log-likelihood by concentrating out the ϕ and σ_ε^2 . Given $\boldsymbol{\delta} = (\lambda, \boldsymbol{\beta}^*)'$, maximizing (3.2) leads to the following QMLEs of ϕ and σ_ε^2 ,

$$\hat{\phi}(\boldsymbol{\delta}) = \left(\tilde{Z}'\Omega_{NT}^{-1}(\boldsymbol{\beta}^*)\tilde{Z}\right)^{-1} \tilde{Z}'\Omega_{NT}^{-1}(\boldsymbol{\beta}^*)S_{NT}(\lambda)\mathbf{Y}, \quad (3.3)$$

and

$$\hat{\sigma}_\varepsilon^2(\boldsymbol{\delta}) = \frac{1}{NT} \hat{\mathbf{V}}'_{NT}(\boldsymbol{\delta})\Omega_{NT}^{-1}(\boldsymbol{\beta}^*)\hat{\mathbf{V}}_{NT}(\boldsymbol{\delta}), \quad (3.4)$$

where $\hat{\mathbf{V}}_{NT}(\boldsymbol{\delta}) = S_{NT}(\lambda)\mathbf{Y} - \tilde{Z}\hat{\phi}(\boldsymbol{\delta})$. Then by plugging (3.3) and (3.4) in (3.2), the concen-

trated log-likelihood function of $\boldsymbol{\delta}$ is

$$\ln \ell_{NT}(\boldsymbol{\delta}) = -\frac{NT}{2} (\ln(2\pi) + 1) - \frac{NT}{2} \ln \hat{\sigma}_\varepsilon^2(\boldsymbol{\delta}) - \frac{1}{2} \ln |\Omega_{NT}(\boldsymbol{\beta}^*)| + \ln |S_{NT}(\lambda)|. \quad (3.5)$$

The QMLE $\hat{\boldsymbol{\delta}}$ which maximizes the function (3.5) is given by

$$\hat{\boldsymbol{\delta}} = \arg \max_{\boldsymbol{\delta} \in \Delta} \ln \ell_{NT}(\boldsymbol{\delta}),$$

where Δ is the parameter space of $\boldsymbol{\delta}$, which is assumed to be compact and the true value $\boldsymbol{\delta}_0 = (\lambda_0, \boldsymbol{\beta}_0^*)'$ is in the interior of Δ . Then the QMLEs of $\boldsymbol{\phi}$ and σ_ε^2 are $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}(\hat{\boldsymbol{\delta}})$ and $\hat{\sigma}_\varepsilon^2 = \hat{\sigma}_\varepsilon^2(\hat{\boldsymbol{\delta}})$, respectively. As a result, the QMLE of $\boldsymbol{\zeta}$ is given by $\hat{\boldsymbol{\zeta}} = (\hat{\boldsymbol{\phi}}', \hat{\sigma}_\varepsilon^2, \hat{\boldsymbol{\delta}}')' = (\hat{\boldsymbol{\phi}}', \hat{\sigma}_\varepsilon^2, \hat{\lambda}, \hat{\boldsymbol{\beta}}^*)'$.

To establish the consistency and asymptotic normality for the QMLE, we introduce the following assumptions.

Assumption 3.4. (i) $E|\varepsilon_{it}|^{4+\epsilon_0} < \infty$ and $E|\eta_i|^{4+\epsilon_0} < \infty$ for some $\epsilon_0 > 0$; (ii) ε_{it} and η_i are mutually independent.

Assumption 3.5. (i) The elements $w_{N,ij}$ of W_N are bounded constants uniformly for all N in all i and j . As a normalization, $w_{N,ii} = 0$ for all i ; (ii) The matrix B_{0N} is nonsingular; (iii) The sequences of matrices $\{W_N\}$ and $\{B_{0N}^{-1}\}$ are bounded in both row and column sums uniformly for all N ; (iv) $\{B_N^{-1}(\lambda)\}$ are bounded in either row or column sums uniformly for all N , uniformly in $\lambda \in \Delta(\lambda)$; (v) The sequence of matrices $\{\sum_{t=1}^T \alpha_0^{t-1} (B_{0N}^{-1})^t\}$ are uniformly bounded for all N and T in both row and column sums.

Assumption 3.6. (i) $x_{\ell i}$'s are uniformly bounded constants for all N ; (ii) $E(z_{\ell it}) = 0$ and $E|z_{\ell it}|^{4+\epsilon_0} < \infty$ for some $\epsilon_0 > 0$; (iii) For any $1 \leq \ell \leq q$, the covariance matrix of $\{z_{\ell it}, 1 \leq i \leq N, 1 \leq t \leq T\}$ is uniformly bounded in row sums, and it holds that

$$\sum_{1 \leq i_3, i_4 \leq N} \sum_{1 \leq t_3, t_4 \leq T} \text{cov}(z_{\ell i_1 t_1}, z_{\ell i_2 t_2}, z_{\ell i_3 t_3}, z_{\ell i_4 t_4}) = O(1) \quad (3.6)$$

for any $1 \leq i_1, i_2 \leq N$ and $1 \leq t_1, t_2 \leq T$; (iv) $\frac{1}{NT} \tilde{Z} \tilde{Z}'$ is positive definite for sufficiently large N and $\lambda_{\min} \left(\frac{1}{NT} \tilde{Z}' \Omega_{NT}^{-1}(\beta^*) \tilde{Z} \right)$ is uniformly bounded away from zero almost surely (a.s.).

Assumption 3.4 provides basic conditions on the disturbances, which are standard in the random effects panel data literature; see Assumptions R(i) and R(ii) of Su and Yang (2015). Note that Assumption 3.4(i) does not require ε_{it} 's to be identically distributed. Since the heterogeneity among individuals is captured by the random effects θ_i 's in model (2.1), we assume ε_{it} 's have a common variance instead of i -dependent variances.

Assumption 3.5 provides the essential features of the weights matrix which are commonly assumed for spatial models; see also Assumption G2 of Su and Yang (2015) and Assumptions 2–5 of Lee (2004). In empirical applications, W_N is usually row normalized in such that $B_N(\lambda)$ is nonsingular for $\lambda \in (-1, 1)$; see also Kelejian and Prucha (2010). Invertibility of B_{0N} in Assumption 3.5(ii) guarantees that (2.3) has an equilibrium and the disturbance term is well defined. Assumption 3.5(iii) limits the spatial correlation to some degree but facilitates the study of the asymptotic properties of the spatial parameter estimators, and Assumption 3.5(iv) is required to establish the consistency. Assumption 3.5(v) is implied by Assumption 3.5(iii) when T is fixed. When both N and T go to infinity, Assumption 3.5(v) is necessary to guarantee that the variances of y_{it} 's are bounded, and it is an extension of Assumption 3.5(iii) which guarantees that the variances of y_i 's in the SAR model are bounded as N goes to infinity (Lee, 2004). Note that Assumption 3.5(iii) implies that both row and column sums of $(B_{0N}^{-1})^t$ are bounded for an arbitrary t . This together with $|\alpha_0| < 1$, implies that Assumption 3.5(v) holds as long as α_0^{t-1} decreases faster than the row or column sum of $(B_{0N}^{-1})^t$ increases as t increases, which is usually satisfied in practice. Moreover, if W_N is a row-normalized matrix, then both $\{B_N^{-1}(\lambda)\}$ and $\{\sum_{t=1}^T \alpha^{t-1} [B_N^{-1}(\lambda)]^t\}$ are uniformly

bounded in row sums for $|\alpha| + |\lambda| < 1$; see also Lee (2004). In addition, Assumption 3.5 actually implies some restrictions on the process $\{\mathbf{y}_t\}$ by imposing conditions on the weights matrix W_N , spatial effect parameter λ and autoregressive coefficient α . But both stationary and nonstationary cases of $\{\mathbf{y}_t\}$ are allowable under Assumption 3.5.

The conditions on regressors in Assumptions 3.6(i), (ii) and (iv) are general and parallel Assumption 6 in Lee (2004), Assumption 4 in Yu et al. (2008) and Assumptions G1(iii) and R(iv*) in Su and Yang (2015), where regressors in Lee (2004) are time-invariant constants, in Yu et al. (2008) are time-variant constants, and in Su and Yang (2015) are time-invariant or time-variant random variables. Note that Theorems 1 and 2 still hold if $x_{\ell i}$'s ($z_{\ell it}$'s) are assumed either constants or random variables. Here we assume time-variant random $z_{\ell it}$'s but time-invariant constant $x_{\ell i}$'s to emphasize the randomness by time t , and this also allows $x_{\ell i}$ to be arbitrarily related to $z_{\ell it}$. Assumption 3.6(iii) is to guarantee that variances associated with $\{z_{\ell it}, 1 \leq i \leq N, 1 \leq t \leq T\}$ are $O(NT)$; see details in Lemmas 4 – 6. Particularly, (3.6) implies that given (i_1, t_1) and (i_2, t_2) , $z_{\ell i_1 t_1} z_{\ell i_2 t_2}$ is correlated with $z_{\ell i_3 t_3} z_{\ell i_4 t_4}$ for only finite terms of (i_3, t_3) and (i_4, t_4) . It is noteworthy that Assumption 3.6(iii) allows for individual dependence among finite neighbors and time dependence up to finite lags, which makes it weaker than the independence assumption of regressors across individuals in Su and Yang (2015).

Recall that $\boldsymbol{\zeta} = (\boldsymbol{\phi}', \sigma_\varepsilon^2, \boldsymbol{\delta}')'$ and $\widehat{\boldsymbol{\zeta}} = (\widehat{\boldsymbol{\phi}}'(\widehat{\boldsymbol{\delta}}), \widehat{\sigma}_\varepsilon^2(\widehat{\boldsymbol{\delta}}), \widehat{\boldsymbol{\delta}}')'$. To show the consistency of QMLE $\widehat{\boldsymbol{\zeta}}$, we first verify that $\boldsymbol{\zeta}_0 = (\boldsymbol{\phi}_0', \sigma_{\varepsilon 0}^2, \boldsymbol{\delta}_0')'$ is identifiable. For the quasi-log-likelihood function at (3.2), given $\boldsymbol{\delta}$, its expectation $E[\ln \mathcal{L}_{NT}(\boldsymbol{\zeta})]$ is maximized at

$$\widetilde{\boldsymbol{\phi}}(\boldsymbol{\delta}) = \left[E \left(\widetilde{\mathbf{Z}}' \Omega_{NT}^{-1}(\boldsymbol{\beta}^*) \widetilde{\mathbf{Z}} \right) \right]^{-1} E \left(\widetilde{\mathbf{Z}}' \Omega_{NT}^{-1}(\boldsymbol{\beta}^*) S_{NT}(\lambda) \mathbf{Y} \right), \quad (3.7)$$

and

$$\tilde{\sigma}_\varepsilon^2(\boldsymbol{\delta}) = \frac{1}{NT} E \left(\tilde{\mathbf{V}}_{NT}'(\boldsymbol{\delta}) \Omega_{NT}^{-1}(\boldsymbol{\beta}^*) \tilde{\mathbf{V}}_{NT}(\boldsymbol{\delta}) \right), \quad (3.8)$$

where $\tilde{\mathbf{V}}_{NT}(\boldsymbol{\delta}) = S_{NT}(\lambda) \mathbf{Y} - \tilde{Z} \tilde{\boldsymbol{\phi}}(\boldsymbol{\delta})$. Define $\ln \ell_{NT}^*(\boldsymbol{\delta}) = \max_{\boldsymbol{\phi}, \sigma_\varepsilon^2} E[\ln \mathcal{L}_{NT}(\boldsymbol{\zeta})]$, then by (3.7)

and (3.8), we have

$$\ln \ell_{NT}^*(\boldsymbol{\delta}) = -\frac{NT}{2} (\ln(2\pi) + 1) - \frac{NT}{2} \ln \tilde{\sigma}_\varepsilon^2(\boldsymbol{\delta}) - \frac{1}{2} \ln |\Omega_{NT}(\boldsymbol{\beta}^*)| + \ln |S_{NT}(\lambda)|.$$

Denote $\Omega_{0NT} = \Omega_{NT}(\boldsymbol{\beta}_0^*)$, since $E \left(\tilde{Z}' \Omega_{0NT}^{-1} \mathbf{V}_{NT} \right) = 0$ by Lemma 8 in the Supplementary Material, it can be shown that $\tilde{\boldsymbol{\phi}}(\boldsymbol{\delta}_0) = \boldsymbol{\phi}_0 + [E(\tilde{Z}' \Omega_{0NT}^{-1} \tilde{Z})]^{-1} E(\tilde{Z}' \Omega_{0NT}^{-1} \mathbf{V}_{NT}) = \boldsymbol{\phi}_0$. Then it follows that $\tilde{\mathbf{V}}_{NT}(\boldsymbol{\delta}_0) = \mathbf{V}_{NT}$ and $\tilde{\sigma}_\varepsilon^2(\boldsymbol{\delta}_0) = \sigma_{\varepsilon_0}^2$. Identification of $\boldsymbol{\delta}_0$ can be based on the maximum values of $\frac{1}{NT} \ln \ell_{NT}^*(\boldsymbol{\delta})$, and hence that of $\boldsymbol{\phi}_0$ and $\sigma_{\varepsilon_0}^2$ follows. The following condition is imposed for identification.

Assumption 3.7. *For any $\boldsymbol{\delta} \neq \boldsymbol{\delta}_0$, it holds that*

$$\lim_{N \rightarrow \infty} \frac{1}{NT} [\ln |\tilde{\sigma}_\varepsilon^{-2}(\boldsymbol{\delta}) \Omega_{NT}^{-1}(\boldsymbol{\beta}^*) S_{NT}(\lambda)| - \ln |\sigma_{\varepsilon_0}^{-2} \Omega_{0NT}^{-1} S_{0NT}|] \neq 0.$$

Assumption 3.7 is equivalent to that $\lim_{N \rightarrow \infty} \frac{1}{NT} [\ln \ell_{NT}^*(\boldsymbol{\delta}) - \ln \ell_{NT}^*(\boldsymbol{\delta}_0)] \neq 0$ for any $\boldsymbol{\delta} \neq \boldsymbol{\delta}_0$, which ensures the identification of $\boldsymbol{\delta}_0$; see also Assumption 9 of Lee (2004) and Assumption R(iv) of Su and Yang (2015). Finally, the consistency of $\hat{\boldsymbol{\delta}}$ holds by the identification and uniform convergence of $\frac{1}{NT} [\ln \ell_{NT}(\boldsymbol{\delta}) - \ln \ell_{NT}^*(\boldsymbol{\delta})]$ to zero on Δ , and consequently the consistency of $\hat{\boldsymbol{\phi}}$ and $\hat{\sigma}_\varepsilon^2$ follows.

Theorem 1. *Under Assumptions 3.1, 3.4 – 3.7, then $\hat{\boldsymbol{\zeta}} \rightarrow_p \boldsymbol{\zeta}_0$ as $N \rightarrow \infty$.*

Let $\Sigma = \lim_{N \rightarrow \infty} \Sigma_N$ and $\Sigma^* = \lim_{N \rightarrow \infty} \Sigma_N^*$, where $\Sigma_N = -E \left(\frac{1}{NT} \frac{\partial^2 \ln \mathcal{L}_{NT}(\boldsymbol{\zeta}_0)}{\partial \boldsymbol{\zeta} \partial \boldsymbol{\zeta}'} \right)$ and

$$\Sigma_N^* = E \left(\frac{1}{\sqrt{NT}} \frac{\partial \ln \mathcal{L}_{NT}(\boldsymbol{\zeta}_0)}{\partial \boldsymbol{\zeta}} \frac{1}{\sqrt{NT}} \frac{\partial \ln \mathcal{L}_{NT}(\boldsymbol{\zeta}_0)}{\partial \boldsymbol{\zeta}'} \right) + E \left(\frac{1}{NT} \frac{\partial^2 \ln \mathcal{L}_{NT}(\boldsymbol{\zeta}_0)}{\partial \boldsymbol{\zeta} \partial \boldsymbol{\zeta}'} \right).$$

By the consistency result given in Theorem 1, the asymptotic normality of $\hat{\boldsymbol{\zeta}}$ can be obtained by Taylor expansion and the central limit theorem for linear-quadratic forms in Kelejian and Prucha (2001).

Theorem 2. *Under Assumptions 3.2, 3.4 – 3.7, if Σ and Σ^* exist, and Σ is positive definite, then $\sqrt{NT}(\hat{\boldsymbol{\zeta}} - \boldsymbol{\zeta}_0) \rightarrow_d N(\mathbf{0}, \Sigma^{-1} + \Sigma^{-1}\Sigma^*\Sigma^{-1})$ as $N \rightarrow \infty$.*

Theorem 2 provides the asymptotic distribution of the QMLE $\hat{\boldsymbol{\zeta}}$, where the matrices Σ_N and Σ_N^* are relegated to Section S3.1 of the Supplementary Material. If ε_{it} and η_i are normally distributed, then the QMLE reduces to the MLE and its limiting distribution can be simplified to $\sqrt{NT}(\hat{\boldsymbol{\zeta}} - \boldsymbol{\zeta}_0) \rightarrow_d N(\mathbf{0}, \Sigma^{-1})$ as $N \rightarrow \infty$. Note that Theorems 1 and 2 allow T to be fixed or tend to infinity, which makes QMLE applicable to both the short or long panel data.

To calculate the covariance matrix of $\hat{\boldsymbol{\zeta}}$ in practice, consistent estimates of Σ and Σ^* are necessary. Obviously, the matrix $\Sigma = \lim_{N \rightarrow \infty} -E \left(\frac{1}{NT} \frac{\partial^2 \ln \mathcal{L}_{NT}(\boldsymbol{\zeta}_0)}{\partial \boldsymbol{\zeta} \partial \boldsymbol{\zeta}'} \right)$ can be consistently estimated by $\hat{\Sigma} = -\frac{1}{NT} \frac{\partial^2 \ln \mathcal{L}_{NT}(\hat{\boldsymbol{\zeta}})}{\partial \boldsymbol{\zeta} \partial \boldsymbol{\zeta}'}$. However, for the following matrix

$$\Gamma = \Sigma^* + \Sigma = \lim_{N \rightarrow \infty} E \left(\frac{1}{\sqrt{NT}} \frac{\partial \ln \mathcal{L}_{NT}(\boldsymbol{\zeta}_0)}{\partial \boldsymbol{\zeta}} \frac{1}{\sqrt{NT}} \frac{\partial \ln \mathcal{L}_{NT}(\boldsymbol{\zeta}_0)}{\partial \boldsymbol{\zeta}'} \right), \quad (3.9)$$

its closed-form estimate is not readily available; see also Su and Yang (2015). Alternatively, we resort to the residual-based bootstrap and estimate Γ by its bootstrap estimate $\hat{\Gamma}_B$; see its detailed procedure and validity in the Supplementary Material. Consequently, the asymptotic covariance matrix of $\hat{\boldsymbol{\zeta}}$ can be estimated by $\hat{\Sigma}^{-1} \hat{\Gamma}_B \hat{\Sigma}^{-1}$.

To estimate $\boldsymbol{\beta}_0 = (\beta_{00}, \beta_{10}, \dots, \beta_{p0})'$ in model (2.2), the constraint $\beta_{00} \equiv 1$ should be imposed for identification of $\boldsymbol{\beta}_0$ as in Koenker and Zhao (1994). Note that $\boldsymbol{\beta}_0^* = \boldsymbol{\beta}_0 \sigma_{\eta_0} / \sigma_{\varepsilon_0}$, we have $\beta_{\ell 0} = \beta_{\ell 0} / \beta_{00} = \beta_{\ell 0}^* / \beta_{00}^*$ for $\ell = 1, \dots, p$. Then based on the QMLE $\hat{\boldsymbol{\zeta}} = (\hat{\boldsymbol{\phi}}', \hat{\sigma}_{\varepsilon}^2, \hat{\lambda}, \hat{\boldsymbol{\beta}}^*)'$

with $\widehat{\boldsymbol{\beta}}^* = (\widehat{\beta}_0^*, \widehat{\beta}_1^*, \dots, \widehat{\beta}_p^*)'$, we can construct a consistent estimator for $\boldsymbol{\beta}_0$, denoted by $\check{\boldsymbol{\beta}} = (1, \check{\beta}_1, \dots, \check{\beta}_p)'$, where

$$\check{\beta}_\ell = \widehat{\beta}_\ell^* / \widehat{\beta}_0^*, \quad \ell = 1, \dots, p. \quad (3.10)$$

Under the conditions of Theorem 2, by the delta method (Doob, 1935), we can show that $\sqrt{NT}(\check{\beta}_\ell - \beta_{\ell 0}) \rightarrow_d N(0, \nabla h_\ell(\boldsymbol{\zeta}_0)' \Sigma_\zeta \nabla h_\ell(\boldsymbol{\zeta}_0))$ as $N \rightarrow \infty$, where $\Sigma_\zeta = \Sigma^{-1} + \Sigma^{-1} \Sigma^* \Sigma^{-1}$, and $\nabla h_\ell(\boldsymbol{\zeta}) = \partial h_\ell(\boldsymbol{\zeta}) / \partial \boldsymbol{\zeta} = (\mathbf{0}'_{4+p+q}, -\beta_\ell^* / \beta_0^{*2}, \mathbf{0}'_{\ell-1}, 1 / \beta_0^*, \mathbf{0}'_{p-1})'$ with $h_\ell(\boldsymbol{\zeta}) = \beta_\ell^* / \beta_0^*$. And we can calculate the asymptotic variance of $\check{\beta}_\ell$ in practice by plugging in $\nabla h_\ell(\widehat{\boldsymbol{\zeta}})$ and $\widehat{\Sigma}_\zeta = \widehat{\Sigma}^{-1} \widehat{\Gamma}_B \widehat{\Sigma}^{-1}$. However, $\check{\boldsymbol{\beta}}$ performs very poorly when the sample size is small or even moderate; see details in Table 3. As a result, we propose another estimator for $\boldsymbol{\beta}_0$ in Section 3.2.2, which is much more efficient for small and moderate samples.

3.2 Conditional Quantile Estimation

In Section 3.2.1, we first investigate the conditional quantile estimation for ϑ_i 's. Then we construct an efficient estimator for the conditional scale coefficient $\boldsymbol{\beta}$ of the random effects by optimally combining information over multiple quantile levels τ 's in Section 3.2.2.

3.2.1 Weighted conditional quantile estimation

Since the scale effect ϑ_i in model (3.1) is unobservable, to study its conditional quantiles, we first consider to obtain its observable approximate using the QMLE $\widehat{\boldsymbol{\zeta}}$ from the first stage. Replacing $\boldsymbol{\zeta}$ by $\widehat{\boldsymbol{\zeta}}$ in model (2.3), one can get T random approximates of ϑ_i for each i . To fully make use of the information among these approximates, a natural way is to take the average of T random approximates as the final approximate for ϑ_i . Therefore, the pseudo

scale effect denoted by $\widehat{\boldsymbol{\vartheta}}$, that is the approximate of $\boldsymbol{\vartheta}$, is given by

$$\widehat{\boldsymbol{\vartheta}} = (\widehat{\vartheta}_1, \dots, \widehat{\vartheta}_N)' = \frac{1}{T} \sum_{t=1}^T \left[B_N(\widehat{\lambda}) \mathbf{y}_t - \widehat{\alpha} \mathbf{y}_{t-1} - Z_t \widehat{\gamma} - X \widehat{\boldsymbol{\psi}} \right]. \quad (3.11)$$

Thus, the pseudo random effect follows by $\widehat{\boldsymbol{\theta}} = X \widehat{\boldsymbol{\psi}} + \widehat{\boldsymbol{\vartheta}}$. Next the conditional quantile estimation can be applied to model (3.1) with the latent variable $\boldsymbol{\vartheta}$ replaced by its observable pseudo scale effect $\widehat{\boldsymbol{\vartheta}}$. Due to the presence of conditional heteroscedasticity in ϑ_i , we consider the WCQE given by

$$\widehat{\boldsymbol{\varphi}}(\tau) = (\widehat{\varphi}_0(\tau), \widehat{\varphi}_1(\tau), \dots, \widehat{\varphi}_p(\tau))' = \underset{\boldsymbol{\varphi} \in \Phi}{\operatorname{argmin}} \sum_{i=1}^N \frac{1}{\mathbf{x}_{ai}' \widetilde{\boldsymbol{\beta}}_c} \rho_{\tau}(\widehat{\vartheta}_i - \mathbf{x}_{ai}' \boldsymbol{\varphi}), \quad \tau \in \mathcal{T}, \quad (3.12)$$

where Φ is the parameter space of $\boldsymbol{\varphi}(\tau)$, $\mathcal{T}$ is a closed subinterval of $(0, 1)$, $\rho_{\tau}(u) = u(\tau - I(u < 0))$ is the check function, and $\widetilde{\boldsymbol{\beta}}_c$ is a consistent estimator of $\boldsymbol{\beta}_0$ such that $\mathbf{x}_{ai}' \widetilde{\boldsymbol{\beta}}_c$ is bounded away from zero for all $1 \leq i \leq N$; see also Koenker and Zhao (1994) and Zhao and Xiao (2014). For the choice of $\widetilde{\boldsymbol{\beta}}_c$, see more details in Remark 3.

Theorem 3. *Under Assumptions 3.3 – 3.7, as $N \rightarrow \infty$, then the following results hold,*

$$(i) \quad \widehat{\vartheta}_i - \vartheta_i = o_p(1) \text{ for } 1 \leq i \leq N;$$

$$(ii) \quad \frac{1}{\sqrt{N}} \sum_{i=1}^N (\widehat{\vartheta}_i - \vartheta_i) = o_p(1).$$

Theorem 3 plays an important role in verifying the validity of the proposed hybrid estimation procedure. Specifically, Theorem 3(i) guarantees that replacing $\boldsymbol{\vartheta}$ with $\widehat{\boldsymbol{\vartheta}}$ will not affect the theoretical properties of the WCQE $\widehat{\boldsymbol{\varphi}}(\tau)$, and its proof requires $T \rightarrow \infty$ which is implied by $N \rightarrow \infty$ and Assumption 3.3. Theorem 3(ii) largely simplifies the derivation of the Bahadur representation for $\widehat{\boldsymbol{\varphi}}(\tau)$ below; see details in equation (S5.16) of the Supplementary Material.

Denote the true value of $\boldsymbol{\varphi}(\tau)$ as $\boldsymbol{\varphi}_0(\tau) = (\varphi_{00}(\tau), \varphi_{10}(\tau), \dots, \varphi_{p0}(\tau))'$. Assume that the parameter space Φ is compact and $\boldsymbol{\varphi}_0(\tau)$ is its interior point; see also Assumption R2 of Chernozhukov and Hansen (2006) and Assumption 4.1 (b) of Canay (2011). To establish the theoretical properties of $\widehat{\boldsymbol{\varphi}}(\tau)$, we need the following basic assumptions.

Assumption 3.8. For $1 \leq i \leq N$, $\vartheta_i \in \mathbb{R}$ has a bounded conditional density a.s., that is, $\sup_{\vartheta \in \mathbb{R}} f_{\vartheta_i|\mathbf{x}_i}(\vartheta) < K$ for some $K > 0$ a.s., where $\mathbb{R} = (-\infty, \infty)$, and $f_{\vartheta_i|\mathbf{x}_i}(\vartheta)$ is bounded away from zero at the point $\mathbf{x}_i'\boldsymbol{\varphi}_0(\tau)$.

Assumption 3.9. For the function $\boldsymbol{\Pi}_i(\tau, \boldsymbol{\varphi}, r_i) = E \left[\frac{1}{\mathbf{x}_{ai}'\boldsymbol{\beta}} (\tau - I(\vartheta_i < \mathbf{x}_{ai}'\boldsymbol{\varphi} - r_i)) \mathbf{x}_{ai} \right]$ with $(\tau, \boldsymbol{\varphi}, r_i) \in \mathcal{T} \times \Phi \times \mathbb{R}$, the Jacobian matrices $\frac{\partial}{\partial \boldsymbol{\varphi}'} \boldsymbol{\Pi}_i(\tau, \boldsymbol{\varphi}, r_i)$ and $\frac{\partial}{\partial r_i} \boldsymbol{\Pi}_i(\tau, \boldsymbol{\varphi}, r_i)$ for each $1 \leq i \leq N$ are continuous and have full rank, uniformly over $\mathcal{T} \times \Phi \times \mathbb{R}$.

Assumption 3.8 is needed to establish the uniform convergence of $\widehat{\boldsymbol{\varphi}}(\tau)$, and this together with Assumption 3.9 is used for weak convergence of $\widehat{\boldsymbol{\varphi}}(\tau)$; see Assumption R3 of Chernozhukov and Hansen (2006) and Assumption 4.1(c) of Canay (2011).

Theorem 4. Suppose the conditions of Theorem 3 hold. Then under Assumption 3.8, $\sup_{\tau \in \mathcal{T}} \|\widehat{\boldsymbol{\varphi}}(\tau) - \boldsymbol{\varphi}_0(\tau)\| \rightarrow_p 0$ as $N \rightarrow \infty$.

Theorem 4 together with Theorems 1 and 3, ensures that the conditional quantile of the random effect $\boldsymbol{\theta}$ can be consistently estimated by

$$\widehat{Q}_{\boldsymbol{\theta}}(\tau|X) = X\widehat{\boldsymbol{\psi}} + X_a\widehat{\boldsymbol{\varphi}}(\tau). \quad (3.13)$$

Denote the density function of η_i by $f_{\eta}(\cdot)$. Let $S(\tau, \tau') = \min\{\tau, \tau'\} - \tau\tau'$ and $\Xi(\tau) = f_{\eta}(Q_{\eta}(\tau))$. Define the $(p+1) \times (p+1)$ matrices

$$D_N = \frac{1}{N} \sum_{i=1}^N \frac{\mathbf{x}_{ai}\mathbf{x}_{ai}'}{(\mathbf{x}_{ai}'\boldsymbol{\beta}_0)^2}, D_0 = \lim_{N \rightarrow \infty} D_N, \text{ and } D_{1N}(\tau) = \Xi(\tau)D_N.$$

Theorem 5. *Suppose the conditions of Theorem 4 hold. If D_0 exists and is positive definite, then under Assumption 3.9, as $N \rightarrow \infty$,*

$$\sqrt{N}(\hat{\varphi}(\cdot) - \varphi_0(\cdot)) = D_{1N}^{-1}(\cdot) \mathbb{G}_N \left(\frac{\mathbf{x}_{ai}}{\mathbf{x}'_{ai} \boldsymbol{\beta}_0} \Psi \cdot (\eta_i - Q_\eta(\cdot)) \right) + o_p(1) \rightsquigarrow \mathbb{G}(\cdot) \text{ in } \ell^\infty(\mathcal{T}),$$

where $\Psi_\tau(u) = \tau - I(u < 0)$, and $\mathbb{G}(\cdot)$ is a zero mean Gaussian process with covariance kernel $\Xi^{-1}(\tau)S(\tau, \tau')\Xi^{-1}(\tau')D_0^{-1}$.

Theorem 5 states the weak convergence of the quantile regression process $\hat{\varphi}(\cdot)$. For any fixed quantile level τ , by Theorem 5, it follows that, as $N \rightarrow \infty$,

$$\sqrt{N}(\hat{\varphi}(\tau) - \varphi_0(\tau)) \rightarrow_d N(\mathbf{0}, \tau(1 - \tau)\Xi^{-2}(\tau)D_0^{-1}). \quad (3.14)$$

To calculate the covariance matrix of $\hat{\varphi}(\tau)$ in practice, consistent estimators of $\Xi(\tau) = f_\eta(Q_\eta(\tau))$ and D_0 are required. Define the error function $\hat{\eta}_i(\boldsymbol{\beta}) = \hat{v}_i/(\mathbf{x}'_{ai}\boldsymbol{\beta})$, then the residuals can be computed by $\tilde{\eta}_i = \hat{\eta}_i(\tilde{\boldsymbol{\beta}}_c)$, where $\tilde{\boldsymbol{\beta}}_c$ is a consistent estimate of $\boldsymbol{\beta}_0$ as in (3.12). The density function $f_\eta(\cdot)$ can be estimated by the kernel density estimator $\tilde{f}_\eta(x) = (Nh)^{-1} \sum_{i=1}^N K((x - \tilde{\eta}_i)/h)$, where $K(\cdot)$ is the kernel function and h is the bandwidth. In practice, we suggest using the Gaussian kernel for $K(\cdot)$ and its rule-of-thumb bandwidth, $h = 0.9N^{-1/5} \min\{s, \hat{R}/1.34\}$, where s and $\hat{R}$ are the sample standard deviation and interquartile of the residuals, respectively; see also Zhao and Xiao (2014). As a result, $\Xi(\tau)$ can be estimated by $\tilde{f}_\eta(\bar{Q}_\eta(\tau))$, where $\bar{Q}_\eta(\tau)$ is the sample τ -th quantile of $\{\tilde{\eta}_i\}_{i=1}^N$. Moreover, D_0 can be approximated by D_N with $\boldsymbol{\beta}_0$ replaced by $\tilde{\boldsymbol{\beta}}_c$, i.e. $\tilde{D}_N = \frac{1}{N} \sum_{i=1}^N \frac{\mathbf{x}_{ai}\mathbf{x}'_{ai}}{(\mathbf{x}'_{ai}\tilde{\boldsymbol{\beta}}_c)^2}$. Consequently, the asymptotic covariance matrix of $\hat{\varphi}(\tau)$ can be estimated by plugging in $\tilde{\Xi}(\tau) = \tilde{f}_\eta(\bar{Q}_\eta(\tau))$ and $\tilde{D}_N$ to replace $\Xi(\tau)$ and D_0 , respectively.

Remark 1. (Discussion on the estimation of $\boldsymbol{\psi}$) As the conditional mean coefficient of random effects, $\boldsymbol{\psi}$ can be estimated either by the quasi-maximum likelihood estimation in the

first stage or the conditional quantile estimation for model $Q_{\boldsymbol{\theta}}(\tau|\mathbf{X}) = \mathbf{X}\boldsymbol{\psi} + \mathbf{X}_a\boldsymbol{\varphi}(\tau)$ in the second stage. However, by comparing Theorems 2 and 5, it can be found that the QMLE $\hat{\boldsymbol{\psi}}$ of $\boldsymbol{\psi}$ has faster convergence rate of $\sqrt{NT}$ than that of $\sqrt{N}$ for the quantile estimator obtained using model $Q_{\boldsymbol{\theta}}(\tau|\mathbf{X}) = \mathbf{X}\boldsymbol{\psi} + \mathbf{X}_a\boldsymbol{\varphi}(\tau)$. This is another reason that the proposed two-stage hybrid estimation procedure does not consider the quantile estimation for both $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}(\tau)$ simultaneously.

Remark 2. (Discussion on an alternative estimation procedure) For estimation of DSAR model (2.1) with the random effect θ_i specified by (2.2), we can consider the following alternative estimation procedure.

Stage 1': Estimate $(\alpha, \boldsymbol{\gamma}', \sigma_\varepsilon^2, \boldsymbol{\delta})'$ by QMLE using the transformation approach or first-differencing to eliminate individual effects θ_i 's (Lee and Yu, 2010; Yu et al., 2008).

Stage 2': Define the pseudo random effect $\tilde{\boldsymbol{\theta}}$ similarly as in (3.11). Then apply the WCQE to model (2.4) with $\boldsymbol{\theta}$ replaced by its approximate $\tilde{\boldsymbol{\theta}}$. And τ -independent $\boldsymbol{\psi}$ and $\boldsymbol{\beta}$ can be estimated by $\tilde{\boldsymbol{\psi}}$ and $\tilde{\boldsymbol{\beta}}$ via combining the information of quantile estimators at multiple quantile levels, respectively.

Note that the transformation approach or first-differencing in Stage 1' ignores the structure of $\boldsymbol{\theta}$ in (2.2), which makes the estimation less efficient. Moreover, $\tilde{\boldsymbol{\psi}}$ obtained by quantile regression in Stage 2' will also result in efficiency loss as discussed in Remark 1. Thus we prefer the two-stage hybrid estimation procedure proposed in Section 3.

Remark 3. (Choice of $\tilde{\boldsymbol{\beta}}_c$ in (3.12) for the WCQE) To conduct the weighted quantile regression in (3.12), we can employ $\tilde{\boldsymbol{\beta}}$ in (3.10) as $\tilde{\boldsymbol{\beta}}_c$. However, $\tilde{\boldsymbol{\beta}}$ is not stable in finite sample cases, so we construct an alternative choice for $\tilde{\boldsymbol{\beta}}_c$. Note that $\boldsymbol{\varphi}_0(\tau) = Q_\eta(\tau)\boldsymbol{\beta}_0$, we can construct $\tilde{\boldsymbol{\beta}}_c$ by combining the information of the estimator for $\boldsymbol{\varphi}_0(\tau)$ over multi-

ple quantile levels. Furthermore, $\varphi_0(\tau)$ and $Q_\eta(\tau)$ have the same sign, thus we consider $\sum_{k=1}^K |\varphi_0(\tau_k)| = \beta_0 \sum_{k=1}^K |Q_\eta(\tau_k)|$ alternatively to avoid the offset among positive and negative values for both terms at different quantile levels. Following Koenker and Zhao (1994), we adopt the constraint $\beta_{00} \equiv 1$ to ensure the identification of β_0 and $Q_\eta(\tau)$. As a result, we suggest a choice for $\tilde{\beta}_c$ in (3.12) as follows

$$\tilde{\beta}_c = \frac{\sum_{k=1}^K |\tilde{\varphi}(\tau_k)|}{\sum_{k=1}^K |\tilde{Q}_\eta(\tau_k)|}, \quad (3.15)$$

where $\tilde{\varphi}(\tau_k)$'s with $\tau_k = k/(K+1)$ for $1 \leq k \leq K$, are the unweighted conditional quantile estimators defined by $\tilde{\varphi}(\tau_k) = (\tilde{\varphi}_0(\tau_k), \dots, \tilde{\varphi}_p(\tau_k))' = \operatorname{argmin}_{\varphi \in \Phi} \sum_{i=1}^N \rho_{\tau_k}(\hat{v}_i - \mathbf{x}_{ai}'\varphi)$ and $\tilde{Q}_\eta(\tau_k) = \tilde{\varphi}_0(\tau_k)$; see also Zhu et al. (2018). Clearly, (3.15) guarantees the positivity of $\tilde{\beta}_c$.

3.2.2 Weighted quantile average estimation

For the conditional scale coefficient β of the random effects, we have proposed two consistent estimators $\check{\beta}$ and $\tilde{\beta}_c$ in (3.10) and (3.15), respectively. However, $\check{\beta}$ is not suggested as it is not stable in small or even moderate samples. For $\tilde{\beta}_c$, it is actually an equally weighted quantile average estimator by assigning equal weight $[\sum_{k=1}^K |\tilde{Q}_\eta(\tau_k)|]^{-1}$ to the WCQE $\hat{\varphi}(\cdot)$ at each quantile level, which may lead to a less efficient estimator. We next introduce an efficient estimator of β by optimally combining the information of $\hat{\varphi}(\cdot)$ across K quantile levels: $\tau_k = k/(K+1)$ for $1 \leq k \leq K$, where K is a fixed integer.

Recall that $\beta_0 = (\beta_{00}, \beta_{10}, \dots, \beta_{p0})'$ is the true value of β , and $\varphi_0(\tau) = Q_\eta(\tau)\beta_0$ with $\varphi_{\ell 0}(\tau) = Q_\eta(\tau)\beta_{\ell 0}$ holds for $\ell = 0, 1, \dots, p$. Note that for any weight vector $\pi_K = (\pi_1, \dots, \pi_K)'$ satisfying $\sum_{k=1}^K \pi_k Q_\eta(\tau_k) = 1$, it holds that $\beta_0 = \sum_{k=1}^K \pi_k \varphi_0(\tau_k)$ under the constraint $\beta_{00} \equiv 1$. This motivates us to combine $\{\hat{\varphi}(\tau_k), 1 \leq k \leq K\}$ linearly to define the

WQAE of β as follows

$$\widehat{\beta}(\pi_K) = \sum_{k=1}^K \pi_k \widehat{\varphi}(\tau_k) \quad \text{with} \quad \sum_{k=1}^K \pi_k Q_\eta(\tau_k) = 1, \quad (3.16)$$

see also Zhao and Xiao (2014).

Denote $\mathcal{B}$ as the parameter space of β , and assume $\mathcal{B}$ is compact and β_0 is an interior point. Define the $K \times K$ matrix $H = \{h_{ij}, 1 \leq i, j \leq K\}$ with $h_{ij} = \Xi^{-1}(\tau_i)S(\tau_i, \tau_j)\Xi^{-1}(\tau_j)$.

The asymptotics of $\widehat{\beta}(\pi_K)$ and the optimal choice for π_K are provided below.

Theorem 6. *Suppose the conditions of Theorem 5 hold, then as $N \rightarrow \infty$,*

$$\sqrt{N} \left(\widehat{\beta}(\pi_K) - \beta_0 \right) \rightarrow_d N(0, \Upsilon(\pi_K)),$$

where $\Upsilon(\pi_K) = \pi_K' H \pi_K D_0^{-1}$. Moreover, the optimal weight is

$$\pi_{opt,K} = \underset{\pi, s.t. \pi' \mathbf{q} = 1}{\operatorname{argmin}} \Upsilon(\pi_K) = \frac{H^{-1} \mathbf{q}}{\mathbf{q}' H^{-1} \mathbf{q}}, \quad (3.17)$$

where $\mathbf{q} = (Q_\eta(\tau_1), \dots, Q_\eta(\tau_K))'$. Thus the asymptotic covariance of the optimal WQAE

$$\widehat{\beta}(\pi_{opt,K}) \text{ is } \Upsilon(\pi_{opt,K}) = (\mathbf{q}' H^{-1} \mathbf{q})^{-1} D_0^{-1}.$$

To estimate the optimal weight $\pi_{opt,K}$, consistent estimators of $\mathbf{q}$ and H are necessary. We can approximate $\mathbf{q}$ by $\widehat{\mathbf{q}} = (\widehat{Q}_\eta(\tau_1), \dots, \widehat{Q}_\eta(\tau_K))'$ with $\widehat{Q}_\eta(\tau) = \widehat{\varphi}_0(\tau)$, and estimate H by $\widehat{H} = \{\widehat{h}_{ij}, 1 \leq i, j \leq K\}$ with $\widehat{h}_{ij} = \widehat{\Xi}^{-1}(\tau_i)S(\tau_i, \tau_j)\widehat{\Xi}^{-1}(\tau_j)$ and $\widehat{\Xi}(\tau) = \widehat{f}_\eta(\widehat{Q}_\eta(\tau))$. Here using $\widehat{Q}_\eta(\tau)$ instead of the empirical quantile $\bar{Q}_\eta(\tau)$ to estimate $\mathbf{q}$ can ensure the estimate of β_{00} being one. Then a consistent estimator $\widehat{\pi}_{opt,K} = (\widehat{\pi}_{1,opt}, \dots, \widehat{\pi}_{K,opt})'$ of $\pi_{opt,K} = (\pi_{1,opt}, \dots, \pi_{K,opt})'$ can be obtained by plugging in $\widehat{H}$ and $\widehat{\mathbf{q}}$. Finally, the optimal WQAE can be calculated by $\widehat{\beta}(\widehat{\pi}_{opt,K}) = \sum_{k=1}^K \widehat{\pi}_{k,opt} \widehat{\varphi}(\tau_k)$. It can be shown that $\sqrt{N}(\widehat{\beta}(\widehat{\pi}_{opt,K}) - \beta_0) \rightarrow_d N(0, \Upsilon(\pi_{opt,K}))$ as $N \rightarrow \infty$, which implies that $\widehat{\beta}(\widehat{\pi}_{opt,K})$ and $\widehat{\beta}(\pi_{opt,K})$ have the same asymptotic efficiency.

Remark 4. (Comparison of $\check{\beta}$, $\tilde{\beta}_c$ and $\hat{\beta}(\hat{\pi}_{opt,K})$) For $\check{\beta}$ and $\hat{\beta}(\hat{\pi}_{opt,K})$, they are estimated by different methods and thus their theoretical comparison is infeasible. Alternatively, we compare their finite sample performance by simulation study. Simulation results in Section 4 indicate that $\hat{\beta}(\hat{\pi}_{opt,K})$ outperforms $\check{\beta}$ in biases, empirical and asymptotic standard deviations when the sample size is small or even moderate. In contrast, notice that the convergence rates of $\hat{\beta}(\hat{\pi}_{opt,K})$ and $\check{\beta}$ are $\sqrt{N}$ and $\sqrt{NT}$, respectively, thus $\check{\beta}$ can be more efficient than $\hat{\beta}(\hat{\pi}_{opt,K})$ for large samples. We also compare the finite sample performance of $\hat{\beta}(\hat{\pi}_{opt,K})$ and $\tilde{\beta}_c$, and simulation results show that $\hat{\beta}(\hat{\pi}_{opt,K})$ has smaller biases and empirical standard deviations than $\tilde{\beta}_c$. Therefore, we suggest to improve efficiency using $\hat{\beta}(\hat{\pi}_{opt,K})$ when sample size is small or moderate, whereas $\check{\beta}$ is preferred in large sample settings owing to its faster convergence rate.

4. Simulation Study

This section conducts a simulation experiment to evaluate the finite sample performance of the proposed two-stage hybrid estimation procedure in Section 3, including the Gaussian QMLE $\hat{\zeta}$, the WCQE $\hat{\varphi}(\tau)$ and the WQAE $\hat{\beta}(\hat{\pi}_{opt,K})$.

The data are generated from the following model

$$\begin{cases} y_{it} = \theta_i + 0.5y_{i,t-1} + 0.5 \sum_{j=1}^N w_{N,ij} y_{jt} + z_{it} + \varepsilon_{it}, \\ \theta_i = 1 + x_i + (1 + |x_i|) \eta_i, \end{cases}$$

where the innovations $\{\varepsilon_{it}\}$ are independently standard normal, $\{\eta_i\}$ are independently standard normal or follow the uniform distribution $U(-\sqrt{3}, \sqrt{3})$, and $\{x_i\}$ and $\{z_{it}\}$ are independently generated randomly from the uniform distribution $U(-1, 1)$ and standard normal distribution, respectively. The spatial weights matrix is generated according to Rook con-

tiguity and row normalized (Yu et al., 2008; Su and Yang, 2015). Note that the QMLE at Stage 1 has $\sqrt{NT}$ -convergence rate under Assumption 3.2, which allows two cases: (i) both N and T are large; (ii) N is large but T is fixed. Whereas both WCQE and WQAE at Stage 2 have $\sqrt{N}$ -convergence rate under Assumption 3.3, which only allows case (i). Hence, we consider three sample settings $(N, T) = (20, 20), (20, 50)$ and $(100, 50)$ for QMLE, and $(N, T) = (20, 20), (100, 50)$ and $(300, 200)$ for WCQE and WQAE, with 1000 replications generated for each sample setting.

We aim to estimate the parameters $\boldsymbol{\zeta}_0 = (\alpha_0, \gamma_{10}, \psi_{00}, \psi_{10}, \sigma_{\varepsilon 0}^2, \lambda_0, \beta_{00}^*, \beta_{10}^*)'$, $\boldsymbol{\varphi}_0(\tau) = (\varphi_{00}(\tau), \varphi_{10}(\tau))'$ and β_{10} using the QMLE $\hat{\boldsymbol{\zeta}}$, the WCQE $\hat{\boldsymbol{\varphi}}(\tau)$ and the WQAE $\hat{\beta}_1(\hat{\boldsymbol{\pi}}_{opt,K})$, respectively, where $(\beta_{00}^*, \beta_{10}^*) = (1, \beta_{10})\sigma_{\eta 0}/\sigma_{\varepsilon 0}$ and $\boldsymbol{\varphi}_0(\tau) = (1, \beta_{10})Q_{\eta}(\tau)$. The quasi-Newton method (Doob, 1935) is employed for solving the QMLE. For the WCQE, we use the R package “nloptr” (Johnson, 2021) with the same sign restriction imposed on each element of $\hat{\boldsymbol{\varphi}}(\tau)$ owing to $\boldsymbol{\varphi}_0(\tau) = \boldsymbol{\beta}_0 Q_{\eta}(\tau)$ and $\boldsymbol{\beta}_0 \geq 0$. For the WQAE, we employ $K = 9$ quantile levels $\tau_k = k/10$ with $k = 1, \dots, 9$, and the resulting $\hat{\beta}_1(\hat{\boldsymbol{\pi}}_{opt,K})$ is almost nonnegative under the same sign restriction for $\hat{\boldsymbol{\varphi}}(\tau)$.

Table 1 reports the biases, empirical standard deviations (ESDs) and asymptotic standard deviations (ASDs) of the QMLE $\hat{\boldsymbol{\zeta}}$ at Stage 1, where ASDs are calculated by the residual-based bootstrap with bootstrap sample size $B = 500$. Recall that we require $N \rightarrow \infty$ theoretically, we only focus on the comparison of (N, T) settings with N increasing. Comparing the results of settings $(N, T) = (20, 20)$ and $(100, 50)$ in which T increases at a slower rate than N , we can see that most of the biases, ESDs and ASDs become smaller as N increases. We have the same finding when compares the results of setting $(N, T) = (20, 50)$ with that of setting $(100, 50)$ where T is fixed as N increases. Moreover, the biases for the

normal distribution of η_i are mostly smaller than those for the uniform distribution. This is as expected as the QMLE reduces to the MLE when η_i 's are normally distributed.

Table 2 summarizes the biases, ESDs and ASDs of the WCQE $\hat{\varphi}(\tau)$ at Stage 2 for three quantile levels $\tau = 0.25, 0.5$ and 0.75 , and three sample settings $(N, T) = (20, 20), (100, 50)$ and $(300, 200)$. We have the following findings: (i) for both normally and uniformly distributed innovations $\{\eta_i\}$, as N increases, most of the biases, ESDs and ASDs become smaller, and the ESDs get closer to their corresponding ASDs; (ii) the performance of $\hat{\varphi}(\tau)$ gets better as the quantile level τ gets closer to the center. In addition, the ESDs are obviously smaller than the corresponding ASDs at $\tau = 0.5$ because $Q_\eta(0.5) \equiv 0$ holds for symmetric distributions centered at zero and thus $\varphi_{00}(0.5) = \varphi_{10}(0.5) \equiv 0$, which makes ESDs abnormally small under the same sign constraint. However the ESDs are closer to their corresponding ASDs for larger N ; see Table S.1 for evidence in the Supplementary Material.

Table 3 provides the biases, ESDs and ASDs of the WQAE $\hat{\beta}_1(\hat{\pi}_{opt,K})$ at Stage 2 for three sample settings $(N, T) = (20, 20), (100, 50)$ and $(300, 200)$. For comparison, the results of $\check{\beta}_1$ in (3.10) and $\tilde{\beta}_{1c}$ in (3.15) are also reported, where the extra setting $(N, T) = (50, 20)$ is added to show the performance of $\check{\beta}_1$ in cases with moderate sample sizes. It can be seen that, as N increases, the biases, ESDs and ASDs of all estimators become smaller, and the ESDs get closer to the corresponding ASDs. Moreover, the ESDs of $\hat{\beta}_1(\hat{\pi}_{opt,K})$ will move closer to their corresponding ASDs for larger N ; see Table S.2 in the Supplementary Material. In addition, the estimate $\check{\beta}_1$ is not satisfactory as the sample size is not large, and $\tilde{\beta}_{1c}$ is usually less efficient than $\hat{\beta}_1(\hat{\pi}_{opt,K})$ in terms of ESDs and ASDs.

In summary, the finite sample performance of the two-stage hybrid estimation procedure

is reasonable, which validates the accuracy of the asymptotics in Section 3.

5. Empirical Analysis of Air Quality

This section illustrates the usefulness of the proposed model and its estimation procedure by analysing the pure influence of socio-economic factors on the air quality in China after the spatial-temporal effects are eliminated.

We choose the air quality index (AQI) as the indicator for air quality, downloaded from China National Environmental Monitoring Centre (<http://www.cnemc.cn/>), which spans from January 1, 2018 to December 16, 2018 among 143 cities available in China. Since AQI is closely related to the meteorological factors (Yang et al., 2007; Lee et al., 2012), we select temperature (TEM), precipitation (PRE) and wind speed (WIN) to represent the exogenous spatial-temporal effects in our model, which is downloaded from National Meteorological Information Center (<http://data.cma.cn/en>). Moreover, AQI can also be affected by the socio-economic factors related to each city (Chen and Xu, 2017; Yu and Liu, 2020). Hence, we use Gross Regional Product (GRP) and secondary industry share in GRP (Industry) as the space-specific factors, and the data are obtained from China City Statistical Yearbook. Many references indicate that population or population density is also an important socio-economic factor that can influence air quality, but Figure 1 shows that there is a significant positive correlation between GRP and population (or population density) in China, hence we only include GRP to avoid multicollinearity. To balance the size and information of data in time dimension, we transform the original average hourly data of AQI and average daily data of TEM and WIN into average weekly data by average, and transform the original cumulative daily data of PRE into cumulative weekly data by summation. Finally we obtain

the spatial-temporal dataset with $N = 143$ and $T = 49$.

In Figure 2, the heat maps of the AQI of some local areas of China from the first to ninth weeks in 2018 imply that there are spatial and temporal effects in the air quality. This motivates us to consider the spatial and temporal effects with our model. In this study, we aim to understand the influence of the local socio-economic factors on the air quality with those spatial-temporal effects controlled so that the governments can evaluate how their local policies work in improving air qualities.

Note that the scales of AQI, TEM, PRE, WIN, GRP and Industry differ from each other as clearly demonstrated in Table 4, so we divide them by the estimated standard deviations, respectively. Moreover, we centralize TEM, PRE and WIN owing to the zero mean requirement by Assumption 3.6(ii). In accordance with the model setting in (2.1), for $i = 1, \dots, 143$ and $t = 0, \dots, 49$, we denote the processed variables AQI, TEM, PRE, WIN, GRP and Industry as $\{y_{it}\}$, $\{z_{1it}\}$, $\{z_{2it}\}$, $\{z_{3it}\}$, $\{x_{1i}\}$ and $\{x_{2i}\}$, respectively. We consider the proposed two-stage hybrid estimation procedure in Section 3. The fitted model of (2.1) is then given by

$$y_{it} = \hat{\theta}_i + 0.178 y_{i,t-1} + 0.648 \sum_{j=1}^N w_{N,ij} y_{jt} - 0.096 z_{1it} - 0.065 z_{2it} - 0.078 z_{3it} + \hat{\varepsilon}_{it}, \quad (5.1)$$

where the spatial weights matrix $W_N = \{w_{N,ij} = a_{ij}/d_i\}$ is a row-normalized binary adjacency matrix with $d_i = \sum_{j=1}^N a_{ij}$, $a_{ii} = 0$ and $a_{ij} = 1$ if the i -th city is adjacent to the j -th city geographically, otherwise $a_{ij} = 0$ for $i \neq j$. The summary information of fitted coefficients in model (5.1) is provided in Table 5, where the standard errors are calculated based on the residual-based bootstrap with bootstrap sample size $B = 500$. We observe that the coefficients are significantly nonzeros at the 10% significance level. The fitted model (5.1) indicates that the first lag of AQI and its neighbors have positive impacts on AQI, while the

temperature, precipitation and wind speed have negative impacts on AQI. These findings are consistent with our common sense.

We are particularly interested in the influence of x_{1i} and x_{2i} on the low and high quantile levels of individual random effects θ_i 's, that is the influence of socio-economic factors on the air quality in good or poor status after the spatial-temporal effects are removed. For this purpose, we choose two quantile levels $\tau = 0.25$ and 0.75 for illustration. The estimated conditional quantile functions of θ_i in model (2.4) are

$$\begin{cases} \widehat{Q}_{\theta_i}(0.25|\mathbf{x}_i) = (0.171 + 0.078 x_{1i} + 0.063 x_{2i}) + (-0.066 - 0.016 |x_{1i}| - 0.046 |x_{2i}|), \\ \widehat{Q}_{\theta_i}(0.75|\mathbf{x}_i) = (0.171 + 0.078 x_{1i} + 0.063 x_{2i}) + (0.030 + 0.021 |x_{1i}| + 0.014 |x_{2i}|), \end{cases} \quad (5.2)$$

where the summary information of fitted coefficients is reported in Table 5. For each equation of (5.2), the first parenthesis corresponds to the estimated conditional location of θ_i while the second parenthesis represents the conditional quantile estimate of the scale effect. Since the conditional location of θ_i is actually the conditional expectation of θ_i , the fitted conditional expectation of θ_i is $\widehat{E}(\theta_i|\mathbf{x}_i) = 0.171 + 0.078x_{1i} + 0.063x_{2i}$. Notice that $x_{1i}, x_{2i} > 0$ for all i in our situation, then $|x_{\ell i}|$ equal to $x_{\ell i}$ for $\ell = 1, 2$. As a result, based on model (5.2) and Table 5, we have the following conclusions: (i) both the secondary industry share and GRP may impose positive impacts on the conditional mean of θ_i as well as its low and high quantiles; (ii) with the spatial-temporal effects eliminated, the insignificant positive impact on AQI of GRP implies that China may locate near the apex of the environmental Kuznets curve (Stern et al., 1996), that is, the adverse impact of China's economic development on the environment diminishes gradually; (iii) the secondary industry share has statistically significant positive impact on the conditional location of θ_i at 10% significance level, which indicates that the secondary industry may cause the deterioration of air quality in China.

Moreover, the variable x_{2i} is significantly related with the conditional scale of θ_i at the lower quantile level while no significant effect is observed at the higher level under 10% significance level, which provides evidence of possible heterogeneity in AQI due to the secondary industry share.

6. Conclusion and Discussion

This paper introduces a dynamic spatial autoregressive model with heterogeneous random effects, which is useful in many cases with complicated correlation structures. Although we focus on the space-specific factors in this paper, this idea can be similarly used in other random effect models to account for possible heterogeneity based on quantile regression.

There are several potential research topics in the framework of the proposed new model. First, note that the asymptotic covariance of the QMLE in Theorem 2 is infeasible to compute, and the limiting distributions of the WCQE and WQAE in (3.14) and Theorem 6 are not convenient to estimate due to the unknown density function $f_\eta(\cdot)$ of innovations whose estimation needs some non-parametric methods with tuning parameters. The bootstrap method we used to approximate the covariance matrix of QMLE only introduces the perturbation into the individual level but ignores the time level, which can make it less efficient. As a result, we can consider a new method perturbing the information of individual and time levels simultaneously for the proposed model, and hopefully the bootstrapping can simplify the estimation of asymptotic distributions and lead to more accurate inference results when the sample size is moderate or even small. Second, missing values in individuals or time points are very common for the spatial panel data in practice. For example, in our real application, there are missing observations of AQI for certain cities (for example, Yantai),

and we adopt the likewise deletion method for simplicity. However, the likewise deletion may result in information loss. Hence, it is beneficial to consider an estimation procedure in the presence of missing values at the response. Third, a location-scale model as given in (2.2) is specified in this paper, and a more general quantile regression model can be considered in the following form

$$Q_{\theta_i}(\tau|x_{1i}, \dots, x_{pi}) = \tilde{\psi}_0(\tau) + \sum_{\ell=1}^p \tilde{\psi}_\ell(\tau)x_{\ell i},$$

where $\tilde{\psi}_\ell(\cdot)$'s are the quantile dependent coefficients. We leave these topics for future research.

Supplementary Materials

The supplementary material contains a bootstrapping procedure for QMLE, additional results for simulation, detailed forms of the matrices Σ_N , Σ_N^* , $\Omega_{NT}^{-1}(\beta^*)$ and the vector $\mathbf{Y}_{-1}$, as well as technical details for Theorems 1 – 6 and Propositions 1 – 2 with Lemmas 1 – 10.

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Figure 1: Scatterplots of log-transformed variables “GRP”, “population” and “population density”.

Figure 2: Heat maps of the AQI of a part of central and eastern China from the first to ninth weeks in 2018.

Table 1: Biases, ESDs and ASDs of the QMLE $\hat{\zeta}$ when the innovations $\{\eta_i\}$ follow the standard normal or uniform distribution.

	N	T	$N(0, 1)$			$U(-\sqrt{3}, \sqrt{3})$		
			Bias	ESD	ASD	Bias	ESD	ASD
α	20	20	0.0021	0.0234	0.0398	0.0029	0.0232	0.0341
	20	50	0.0007	0.0146	0.0400	0.0008	0.0141	0.0332
	100	50	0.0006	0.0080	0.0089	0.0006	0.0079	0.0088
γ_1	20	20	0.0020	0.0511	0.0561	0.0013	0.0516	0.0571
	20	50	0.0019	0.0336	0.0381	0.0026	0.0318	0.0371
	100	50	0.0002	0.0145	0.0147	0.0005	0.0144	0.0147
ψ_0	20	20	0.0089	0.3737	1.2303	0.0091	0.3930	0.8101
	20	50	0.0292	0.3561	1.8826	0.0201	0.3966	2.1397
	100	50	0.0040	0.1537	0.1609	0.0045	0.1519	0.1596
ψ_1	20	20	0.0404	0.6824	1.2488	0.0161	0.6284	0.9061
	20	50	0.0018	0.6537	4.9535	0.0149	0.6501	3.0061
	100	50	0.0072	0.2782	0.2731	0.0100	0.2742	0.2735
σ_ε^2	20	20	0.0039	0.0714	0.1266	0.0086	0.0752	0.1138
	20	50	0.0013	0.0464	0.1113	0.0027	0.0449	0.0913
	100	50	0.0001	0.0201	0.0377	0.0003	0.0193	0.0366
λ	20	20	0.0030	0.0237	0.0435	0.0037	0.0235	0.0375
	20	50	0.0008	0.0147	0.0428	0.0010	0.0141	0.0346
	100	50	0.0006	0.0080	0.0089	0.0006	0.0079	0.0088
β_0^*	20	20	0.1805	0.5059	2.7118	0.1295	0.4365	1.6301
	20	50	0.1834	0.4967	7.5333	0.1041	0.4295	4.2356
	100	50	0.0198	0.2087	0.2581	0.0177	0.1360	0.1718
β_1^*	20	20	0.1256	0.9422	5.7388	0.0584	0.7995	3.0798
	20	50	0.1419	0.9644	15.7608	0.0312	0.8020	9.5194
	100	50	0.0015	0.4065	0.4525	0.0043	0.2630	0.2991

Table 2: Biases, ESDs and ASDs of the WCQE $\hat{\varphi}(\tau)$ at $\tau = 0.25, 0.5, 0.75$, when the innovations $\{\eta_i\}$ follow the standard normal or uniform distribution.

				$N(0, 1)$			$U(-\sqrt{3}, \sqrt{3})$		
				Bias	ESD	ASD	Bias	ESD	ASD
$\tau = 0.25$	φ_0	20	20	0.0303	0.4880	0.9059	0.1333	0.5239	0.9455
		100	50	0.0219	0.3287	0.3713	0.0479	0.3667	0.3934
		300	200	0.0081	0.1935	0.1970	0.0144	0.2116	0.2101
	φ_1	20	20	0.0913	0.7667	1.6977	0.0247	0.8428	1.7688
		100	50	0.0585	0.5937	0.7213	0.0496	0.6657	0.7625
		300	200	0.0225	0.3896	0.4091	0.0056	0.4270	0.4368
$\tau = 0.5$	φ_0	20	20	0.0164	0.2520	0.8681	0.0037	0.2654	1.0016
		100	50	0.0038	0.1092	0.3529	0.0035	0.1334	0.4541
		300	200	0.0045	0.0589	0.1851	0.0041	0.0750	0.2427
	φ_1	20	20	0.0143	0.3411	1.6275	0.0253	0.4375	1.8757
		100	50	0.0104	0.1718	0.6852	0.0028	0.2384	0.8802
		300	200	0.0052	0.1125	0.3844	0.0015	0.1558	0.5042
$\tau = 0.75$	φ_0	20	20	0.0336	0.4843	0.8838	0.1093	0.5027	0.9572
		100	50	0.0141	0.3197	0.3737	0.0511	0.3666	0.3981
		300	200	0.0082	0.1913	0.1960	0.0063	0.2080	0.2101
	φ_1	20	20	0.0864	0.7897	1.6561	0.0118	0.8468	1.7927
		100	50	0.0300	0.5683	0.7254	0.0517	0.6734	0.7715
		300	200	0.0126	0.3703	0.4069	0.0119	0.4215	0.4368

Table 3: Biases, ESDs and ASDs of the three estimators of β_0 , i.e. $\check{\beta}$ by the QMLE, the initial estimator $\tilde{\beta}_c$ and the WQAE $\hat{\beta}(\hat{\pi}_{opt,K})$, when the innovations $\{\eta_i\}$ follow the standard normal or uniform distribution.

	N	T	$N(0, 1)$			$U(-\sqrt{3}, \sqrt{3})$		
			Bias	ESD	ASD	Bias	ESD	ASD
$\check{\beta}_1$	20	20	276805	827496	5e+12	146686	609974	3e+12
	50	20	3181	100581	4e+9	3281	103752	4e+9
	100	50	0.1638	0.7819	0.8772	0.0779	0.4386	0.5052
	300	200	0.0340	0.3032	0.3548	0.0167	0.1942	0.2278
$\tilde{\beta}_{1c}$	20	20	0.5346	1.2984	-	0.4840	1.2580	-
	100	50	0.2714	0.7978	-	0.2020	0.5663	-
	300	200	0.0668	0.3621	-	0.0637	0.3044	-
$\hat{\beta}_1$	20	20	0.5116	1.2505	1.1530	0.3418	1.1946	1.0260
	100	50	0.1165	0.8976	0.4603	0.0669	0.5035	0.3477
	300	200	0.0348	0.3578	0.2455	0.0606	0.2372	0.1730

* $e + 12$ denotes 10^{12}

Table 4: Summary statistics for AQI, TEM (0.1°C), PRE (0.1mm), WIN (0.1m/s), GRP (10000 yuan) and Industry (%).

	AQI	TEM	PRE	WIN	GRP	Industry
Min	13.264	-283.000	0.000	4.714	1500100	18.270
Max	312.845	325.714	5113.000	78.286	326798700	72.900

Table 5: Summary information of fitted coefficients for models (5.1) and (5.2).

	Estimate	Std. Error	z statistic	p-value
α	0.178	0.027	6.593	0.000
λ	0.648	0.022	29.455	0.000
γ_1	-0.096	0.015	-6.400	0.000
γ_2	-0.065	0.006	-10.833	0.000
γ_3	-0.078	0.013	-6.000	0.000
ψ_0	0.171	0.194	0.881	0.378
ψ_1	0.078	0.088	0.886	0.375
ψ_2	0.063	0.036	1.750	0.080
$\varphi_0(0.25)$	-0.066	0.103	-0.641	0.522
$\varphi_1(0.25)$	-0.016	0.028	-0.571	0.568
$\varphi_2(0.25)$	-0.046	0.024	-1.917	0.055
$\varphi_0(0.75)$	0.030	0.216	0.139	0.890
$\varphi_1(0.75)$	0.021	0.058	0.362	0.717
$\varphi_2(0.75)$	0.014	0.049	0.286	0.775