**ON AN APPLICATION OF PARSEVAL'S FORMULA TO
PROBLEMS OF $G_\theta^\times$ -SUMMABILITY OF EIGENFUNCTION
EXPANSION OF THE LAPLACIAN OPERATOR**

M. VUKOVIĆ, E. ILIĆ-GEORGIJEVIĆ AND O. STEVANOVIĆ

To the memory of deeply respected Academician Mahmut Bajraktarević

ABSTRACT. In this paper, applying Parseval's formula, we prove a $G_\theta^\times$ -summability analogue of Avadhani's theorem for the Riesz-summability of the eigenfunction expansion. A crucial step in our proof of this theorem was to find a function $g(x)$ that would lead us to the kernel of the $G_\theta^\times$ -summability, which is more complex than the kernel of the Riesz-summability.

1. INTRODUCTION

Let D be a bounded open domain in the Euclidean space E_k of dimension k ($k \geq 2$) and $x = (x_1, \dots, x_k)$, $x_0 = (x_1^0, \dots, x_k^0)$ be two points in D . Also, let the boundary ∂D of D be sufficiently regular in order that the following eigenvalue problem

$$\begin{aligned}\Delta u + \lambda u &= 0 && \text{in } D, \\ u &= 0 && \text{on } \partial D,\end{aligned}$$

where

$$u = u(x) = u(x_1, \dots, x_k), \quad \Delta u = \sum_{i=1}^k \frac{\partial^2 u}{\partial x_i^2},$$

possesses an infinite number of positive eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \rightarrow \infty, \quad \text{as } n \rightarrow \infty$$

with the corresponding eigenfunctions

$$w_1(x), w_2(x), \dots, w_n(x), \dots$$

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We assume that these eigenfunctions $w_n(x)$ form a complete orthonormal set in the space $L^2(D)$ of functions $f(x)$ which are square integrable in D . The $G_\theta^\varkappa$ -summability method was introduced by Avakumović [4], and is defined as

$$G_\theta^\varkappa(\lambda; w) = \sum_{\lambda_m \leq w} \{1 - e^{(\lambda_m - w)w^{-\theta}}\}^\varkappa a_m, \quad 0 < \theta < 1, \quad \varkappa > 0$$

$$(0 < \lambda_1 < \dots < \lambda_m \rightarrow \infty, \quad m \rightarrow \infty).$$

This paper concerns the problem of the $G_\theta^\varkappa$ -summability of the eigenfunction expansion

$$f(x) \sim \sum a_n w_n(x), \quad (1.1)$$

of $f \in L^2(D)$, where

$$a_n = \int_D f(x) w_n(x) dV, \quad (1.2)$$

and dV denotes the volume element in E_k .

The Riesz method of summability has been studied by many authors: S. Minakshisundaram [14] proved that the Riesz-summability $(\lambda, \varkappa)$ of the series (1.1) at the point $x_0 \in D$ depended only on the behaviour of $f(x)$ in a neighborhood of x_0 , no matter how small, if $\varkappa > \frac{k}{2}$, i.e., that this summability was a local property of the function $f(x)$ at the point x_0 .

E. C. Titchmarsh [15] established that this result could be extended to the case $\varkappa \geq \frac{k}{2}$.

Following Titchmarsh's idea [15] and exploring Avakumović's estimation [3] of the sum of squares of orthonormal eigenfunctions, which has a prominent role in the proofs of theorems on the Riesz-summability of the eigenfunction expansion (1.1), B. M. Levitan [7, 8] and T. V. Avadhani [2] proved that the Riesz-summability of order $\varkappa$ of the eigenfunction expansion (1.1) at $x_0 \in D$ depended only on the behaviour of f in a neighborhood of x_0 if $\varkappa \geq \frac{k-1}{2}$, while for $\varkappa < \frac{k-1}{2}$, the Riesz-summability was no longer a local property of $f(x)$. The $G_\theta^\varkappa$ -method of summation [4], has been also successfully applied to the eigenfunction expansion (1.1).

M. Maravić in [9] gave a connection between the Riesz and $G_\theta^\varkappa$ -summabilities, and starting from this result in [12] proved that the $G_\theta^\varkappa$ -summability of the series (1.1) of the function $f(x) \in L^2(D)$ at the point $x_0 \in D$ depended only on the behaviour of $f(x)$ in a neighborhood of x_0 if $\varkappa > \frac{k-1}{2}(2\theta - 1)^{-1} + 1$, $\frac{1}{2} < \theta < 1$ (we also refer to [10, 11, 13]).

Here we improve this result using the $G_\theta^\varkappa$ -summability directly, by constructing the function $g(x)$ that will lead to the kernel of the $G_\theta^\varkappa$ -summability, which was a crucial step as well as the main difficulty in our proof. Namely, we prove the following theorem:

Theorem 1.1. *The $G_\theta^\varkappa$ -summability of the series (1.1) of the function $f(x) \in L^2(D)$ at $x_0 \in D$ depends only on the behaviour of $f(x)$ in a neighborhood of x_0 if $\varkappa \geq \frac{k}{2}$.*

It is expected that Theorem 1.1 could be extended to the case when $\varkappa \geq \frac{k-1}{2}$.

2. PRELIMINARIES

We will make use of the well known properties of Bessel functions.

Lemma 2.1 ([17]). *If a and b are positive real numbers, $\nu > -1$, and $\mu - \nu > 0$, then we have*

$$\int_0^\infty \mathcal{J}_{\mu+1}(at) \mathcal{J}_\nu(bt) t^{\nu-\mu} dt = \begin{cases} 0, & a < b; \\ \frac{b^\nu (a^2 - b^2)^{\mu-\nu}}{2^{\mu-\nu} a^{\mu+1} \Gamma(\mu-\nu+1)}, & a \geq b, \end{cases} \quad (2.1)$$

where $\mathcal{J}_\mu(x)$ is the Bessel function of the first kind, and

$$\mathcal{J}_\mu(x) = O(x^{-\frac{1}{2}}), \quad x \rightarrow \infty.$$

Throughout the paper we will use the following notations and formulae.

The spherical mean of the function f over a sphere of radius t and center x is given by

$$f(x; t) = \frac{1}{\sigma_{k-1}} \int_S f(x + \epsilon t) d\sigma, \quad (2.2)$$

where $x + \epsilon t = (x_1 + \epsilon_1 t, \dots, x_k + \epsilon_k t)$, S is the sphere $\epsilon_1^2 + \dots + \epsilon_k^2 = t^2$, $d\sigma$ the $(k-1)$ -dimensional volume element of S , and σ_{k-1} the volume of S [5]. If σ is the surface of the sphere of radius t with center x , and $|\sigma|$ its $(k-1)$ -dimensional volume, then, according to [14], we have

$$\frac{1}{|\sigma|} \int_\sigma w_n d\sigma = \frac{w_n(x) \Gamma(\frac{k}{2}) \mathcal{J}_{\frac{k}{2}-1}(t\sqrt{\lambda_n})}{(\frac{1}{2}t\sqrt{\lambda_n})^{\frac{k}{2}-1}}. \quad (2.3)$$

Lemma 2.2 ([6]). *Let $R > 1$ and $\rho \in [1, R]$. Then for the eigenfunctions of the Laplacian operator*

$$\sum_{|\lambda_n - R| \leq \rho} w_n^2(x) = O(\rho R^{k-1}) \quad (2.4)$$

uniformly with respect to x in any subdomain D' of D .

Lemma 2.3. *If (a_n) is the Fourier sequence of $f \in L^2(D)$, and R, ρ the same as in Lemma 2.2, then*

$$\sum_{|\lambda_n - R| \leq \rho} |a_n w_n(x)| = O(\sqrt{\rho} R^{\frac{k-1}{2}}) \quad (2.5)$$

uniformly with respect to x in any subdomain D' of D .

Lemma 2.3 is a corollary to Lemma 2.2. In order to prove it, it is sufficient to apply to the sum on the left-hand side of (2.5), first, the Cauchy–Bunjakovski inequality, then the estimation (2.4), and finally, the Bessel’s inequality

$$\sum_{n=1}^{\infty} f_n^2 \leq \int_D f^2(x) d\sigma.$$

Lemma 2.4 ([1]). *If $\varkappa \geq 0$, $\delta > 0$, and*

$$I_k(\lambda_n) = \int_{\delta}^{\infty} \frac{\mathcal{J}_{\frac{k}{2}+\varkappa}(\sqrt{\lambda_n}t) \mathcal{J}_{\frac{k}{2}-1}(\sqrt{\lambda_n}t)}{t^{\varkappa}} dt, \quad (2.6)$$

then, for all $\lambda, \lambda_n > 0$, we have

$$|I_k(\lambda_n)| < C \lambda_n^{-\frac{1}{4}} \lambda^{-\frac{1}{4}}, \quad (2.7)$$

where C is a constant.

Lemma 2.5. *For every $\alpha > 0$ the series*

$$\sum_{n=1}^{\infty} \frac{w_n^2(x)}{\lambda_n^{\frac{k}{2}+\alpha}} \quad (2.8)$$

converges uniformly in an arbitrary proper subdomain D' of D .

Finally, let us recall the Parseval’s formula

$$\int_D f(x)g(x)dx = \sum a_n b_n, \quad (2.9)$$

where

$$f \sim \sum a_n w_n, \quad f \in L^2, \quad (2.10)$$

$$g \sim \sum b_n w_n, \quad g \in L^2. \quad (2.11)$$

3. MAIN RESULTS

In this section we prove Theorem 1.1. We first construct the function which leads to the kernel of the $G_{\theta}^{\varkappa}$ –summability method, and then establish two lemmas.

In order to obtain the kernel of the $G_\theta^\varkappa$ -summability method as a factor in front of $a_n w_n(x)$, we chose the function $g(x)$ such that

$$g(x) = \begin{cases} 0, & \text{if } |x - x_0|^2 = t^2 > \delta^2, \\ \frac{(\lambda_n R)^{\frac{k}{2}} V_{\varkappa + \frac{k}{2}} \left(t e^{-\frac{\lambda_n - R}{4R^\theta}} \sqrt{\lambda_n R} \right) V_{\frac{k}{2} - 1} \left(t e^{\frac{\lambda_n - R}{4R^\theta}} \sqrt{\lambda_n R} \right)}{2^{-\varkappa} (\Gamma(1 + \varkappa))^{-1} \left(e^{\frac{\lambda_n - R}{4R^\theta}} \right)^k (2\pi)^{\frac{k}{2}} V_{\frac{k}{2} - 1}(t\sqrt{\lambda_n})}, \\ \text{if } |x - x_0|^2 = t^2 \leq \delta^2, \varkappa > -\frac{1}{2}, \end{cases}$$

where $|x - y|^2 = (x_1 - y_1)^2 + \dots + (x_k - y_k)^2$, $0 < \delta < \delta'$, δ' the minimal distance between x_0 and ∂D , and $V_\mu(x) = x^{-\mu} \mathcal{J}_\mu(x)$.

Then we have

$$\begin{aligned} \int_D f(x) g(x) dx &= \frac{2^\varkappa \Gamma(1 + \varkappa) (\lambda_n R)^{\frac{k}{2}}}{(2\pi)^{\frac{k}{2}} \left(e^{\frac{\lambda_n - R}{4R^\theta}} \right)^k} \\ &\cdot \int_{\Omega_\delta} \frac{V_{\varkappa + \frac{k}{2}} \left(t e^{-\frac{\lambda_n - R}{4R^\theta}} \sqrt{\lambda_n R} \right) V_{\frac{k}{2} - 1} \left(t e^{\frac{\lambda_n - R}{4R^\theta}} \sqrt{\lambda_n R} \right)}{V_{\frac{k}{2} - 1}(t\sqrt{\lambda_n})} f(x) dx. \end{aligned} \quad (3.1)$$

Instead of Cartesian coordinates $x_1, \dots, x_k$ we will use spherical ones

$$t, \theta_1, \dots, \theta_{k-1}, \quad 0 \leq \theta_m \leq \pi, \quad m = 1, \dots, k-2, \quad 0 \leq \theta_{k-1} \leq 2\pi.$$

Let us insert the substitution

$$x_1 = x_1^0 + \alpha_1 t, \dots, x_k = x_k^0 + \alpha_k t,$$

where

$$\alpha_1 = \sin \theta_1, \quad \alpha_2 = \sin \theta_1 \cos \theta_2, \quad \dots, \quad \alpha_k = \sin \theta_1 \sin \theta_2 \dots \sin \theta_{k-2} \cos \theta_{k-1}.$$

Volume element of a unit sphere has the form

$$d_1 \sigma = \sin^{k-2} \theta_1 \sin^{k-3} \theta_2 \dots \sin \theta_{k-2} d\theta_1 d\theta_2 \dots d\theta_{k-1},$$

and of a sphere with half diameter t the volume element is

$$d_t \sigma = t^{k-1} d_1 \sigma.$$

After these substitutions, we obtain

$$\begin{aligned}
 \int_D f(x)g(x)dx &= \frac{2^\varkappa \Gamma(1+\varkappa)(\lambda_n R)^{\frac{k}{2}}}{(2\pi)^{\frac{k}{2}} \left(e^{\frac{\lambda_n - R}{4R^\theta}}\right)^k} \\
 &\cdot \int_0^\delta t^{k-1} \frac{V_{\varkappa+\frac{k}{2}}\left(te^{-\frac{\lambda_n - R}{4R^\theta}}\sqrt{\lambda_n R}\right) V_{\frac{k}{2}-1}\left(te^{\frac{\lambda_n - R}{4R^\theta}}\sqrt{\lambda_n R}\right)}{V_{\frac{k}{2}-1}(t\sqrt{\lambda_n})} \cdot \frac{2\pi^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right)} f(x_0; t) dt \\
 &= \frac{2^{\varkappa-\frac{k}{2}+1} \Gamma(1+\varkappa)(\lambda_n R)^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right) \left(e^{\frac{\lambda_n - R}{4R^\theta}}\right)^k} \\
 &\cdot \int_0^\delta t^{k-1} \frac{V_{\varkappa+\frac{k}{2}}\left(te^{-\frac{\lambda_n - R}{4R^\theta}}\sqrt{\lambda_n R}\right) V_{\frac{k}{2}-1}\left(te^{\frac{\lambda_n - R}{4R^\theta}}\sqrt{\lambda_n R}\right)}{V_{\frac{k}{2}-1}(t\sqrt{\lambda_n})} f(x_0; t) dt. \quad (3.2)
 \end{aligned}$$

According to (2.1), (2.2) and (2.3), we have

$$\begin{aligned}
 b_n &= \int_D g(x)w_n(x)dx = \frac{2^\varkappa \Gamma(1+\varkappa)(\lambda_n R)^{\frac{k}{2}}}{(2\pi)^{\frac{k}{2}} \left(e^{\frac{\lambda_n - R}{4R^\theta}}\right)^k} \\
 &\cdot \int_{\Omega_\delta} \frac{V_{\varkappa+\frac{k}{2}}\left(te^{-\frac{\lambda_n - R}{4R^\theta}}\sqrt{\lambda_n R}\right) V_{\frac{k}{2}-1}\left(te^{\frac{\lambda_n - R}{4R^\theta}}\sqrt{\lambda_n R}\right)}{V_{\frac{k}{2}-1}(t\sqrt{\lambda_n})} w_n(x) dx \\
 &= \frac{2^{\varkappa-\frac{k}{2}+1} \Gamma(1+\varkappa)(\lambda_n R)^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right) \left(e^{\frac{\lambda_n - R}{4R^\theta}}\right)^k} \\
 &\cdot \int_0^\delta t^{k-1} \frac{V_{\varkappa+\frac{k}{2}}\left(te^{-\frac{\lambda_n - R}{4R^\theta}}\sqrt{\lambda_n R}\right) V_{\frac{k}{2}-1}\left(te^{\frac{\lambda_n - R}{4R^\theta}}\sqrt{\lambda_n R}\right)}{V_{\frac{k}{2}-1}(t\sqrt{\lambda_n})} \left(\frac{1}{|\sigma|} \int_\sigma w_n d\sigma\right) dt \\
 &= \frac{2^\varkappa \Gamma(1+\varkappa)(\lambda_n R)^{\frac{1-\varkappa}{2}}}{\left(e^{\frac{\lambda_n - R}{4R^\theta}}\right)^{k-\varkappa-1}} w_n(x_0) \cdot \\
 &\quad \cdot \int_0^\delta t^{-\varkappa} \mathcal{J}_{\varkappa+\frac{k}{2}}\left(te^{-\frac{\lambda_n - R}{4R^\theta}}\sqrt{\lambda_n R}\right) \mathcal{J}_{\frac{k}{2}-1}\left(te^{\frac{\lambda_n - R}{4R^\theta}}\sqrt{\lambda_n R}\right) dt \\
 &= \frac{2^\varkappa \Gamma(1+\varkappa)(\lambda_n R)^{\frac{1-\varkappa}{2}}}{\left(e^{\frac{\lambda_n - R}{4R^\theta}}\right)^{k-\varkappa-1}} w_n(x_0) \left(\int_0^\infty - \int_\delta^\infty\right)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{2^\varkappa \Gamma(1 + \varkappa) (\lambda_n R)^{\frac{1-\varkappa}{2}}}{\left(e^{\frac{\lambda_n - R}{4R^\theta}}\right)^{k-\varkappa-1}} w_n(x_0) \cdot \\
&\quad \cdot \left(\frac{\left(e^{\frac{\lambda_n - R}{4R^\theta}}\right)^{\frac{k}{2}-1} \left(e^{-\frac{\lambda_n - R}{2R^\theta}} - e^{\frac{\lambda_n - R}{2R^\theta}}\right)^\varkappa}{2^\varkappa \Gamma(1 + \varkappa) \left(e^{-\frac{\lambda_n - R}{4R^\theta}}\right)^{\frac{k}{2}+\varkappa}} (\lambda_n R)^{\frac{\varkappa-1}{2}} - \int_\delta^\infty \right) \\
&= \left(1 - e^{\frac{\lambda_n - R}{R^\theta}}\right)^\varkappa w_n(x_0) \delta_n^\lambda - \frac{2^\varkappa \Gamma(1 + \varkappa) (\lambda_n R)^{\frac{1-\varkappa}{2}}}{\left(e^{\frac{\lambda_n - R}{4R^\theta}}\right)^{k-\varkappa-1}} w_n(x_0) \cdot \\
&\quad \cdot \int_\delta^\infty t^{-\varkappa} \mathcal{J}_{\varkappa+\frac{k}{2}} \left(t e^{-\frac{\lambda_n - R}{4R^\theta}} \sqrt{\lambda_n R} \right) \mathcal{J}_{\frac{k}{2}-1} \left(t e^{\frac{\lambda_n - R}{4R^\theta}} \sqrt{\lambda_n R} \right) dt, \quad (3.3)
\end{aligned}$$

where

$$\delta_n^\lambda = \begin{cases} 1, & \lambda_n < R; \\ 0, & \lambda_n \geq R. \end{cases}$$

From (2.9), (3.2) and (3.3), we obtain

$$\begin{aligned}
&\sum_{\lambda_n \leq R} \left(1 - e^{\frac{\lambda_n - R}{R^\theta}}\right)^\varkappa a_n w_n(x_0) - \frac{2^{\varkappa-\frac{k}{2}+1} \Gamma(1 + \varkappa) (\lambda_n R)^{\frac{k}{2}}}{\Gamma\left(\frac{k}{2}\right) \left(e^{\frac{\lambda_n - R}{4R^\theta}}\right)^k} \cdot \\
&\quad \cdot \int_0^\delta t^{k-1} \frac{V_{\varkappa+\frac{k}{2}} \left(t e^{-\frac{\lambda_n - R}{4R^\theta}} \sqrt{\lambda_n R} \right) V_{\frac{k}{2}-1} \left(t e^{\frac{\lambda_n - R}{4R^\theta}} \sqrt{\lambda_n R} \right)}{V_{\frac{k}{2}-1}(t\sqrt{\lambda_n})} f(x_0; t) dt \\
&= \frac{2^\varkappa \Gamma(1 + \varkappa)}{R^{\frac{\varkappa-1}{2}}} \sum \frac{a_n w_n(x_0)}{\lambda_n^{\frac{\varkappa-1}{2}} \left(e^{\frac{\lambda_n - R}{4R^\theta}}\right)^{k-\varkappa-1}} I_k(\lambda_n R),
\end{aligned}$$

where

$$I_k(\lambda_n R) = \int_\delta^\infty t^{-\varkappa} \mathcal{J}_{\varkappa+\frac{k}{2}} \left(t e^{-\frac{\lambda_n - R}{4R^\theta}} \sqrt{\lambda_n R} \right) \mathcal{J}_{\frac{k}{2}-1} \left(t e^{\frac{\lambda_n - R}{4R^\theta}} \sqrt{\lambda_n R} \right) dt.$$

Lemma 3.1. *If $m \geq \lceil \frac{1}{R-1} \rceil$, then*

$$\frac{2^\varkappa \Gamma(1 + \varkappa)}{R^{\frac{\varkappa-1}{2}}} \sum_{|\lambda_n - R| \leq \frac{Rm}{m+1}} \frac{a_n w_n(x_0)}{\lambda_n^{\frac{\varkappa-1}{2}} \left(e^{\frac{\lambda_n - R}{4R^\theta}}\right)^{k-\varkappa-1}} I_k(\lambda_n R) = o(1), \text{ as } R \rightarrow \infty$$

if $\varkappa \geq \frac{k}{2}$. Here $[x]$ denotes the least integer number not less than x , i.e., $[x] = [x] + 1$.

Proof. Since $|\lambda_n - R| \leq \frac{Rm}{m+1}$, we have

$$\frac{1}{\lambda_n} = O\left(\frac{1}{R}\right) \quad \text{and} \quad \left(e^{\frac{\lambda_n - R}{4R^\theta}}\right)^{\varkappa+1-k} \leq \left(e^{-\frac{m}{4R^{\theta-1}(m+1)}}\right)^{|k-\varkappa-1|}.$$

According to (2.7), we have

$$I_k(\lambda_n R) = O(R^{-\frac{1}{2}} \lambda_n^{-\frac{1}{2}}).$$

The assertion now follows from Lemma 2.3. Indeed,

$$\sum_{|\lambda_n - R| \leq \frac{Rm}{m+1}} |a_n w_n(x_0)| = O\left(\sqrt{\frac{Rm}{m+1}} R^{\frac{k-1}{2}}\right),$$

hence

$$\begin{aligned} & \left| \frac{2^\varkappa \Gamma(1+\varkappa)}{R^{\frac{\varkappa-1}{2}}} \sum_{|\lambda_n - R| \leq \frac{Rm}{m+1}} \frac{a_n w_n(x_0)}{\lambda_n^{\frac{\varkappa-1}{2}} \left(e^{\frac{\lambda_n - R}{4R^\theta}}\right)^{k-\varkappa-1}} I_k(\lambda_n R) \right| \leq \\ & \leq \frac{C}{R^{\frac{\varkappa}{2}}} \left(e^{-\frac{m}{4R^{\theta-1}(m+1)}}\right)^{|k-\varkappa-1|} \sum_{|\lambda_n - R| \leq \frac{Rm}{m+1}} \frac{|a_n w_n(x_0)|}{\lambda_n^{\frac{\varkappa}{2}}} \\ & \leq \frac{C}{R^\varkappa} \left(e^{-\frac{m}{4R^{\theta-1}(m+1)}}\right)^{|k-\varkappa-1|} \sum_{|\lambda_n - R| \leq \frac{Rm}{m+1}} |a_n w_n(x_0)| \\ & \leq C \sqrt{\frac{m}{m+1}} R^{\frac{k}{2}-\varkappa} \left(e^{-\frac{m}{4R^{\theta-1}(m+1)}}\right)^{|k-\varkappa-1|}, \end{aligned}$$

where C is a constant. $\square$

Lemma 3.2. *If $m \geq 0$, then*

$$\frac{2^\varkappa \Gamma(1+\varkappa)}{R^{\frac{\varkappa-1}{2}}} \sum_{R+m < \lambda_n - R \leq R+m+1} \frac{a_n w_n(x_0)}{\lambda_n^{\frac{\varkappa-1}{2}} \left(e^{\frac{\lambda_n - R}{4R^\theta}}\right)^{k-\varkappa-1}} I_k(\lambda_n R) = o(1),$$

as $R \rightarrow \infty$ if $\varkappa \geq \frac{k+1}{4}$.

Proof. From $R+m < \lambda_n - R \leq R+m+1$, $m \geq 0$, we obtain

$$\frac{1}{\lambda_n} = O\left(\frac{1}{R}\right) \quad \text{and} \quad \left(e^{\frac{\lambda_n - R}{4R^\theta}}\right)^{\varkappa+1-k} \leq \left(e^{-\frac{R+m+1}{4R^\theta}}\right)^{|k-\varkappa-1|}.$$

Now we proceed analogously as in the proof of the previous lemma, except after exploiting (2.7) in order to estimate $I_k(\lambda_n R)$, instead of Lemma 2.3,

we use Lemma 2.5. Namely,

$$\sum_{R+m < \lambda_n - R \leq R+m+1} \frac{|a_n w_n(x_0)|}{\lambda_n^{\frac{k}{4} + \frac{1}{4}}} < \infty \text{ as } R \rightarrow \infty,$$

hence

$$\begin{aligned} & \left| \frac{2^\varkappa \Gamma(1 + \varkappa)}{R^{\frac{\varkappa-1}{2}}} \sum_{R+m < \lambda_n - R \leq R+m+1} \frac{a_n w_n(x_0)}{\lambda_n^{\frac{\varkappa-1}{2}} \left(e^{\frac{\lambda_n - R}{4R^\theta}} \right)^{k-\varkappa-1}} I_k(\lambda_n R) \right| \leq \\ & \leq \frac{C}{R^{\frac{\varkappa}{2}}} \left(e^{-\frac{R+m+1}{4R^\theta}} \right)^{|k-\varkappa-1|} \sum_{R+m < \lambda_n - R \leq R+m+1} \frac{|a_n w_n(x_0)|}{\lambda_n^{\frac{\varkappa}{2}}} \\ & \leq \frac{C}{R^{\varkappa - \frac{k+1}{4}}} \left(e^{-\frac{R+m+1}{4R^\theta}} \right)^{|k-\varkappa-1|} \sum_{R+m < \lambda_n - R \leq R+m+1} \frac{|a_n w_n(x_0)|}{\lambda_n^{\frac{k}{4} + \frac{1}{4}}} = o(1), \end{aligned}$$

as $R \rightarrow \infty$ for $\varkappa \geq \frac{k+1}{4}$. $\square$

Proof of Theorem 1.1. It is enough to prove

$$\frac{2^\varkappa \Gamma(1 + \varkappa)}{R^{\frac{\varkappa-1}{2}}} \sum_{n=1}^{\infty} \frac{a_n w_n(x_0)}{\lambda_n^{\frac{\varkappa-1}{2}} \left(e^{\frac{\lambda_n - R}{4R^\theta}} \right)^{k-\varkappa-1}} I_k(\lambda_n R) = o(1), \text{ as } R \rightarrow \infty \quad (3.4)$$

if $\varkappa \geq \frac{k}{2}$. We may assume that $R \geq 2$. Then $\lceil \frac{1}{R-1} \rceil = 1$. Let us rewrite (3.4) as

$$\begin{aligned} & \frac{2^\varkappa \Gamma(1 + \varkappa)}{R^{\frac{\varkappa-1}{2}}} \sum_{n=1}^{\infty} \frac{a_n w_n(x_0)}{\lambda_n^{\frac{\varkappa-1}{2}} \left(e^{\frac{\lambda_n - R}{4R^\theta}} \right)^{k-\varkappa-1}} I_k(\lambda_n R) \\ & = \frac{2^\varkappa \Gamma(1 + \varkappa)}{R^{\frac{\varkappa-1}{2}}} \left(\sum_{m=1}^{\infty} \sum_{|\lambda_n - R| \leq \frac{Rm}{m+1}} + \sum_{m=0}^{\infty} \sum_{R+m \leq \lambda_n - R \leq R+m+1} \right). \quad (3.5) \end{aligned}$$

Since the series with respect to m on the right-hand side in (3.5) converges uniformly with respect to R , and the assertion follows from Lemma 3.1 and Lemma 3.2, i.e., $G_\theta^\varkappa$ -summability of the series (1.1) of $f \in L^2(D)$ at $x_0 \in D$ depends only on the behaviour of f in the neighborhood of x_0 if $\varkappa \geq \frac{k}{2}$. $\square$

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Mirjana Vuković
Academy of Sciences and Arts of
Bosnia and Herzegovina
Bistrik 7, 71000 Sarajevo
Bosnia and Herzegovina
mvukovic@anubih.ba

Emil Ilić-Georgijević
Faculty of Civil Engineering
University of Sarajevo
Patriotske lige 30, 71000 Sarajevo
Bosnia and Herzegovina
emil.ilic.georgijevic@gmail.com

Olivera Stevanović
Synergy University
Raje Baničića bb, 76300 Bijeljina
Bosnia and Herzegovina
ol.stevanovic@gmail.com