GENERAL OPIAL TYPE INEQUALITY FOR QUOTIENT OF FUNCTIONS

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ABSTRACT. In this paper we give the general Opial type inequality for quotient of functions. Our result contains two functions, convex and concave functions. We prove a new general inequality on a measure space (Ω, Σ, μ) . We apply our result to numerous symmetric functions and obtain new results that involve Green's functions, Lidstone series and the Hermite's interpolating polynomials.

1. INTRODUCTION

In 1960, Opial [5] proved the following inequality:

Let $f \in C^1[0, h]$ be such that $f(0) = f(h) = 0$ and $f(x) > 0$ for $x \in (0, h)$. Then

$$\int_0^h |f(x)f'(x)|dx \leq \frac{h}{4} \int_0^h [f'(x)]^2 dx, \quad (1.1)$$

where $h/4$ is the best possible.

This inequality has been generalized and extended in several directions (for more details see e.g [1], [3]).

In the sequel let (Ω, Σ, μ) be a measure space and let $k : \Omega \times \Omega \rightarrow \mathbb{R}$ be a symmetric nonnegative or nonpositive function such that

$$K(x) := \int_{\Omega} k(x, y) d\mu(y), \quad K(x) \neq 0, \quad a.e. x \in \Omega, \quad (1.2)$$

such that $|K(x)| < \infty$. In the rest of the paper we assume that all integrals are well defined. The following result is given in [4].

Theorem 1.1. *Let $k : \Omega \times \Omega \rightarrow \mathbb{R}$ be a symmetric nonnegative or nonpositive function. If f is a positive convex function, g a positive concave*

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function on an interval $I \subseteq \mathbb{R}$, $v : \Omega \rightarrow \mathbb{R}$ is either nonnegative or non-positive, such that $\text{Im}|v| \subseteq I$ and u defined by

$$u(x) := \int_{\Omega} k(x, y)v(y)d\mu(y) < \infty, \quad (1.3)$$

then the following inequality

$$\int_{\Omega} |K(x)|f\left(\left|\frac{u(x)}{K(x)}\right|\right)g(|v(x)|)d\mu(x) \leq \int_{\Omega} |K(x)|f(|v(x)|)g\left(\left|\frac{u(x)}{K(x)}\right|\right)d\mu(x) \quad (1.4)$$

holds, where K is defined by (1.2).

The obtained inequalities are not direct generalizations of the Opial inequality but are of Opial type because the integrals contain function and its integral representation. In this paper we give a generalization of inequality (1.4).

2. THE MAIN RESULTS

Our first result is given in the next theorem.

Theorem 2.1. *Let $k : \Omega \times \Omega \rightarrow \mathbb{R}$ be a symmetric non-negative or non-positive function and K be defined by (1.2). If f is a positive convex function, g a positive concave function on an interval $I \subseteq \mathbb{R}$, $v, n : \Omega \rightarrow \mathbb{R}$ are either non-negative or non-positive, such that $\text{Im}|v|, \text{Im}|\frac{v}{n}| \subseteq I$, u defined by*

$$u(x) := \int_{\Omega} k(x, y)v(y)d\mu(y) < \infty, \quad (2.1)$$

and m defined by

$$m(x) := \int_{\Omega} k(x, y)n(y)d\mu(y) < \infty, \quad (2.2)$$

then the following inequality

$$\begin{aligned} & \int_{\Omega} |m(x)|f\left(\left|\frac{u(x)}{m(x)}\right|\right)g(|v(x)|)d\mu(x) \\ & \leq \int_{\Omega} |n(x)||K(x)|f\left(\left|\frac{v(x)}{n(x)}\right|\right)g\left(\left|\frac{u(x)}{K(x)}\right|\right)d\mu(x) \end{aligned} \quad (2.3)$$

holds, where K is defined by (1.2).

Proof. By using Jensen's inequality and the Fubini theorem we find that

$$\begin{aligned}
 & \int_{\Omega} |m(x)| f\left(\left|\frac{u(x)}{m(x)}\right|\right) g(|v(x)|) d\mu(x) \\
 &= \int_{\Omega} |m(x)| f\left(\frac{|\int_{\Omega} k(x, y) v(y) d\mu(y)|}{|m(x)|}\right) g(|v(x)|) d\mu(x) \\
 &\leq \int_{\Omega} |m(x)| f\left(\int_{\Omega} \frac{|k(x, y) n(y)|}{|m(x)|} \frac{|v(y)|}{|n(y)|} d\mu(y)\right) g(|v(x)|) d\mu(x) \\
 &\leq \int_{\Omega} \left(\int_{\Omega} |k(x, y) n(y)| f\left(\frac{|v(y)|}{|n(y)|}\right) d\mu(y)\right) g(|v(x)|) d\mu(x) \\
 &= \int_{\Omega} f\left(\frac{|v(y)|}{|n(y)|}\right) |n(y)| \left(\int_{\Omega} |k(x, y)| g(|v(x)|) d\mu(x)\right) d\mu(y).
 \end{aligned}$$

Since k is a symmetric function we get that $|k(x, y)| = |k(y, x)|$ so by using Jensen's inequality we obtain that

$$\begin{aligned}
 & \int_{\Omega} f\left(\frac{|v(y)|}{|n(y)|}\right) |n(y)| \left(\int_{\Omega} |k(y, x)| g(|v(x)|) d\mu(x)\right) d\mu(y) \\
 &\leq \int_{\Omega} |n(y)| f\left(\frac{|v(y)|}{|n(y)|}\right) |K(y)| g\left(\frac{1}{|K(y)|} \int_{\Omega} |k(y, x) v(x)| d\mu(x)\right) d\mu(y) \\
 &= \int_{\Omega} |n(y)| f\left(\frac{|v(y)|}{|n(y)|}\right) |K(y)| g\left(\frac{|u(y)|}{|K(y)|}\right) d\mu(y)
 \end{aligned}$$

and the proof is complete. $\square$

Remark 2.1. If $n(y) = 1$, then $m(x) = K(x)$ and inequality (2.4) reduces to (1.4).

We apply our result to numerous symmetric functions and obtain new results that involve Green's functions, Lidstone series and the Hermite's interpolating polynomials.

Consider the Green function G defined on $[a, b] \times [a, b]$ by

$$G(t, s) = \begin{cases} \frac{(t-b)(s-a)}{b-a}, & a \leq s \leq t; \\ \frac{(t-a)(s-b)}{b-a}, & t \leq s \leq b. \end{cases} \quad (2.4)$$

The function G is convex under s , it is symmetric non-positive function and it is also convex under t . It is continuous under s and continuous under t . For any function $\varphi : [a, b] \rightarrow \mathbb{R}$, $\varphi \in C^2([a, b])$, we can easily show by integrating by parts that the following is valid

$$\varphi(x) = \frac{b-x}{b-a} \varphi(a) + \frac{x-a}{b-a} \varphi(b) + \int_a^b G(x, s) \varphi''(s) ds, \quad (2.5)$$

where the function G is defined as above in (2.4). Since function G is a non-positive symmetric function we can apply Theorem 2.1 and obtain the following corollary.

Corollary 2.1. *If f is a positive convex function and g a positive concave function on an interval $I \subseteq \mathbb{R}$, then the inequality*

$$\begin{aligned} & \int_a^b \left| m(x) - \frac{b-x}{b-a}m(a) - \frac{x-a}{b-a}m(b) \right| \\ & \quad \times f \left(\frac{|\varphi(x) - \frac{b-x}{b-a}\varphi(a) - \frac{x-a}{b-a}\varphi(b)|}{|m(x) - \frac{b-x}{b-a}m(a) - \frac{x-a}{b-a}m(b)|} \right) g(|\varphi''(x)|) dx \\ & \leq \frac{1}{2} \int_a^b |m''(x)|(b-x)(x-a) f \left(\frac{|\varphi''(x)|}{|m''(x)|} \right) \\ & \quad \times g \left(\frac{2|\varphi(x) - \frac{b-x}{b-a}\varphi(a) - \frac{x-a}{b-a}\varphi(b)|}{(b-x)(x-a)} \right) dx \end{aligned} \quad (2.6)$$

holds for all functions $\varphi : [a, b] \rightarrow \mathbb{R}$, $\varphi \in C^2([a, b])$ such that φ'' is either non-negative or non-positive function and $m : [a, b] \rightarrow \mathbb{R}$ such that m'' is either non-negative or non-positive function.

Proof. Since function G defined by (2.4) is a non-positive symmetric function we can apply Theorem 2.1. Let $\Omega = [a, b]$, $k(x, s) = G(x, s)$, $v(s) = \varphi''(s)$, $n(x) = m''(x)$. Then

$$u(x) = \int_a^b G(x, s)\varphi''(s)ds = \varphi(x) - \frac{b-x}{b-a}\varphi(a) - \frac{x-a}{b-a}\varphi(b),$$

$$|K(x)| = \int_a^b |G(x, s)|ds = \frac{(b-x)(x-a)}{2},$$

$$m(x) = \int_a^b G(x, s)m''(s)ds = m(x) - \frac{b-x}{b-a}m(a) - \frac{x-a}{b-a}m(b)$$

and inequality (2.3) becomes

$$\begin{aligned} & \int_a^b \left| m(x) - \frac{b-x}{b-a}m(a) - \frac{x-a}{b-a}m(b) \right| \\ & \quad \times f \left(\frac{|\varphi(x) - \frac{b-x}{b-a}\varphi(a) - \frac{x-a}{b-a}\varphi(b)|}{|m(x) - \frac{b-x}{b-a}m(a) - \frac{x-a}{b-a}m(b)|} \right) g(|\varphi''(x)|) dx \\ & \leq \frac{1}{2} \int_a^b |m''(x)|(b-x)(x-a) f \left(\frac{|\varphi''(x)|}{|m''(x)|} \right) \end{aligned}$$

$$\times g \left(\frac{2|\varphi(x) - \frac{b-x}{b-a}\varphi(a) - \frac{x-a}{b-a}\varphi(b)|}{(b-x)(x-a)} \right) dx \quad (2.7)$$

so the proof is complete. $\square$

Remark 2.2. If $\varphi(a) = \varphi(b) = 0$, then (2.7) reduces to

$$\begin{aligned} & \int_a^b \left| m(x) - \frac{b-x}{b-a}m(a) - \frac{x-a}{b-a}m(b) \right| \\ & \quad \times f \left(\frac{|\varphi(x)|}{|m(x) - \frac{b-x}{b-a}m(a) - \frac{x-a}{b-a}m(b)|} \right) g(|\varphi''(x)|) dx \\ & \leq \frac{1}{2} \int_a^b |m''(x)|(b-x)(x-a) f \left(\frac{|\varphi''(x)|}{|m''(x)|} \right) g \left(\frac{2|\varphi(x)|}{(b-x)(x-a)} \right) dx \end{aligned}$$

Remark 2.3. If $m''(x) = 1$, then $m(x) = K(x)$ and (2.7) reduces to

$$\begin{aligned} & \int_a^b (b-x)(x-a) f \left(\frac{2|\varphi(x) - \frac{b-x}{b-a}\varphi(a) - \frac{x-a}{b-a}\varphi(b)|}{(b-x)(x-a)} \right) g(|\varphi''(x)|) dx \\ & \leq \int_a^b (b-x)(x-a) f(|\varphi''(x)|) g \left(\frac{2|\varphi(x) - \frac{b-x}{b-a}\varphi(a) - \frac{x-a}{b-a}\varphi(b)|}{(b-x)(x-a)} \right) dx \end{aligned} \quad (2.8)$$

that is given in [4].

We continue with the definition of Lidstone series. Lidstone series is a generalization of Taylor's series. It approximates to a given function in the neighborhood of two points (instead of one). Such series have been studied by G. J. Lidstone (1929), H. Poritsky (1932), J. M. Wittaker (1934) and others.

Definition 2.1. Let $\varphi \in C^\infty([0, 1])$, then Lidstone series has the form

$$\sum_{k=0}^{\infty} \left(\varphi^{(2k)}(0)\Lambda_k(1-x) + \varphi^{(2k)}(1)\Lambda_k(x) \right),$$

where Λ_n is a polynomial of degree $2n+1$ defined by the relations

$$\begin{aligned} \Lambda_0(t) &= t, \\ \Lambda_n''(t) &= \Lambda_{n-1}(t), \\ \Lambda_n(0) &= \Lambda_n(1) = 0, \quad n \geq 1. \end{aligned} \quad (2.9)$$

Another explicit representations of Lidstone polynomial are given by [2] and [6],

$$\begin{aligned}\Lambda_n(t) &= (-1)^n \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2n+1}} \sin k\pi t, \\ \Lambda_n(t) &= \frac{1}{6} \left[\frac{6t^{2n+1}}{(2n+1)!} - \frac{t^{2n-1}}{(2n-1)!} \right] \\ &\quad - \sum_{k=0}^{n-2} \frac{2(2^{2k+3}-1)}{(2k+4)!} B_{2k+4} \frac{t^{2n-2k-3}}{(2n-2k-3)!}, \quad n = 1, 2, \dots, \\ \Lambda_n(t) &= \frac{2^{2n+1}}{(2n+1)!} B_{2n+1} \left(\frac{1+t}{2} \right), \quad n = 1, 2, \dots,\end{aligned}$$

where B_{2k+4} is the $(2k+4)$ -th Bernoulli number and $B_{2n+1} \left(\frac{1+t}{2} \right)$ is a Bernoulli polynomial.

In [7], Widder proved the fundamental lemma:

Lemma 2.1. *If $\varphi \in C^{(2n)}([0, 1])$, then*

$$\varphi(t) = \sum_{k=0}^{n-1} \left[\varphi^{(2k)}(0) \Lambda_k(1-t) + \varphi^{(2k)}(1) \Lambda_k(t) \right] + \int_0^1 G_n(t, s) \varphi^{(2n)}(s) ds, \quad (2.10)$$

where

$$G_1(t, s) = G(t, s) = \begin{cases} (t-1)s, & \text{if } s \leq t, \\ (s-1)t, & \text{if } t \leq s, \end{cases} \quad (2.11)$$

is the homogeneous Green's function of the differential operator $\frac{d^2}{ds^2}$ on $[0, 1]$, and with the successive iterates of $G(t, s)$

$$G_n(t, s) = \int_0^1 G_1(t, p) G_{n-1}(p, s) dp, \quad n \geq 2. \quad (2.12)$$

Lidstone polynomial can be expressed, in terms of $G_n(t, s)$ as

$$\Lambda_n(t) = \int_0^1 G_n(t, s) s ds.$$

Notice that $G_n(t, s)$ is a symmetric function. Now we give the following special case of Theorem 1.1.

Corollary 2.2. *Let $\varphi : [a, b] \rightarrow \mathbb{R}$ be of $C^{(2n)}$ on $[a, b]$, $n \geq 1$ such that $\varphi^{(2n)}$ is either non-negative or non-positive function and $m : [a, b] \rightarrow \mathbb{R}$ be of $C^{(2n)}$ on $[a, b]$, $n \geq 1$ such that $m^{(2n)}$ is either non-negative or non-positive*

function. If f is a positive convex function and g a positive concave function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\begin{aligned}
 & \int_a^b \left| m(x) - \sum_{k=0}^{n-1} (b-a)^{2k} \left[m^{(2k)}(a) \Lambda_k \left(\frac{b-x}{b-a} \right) + m^{(2k)}(b) \Lambda_k \left(\frac{x-a}{b-a} \right) \right] \right| \\
 & \times f \left(\frac{|\varphi(x) - \sum_{k=0}^{n-1} (b-a)^{2k} \left[\varphi^{(2k)}(a) \Lambda_k \left(\frac{b-x}{b-a} \right) + \varphi^{(2k)}(b) \Lambda_k \left(\frac{x-a}{b-a} \right) \right]|}{|m(x) - \sum_{k=0}^{n-1} (b-a)^{2k} \left[m^{(2k)}(a) \Lambda_k \left(\frac{b-x}{b-a} \right) + m^{(2k)}(b) \Lambda_k \left(\frac{x-a}{b-a} \right) \right]|} \right) \\
 & \quad \times g(|\varphi^{(2n)}(x)|) dx \\
 & \leq \int_a^b E_{2n}(x) |m^{(2n)}(x)| \\
 & \times g \left(\frac{|\varphi(x) - \sum_{k=0}^{n-1} (b-a)^{2k} \left[\varphi^{(2k)}(a) \Lambda_k \left(\frac{b-x}{b-a} \right) + \varphi^{(2k)}(b) \Lambda_k \left(\frac{x-a}{b-a} \right) \right]|}{E_{2n}(x)} \right) \\
 & \quad \times f \left(\left| \frac{\varphi^{(2n)}(x)}{m^{(2n)}(x)} \right| \right) dx \tag{2.13}
 \end{aligned}$$

holds, where E_{2n} is Euler polynomial.

Proof. By Widder's lemma we can represent every function $\varphi \in C^{(2n)}([a, b])$ in the form:

$$\begin{aligned}
 \varphi(x) &= \sum_{k=0}^{n-1} (b-a)^{2k} \left[\varphi^{(2k)}(a) \Lambda_k \left(\frac{b-x}{b-a} \right) + \varphi^{(2k)}(b) \Lambda_k \left(\frac{x-a}{b-a} \right) \right] \\
 &+ (b-a)^{2n-1} \int_a^b G_n \left(\frac{x-a}{b-a}, \frac{s-a}{b-a} \right) \varphi^{(2n)}(s) ds. \tag{2.14}
 \end{aligned}$$

Let

$$\begin{aligned}
 & \varphi(x) - \sum_{k=0}^{n-1} (b-a)^{2k} \left[\varphi^{(2k)}(a) \Lambda_k \left(\frac{b-x}{b-a} \right) + \varphi^{(2k)}(b) \Lambda_k \left(\frac{x-a}{b-a} \right) \right] \\
 &= (b-a)^{2n-1} \int_a^b G_n \left(\frac{x-a}{b-a}, \frac{s-a}{b-a} \right) \varphi^{(2n)}(s) ds, \tag{2.15}
 \end{aligned}$$

Now apply Theorem 2.1 with $k(x, s) = (b-a)^{2n-1} G_n \left(\frac{x-a}{b-a}, \frac{s-a}{b-a} \right)$, $v(s) = \varphi^{(2n)}(s)$. Then

$$K(x) = (b-a)^{2n-1} \int_a^b G_n \left(\frac{x-a}{b-a}, \frac{s-a}{b-a} \right) ds = E_{2n}(x),$$

where E_{2n} is Euler polynomial.

$$\begin{aligned}
 u(x) &= (b-a)^{2n-1} \int_a^b G_n \left(\frac{x-a}{b-a}, \frac{s-a}{b-a} \right) \varphi^{(2n)}(s) ds \\
 &= \varphi(x) - \sum_{k=0}^{n-1} (b-a)^{2k} \left[\varphi^{(2k)}(a) \Lambda_k \left(\frac{b-x}{b-a} \right) + \varphi^{(2k)}(b) \Lambda_k \left(\frac{x-a}{b-a} \right) \right] \\
 m(x) &= (b-a)^{2n-1} \int_a^b G_n \left(\frac{x-a}{b-a}, \frac{s-a}{b-a} \right) m^{(2n)}(s) ds \\
 &= m(x) - \sum_{k=0}^{n-1} (b-a)^{2k} \left[m^{(2k)}(a) \Lambda_k \left(\frac{b-x}{b-a} \right) + m^{(2k)}(b) \Lambda_k \left(\frac{x-a}{b-a} \right) \right]
 \end{aligned}$$

and inequality (2.3) becomes (2.13) and the proof is complete. $\square$

We continue with the following special case of inequality (2.13).

Remark 2.4. If $\varphi^{(2k)}(a) = \varphi^{(2k)}(b) = 0$, then (2.13) reduces to

$$\begin{aligned}
 & \int_a^b \left| m(x) - \sum_{k=0}^{n-1} (b-a)^{2k} \left[m^{(2k)}(a) \Lambda_k \left(\frac{b-x}{b-a} \right) + m^{(2k)}(b) \Lambda_k \left(\frac{x-a}{b-a} \right) \right] \right| \\
 & \times f \left(\frac{|\varphi(x)|}{\left| m(x) - \sum_{k=0}^{n-1} (b-a)^{2k} \left[m^{(2k)}(a) \Lambda_k \left(\frac{b-x}{b-a} \right) + m^{(2k)}(b) \Lambda_k \left(\frac{x-a}{b-a} \right) \right] \right|} \right) \\
 & \times g(|\varphi^{(2n)}(x)|) dx \leq \int_a^b E_{2n}(x) |m^{(2n)}(x)| g \left(\frac{|\varphi(x)|}{E_{2n}(x)} \right) f \left(\left| \frac{\varphi^{(2n)}(x)}{m^{(2n)}(x)} \right| \right) dx.
 \end{aligned}$$

Remark 2.5. If $n(x) = 1$, then $m(x) = K(x)$ and (2.13) reduces to

$$\begin{aligned}
 & \int_a^b E_{2n}(x) \\
 & \times f \left(\frac{|\varphi(x) - \sum_{k=0}^{n-1} (b-a)^{2k} \left[\varphi^{(2k)}(a) \Lambda_k \left(\frac{b-x}{b-a} \right) + \varphi^{(2k)}(b) \Lambda_k \left(\frac{x-a}{b-a} \right) \right]|}{E_{2n}(x)} \right) \\
 & \times g(|\varphi^{(2n)}(x)|) dx \leq \int_a^b E_{2n}(x) \\
 & \times g \left(\frac{|\varphi(x) - \sum_{k=0}^{n-1} (b-a)^{2k} \left[\varphi^{(2k)}(a) \Lambda_k \left(\frac{b-x}{b-a} \right) + \varphi^{(2k)}(b) \Lambda_k \left(\frac{x-a}{b-a} \right) \right]|}{E_{2n}(x)} \right) \\
 & \times f(|\varphi^{(2n)}(x)|) dx
 \end{aligned} \tag{2.16}$$

that is given in [4].

We continue with the Hermite interpolating polynomial for the two-point Taylor conditions see [2].

Lemma 2.2. *Let $\varphi \in C^n[a, b]$, ($n \geq 2$, $n = 2\mathbf{m}$) then holds*

$$\begin{aligned} \varphi(x) = \sum_{i=0}^{\mathbf{m}-1} \sum_{k=0}^{\mathbf{m}-1-i} \binom{\mathbf{m}+k-1}{k} \cdot [\varphi^{(i)}(a)\tau_i(x) + \varphi^{(i)}(b)\nu_i(x)] \\ + \int_a^b \varphi^{(2\mathbf{m})}(s)G_{2T}(x, s) ds \end{aligned} \quad (2.17)$$

where τ_i and ν_i are defined on $[a, b]$:

$$\tau_i(x) = \frac{(x-a)^i}{i!} \left(\frac{x-b}{a-b} \right)^{\mathbf{m}} \left(\frac{x-a}{b-a} \right)^k, \quad (2.18)$$

$$\nu_i(x) = \frac{(x-b)^i}{i!} \left(\frac{x-a}{b-a} \right)^{\mathbf{m}} \left(\frac{x-b}{a-b} \right)^k \quad (2.19)$$

G_{2T} is the Green's function of the two-point Taylor problem:

$$G_{2T}(t, s) = \begin{cases} \frac{(-1)^{\mathbf{m}}}{(2\mathbf{m}-1)!} p^{\mathbf{m}}(t, s) \sum_{j=0}^{\mathbf{m}-1} \binom{\mathbf{m}-1+j}{j} (t-s)^{\mathbf{m}-1-j} q^j(t, s), & s \leq t \\ \frac{(-1)^{\mathbf{m}}}{(2\mathbf{m}-1)!} q^{\mathbf{m}}(t, s) \sum_{j=0}^{\mathbf{m}-1} \binom{\mathbf{m}-1+j}{j} (s-t)^{\mathbf{m}-1-j} p^j(t, s), & s \geq t \end{cases} \quad (2.20)$$

and

$$p(t, s) = \frac{(s-a)(b-t)}{b-a}, \quad q(t, s) = p(s, t), \quad \forall t, s \in [a, b].$$

Now, we give a special case of Theorem 1.1 that involves Hermite interpolating polynomial for two-point Taylor conditions.

Corollary 2.3. *Let $\varphi : [a, b] \rightarrow \mathbb{R}$ be of $C^{(2\mathbf{m})}$ on $[a, b]$ such that $\varphi^{(2\mathbf{m})}$ is either non-negative or non-positive function. Let $m : [a, b] \rightarrow \mathbb{R}$ be of $C^{(2\mathbf{m})}$ on $[a, b]$ such that $m^{(2\mathbf{m})}$ is either non-negative or non-positive function. If f is a positive convex function and g a positive concave function on an interval $I \subseteq \mathbb{R}$, then the inequality*

$$\begin{aligned} \int_a^b \left| m(x) - \sum_{i=0}^{\mathbf{m}-1} \sum_{k=0}^{\mathbf{m}-1-i} \binom{\mathbf{m}+k-1}{k} [m^{(i)}(a)\tau_i(x) + m^{(i)}(b)\nu_i(x)] \right| \\ \times f \left(\frac{|\varphi(x) - \sum_{i=0}^{\mathbf{m}-1} \sum_{k=0}^{\mathbf{m}-1-i} \binom{\mathbf{m}+k-1}{k} [\varphi^{(i)}(a)\tau_i(x) + \varphi^{(i)}(b)\nu_i(x)]|}{|m(x) - \sum_{i=0}^{\mathbf{m}-1} \sum_{k=0}^{\mathbf{m}-1-i} \binom{\mathbf{m}+k-1}{k} [m^{(i)}(a)\tau_i(x) + m^{(i)}(b)\nu_i(x)]|} \right) \\ \times g(|\varphi^{(2\mathbf{m})}(x)|) dx \leq \int_a^b \frac{1}{(2\mathbf{m})!} (x-a)^{\mathbf{m}} (b-x)^{\mathbf{m}} |m^{(2\mathbf{m})}(x)| f \left(\left| \frac{\varphi^{(2\mathbf{m})}(x)}{m^{(2\mathbf{m})}(x)} \right| \right) \end{aligned}$$

$$\times g \left(\frac{(2\mathbf{m})! |\varphi(x) - \sum_{i=0}^{\mathbf{m}-1} \sum_{k=0}^{\mathbf{m}-1-i} \binom{\mathbf{m}+k-1}{k} [\varphi^{(i)}(a)\tau_i(x) + \varphi^{(i)}(b)\nu_i(x)]|}{(x-a)^{\mathbf{m}}(b-x)^{\mathbf{m}}} \right) dx \quad (2.21)$$

holds.

Proof. Since $G_{2T}(t, s)$ is a symmetric function, we define

$$\begin{aligned} \varphi(x) - \sum_{i=0}^{\mathbf{m}-1} \sum_{k=0}^{\mathbf{m}-1-i} \binom{\mathbf{m}+k-1}{k} \cdot [\varphi^{(i)}(a)\tau_i(x) + \varphi^{(i)}(b)\nu_i(x)] \\ = \int_a^b \varphi^{(2\mathbf{m})}(s) G_{2T}(x, s) ds. \end{aligned} \quad (2.22)$$

Now apply Theorem 2.1 with $k(x, s) = G_{2T}(x, s)$, $v(s) = \varphi^{(2\mathbf{m})}(s)$, $n(x) = m^{(2\mathbf{m})}(x)$. We obtain

$$|K(x)| = \int_a^b |G_{2T}(x, s)| ds = \frac{1}{(2\mathbf{m})!} |(x-a)^{\mathbf{m}}(x-b)^{\mathbf{m}}| = \frac{1}{(2\mathbf{m})!} (x-a)^{\mathbf{m}}(b-x)^{\mathbf{m}},$$

$$\begin{aligned} u(x) &= \int_a^b G_{2T}(x, s) \varphi^{(2\mathbf{m})}(s) ds \\ &= \varphi(x) - \sum_{i=0}^{\mathbf{m}-1} \sum_{k=0}^{\mathbf{m}-1-i} \binom{\mathbf{m}+k-1}{k} \cdot [\varphi^{(i)}(a)\tau_i(x) + \varphi^{(i)}(b)\nu_i(x)] \end{aligned}$$

and inequality (1.4) becomes (2.21), so the proof is complete. $\square$

Remark 2.6. If $\varphi^{(i)}(a) = \varphi^{(i)}(b) = 0$, then inequality (2.21) reduces to

$$\begin{aligned} & \int_a^b \left| m(x) - \sum_{i=0}^{\mathbf{m}-1} \sum_{k=0}^{\mathbf{m}-1-i} \binom{\mathbf{m}+k-1}{k} [m^{(i)}(a)\tau_i(x) + m^{(i)}(b)\nu_i(x)] \right| \\ & \times f \left(\frac{|\varphi(x)|}{|m(x) - \sum_{i=0}^{\mathbf{m}-1} \sum_{k=0}^{\mathbf{m}-1-i} \binom{\mathbf{m}+k-1}{k} [m^{(i)}(a)\tau_i(x) + m^{(i)}(b)\nu_i(x)]|} \right) \\ & \times g(|\varphi^{(2\mathbf{m})}(x)|) dx \leq \int_a^b \frac{1}{(2\mathbf{m})!} (x-a)^{\mathbf{m}}(b-x)^{\mathbf{m}} |m^{(2\mathbf{m})}(x)| \\ & \times f \left(\left| \frac{\varphi^{(2\mathbf{m})}(x)}{m^{(2\mathbf{m})}(x)} \right| \right) g \left(\frac{(2\mathbf{m})! |\varphi(x)|}{(x-a)^{\mathbf{m}}(b-x)^{\mathbf{m}}} \right) dx. \end{aligned} \quad (2.23)$$

Remark 2.7. If $n(x) = 1$, then $m(x) = K(x)$ and (2.21) reduces to

$$\begin{aligned}
 & \int_a^b (x-a)^{\mathbf{m}}(b-x)^{\mathbf{m}} \\
 & \times f\left(\frac{(2\mathbf{m})!|\varphi(x) - \sum_{i=0}^{\mathbf{m}-1} \sum_{k=0}^{\mathbf{m}-1-i} \binom{\mathbf{m}+k-1}{k} [\varphi^{(i)}(a)\tau_i(x) + \varphi^{(i)}(b)\nu_i(x)]|}{(x-a)^{\mathbf{m}}(b-x)^{\mathbf{m}}}\right) \\
 & \times g(|\varphi^{(2\mathbf{m})}(x)|)dx \\
 & \leq \int_a^b (x-a)^{\mathbf{m}}(b-x)^{\mathbf{m}} \\
 & \times g\left(\frac{(2\mathbf{m})!|\varphi(x) - \sum_{i=0}^{\mathbf{m}-1} \sum_{k=0}^{\mathbf{m}-1-i} \binom{\mathbf{m}+k-1}{k} [\varphi^{(i)}(a)\tau_i(x) + \varphi^{(i)}(b)\nu_i(x)]|}{(x-a)^{\mathbf{m}}(b-x)^{\mathbf{m}}}\right) \\
 & \times f(|\varphi^{(2\mathbf{m})}(x)|)dx. \tag{2.24}
 \end{aligned}$$

that is given in [4].

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