SOME FUNCTIONAL EQUATIONS AND THEIR CORRESPONDING SUM FORMS

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ABSTRACT. The general solutions of some functional equations containing at least two unknown mappings have been obtained without imposing any regularity condition on any of the mappings.

1. INTRODUCTION

Let $\mathbb{R}$ denote the set of all real numbers; $I = \{x \in \mathbb{R} : 0 \leq x \leq 1\} = [0, 1]$, the unit closed interval; $]0, 1[= \{x \in \mathbb{R} : 0 < x < 1\}$, the unit open interval and $]0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$. Below we mention some functional equations:

$$f(pq) = qf(p) + pf(q), \tag{FE1}$$

$$f(pq) = qf(p) + pg(q), \tag{FE2}$$

$$f(pq) = qf(p) + pg(q) + g(p)q, \tag{FE3}$$

$$f(pq) = qf(p) + pg(q) + g(p)h(q) \tag{FE4}$$

where $f : I \rightarrow \mathbb{R}$, $g : I \rightarrow \mathbb{R}$ and $h : I \rightarrow \mathbb{R}$ are unknown mappings.

If we choose respectively $h(q) = 0$ and $h(q) = q$, then (FE4) reduces respectively to (FE2) and (FE3).

Let $\Gamma_n = \left\{ (p_1, \dots, p_n) : p_i \geq 0, i = 1, \dots, n; \sum_{i=1}^n p_i = 1 \right\}$, $n = 1, 2, \dots$; denote the set of all n -component complete discrete probability distributions with nonnegative elements.

Let us consider (FE1). In this functional equation, if we replace p by p_i ; q by q_j and then sum both sides of the resulting functional equation with respect to $i = 1, \dots, n$; $j = 1, \dots, m$ such that $(p_1, \dots, p_n) \in \Gamma_n$,

2010 *Mathematics Subject Classification.* 39B22, 39B52.

Key words and phrases. Functional equation, additive mapping, multiplicative mapping, logarithmic mapping.

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$(q_1, \dots, q_m) \in \Gamma_m$, n and m being positive integers, we obtain the sum form functional equation

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i q_j) = \sum_{i=1}^n f(p_i) + \sum_{j=1}^m f(q_j) \quad (1.1)$$

which is due to Chaundy and Mcleod [1]. The functional equation (1.1) is useful in characterizing the Shannon entropies [10] $H_n : \Gamma_n \rightarrow \mathbb{R}$, $n = 1, 2, \dots$ defined as

$$H_n(p_1, \dots, p_n) = - \sum_{i=1}^n p_i \log_2 p_i \quad (1.2)$$

for all $(p_1, \dots, p_n) \in \Gamma_n$, $n = 1, 2, \dots$; assuming that $0 \log_2 0 := 0$. Thus, (1.1) is useful in information theory.

Tabor [9, p-172] has discussed the stability of (FE1), in a somewhat broader context, by taking $f :]0, 1] \rightarrow E$, E a Banach space and also assuming some additional conditions.

By following the method adopted to obtain (1.1) from (FE1), the sum form functional equation

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i q_j) = \sum_{i=1}^n f(p_i) + \sum_{j=1}^m g(q_j), \quad (1.3)$$

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i q_j) = \sum_{i=1}^n f(p_i) + \sum_{j=1}^m g(q_j) + \sum_{i=1}^n g(p_i), \quad (1.4)$$

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i q_j) = \sum_{i=1}^n f(p_i) + \sum_{j=1}^m g(q_j) + \sum_{i=1}^n g(p_i) \sum_{j=1}^m h(q_j) \quad (1.5)$$

valid for all $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$, n and m being integers, can be derived respectively from (FE2), (FE3) and (FE4).

Nath and Singh [4], [5] discussed the functional equations

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i q_j) = \sum_{i=1}^n g(p_i) \sum_{j=1}^m h(q_j), \quad (1.6)$$

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i q_j) = \sum_{i=1}^n g(p_i) \sum_{j=1}^m h(q_j) + \sum_{j=1}^m f(q_j). \quad (1.7)$$

The functional equation (1.5) is an enlargement of (1.6) but not of (1.7). Also, (1.7) is not a particular case of (1.5). In fact, (1.5) should be regarded

as a Pexider-type generalization of the functional equation [8]

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i q_j) = \sum_{i=1}^n f(p_i) + \sum_{j=1}^m f(q_j) + \sum_{i=1}^n g(p_i) \sum_{j=1}^m h(q_j) \quad (1.8)$$

obtained from it by changing $\sum_{j=1}^m f(q_j)$ to $\sum_{j=1}^m g(q_j)$. Hence, the general solutions of (1.5) are bound to be different from those of (1.8). Surprisingly, the solutions (α_1) and (α_2) , of (1.5), are independent of the fixed positive integers n, m ; and these solutions do not satisfy (1.8). On the other hand, every general solution of (1.8) depends upon at least one of the fixed positive integers n and m .

The functional equation (1.5) may also be regarded as a Pexider-type generalization of the equation ([6], [8])

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i q_j) = \sum_{i=1}^n f(p_i) + \sum_{j=1}^m f(q_j) + c \sum_{i=1}^n g(p_i) \sum_{j=1}^m g(q_j), \quad c \neq 0. \quad (1.9)$$

The general solutions of (1.9) (with restrictions on $f(0), g(0), f(1), g(1)$) have been utilized to obtain the general solutions of (1.5). Also, the general solutions of (1.9) are needed to obtain some solutions of the equation [7]

$$\sum_{i=1}^n \sum_{j=1}^m F(p_i q_j) = \sum_{i=1}^n G(p_i) + \sum_{j=1}^m H(q_j) + \sum_{i=1}^n K(p_i) \sum_{j=1}^m L(q_j)$$

in which F, G, H, K, L are real-valued mappings each having domain $I = [0, 1]$.

This paper is organised as follows:

In Section 2, some known definitions and results needed for the development of Sections 3 and 4 have been mentioned. In Section 3, we have obtained the general solutions of (FE4) and then deduced, from these general solutions, the general solutions of (FE2) and (FE3). In Section 4, we have obtained the general solutions of the sum form functional equations (1.3), (1.4) and (1.5) for fixed integers $n \geq 3$ and $m \geq 3$. Section 5 deals with the importance of some functional equations discussed in this paper.

2. SOME DEFINITIONS AND RESULTS

In this section, we mention some known definitions and results which are needed in the subsequent sections.

A mapping $A : \mathbb{R} \rightarrow \mathbb{R}$ is said to be additive on $\mathbb{R}$ if it satisfies the equation $A(x + y) = A(x) + A(y)$ for all $x \in \mathbb{R}, y \in \mathbb{R}$. A mapping $a : I \rightarrow \mathbb{R}$ is said to be additive on the unit triangle $\Delta = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq x + y \leq 1\}$ if it satisfies the equation $a(x + y) = a(x) + a(y)$ for all $(x, y) \in \Delta$. Daróczy and Losonczi [2] have proved that if a mapping

$a : I \rightarrow \mathbb{R}$ is additive on the unit triangle Δ , then it has a unique additive extension $A : \mathbb{R} \rightarrow \mathbb{R}$ in the sense that $A : \mathbb{R} \rightarrow \mathbb{R}$ is additive on $\mathbb{R}$ and $A(x) = a(x)$ for all $x \in I$.

A mapping $M : I \rightarrow \mathbb{R}$ is said to be multiplicative if $M(pq) = M(p)M(q)$ holds for all $p \in I, q \in I$.

A mapping $\ell : I \rightarrow \mathbb{R}$ is said to be logarithmic if $\ell(0) = 0$ and $\ell(pq) = \ell(p) + \ell(q)$ for all $p \in]0, 1], q \in]0, 1]$.

Result 2.1 ([3]). *Let $\psi : I \rightarrow \mathbb{R}$ be a mapping which satisfies the equation $\sum_{i=1}^n \psi(x_i) = c$ for all $(x_1, \dots, x_n) \in \Gamma_n$; c a given real constant and $n \geq 3$ a fixed integer. Then, there exists an additive mapping $b : \mathbb{R} \rightarrow \mathbb{R}$ such that $\psi(x) = b(x) - \frac{1}{n}b(1) + \frac{c}{n}$ for all $x \in I$.*

Result 2.2 ([3]). *Suppose a mapping $f : I \rightarrow \mathbb{R}$ satisfies the functional equation (1.1) for all $(p_1, \dots, p_n) \in \Gamma_n, (q_1, \dots, q_m) \in \Gamma_m$; $n \geq 3, m \geq 3$ being fixed integers. Then, f is of the form*

$$f(p) = \begin{cases} f(0) + f(0)(nm - n - m)p + a(p) + D(p, p) & \text{if } 0 < p \leq 1 \\ f(0) & \text{if } p = 1 \end{cases}$$

where $f(0)$ is an arbitrary real constant; $a : \mathbb{R} \rightarrow \mathbb{R}$ is an additive mapping; the mapping $D : \mathbb{R} \times]0, 1] \rightarrow \mathbb{R}$ is additive in the first variable; there exists a mapping $E : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ additive in both variables such that $a(1) = E(1, 1)$ and $D(pq, pq) = D(pq, p) + D(pq, q) + E(p, q)$ for all $p \in]0, 1], q \in]0, 1]$.

Modified form of Result 2.2 ([8]). *If a mapping $f : I \rightarrow \mathbb{R}$ satisfies (1.1) for all $(p_1, \dots, p_n) \in \Gamma_n, (q_1, \dots, q_m) \in \Gamma_m$; $n \geq 3, m \geq 3$ being fixed integers, then f is of the form*

$$f(p) = f(0) + f(0)(nm - n - m)p + a(p) + D(p, p) \quad (2.1)$$

for all $p \in I$; $f(0)$ is an arbitrary real constant; $a : \mathbb{R} \rightarrow \mathbb{R}$ is an additive mapping; the mapping $D : \mathbb{R} \times I \rightarrow \mathbb{R}$ is additive in the first variable; there exists a mapping $E : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ additive in both variables such that $a(1) = E(1, 1)$ and

$$D(pq, pq) = D(pq, p) + D(pq, q) + E(p, q) \quad (2.2)$$

for all $p \in I, q \in I$.

Using the fact that $a(1) = E(1, 1)$, it can be easily deduced from (2.2) that

$$a(1) + D(1, 1) = 0. \quad (2.3)$$

Result 2.3 ([6]). *Let c^* be a nonzero given real constant and $F_2 : I \rightarrow \mathbb{R}, K_2 : I \rightarrow \mathbb{R}$ be mappings which satisfy the functional equation*

$$\sum_{i=1}^n \sum_{j=1}^m F_2(p_i q_j) = \sum_{i=1}^n F_2(p_i) + \sum_{j=1}^m F_2(q_j) + c^* \sum_{i=1}^n K_2(p_i) \sum_{j=1}^m K_2(q_j) \quad (2.4)$$

for all $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$, $n \geq 3$, $m \geq 3$ being fixed integers. Further, suppose that

$$\left. \begin{array}{ll} \text{(i)} & F_2(1) = 0, \quad F_2(0) = 0, \\ \text{(ii)} & K_2(1) = 0, \quad K_2(0) = 0. \end{array} \right\} \quad (2.5)$$

Then, for all $p \in I$, any general solution (F_2, K_2) of (2.4) satisfying (2.5), is of the form

$$\begin{cases} F_2(p) = a(p) + D(p, p) \\ K_2(p) = A(p) \end{cases} \quad (2.6)$$

or

$$\begin{cases} F_2(p) = a(p) + D(p, p) + \frac{1}{2}c^*p[\ell(p)]^2 \\ K_2(p) = p\ell(p) \end{cases} \quad (2.7)$$

or

$$\begin{cases} F_2(p) = a(p) + D(p, p) + c^*\lambda^2(M(p) - p) \\ K_2(p) = \lambda(M(p) - p) \end{cases} \quad (2.8)$$

where $\lambda \neq 0$ is an arbitrary real constant; $A : \mathbb{R} \rightarrow \mathbb{R}$ is an additive mapping such that $A(1) = 0$; the mappings $a : \mathbb{R} \rightarrow \mathbb{R}$ and $D : \mathbb{R} \times I \rightarrow \mathbb{R}$ are as described in Modified form of Result 2.2; $M : I \rightarrow \mathbb{R}$ is a multiplicative mapping which is not additive and $M(0) = 0$, $M(1) = 1$; and $\ell : I \rightarrow \mathbb{R}$ is a logarithmic mapping.

Remark. Any general solution f of (FE1) must be of the form

$$f(p) = p\ell(p) \quad (2.9)$$

for all $p \in I$; $\ell : I \rightarrow \mathbb{R}$ being a logarithmic mapping. It is easy to verify that (2.9) is also a solution of (1.1). Any general solution of (1.1); $n \geq 3$, $m \geq 3$ being fixed integers, is of the form (2.1) which is remarkably different from (2.9). If we take $f(0) = 0$; $a(p) \equiv 0$; $D : \mathbb{R} \times I \rightarrow \mathbb{R}$ of the form $D(x, p) = x\ell(p)$, $x \in \mathbb{R}$, $p \in I$; and $E(p, q) \equiv 0$ on $I \times I$, then (2.1) reduces to (2.9).

Let us look upon (1.1) as a sum form analogue of (FE1). The above observations justify the need of having found separately the general solutions of (FE1) and its sum form analogue (1.1); $n \geq 3$, $m \geq 3$ being integers. This motivates us to investigate the general solutions of the functional equation (FE4) and its corresponding sum form analogue (1.5); $n \geq 3$, $m \geq 3$ being integers; in Sections 3 and 4.

3. ON THE FUNCTIONAL EQUATION (FE4)

The main result of this section is the following:

Theorem 3.1. *Suppose $f : I \rightarrow \mathbb{R}$, $g : I \rightarrow \mathbb{R}$ and $h : I \rightarrow \mathbb{R}$ are mappings which satisfy the functional equation (FE4) for all $p \in I$, $q \in I$. Then, any general solution (f, g, h) of (FE4) is one of the forms (for all $p \in I$) mentioned below:*

$$f(p) = f(1)p \quad g(p) = 0 \quad h \text{ arbitrary} \quad (3.1)$$

$$f(p) = f(1)p + p\ell(p) \quad g(p) = p\ell(p) \quad h(p) = 0 \quad (3.2)$$

$$f(p) = f(1)p + \frac{1}{\lambda}[M(p) - p] \quad g(p) = \frac{1}{\lambda}[M(p) - p] \quad h(p) = M(p) - p \quad (3.3)$$

$$f(p) = f(1)p + g(1)p\ell(p) \quad g(p) = g(1)p \quad h(p) = p\ell(p) - p \quad (3.4)$$

where $\lambda \neq 0$, $g(1) \neq 0$ and $f(1)$ are arbitrary real constants; $\ell : I \rightarrow \mathbb{R}$ is a logarithmic mapping and $M : I \rightarrow \mathbb{R}$ is a multiplicative mapping.

Proof. Let us put $p = q = 1$ in (FE4). We obtain

$$g(1)[1 + h(1)] = 0. \quad (3.5)$$

Now we divide our discussion into two cases.

Case 1. $g(1) = 0$.

Let us put $p = 1$ in (FE4). Since $g(1) = 0$, we obtain

$$f(q) = f(1)q + g(q). \quad (3.6)$$

From (FE4) and (3.6), it follows that

$$g(pq) = qg(p) + pg(q) + g(p)h(q) \quad (3.7)$$

valid for all $p \in I$, $q \in I$. Let us write (3.7) in the form

$$g(pq) - qg(p) - pg(q) = g(p)h(q). \quad (3.8)$$

The left hand side of (3.8) is symmetric in p and q . So, should also be the right hand side of (3.8). This fact gives

$$g(p)h(q) = g(q)h(p). \quad (3.9)$$

Case 1.1. $g(q) = 0$ for all $q \in I$.

In this case, it is clear from (FE4) that h is an arbitrary mapping. Moreover, (FE4) reduces to $f(pq) = qf(p)$ valid for all $p \in I$, $q \in I$. Since $f(pq) = f(qp)$, it follows that $f(p) = f(1)p$ for all $p \in I$, $f(1)$ being an arbitrary real constant. Thus, we have obtained the solution (3.1) of (FE4).

Case 1.2. g does not vanish identically on I .

In this case, there exists an element $q_0 \in [0, 1[$ such that $g(q_0) \neq 0$. Putting $q = q_0$ in (3.9), we obtain

$$h(p) = \lambda g(p) \quad (3.10)$$

for all $p \in I$ with $\lambda = [g(q_0)]^{-1}h(q_0)$. From (FE4) and (3.10), we get the equation

$$f(pq) = qf(p) + pg(q) + \lambda g(p)g(q) \quad (3.11)$$

for all $p \in I, q \in I$. From (3.6) and (3.11), it follows that

$$g(pq) = qg(p) + pg(q) + \lambda g(p)g(q). \quad (3.12)$$

Case 1.2.1. $\lambda = 0$.

In this case, (3.10) gives $h(p) = 0$ for all $p \in I$. Now, (3.7) reduces to the equation $g(pq) = qg(p) + pg(q)$. So, $g(p) = p\ell(p)$ for all $p \in I$; $\ell : I \rightarrow \mathbb{R}$ being a logarithmic mapping. Now, (3.6) gives $f(q) = f(1)q + q\ell(q)$ for all $q \in I$, $f(1)$ being an arbitrary real constant. Thus, we have obtained the solution (3.2) of (FE4).

Case 1.2.2. $\lambda \neq 0$.

Let us multiply both sides of (3.12) by λ and write the resulting equation in the form

$$\lambda g(pq) + pq = [\lambda g(p) + p][\lambda g(q) + q]. \quad (3.13)$$

Define a mapping $M : I \rightarrow \mathbb{R}$ as

$$M(x) = \lambda g(x) + x \quad (3.14)$$

for all $x \in I$. Now (3.13) reduces to the equation $M(pq) = M(p)M(q)$ valid for all $p \in I, q \in I$. Thus M , defined by (3.14), is a multiplicative mapping. Equation (3.14) gives

$$g(p) = \frac{1}{\lambda}[M(p) - p], \quad \lambda \neq 0 \quad (3.15)$$

for all $p \in I$. From (3.10) and (3.15), we obtain

$$h(p) = M(p) - p$$

for all $p \in I$. From (3.6) and (3.15), it follows that

$$f(p) = f(1)p + \frac{1}{\lambda}[M(p) - p]$$

for all $p \in I$. Thus, we have obtained the solution (3.3) of (FE4).

Case 2. $g(1) \neq 0$.

Now, from (3.5), it follows that $h(1) = -1$. Putting $q = 1$ in (FE4) and using $h(1) = -1$, it follows that $g(p) = g(1)p$ for all $p \in I$, $g(1)$ being an

arbitrary nonzero real constant. Making use of this form of g in (FE4), we obtain the equation

$$f(pq) = qf(p) + g(1)pq + g(1)ph(q). \quad (3.16)$$

Let us write (3.16) in the form

$$f(pq) - g(1)pq = qf(p) + g(1)ph(q). \quad (3.17)$$

The left hand side of (3.17) is symmetric in p and q . So should be its right hand side. This gives us

$$qf(p) + g(1)ph(q) = pf(q) + g(1)qh(p).$$

Putting $q = 1$ in this equation and using $h(1) = -1$, we obtain

$$f(p) = f(1)p + g(1)p + g(1)h(p). \quad (3.18)$$

From (FE4), (3.18) and the fact that $g(p) = g(1)p$ for all $p \in I$, it follows that

$$h(pq) = qh(p) + ph(q) + pq$$

which can be written in the form

$$h(pq) + pq = q[h(p) + p] + p[h(q) + q]. \quad (3.19)$$

Define a mapping $H : I \rightarrow \mathbb{R}$ as

$$H(x) = h(x) + x \quad (3.20)$$

for all $x \in I$. Now (3.19) reduces to

$$H(pq) = qH(p) + pH(q).$$

So, $H(p) = p\ell(p)$ for all $p \in I$, $\ell : I \rightarrow \mathbb{R}$ being a logarithmic mapping. Thus $h(p) + p = p\ell(p)$ from which it follows that $h(p) = p\ell(p) - p$ for all $p \in I$. Making use of this form of h in (3.18), it follows that $f(p) = f(1)p + g(1)p\ell(p)$ for all $p \in I$. Thus, we have obtained the solution (3.4) of (FE4). $\square$

Theorem 3.2. *Suppose $f : I \rightarrow \mathbb{R}$ and $g : I \rightarrow \mathbb{R}$ are mappings which satisfy the functional equation (FE2) for all $p \in I, q \in I$. Then, any general solution (f, g) of (FE2) is only of the form (for all $p \in I$)*

$$f(p) = f(1)p + p\ell(p) \quad g(p) = p\ell(p) \quad (3.21)$$

where $f(1)$ is an arbitrary real constant and $\ell : I \rightarrow \mathbb{R}$ is a logarithmic mapping.

Proof. The equation (FE2) is a particular case of (FE4) (when $h(p) = 0$). The solution (3.21) follows from (3.2). $\square$

Theorem 3.3. *Suppose the mappings $f : I \rightarrow \mathbb{R}$ and $g : I \rightarrow \mathbb{R}$ satisfy (FE3) for all $p \in I$, $q \in I$. Then, any general solution (f, g) of (FE3) is only of the form (for all $p \in I$)*

$$f(p) = f(1)p \quad g(p) = 0 \quad (3.22)$$

where $f(1)$ is an arbitrary real constant.

Proof. The solution (3.22) follows from (3.1). For this, we need to choose $h(p) = p$ for all $p \in I$. $\square$

4. ON THE FUNCTIONAL EQUATION (1.5)

The main result of this section is the following:

Theorem 4.1. *Suppose $f : I \rightarrow \mathbb{R}$, $g : I \rightarrow \mathbb{R}$, $h : I \rightarrow \mathbb{R}$ are mappings which satisfy (1.5) for all $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers. Then, for all $p \in I$, any general solution (f, g, h) of (1.5) is one of the following forms:*

$$\left. \begin{array}{ll} \text{(i)} & f(p) = [f(1) - f(0)]p + a(p) + D(p, p) + f(0) \\ \text{(ii)} & g(p) = [f(1) - f(0)]p + a(p) + D(p, p) + a_1(p) + g(0) \\ \text{(iii)} & h(p) = b_1(p) + h(0) \end{array} \right\} \quad (\alpha_1)$$

or

$$\left. \begin{array}{ll} \text{(i)} & f(p) = [f(1) - f(0)]p + a(p) + D(p, p) + f(0) \\ \text{(ii)} & g(p) = a_2(p) + g(0) \\ \text{(iii)} & h \text{ is an arbitrary real-valued mapping} \end{array} \right\} \quad (\alpha_2)$$

or

$$\left. \begin{array}{ll} \text{(i)} & f(p) = [f(1) - f(0)]p + a(p) + D(p, p) + f(0) \\ \text{(ii)} & g(p) = -[g(1) + (n-1)g(0)]A_2(p) + A_1(p) + g(0) \\ \text{(iii)} & h(p) = \frac{1}{c}\{[f(1) - f(0)]p + a(p) + D(p, p)\} + A_2(p) + h(0) \end{array} \right\} \quad (\alpha_3)$$

or

$$\left. \begin{array}{ll} \text{(i)} & f(p) = [f(1) - f(0)]p + a(p) + D(p, p) + f(0) \\ \text{(ii)} & g(p) = [f(1) - f(0)]p + a(p) + D(p, p) \\ & \quad - [g(1) + (n-1)g(0)]\{[h(1) - h(0)]p + b_2(p)\} + A_1(p) + g(0) \\ \text{(iii)} & h(p) = [h(1) - h(0)]p + b_2(p) + h(0) \end{array} \right\} \quad (\alpha_4)$$

or

$$\left. \begin{array}{ll} \text{(i)} & f(p) = [f(1) - f(0)]p + \frac{1}{2}[c - g(1) - (n-1)g(0)]p[\ell(p)]^2 \\ & \quad + a(p) + D(p, p) + f(0) \\ \text{(ii)} & g(p) = [f(1) - f(0)]p + \frac{1}{2}[c - g(1) - (n-1)g(0)]p[\ell(p)]^2 \\ & \quad + a(p) + D(p, p) - [g(1) + (n-1)g(0)] \\ & \quad \times \{[h(1) - h(0)]p + p\ell(p)\} + A_1(p) + g(0) \\ \text{(iii)} & h(p) = [h(1) - h(0)]p + p\ell(p) + h(0) \end{array} \right\} \quad (\alpha_5)$$

or

$$\left. \begin{aligned} \text{(i)} \quad & f(p) = [f(1) - f(0)]p + [c - g(1) - (n-1)g(0)]\mu^2[M(p) - p] \\ & \quad + a(p) + D(p, p) + f(0) \\ \text{(ii)} \quad & g(p) = [f(1) - f(0)]p + [c - g(1) - (n-1)g(0)]\mu^2[M(p) - p] \\ & \quad + a(p) + D(p, p) - [g(1) + (n-1)g(0)]\{[h(1) - h(0)]p \\ & \quad + [c - g(1) - (n-1)g(0)][M(p) - p]\} + A_1(p) + g(0) \\ \text{(iii)} \quad & h(p) = [h(1) - h(0)]p + [c - g(1) - (n-1)g(0)][M(p) - p] + h(0) \end{aligned} \right\} \quad (\alpha_6)$$

where $c \neq 0$, $\mu \neq 0$ are arbitrary real constants; $a_1 : \mathbb{R} \rightarrow \mathbb{R}$, $a_2 : \mathbb{R} \rightarrow \mathbb{R}$, $b_1 : \mathbb{R} \rightarrow \mathbb{R}$, $b_2 : \mathbb{R} \rightarrow \mathbb{R}$, $A_1 : \mathbb{R} \rightarrow \mathbb{R}$, $A_2 : \mathbb{R} \rightarrow \mathbb{R}$ are additive mappings; $a : \mathbb{R} \rightarrow \mathbb{R}$ and $D : \mathbb{R} \times I \rightarrow \mathbb{R}$ are as described in the Modified form of Result 2.2; $M : I \rightarrow \mathbb{R}$ is a multiplicative mapping which is not additive and $M(0) = 0$, $M(1) = 1$; $\ell : I \rightarrow \mathbb{R}$ is a logarithmic mapping with

$$\left. \begin{aligned} \text{(i)} \quad & b_1(1) = -mh(0) \\ \text{(ii)} \quad & a_1(1) = [-f(1) + (n-1)(m-1)f(0)] + mf(0) - mg(0) \\ \text{(iii)} \quad & A_1(1) = [-f(1) + (n-1)(m-1)f(0)] \\ & \quad - m[g(1) + (n-1)g(0)]h(0) + mf(0) - mg(0) \\ \text{(iv)} \quad & a_2(1) = -ng(0) \\ \text{(v)} \quad & A_2(1) = -mh(0) + \frac{1}{c}\{mf(0) - [g(1) + (m-1)g(0)] \\ & \quad + [-f(1) + (n-1)(m-1)f(0)]\} \\ \text{(vi)} \quad & b_2(1) = 0. \end{aligned} \right\} \quad (4.1)$$

Before giving the proof of Theorem 4.1, we need to prove the following:

Lemma 4.2. Suppose $f : I \rightarrow \mathbb{R}$, $g : I \rightarrow \mathbb{R}$ are mappings which satisfy (1.3) for all $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers. Then, any general solution of (1.3) is of the form (α_1) (i, ii) where $a_1 : \mathbb{R} \rightarrow \mathbb{R}$ is an additive mapping which satisfies (4.1) (ii) and the mappings $a : \mathbb{R} \rightarrow \mathbb{R}$ and $D : \mathbb{R} \times I \rightarrow \mathbb{R}$ are as described in the Modified form of Result 2.2.

Proof. The substitutions $p_1 = 1$, $p_2 = \dots = p_n = 0$ in equation (1.3) give the equation

$$\sum_{j=1}^m \{g(q_j) - f(q_j)\} = [-f(1) + (n-1)(m-1)f(0)] \quad (4.2)$$

valid for all $(q_1, \dots, q_m) \in \Gamma_m$. By Result 2.1, there exists an additive mapping $a_1 : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$g(p) = f(p) + a_1(p) - f(0) + g(0) \quad (4.3)$$

for all $p \in I$ with $a_1(1)$ given by (4.1) (ii). From equations (4.2) and (1.3), we obtain

$$\sum_{i=1}^n \sum_{j=1}^m f(p_i q_j) = \sum_{i=1}^n f(p_i) + \sum_{j=1}^m f(q_j) + [-f(1) + (n-1)(m-1)f(0)]. \quad (4.4)$$

Define a mapping $f_1 : I \rightarrow \mathbb{R}$ as

$$f_1(x) = f(x) - [f(1) - f(0)]x - f(0) \quad (4.5)$$

for all $x \in I$. Then $f_1(0) = 0$ and $f_1(1) = 0$. Moreover, from (4.4) and (4.5), we have

$$\sum_{i=1}^n \sum_{j=1}^m f_1(p_i q_j) = \sum_{i=1}^n f_1(p_i) + \sum_{j=1}^m f_1(q_j) \quad (4.5a)$$

for all $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$. Now by using Modified Form of Result 2.2, (4.5) and (4.3), solution $(\alpha_1)((i),(ii))$ of (1.3) follows. $\square$

Notice that (3.21) is also a solution of (1.3) but it is included in $(\alpha_1)((i),(ii))$.

Proof of Theorem 4.1. Putting $p_1 = 1$, $p_2 = \dots = p_n = 0$ in equation (1.5), we obtain

$$\sum_{j=1}^m \{g(q_j) + [g(1) + (n-1)g(0)]h(q_j) - f(q_j)\} = -f(1) + (n-1)(m-1)f(0). \quad (4.6)$$

By Result 2.1, there exists an additive mapping $A_1 : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$g(p) = [f(p) - f(0)] - [g(1) + (n-1)g(0)][h(p) - h(0)] + A_1(p) + g(0) \quad (4.7)$$

with $A_1(1)$ given by (4.1) (iii). Now (1.5), (4.6), (4.7) and (4.1) (ii) give

$$\begin{aligned} & \sum_{j=1}^m \left\{ \sum_{i=1}^n f(p_i q_j) - q_j \sum_{i=1}^n f(p_i) - f(q_j) - h(q_j) \sum_{i=1}^n f(p_i) \right. \\ & \quad + [g(1) + (n-1)g(0)]h(q_j) \sum_{i=1}^n h(p_i) \\ & \quad - \{(n-m)[g(1) + (n-1)g(0)]h(0) - (n-m)f(0) \\ & \quad \left. - [g(1) + (m-1)g(0)] + [-f(1) + (n-1)(m-1)f(0)]\}h(q_j) \right\} \\ & \quad = [-f(1) + (n-1)(m-1)f(0)]. \end{aligned}$$

By Result 2.1, there exists a mapping $A : \Gamma_n \times \mathbb{R} \rightarrow \mathbb{R}$, additive in second variable such that

$$\begin{aligned}
& \sum_{i=1}^n f(p_i q) - q \sum_{i=1}^n f(p_i) - f(q) - h(q) \sum_{i=1}^n f(p_i) \\
& + [g(1) + (n-1)g(0)]h(q) \sum_{i=1}^n h(p_i) \\
& - \{(n-m)[g(1) + (n-1)g(0)]h(0) - (n-m)f(0) \\
& - [g(1) + (m-1)g(0)] + [-f(1) + (n-1)(m-1)f(0)]\}h(q) \\
& = A(p_1, \dots, p_n; q) - h(0) \sum_{i=1}^n f(p_i) \\
& + [g(1) + (n-1)g(0)]h(0) \sum_{i=1}^n h(p_i) + (n-1)f(0) \\
& - \{(n-m)[g(1) + (n-1)g(0)]h(0) - (n-m)f(0) \\
& - [g(1) + (m-1)g(0)] + [-f(1) + (n-1)(m-1)f(0)]\}h(0) \quad (4.8)
\end{aligned}$$

for all $(p_1, \dots, p_n) \in \Gamma_n$, $q \in I$ with

$$\begin{aligned}
A(p_1, \dots, p_n; 1) &= mh(0) \sum_{i=1}^n f(p_i) - m[g(1) + (n-1)g(0)]h(0) \sum_{i=1}^n h(p_i) \\
& + m\{(n-m)[g(1) + (n-1)g(0)]h(0) - (n-m)f(0) \\
& - [g(1) + (m-1)g(0)] + [-f(1) + (n-1)(m-1)f(0)]\}h(0) \\
& - [f(1) + (n-1)f(0)]. \quad (4.9)
\end{aligned}$$

Putting $q = r_t$, $t = 1, \dots, n$ successively in (4.8), taking summation with respect to t from $t = 1$ to n , and using (4.9), we obtain

$$\begin{aligned}
& \sum_{i=1}^n \sum_{t=1}^n f(p_i r_t) - \sum_{i=1}^n f(p_i) - \sum_{t=1}^n f(r_t) + [g(1) + (n-1)g(0)] \sum_{i=1}^n h(p_i) \sum_{t=1}^n h(r_t) \\
& - (m-n)\{(n-m)[g(1) + (n-1)g(0)]h(0) - (n-m)f(0) \\
& - [g(1) + (m-1)g(0)] + [-f(1) + (n-1)(m-1)f(0)]\}h(0) \\
& - n(n-1)f(0) + [f(1) + (n-1)f(0)] \\
& = \sum_{i=1}^n f(p_i) \sum_{t=1}^n h(r_t) + \{(n-m)[g(1) + (n-1)g(0)]h(0) - (n-m)f(0) \\
& - [g(1) + (m-1)g(0)] + [-f(1) + (n-1)(m-1)f(0)]\} \sum_{t=1}^n h(r_t)
\end{aligned}$$

$$+ (m-n)h(0) \sum_{i=1}^n f(p_i) - (m-n)[g(1) + (n-1)g(0)]h(0) \sum_{i=1}^n h(p_i). \quad (4.10)$$

Since left hand side of equation (4.10) is symmetric in p_i and r_t , so should be its right hand side. This fact gives rise to the equation

$$\begin{aligned} & \left\{ \sum_{i=1}^n f(p_i) - (n-m)f(0) - [g(1) + (m-1)g(0)] \right. \\ & \quad \left. + [-f(1) + (n-1)(m-1)f(0)] \right\} \left[\sum_{t=1}^n h(r_t) + (m-n)h(0) \right] \\ &= \left\{ \sum_{t=1}^n f(r_t) - (n-m)f(0) - [g(1) + (m-1)g(0)] \right. \\ & \quad \left. + [-f(1) + (n-1)(m-1)f(0)] \right\} \left[\sum_{i=1}^n h(p_i) + (m-n)h(0) \right]. \quad (4.11) \end{aligned}$$

Case 1.

$$\sum_{t=1}^n h(r_t) + (m-n)h(0) = 0 \text{ on } \Gamma_n.$$

In this case, by Result 2.1, it follows that there exists an additive mapping $b_1 : \mathbb{R} \rightarrow \mathbb{R}$ such that (α_1) (iii) holds with $b_1(1)$ given by (4.1) (i).

Now, from equations (1.5), (α_1) (iii) and (4.1) (i); equation (1.3) follows. Also, by using Lemma 4.2; (α_1) (i, ii), along with (4.1)(ii); follows. Thus, we have obtained the solution (α_1) .

Case 2.

$$\sum_{t=1}^n h(r_t) + (m-n)h(0) \text{ does not vanish identically on } \Gamma_n.$$

In this case, there exists a probability distribution $(r_1^*, \dots, r_n^*) \in \Gamma_n$ such that $\sum_{t=1}^n h(r_t^*) + (m-n)h(0) \neq 0$. Putting $r_t = r_t^*$, $t = 1, \dots, n$ in (4.11), it follows that

$$\begin{aligned} \sum_{i=1}^n f(p_i) &= c \left[\sum_{i=1}^n h(p_i) + (m-n)h(0) \right] + [g(1) + (m-1)g(0)] \\ & \quad + (n-m)f(0) - [-f(1) + (n-1)(m-1)f(0)] \quad (4.12) \end{aligned}$$

where

$$\begin{aligned} c &= \left[\sum_{t=1}^n h(r_t^*) + (m-n)h(0) \right]^{-1} \left\{ \sum_{t=1}^n f(r_t^*) - (n-m)f(0) \right. \\ & \quad \left. - [g(1) + (m-1)g(0)] + [-f(1) + (n-1)(m-1)f(0)] \right\}. \end{aligned}$$

From equations (4.7), (4.1)(iii) and (4.12), it follows that

$$\sum_{i=1}^n g(p_i) = [c - g(1) - (n-1)g(0)] \left[\sum_{i=1}^n h(p_i) + (m-n)h(0) \right] + [g(1) + (n-1)g(0)] \quad (4.13)$$

for all $(p_1, \dots, p_n) \in \Gamma_n$. Also, from equations (1.5), (4.6) and (4.13), we obtain

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^m f(p_i q_j) &= \sum_{i=1}^n f(p_i) + \sum_{j=1}^m f(q_j) + [-f(1) + (n-1)(m-1)f(0)] \\ &\quad + [c - g(1) - (n-1)g(0)] \left[\sum_{i=1}^n h(p_i) + (m-n)h(0) \right] \sum_{j=1}^m h(q_j). \end{aligned} \quad (4.14)$$

Define the mapping $f_1 : I \rightarrow \mathbb{R}$ as in (4.5) for all $x \in I$. From (4.14) and (4.5), we have

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^m f_1(p_i q_j) &= \sum_{i=1}^n f_1(p_i) + \sum_{j=1}^m f_1(q_j) + [c - g(1) - (n-1)g(0)] \\ &\quad \times \left[\sum_{i=1}^n h(p_i) + (m-n)h(0) \right] \sum_{j=1}^m h(q_j). \end{aligned} \quad (4.15)$$

Case 2.1. $c - g(1) - (n-1)g(0) = 0$.

Now equation (4.15) reduces to (4.5a). By Modified form of Result 2.2 and (4.5), we have

$$f(p) = [f(1) - f(0)]p + a(p) + D(p, p) + f(0) \quad (4.16)$$

which is $(\alpha_2)(i)$ as well as $(\alpha_3)(i)$.

Case 2.1.1. $c = 0$.

In this case, $g(1) + (n-1)g(0) = 0$. Now equation (4.13) gives $\sum_{i=1}^n g(p_i) = 0$. By Result 2.1, there exists an additive mapping $a_2 : \mathbb{R} \rightarrow \mathbb{R}$ such that solution g is of the form $(\alpha_2)(ii)$ with $a_2(1)$ given by (4.1) (iv). Also, from equations (1.5), $(\alpha_2)(ii)$ and (4.1) (iv), we observe that h is an arbitrary real-valued mapping. Thus we have proved $(\alpha_2)(iii)$. So, we have obtained the solution (α_2) .

Case 2.1.2. $c \neq 0$.

In this case, let us rewrite (4.12) in the form

$$\begin{aligned} \sum_{i=1}^n \left\{ h(p_i) - \frac{1}{c} f(p_i) \right\} &= (n-m)h(0) + \frac{1}{c} \{ -(n-m)f(0) \\ &\quad - [g(1) + (m-1)g(0)] + [-f(1) + (n-1)(m-1)f(0)] \}. \end{aligned}$$

By Result 2.1, there exists an additive mapping $A_2 : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$h(p) = \frac{1}{c}[f(p) - f(0)] + A_2(p) + h(0) \quad (4.17)$$

with $A_2(1)$ given by (4.1)(v). From (4.17) and (4.16); (α_3) (iii) follows. Also, from equations (4.7), (4.16) and (α_3) (iii); (α_3) (ii) follows. Thus, we have obtained the solution (α_3) .

Case 2.2. $c - g(1) - (n - 1)g(0) \neq 0$.

In this case, putting $p_1 = 1, p_2 = \dots = p_n = 0; q_1 = 1, q_2 = \dots = q_m = 0$ in equation (4.15) and making use of the fact that $f_1(0) = 0, f_1(1) = 0$, we obtain $h(1) + (m - 1)h(0) = 0$. Define a mapping $h_1 : I \rightarrow \mathbb{R}$ as

$$h_1(x) = h(x) - h(0) - [h(1) - h(0)]x \quad (4.18)$$

for all $x \in I$. From (4.18), we have $h_1(0) = 0, h_1(1) = 0$. Also, from (4.15) and (4.18), it follows that

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^m f_1(p_i q_j) &= \sum_{i=1}^n f_1(p_i) + \sum_{j=1}^m f_1(q_j) \\ &+ [c - g(1) - (n - 1)g(0)] \sum_{i=1}^n h_1(p_i) \sum_{j=1}^m h_1(q_j) \end{aligned}$$

for all $(p_1, \dots, p_n) \in \Gamma_n, (q_1, \dots, q_m) \in \Gamma_m$.

Making use of Result 2.3 with $c^* = [c - g(1) - (n - 1)g(0)] \neq 0$; equations (4.5), (4.18) and (4.7); the solutions $(\alpha_4), (\alpha_5)$ and (α_6) of (1.5); together with (4.1)(vi); can be obtained. $\square$

Theorem 4.3. Suppose $f : I \rightarrow \mathbb{R}, g : I \rightarrow \mathbb{R}$ are mappings which satisfy (1.4) for all $(p_1, \dots, p_n) \in \Gamma_n, (q_1, \dots, q_m) \in \Gamma_m; n \geq 3, m \geq 3$ being fixed integers. Then, for all $p \in I$, any general solution (f, g) of (1.4) is only of the form

$$\left. \begin{array}{l} \text{(i)} \quad f(p) = a(p) + f(0) \\ \text{(ii)} \quad g(p) = b(p) + g(0) \end{array} \right\} \quad (4.19)$$

where $a : \mathbb{R} \rightarrow \mathbb{R}, b : \mathbb{R} \rightarrow \mathbb{R}$ are additive mappings such that

$$\left. \begin{array}{l} \text{(i)} \quad a(1) = f(1) - f(0) \\ \text{(ii)} \quad b(1) = n(m - 1)f(0) - [g(1) + (m + n - 1)g(0)] \end{array} \right\} \quad (4.20)$$

and

$$n(m - 1)f(0) = [g(1) + (n - 1)g(0)] + [g(1) + (m - 1)g(0)]. \quad (4.21)$$

Proof. The substitutions $q_1 = 1, q_2 = \dots = q_m = 0$ in (1.4) give

$$\sum_{i=1}^n g(p_i) = n(m - 1)f(0) - [g(1) + (m - 1)g(0)] \quad (4.22)$$

for all $(p_1, \dots, p_n) \in \Gamma_n$. By Result 2.1, there exists an additive mapping $b : \mathbb{R} \rightarrow \mathbb{R}$ such that (4.19) (ii) with $b(1)$ given by (4.20) (ii). Making the substitutions $p_1 = 1, p_2 = \dots = p_n = 0$ in (4.22); (4.21) follows.

Putting $p_1 = 1, p_2 = \dots = p_n = 0$ in (1.4) and using (4.19)(ii) along with (4.20)(ii), we obtain

$$\sum_{j=1}^m f(q_j) = f(1) + (m-1)f(0).$$

By Result 2.1, there exists an additive mapping $a : \mathbb{R} \rightarrow \mathbb{R}$ such that (4.19) (i) holds with $a(1)$ given by (4.20) (i). $\square$

Notice that (3.22) is also a solution of (1.4) and it is included in (4.19).

5. COMMENTS

(I) Since (1.5) has been derived from (FE4), it is obvious that every solution of (FE4) is also a solution of (1.5) but the possibility is not ruled out that (1.5) may have solutions different from (3.1) to (3.4). This, indeed, is the motivation to carry out separately the research work mentioned in Sections 3 and 4. In fact, the solutions (3.1), (3.2), (3.3) and (3.4) are included respectively in (α_2) , (α_1) , (α_6) and (α_3) ; with the additional information as mentioned in the statement of Theorem 4.1. For instance, (3.2) is included in (α_1) if we choose $f(0) = g(0) = h(0) = 0$, $a(p) \equiv 0$, $b_1(p) \equiv 0$, $a_1(p) = -f(1)p$ and $D(p, p) = p\ell(p)$, $\ell : I \rightarrow \mathbb{R}$ being a logarithmic mapping; and (3.4) is included in (α_3) provided we choose $f(0) = g(0) = h(0) = 0$, $a(p) \equiv 0$, $D(p, p) = cp\ell(p)$, $c = g(1)$, $A_2(p) = [-1 - \frac{f(1)}{g(1)}]p$, $A_1(p) = -f(1)p$.

(II) Let X_n be a discrete random variable which takes the distinct real values $x_1, \dots, x_n$ with respective probabilities $p_1, \dots, p_n$; $(p_1, \dots, p_n) \in \Gamma_n$, $n \geq 1$ an integer, where

$$x_i = \begin{cases} -\ell(p_i) & \text{if } 0 < p_i \leq 1 \\ 0 & \text{if } p_i = 0 \end{cases} \quad (5.1)$$

and $\ell : I \rightarrow \mathbb{R}$ is a logarithmic mapping which does not vanish identically on I . Let

$$\mu'_r(X_n) = \sum_{i=1}^n p_i x_i^r \quad (5.2)$$

denote the r th order moment of X_n about the origin, $r = 1, 2, \dots$. Clearly,

$$\mu'_r(X_n) = \sum_{i=1}^n \varphi_r(p_i) \quad (5.3)$$

where the mappings $\varphi_r : I \rightarrow \mathbb{R}$, $r = 1, 2, \dots$ are defined as

$$\varphi_r(p) = (-1)^r p [\ell(p)]^r \quad (5.4)$$

assuming that

$$0[\ell(0)]^r = 0, \quad r = 1, 2, \dots \quad (5.5)$$

If $r = 1$, then (5.4) and (5.5) give rise to the equation

$$\varphi_1(pq) = q\varphi_1(p) + p\varphi_1(q)$$

which reduces to (FE1) on writing $f : I \rightarrow \mathbb{R}$ in place of $\varphi_1 : I \rightarrow \mathbb{R}$.

If $r = 1, 2$, then (5.4) and (5.5) give

$$\varphi_2(pq) = q\varphi_2(p) + p\varphi_2(q) + 2\varphi_1(p)\varphi_1(q)$$

which may be generalized in the form

$$f(pq) = qf(p) + pf(q) + c^*g(p)g(q) \quad (5.6)$$

where $c^* \neq 0$ is a given real constant and $f : I \rightarrow \mathbb{R}$, $g : I \rightarrow \mathbb{R}$ are now written respectively in place of $\varphi_2 : I \rightarrow \mathbb{R}$ and $\varphi_1 : I \rightarrow \mathbb{R}$. Equation (FE4) is a Pexider-type generalization of (5.6) containing three unknown real-valued mappings f, g, h , each having domain I . An advantage of discussing (FE4) is that the solution(s) of (FE2), which is a Pexider-type generalization of (FE1), have been deduced from those of (FE4). Equation (1.5) is the corresponding sum form analogue of (FE4).

Now consider the integer r , $r \geq 3$. In this case, making use of binomial theorem for positive integers, it can be easily seen that

$$\varphi_r(pq) = q\varphi_r(p) + {}^rC_1\varphi_{r-1}(p)\varphi_1(q) + \dots + {}^rC_{r-1}\varphi_1(p)\varphi_{r-1}(q) + p\varphi_r(q) \quad (5.7)$$

where ${}^rC_1, \dots, {}^rC_{r-1}$ are binomial coefficients. Equation (5.7) may be generalized in the form

$$f_r(pq) = qf_r(p) + a_1f_{r-1}(p)f_1(q) + \dots + a_{r-1}f_1(p)f_{r-1}(q) + pf_r(q) \quad (5.8)$$

where $p \in I$, $q \in I$, $f_i : I \rightarrow \mathbb{R}$ are unknown mappings; $i = 1, \dots, r$, $r \geq 3$ an integer; and $a_1, \dots, a_{r-1}$ are given real constants. The sum form analogue of (5.8) is

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^m f_r(p_i q_j) &= \sum_{i=1}^n f_r(p_i) + a_1 \sum_{i=1}^n \sum_{j=1}^m f_{r-1}(p_i) f_1(q_j) + \\ &\dots + a_{r-1} \sum_{i=1}^n \sum_{j=1}^m f_1(p_i) f_{r-1}(q_j) + \sum_{j=1}^m f_r(q_j) \end{aligned} \quad (5.9)$$

valid for all $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$; $n = 1, 2, \dots$; $m = 1, 2, \dots$

The general solutions of (5.8), $r \geq 3$ an integer, are not known to us. Also, the general solutions of (5.9), $r \geq 3$ an integer, $(p_1, \dots, p_n) \in \Gamma_n$,

$(q_1, \dots, q_m) \in \Gamma_m$, for all integers $n \geq 1$, $m \geq 1$ or with some restrictions(s) on n and m such as $n \geq 3$, $m \geq 3$ fixed integers, are not known.

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(Received: November 29, 2014)

(Revised: June 18, 2015)

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