UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING A CERTAIN NONZERO POLYNOMIAL WITH WEIGHT

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ABSTRACT. In this paper we investigate the uniqueness problem of meromorphic functions when certain non-linear differential polynomials share a nonzero polynomial. We prove a uniqueness result which significantly rectify, improves and generalizes the recent results of P. Sahoo and S. Seikh [12] as well as X. M. Li and L. Cao [10].

1. INTRODUCTION, DEFINITIONS AND RESULTS

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

We adopt the standard notations of value distribution theory (see [4]). For a non-constant meromorphic function f , we denote by $T(r, f)$ the Nevanlinna characteristic of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \rightarrow \infty$ possibly outside a set of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to f , provided that $T(r, a) = S(r, f)$.

We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \rightarrow \infty$, outside of a possible exceptional set of finite linear measure.

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In 1959, W.K. Hayman (see [3], Corollary of Theorem 9) proved the following theorem.

Theorem A. *Let f be a transcendental meromorphic function and $n(\geq 3)$ be an integer. Then $f^n f' = 1$ has infinitely many solutions.*

Yang and Hua [15] got a unicity theorem corresponding to Theorem A.

Theorem B. *Let f and g be two non-constant meromorphic functions, $n \in \mathbb{N}$ such that $n \geq 11$. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where $c_1, c_2, c \in \mathbb{C}$ such that $(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

In 2002, Fang and Qiu [1] proved the following theorem.

Theorem C. *Let f and g be two non-constant meromorphic functions, and $n \in \mathbb{N}$ such that $n \geq 11$. If $f^n f' - z$ and $g^n g' - z$ share 0 CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where $c_1, c_2, c \in \mathbb{C}$ such that $4(c_1 c_2)^{n+1} c^2 = -1$ or $f \equiv tg$ for a constant t such that $t^{n+1} = 1$.*

Recently Li and Gao [10] proved the following result.

Theorem D. *Let f and g be two transcendental meromorphic functions, let $n \in \mathbb{N}$ such that $n \geq 11$ and let $P \not\equiv 0$ be a polynomial with its degree $\gamma_P \leq 11$. If $f^n f' - P$ and $g^n g' - P$ share 0 CM, then either $f \equiv tg$ for a constant t such that $t^{n+1} = 1$ or $f(z) = c_1 e^{cQ}$, $g(z) = c_2 e^{-cQ}$, where $c_1, c_2, c \in \mathbb{C}$ such that $(c_1 c_2)^{n+1} c^2 = -1$, Q is a polynomial satisfying $Q(z) = \int_0^z P(\eta) d\eta$.*

We now explain the notation of weighted sharing as introduced in ([5], [6]).

Definition 1. [5, 6] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a -points of f , where an a -point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k .

We write f, g share (a, k) to mean that f, g share the value a with weight k . Clearly if f, g share (a, k) , then f, g share (a, p) for any integer p , $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Noting that $f^n(z) f'(z) = \frac{1}{n+1} (f^{n+1}(z))'$, P. Sahoo and S. Seikh [12] considered the case of k -th derivative and proved the following result.

Theorem E. *Let f and g be two transcendental meromorphic functions, let n, k be two positive integers such that $n \geq 3k + 9$ and let $P \not\equiv 0$ be a polynomial with its degree $\gamma_P \leq n - 1$. Let $(f^n)^{(k)} - P$ and $(g^n)^{(k)} - P$ share $(0, 2)$. Then*

- (i) if $k = 1$, either $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$ or $f(z) = c_1 e^{cQ(z)}$ and $g(z) = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(z)dz$, c_1, c_2 and c are constants such that $c^2(c_1 c_2)^n = -1$,
- (ii) if $k \geq 2$, either $(f^n)^{(k)}(g^n)^{(k)} \equiv P^2$ or $f \equiv tg$ for a complex number t satisfying $t^n = 1$.

Remark 1. Both Theorem 1.1 [10] and Theorem 1 [12] are new and seem fine. However in proof of both Theorem 1.1 [10] and Theorem 1 [12] there is an error.

The authors declare that Lemma 2.8 [10] can be obtained from Theorem 4.8 of [17]. But the following example shows that Lemma 2.8 of [10] is invalid.

Example 1.1. Let $F = ze^z$, $G = \frac{1}{ze^z}$, then F and G share 1 and -1 and satisfy

$$N(r, 0; F) + N(r, \infty; F) = S(r, F) \quad \text{and} \quad N(r, 0; G) + N(r, \infty; G) = S(r, G).$$

It is clear that F and G share neither 0 nor ∞ .

Since Lemma 2.8 [10] plays an important role in proving Lemma 2.10 [10] and the reason as above, it is thought that there exists a big gap in the proof of Lemma 2.10 [10].

Now existence of Theorem 1.1 [10] as well as Theorem 1 [12] are questionable here. Also it is quite natural to ask the following question:

Question. Keeping all other conditions intact can the first conclusion in Theorem E be removed when $k \geq 2$?

In this paper taking the possible answer of the above question in background we study Theorem 1.1 [10] as well as Theorem 1 [12] again in more general form with out help of Lemma 2.8 [10] as well as Lemma 2.10 of [10] and obtain the following results.

Theorem 1. Let f and g be two transcendental meromorphic functions. Let $p(z)$ be a nonzero polynomial such that either $\deg(p) \leq n-1$ or zeros of $p(z)$ are of multiplicities atmost $n-k-1$, where $n, k \in \mathbb{N}$ such that $n > 3k+8$. Suppose $(f^n)^{(k)} - p$ and $(g^n)^{(k)} - p$ share $(0, 2)$, then one of the following two conclusions holds:

- (1) $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$,
- (2) if $p(z)$ is not a constant, then $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)dt$, $c_1, c_2, c \in \mathbb{C}$ such that $(nc)^2(c_1 c_2)^n = -1$, if $p(z) = b \in \mathbb{C} \setminus \{0\}$, then $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, where $c_3, c_4, d \in \mathbb{C}$ such that $(-1)^k(c_3 c_4)^n(nd)^{2k} = b^2$.

Taking $k = 1$ and choosing $n+1$ instead of n in Theorem 1 we get the following result.

Corollary 1. *Let f and g be two transcendental meromorphic functions, let $n \in \mathbb{N}$ such that $n \geq 11$ and let $P \not\equiv 0$ be a polynomial such that either $\deg(P) \leq n$ or zeros of $P(z)$ are of multiplicities at most $n - 1$. If $f^n f' - P$ and $g^n g' - P$ share $(0, 2)$, then either $f \equiv tg$ for a constant t such that $t^{n+1} = 1$ or $f(z) = c_1 e^{cQ}$, $g(z) = c_2 e^{-cQ}$, where c_1, c_2 and c are three constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$, Q is a polynomial satisfying $Q = \int_0^z P(\eta) d\eta$.*

We now explain the following definitions and notations which are used in the paper.

Definition 2. [7] Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

- (i) $N(r, a; f \mid \geq p)$ ($\overline{N}(r, a; f \mid \geq p)$) denotes the counting function (reduced counting function) of those a -points of f whose multiplicities are not less than p .
- (ii) $N(r, a; f \mid \leq p)$ ($\overline{N}(r, a; f \mid \leq p)$) denotes the counting function (reduced counting function) of those a -points of f whose multiplicities are not greater than p .

Definition 3. [18] For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by $N_p(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid \geq 2) + \cdots + \overline{N}(r, a; f \mid \geq p)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 4. Let $a, b \in \mathbb{C} \cup \{\infty\}$. Let p be a positive integer. We denote by $\overline{N}(r, a; f \mid \geq p \mid g = b)$ ($\overline{N}(r, a; f \mid \geq p \mid g \neq b)$) the reduced counting function of those a -points of f with multiplicities $\geq p$, which are the b -points (not the b -points) of g .

Definition 5. [5, 6] Let f, g share a value a IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a -points of f whose multiplicities differ from the multiplicities of the corresponding a -points of g . Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$.

2. LEMMAS

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H the function as follows:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right). \quad (2.1)$$

Lemma 1. [14] *Let f be a non-constant meromorphic function and let $a_n(z) (\neq 0)$, $a_{n-1}(z)$, $\dots$, $a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, \dots, n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2. [11] *Let f be a non-constant meromorphic function, and $s, k \in \mathbb{N}$. Then*

$$N_s(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{s+k}(r, 0; f) + S(r, f), \quad (2.2)$$

$$N_s(r, 0; f^{(k)}) \leq k\overline{N}(r, \infty; f) + N_{s+k}(r, 0; f) + S(r, f). \quad (2.3)$$

Lemma 3. [8] *If $N(r, 0; f^{(k)} | f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f , where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r, 0; f^{(k)} | f \neq 0) \leq k\overline{N}(r, \infty; f) + N(r, 0; f | < k) + k\overline{N}(r, 0; f | \geq k) + S(r, f).$$

Lemma 4. [13] *Let f and g be two non-constant meromorphic functions, $k, n \in \mathbb{N}$ such that $n > 2k + 1$. If $(f^n)^{(k)} \equiv (g^n)^{(k)}$, then $f \equiv tg$ for a constant t such that $t^n = 1$.*

Lemma 5. [9] *Let f_1 and f_2 be two non-constant meromorphic functions satisfying $\overline{N}(r, 0; f_i) + \overline{N}(r, \infty; f_i) = S(r; f_1, f_2)$ for $i = 1, 2$. If $f_1^s f_2^t - 1$ is not identically zero for arbitrary integers s and t ($|s| + |t| > 0$), then for any positive ε , we have*

$$N_0(r, 1; f_1, f_2) \leq \varepsilon T(r) + S(r; f_1, f_2),$$

where $N_0(r, 1; f_1, f_2)$ denotes the deduced counting function related to the common 1-points of f_1 and f_2 and $T(r) = T(r, f_1) + T(r, f_2)$, $S(r; f_1, f_2) = o(T(r))$ as $r \rightarrow \infty$ possibly outside a set of finite linear measure.

Lemma 6. [4] *Suppose that f is a non-constant meromorphic function, $k \geq 2$ is an integer. If*

$$N(r, \infty; f) + N(r, 0; f) + N(r, 0; f^{(k)}) = S(r, \frac{f'}{f}),$$

then $f(z) = e^{az+b}$, where $a \neq 0$, b are constants.

Lemma 7. [2] *Let $f(z)$ be a non-constant entire function and let $k \geq 2$ be a positive integer. If $f(z)f^{(k)}(z) \neq 0$, then $f(z) = e^{az+b}$, where $a \neq 0$, b are constant.*

Lemma 8. [17] *Let f be a non-constant meromorphic function, and let k be a positive integer. Suppose that $f^{(k)} \not\equiv 0$, then*

$$N(r, 0; f^{(k)}) \leq N(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f).$$

Lemma 9. *Let f, g be two non-constant meromorphic functions, let n and k be two positive integers with $n > k + 2$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share α IM, where $\alpha (\neq 0, \infty)$ is a small function of f and g , then $T(r, f) = O(T(r, g))$ and $T(r, g) = O(T(r, f))$.*

Proof. Let $F = f^n$. By the second fundamental theorem for small functions (see [16]), we have

$$T(r, F^{(k)}) \leq \overline{N}(r, \infty; F^{(k)}) + \overline{N}(r, 0; F^{(k)}) + \overline{N}(r, \alpha; F^{(k)}) + (\varepsilon + o(1))T(r, F),$$

for all $\varepsilon > 0$. Now in view of Lemmas 1 and 2 for $p = 1$ and using above we get

$$\begin{aligned} & n T(r, f) \\ & \leq T(r, [f^n]^{(k)}) - \overline{N}(r, 0; [f^n]^{(k)}) + N_{k+1}(r, 0; f^n) + (\varepsilon + o(1))T(r, f) \\ & \leq \overline{N}(r, 0; [f^n]^{(k)}) + \overline{N}(r, \infty; f) + \overline{N}(r, \alpha; [f^n]^{(k)}) - \overline{N}(r, 0; [f^n]^{(k)}) \\ & \quad + N_{k+1}(r, 0; f^n) + (\varepsilon + o(1))T(r, f) \\ & \leq \overline{N}(r, \infty; f) + \overline{N}(r, \alpha; [f^n]^{(k)}) + (k+1)\overline{N}(r, 0; f) + (\varepsilon + o(1))T(r, f) \\ & \leq (k+2) T(r, f) + \overline{N}(r, \alpha; [g^n]^{(k)}) + (\varepsilon + o(1))T(r, f) \\ & \leq (k+2) T(r, f) + n(k+1) T(r, g) + (\varepsilon + o(1))T(r, f), \end{aligned}$$

i.e., $(n-k-2) T(r, f) \leq n(k+1) T(r, g) + (\varepsilon + o(1))T(r, f)$. Since $n > k+2$, take $\varepsilon < 1$ and we have $T(r, f) = O(T(r, g))$. Similarly we have $T(r, g) = O(T(r, f))$. This completes the proof of the lemma. $\square$

Lemma 10. *Let f, g be two transcendental meromorphic functions and $p(z)$ be a nonzero polynomial such that either $\deg(p) \leq n-1$ or zeros of $p(z)$ are of multiplicities atmost $n-k-1$, where $n, k \in \mathbb{N}$ such that $n > \max\{2k, k+2\}$. Suppose $(f^n)^{(k)}(g^n)^{(k)} \equiv p^2$, where $(f^n)^{(k)} - p(z)$ and $(g^n)^{(k)} - p(z)$ share 0 CM. Now*

(i) *if $p(z)$ is not a constant, then $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)dt$, $c_1, c_2, c \in \mathbb{C}$ such that $(nc)^2(c_1 c_2)^n = -1$, (ii) if $p(z) = b \in \mathbb{C} \setminus \{0\}$, then $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, where $c_3, c_4, d \in \mathbb{C}$ such that $(-1)^k(c_3 c_4)^n (nd)^{2k} = b^2$.*

Proof. Suppose

$$(f^n)^{(k)}(g^n)^{(k)} \equiv p^2. \quad (2.4)$$

We consider the following cases:

Case 1. Let $\deg(p(z)) = l(\geq 1)$.

Let $z_0(p(z_0) \neq 0)$ be a zero of f with multiplicity q . Note that z_0 is a zero of $(f^n)^{(k)}$ with multiplicity $nq - k$. Obviously z_0 will be a pole of g with multiplicity q_1 , say. Note that z_0 is a pole of $(g^n)^{(k)}$ with multiplicity $nq_1 + k$ and so $nq - k = nq_1 + k$. Now $nq - k = nq_1 + k$ implies that

$$n(q - q_1) = 2k. \quad (2.5)$$

Since $n > 2k$, we get a contradiction from (2.5). This shows that z_0 is a zero of $p(z)$ and so we have $N(r, 0; f) = O(\log r)$. Similarly we can prove that

$N(r, 0; g) = O(\log r)$. Thus in general we can take $N(r, 0; f) + N(r, 0; g) = O(\log r)$. We know that

$$N(r, \infty; (f^n)^{(k)}) = n N(r, \infty; f) + k \bar{N}(r, \infty; f).$$

Also by Lemma 8 we have

$$\begin{aligned} N(r, 0; (g^n)^{(k)}) &\leq n N(r, 0; g) + k \bar{N}(r, \infty; g) + S(r, g) \\ &\leq k \bar{N}(r, \infty; g) + O(\log r) + S(r, g). \end{aligned}$$

From (2.4) we get $N(r, \infty; (f^n)^{(k)}) = N(r, 0; (g^n)^{(k)})$, i.e.,

$$n N(r, \infty; f) + k \bar{N}(r, \infty; f) \leq k \bar{N}(r, \infty; g) + O(\log r) + S(r, g). \quad (2.6)$$

Similarly we get

$$n N(r, \infty; g) + k \bar{N}(r, \infty; g) \leq k \bar{N}(r, \infty; f) + O(\log r) + S(r, f). \quad (2.7)$$

Since f and g are transcendental, it follows that

$$S(r, f) + O(\log r) = S(r, f), \quad S(r, g) + O(\log r) = S(r, g).$$

Now combining (2.6) and (2.7) we get $N(r, \infty; f) + N(r, \infty; g) = S(r, f) + S(r, g)$. By Lemma 9 we have $S(r, f) = S(r, g)$ and so we obtain $N(r, \infty; f) = S(r, f)$ and $N(r, \infty; g) = S(r, g)$. Let

$$F_1 = \frac{(f^n)^{(k)}}{p} \quad \text{and} \quad G_1 = \frac{(g^n)^{(k)}}{p}. \quad (2.8)$$

Note that $T(r, F_1) \leq n(k+1)T(r, f) + S(r, f)$ and so $T(r, F_1) = O(T(r, f))$. Also by Lemma 2, one can obtain $T(r, f) = O(T(r, F_1))$. Hence $S(r, F_1) = S(r, f)$. Similarly we get $S(r, G_1) = S(r, g)$. Hence we get $S(r, F_1) = S(r, G_1)$. From (2.4) we get

$$F_1 G_1 \equiv 1. \quad (2.9)$$

If $F_1 \equiv cG_1$, where c is a nonzero constant, then F_1 is a constant and so f is a polynomial, which contradicts our assumption. Hence $F_1 \not\equiv cG_1$ and so in view of (2.9) we see that F_1 and G_1 share -1 IM. Now by Lemma 8 we have

$$N(r, 0; F_1) \leq n N(r, 0; f) + k \bar{N}(r, \infty; f) + S(r, f) \leq S(r, F_1).$$

Similarly we have

$$N(r, 0; G_1) \leq n N(r, 0; g) + k \bar{N}(r, \infty; g) + S(r, g) \leq S(r, G_1).$$

Also we see that $N(r, \infty; F_1) = S(r, F_1)$, $N(r, \infty; G_1) = S(r, G_1)$. Clearly $\bar{N}(r, 1; F_1) \sim T(r, F_1)$ and $\bar{N}(r, 1; G_1) \sim T(r, G_1)$.

Also $T(r, F_1) = T(r, G_1) + O(1)$.

Let

$$f_1 = \frac{F_1}{G_1} \quad \text{and} \quad f_2 = \frac{F_1 - 1}{G_1 - 1}.$$

Clearly f_1 is non-constant. If f_2 is a nonzero constant then F_1 and G_1 share ∞ CM and so from (2.9) we conclude that F_1 and G_1 have no poles.

Next we suppose that f_2 is non-constant. Also we see that

$$F_1 = \frac{f_1(1 - f_2)}{f_1 - f_2}, \quad G_1 = \frac{1 - f_2}{f_1 - f_2}.$$

Clearly $T(r, F_1) \leq 2[T(r, f_1) + T(r, f_2)] + O(1)$ and $T(r, f_1) + T(r, f_2) \leq 4T(r, F_1) + O(1)$. These give $S(r, F_1) = S(r, f_1, f_2)$. Also we see that $\overline{N}(r, 0; f_i) + \overline{N}(r, \infty; f_i) = S(r, f_1, f_2)$ for $i = 1, 2$.

Next we suppose $\overline{N}(r, -1; F_1) \neq S(r, F_1)$, otherwise F_1 will be a constant.

Also we see that $\overline{N}(r, -1; F_1) \leq N_0(r, 1; f_1, f_2)$. Thus we have

$$T(r, f_1) + T(r, f_2) \leq 4 N_0(r, 1; f_1, f_2) + S(r, F_1).$$

Then by Lemma 5 there exist two integers s and t ($|s| + |t| > 0$) such that $f_1^s f_2^t \equiv 1$, i.e.,

$$\left[\frac{F_1}{G_1} \right]^s \left[\frac{F_1 - 1}{G_1 - 1} \right]^t \equiv 1. \quad (2.10)$$

We now consider the following cases:

Case (i) Let $s = 0$ and $t \neq 0$. Then from (2.10) we get $(F_1 - 1)^t \equiv (G_1 - 1)^t$. This shows that F_1 and G_1 share ∞ CM and so from (2.9) we conclude that F_1 and G_1 have no poles.

Case (ii) Suppose $s \neq 0$ and $t = 0$. Then from (2.10) we get $F_1^s \equiv G_1^s$ and so we arrive at a contradiction from (2.9).

Case (iii) Suppose $s > 0$ and $t = -t_1$, where $t_1 > 0$. Then we have

$$\left[\frac{F_1}{G_1} \right]^s \equiv \left[\frac{F_1 - 1}{G_1 - 1} \right]^{t_1}. \quad (2.11)$$

If possible suppose F_1 has a pole. Let z_{p_1} be a pole of F_1 of multiplicity p_1 . Then from (2.9) we see that z_{p_1} must be a zero of G_1 of multiplicity p_1 . Now from (2.11) we get $2s = t_1$ and so

$$\left[\frac{F_1}{G_1} \right]^s \equiv \left[\frac{F_1 - 1}{G_1 - 1} \right]^{2s}.$$

This implies that

$$\frac{F_1^s - G_1^s}{G_1^s} \equiv \frac{(F_1 - 1)^{2s} - (G_1 - 1)^{2s}}{(G_1 - 1)^{2s}},$$

i.e.,

$$F_1^{s-1} + F_1^{s-2}G_1 + F_1^{s-3}G_1^2 + \cdots + F_1G_1^{s-2} + G_1^{s-1}$$

$$\equiv G_1^s \frac{(F_1 - 1)^{2s} - (G_1 - 1)^{2s}}{(G_1 - 1)^{2s}(F_1 - G_1)}. \quad (2.12)$$

If z_p is a zero of $F_1 - 1$ with multiplicity p then the Taylor expansion of $F_1 - 1$ about z_p is

$$F_1 - 1 = a_p(z - z_p)^p + a_{p+1}(z - z_p)^{p+1} + \dots, \quad a_p \neq 0.$$

Since $F_1 - 1$ and $G_1 - 1$ share 0 CM,

$$G_1 - 1 = b_p(z - z_p)^p + b_{p+1}(z - z_p)^{p+1} + \dots, \quad b_p \neq 0.$$

Let

$$\Phi_1 = \frac{F_1'}{F_1} - \frac{G_1'}{G_1} \quad (2.13)$$

and

$$\Phi_2 = \left(\frac{F_1'}{F_1}\right)^{2s} - \left(\frac{G_1'}{G_1}\right)^{2s} \quad (2.14)$$

Since $F_1 \not\equiv cG_1$, where c is a nonzero constant, it follows that $\Phi_1 \not\equiv 0$ and $\Phi_2 \not\equiv 0$. Also $T(r, \Phi_1) = S(r, F_1)$ and $T(r, \Phi_2) = S(r, F_1)$. From (2.13) we find $\bar{N}_{(2)}(r, 1; F_1) = \bar{N}_{(2)}(r, 1; G_1) = S(r, F_1)$. Let $p = 1$. If $a_1 = b_1$, then by an elementary calculation gives that $\Phi_1(z) = O((z - z_1)^k)$, where k is a positive integer. This proves that z_1 is a zero of Φ_1 .

Next we suppose $a_1 \neq b_1$, but $a_1^{2s} = b_1^{2s}$.

Then by an elementary calculation we get $\Phi_2(z) = O((z - z_1)^q)$, where q is a positive integer. This proves that z_1 is a zero of Φ_2 .

Finally we suppose $a_1 \neq b_1$ and $a_1^{2s} \neq b_1^{2s}$. Therefore from (2.12) we arrive at a contradiction. Hence $N_1(r, 1; F_1) = N_1(r, 1; G_1) = S(r, F_1)$ and so $\bar{N}(r, 1; F_1) = \bar{N}(r, 1; G_1) = S(r, F_1)$. But this is impossible as $\bar{N}(r, 1; F_1) \sim T(r, F_1)$ and $\bar{N}(r, 1; G_1) \sim T(r, G_1)$.

Hence F_1 has no pole. Similarly we can prove that G_1 also has no poles.

Case (iv) Suppose either $s > 0$ and $t > 0$ or $s < 0$ and $t < 0$. Then from (2.10) one can easily prove that F_1 and G_1 have no poles.

Finally we conclude that F_1 and G_1 have no poles.

Consequently from (2.9) we see that F_1 and G_1 have no zeros. we deduce from (2.8) that both f and g have no pole. Since F_1 and G_1 have no zeros and poles, we have $F_1 \equiv e^{\gamma_1} G_1$, i.e., $(f^n)^{(k)} \equiv e^{\gamma_1} (g^n)^{(k)}$, where γ_1 is a non-constant entire function. Then from (2.4) we get

$$(f^n)^{(k)} \equiv ce^{\frac{1}{2}\gamma_1} p(z), \quad (g^n)^{(k)} \equiv ce^{-\frac{1}{2}\gamma_1} p(z),$$

where $c \in \{-1, 1\}$. This shows that $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share 0 CM. Also we deduce from (2.8) that both f and g are transcendental entire functions. Since $N(r, 0; f) = O(\log r)$ and $N(r, 0; g) = O(\log r)$, so we can take

$$f(z) = h_1(z)e^{\alpha(z)}, \quad g(z) = h_2(z)e^{\beta(z)}, \quad (2.15)$$

where h_1 and h_2 are nonzero polynomials and α, β are two non-constant entire functions.

We deduce from (2.4) and (2.15) that either both α and β are transcendental entire functions or both α and β are polynomials. We consider the following cases:

Subcase 1.1. Let $k \geq 2$.

First we suppose both α and β are transcendental entire functions.

Let $\alpha_1 = \alpha' + \frac{h_1'}{h_1}$ and $\beta_1 = \beta' + \frac{h_2'}{h_2}$. Clearly both α_1 and β_1 are transcendental functions. Note that

$$S(r, n\alpha_1) = S\left(r, \frac{(f^n)'}{f^n}\right), \quad S(r, n\beta_1) = S\left(r, \frac{(g^n)'}{g^n}\right).$$

Moreover we see that

$$N(r, 0; (f^n)^{(k)}) \leq N(r, 0; p^2) = O(\log r) \text{ and} \\ N(r, 0; (g^n)^{(k)}) \leq N(r, 0; p^2) = O(\log r).$$

From these and using (2.15) we have

$$N(r, \infty; f^n) + N(r, 0; f^n) + N(r, 0; (f^n)^{(k)}) = S(r, n\alpha_1) = S\left(r, \frac{(f^n)'}{f^n}\right) \quad (2.16)$$

and

$$N(r, \infty; g^n) + N(r, 0; g^n) + N(r, 0; (g^n)^{(k)}) = S(r, n\beta_1) = S\left(r, \frac{(g^n)'}{g^n}\right). \quad (2.17)$$

Then from (2.16), (2.17) and Lemma 6 we must have $f(z) = e^{az+b}$, $g(z) = e^{cz+d}$, where $a \neq 0$, $b, c \neq 0$ and d are constants. But these types of f and g do not satisfy relation (2.4).

Next we suppose α and β are both polynomials.

Also from (2.4) we get $\alpha + \beta \equiv C$ i.e., $\alpha' \equiv -\beta'$. Therefore $\deg(\alpha) = \deg(\beta)$. We deduce from (2.15) that

$$(f^n)^{(k)} \equiv Ah_1^{n-k}[h_1^k(\alpha')^k + P_{k-1}(\alpha', h_1')]e^{n\alpha} \equiv A_1pe^{n\alpha}, \quad (2.18)$$

and

$$(g^n)^{(k)} \equiv Bh_2^{n-k}[h_2^k(\beta')^k + Q_{k-1}(\beta', h_2')]e^{n\beta} \equiv B_1pe^{n\beta}, \quad (2.19)$$

where A, B, A_1, B_1 are nonzero constants, $P_{k-1}(\alpha', h_1')$ and $Q_{k-1}(\beta', h_2')$ are differential polynomials in α', h_1' and β', h_2' respectively. By virtue of polynomial p , from (2.18) and (2.19) we conclude that both h_1 and h_2 are nonzero constant. So we can rewrite f and g as follows:

$$f = e^{\gamma_2}, \quad g = e^{\delta_2}, \quad (2.20)$$

where $\gamma_2 + \delta_2 \equiv C$ and $\deg(\gamma_2) = \deg(\delta_2)$. If $\deg(\gamma_2) = \deg(\delta_2) = 1$, then we again get a contradiction from (2.4). Next we suppose $\deg(\gamma_2) = \deg(\delta_2) \geq 2$. We deduce from (2.20) that

$$\begin{aligned}
(f^n)' &= n\gamma_2' e^{n\gamma_2} \\
(f^n)'' &= [n^2(\gamma_2')^2 + n\gamma_2''] e^{n\gamma_2} \\
(f^n)''' &= [n^3(\gamma_2')^3 + 3n^2\gamma_2'\gamma_2'' + n\gamma_2'''] e^{n\gamma_2} \\
(f^n)^{(iv)} &= [n^4(\gamma_2')^4 + 6n^3(\gamma_2')^2\gamma_2'' + 3n^2(\gamma_2'')^2 + 4n^2\gamma_2'\gamma_2''' + n\gamma_2^{(iv)}] e^{n\gamma_2} \\
(f^n)^{(v)} &= [n^5(\gamma_2')^5 + 10n^4(\gamma_2')^3\gamma_2'' + 15n^3\gamma_2'(\gamma_2'')^2 + 10n^3(\gamma_2')^2\gamma_2''' + 10n^2\gamma_2''\gamma_2''' \\
&\quad + 5n^2\gamma_2'\gamma_2^{(iv)} + n\gamma_2^{(v)}] e^{n\gamma_2} \\
&\dots \dots \dots \dots \dots \dots \dots \dots \\
(f^n)^{(k)} &= [n^k(\gamma_2')^k + K(\gamma_2')^{k-2}\gamma_2'' + P_{k-2}(\gamma_2')] e^{n\gamma_2}.
\end{aligned}$$

Similarly we get

$$\begin{aligned}
(g^n)^{(k)} &= [n^k(\delta_2')^k + K(\delta_2')^{k-2}\delta_2'' + P_{k-2}(\delta_2')] e^{n\delta_2} \\
&= [(-1)^k n^k (\gamma_2')^k - K(-1)^{k-2} (\gamma_2')^{k-2} \gamma_2'' + P_{k-2}(-\gamma_2')] e^{n\delta_2},
\end{aligned}$$

where K is a suitably positive integer and $P_{k-2}(\gamma_2')$ is a differential polynomial in γ_2' .

Since $\deg(\gamma_2) \geq 2$, we observe that $\deg((\gamma_2')^k) \geq k \deg(\gamma_2')$ and so $(\gamma_2')^{k-2}\gamma_2''$ is either a nonzero constant or $\deg((\gamma_2')^{k-2}\gamma_2'') \geq (k-1) \deg(\gamma_2') - 1$. Also we see that

$$\deg((\gamma_2')^k) > \deg((\gamma_2')^{k-2}\gamma_2'') > \deg(P_{k-2}(\gamma_2')) \left(\text{or } \deg(P_{k-2}(-\gamma_2')) \right).$$

Since $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 0 CM, the polynomials

$$n^k(\gamma_2')^k + K(\gamma_2')^{k-2}\gamma_2'' + P_{k-2}(\gamma_2')$$

and

$$(-1)^k n^k (\gamma_2')^k - K(-1)^{k-2} (\gamma_2')^{k-2} \gamma_2'' + P_{k-2}(-\gamma_2')$$

must be identical but this is impossible for $k \geq 2$. Actually the terms $n^k(\gamma_2')^k + K(\gamma_2')^{k-2}\gamma_2''$ and $(-1)^k n^k (\gamma_2')^k - K(-1)^{k-2} (\gamma_2')^{k-2} \gamma_2''$ can not be identical for $k \geq 2$.

Subcase 1.2. Let $k = 1$.

Now from (2.4) we get

$$f^{n-1} f' g^{n-1} g' \equiv p_1^2, \quad (2.21)$$

where $p_1^2 = \frac{1}{n^2}p^2$.

First we suppose both α and β are transcendental entire functions.

Let $h = fg$ and we consider the following subcases :

Subcase 1.2.1. Suppose that h is a nonzero polynomial.

Then from (2.15), it is clear that $h = Ah_1h_2$, where $A = e^C$ and $\alpha + \beta \equiv C$ a constant. Therefore $\alpha' \equiv -\beta'$. Now from (2.21) we see that

$$A\alpha'(-h_1'h_2 + h_1h_2' - h_1h_2\alpha') \equiv e^{-(n-1)C} \frac{p_1^2}{(h_1h_2)^{n-1}} - Ah_1'h_2,$$

where $\frac{p_1^2}{(h_1h_2)^{n-1}}$ is a polynomial. It is clear that $N(r, 0; \alpha') = O(\log r)$ and $N(r, 0; -h_1'h_2 + h_1h_2' - h_1h_2\alpha') = O(\log r)$. By the second fundamental theorem for small functions (see [16]), we have

$$\begin{aligned} T(r, \alpha') &\leq \overline{N}(r, \infty; \alpha') + \overline{N}(r, 0; \alpha') + \overline{N}(r, 0; -h_1'h_2 + h_1h_2' - h_1h_2\alpha') \\ &\quad + (\varepsilon + o(1))T(r, \alpha') \leq O(\log r) + (\varepsilon + o(1))T(r, \alpha'), \end{aligned}$$

for all $\varepsilon > 0$. This shows that α' is a polynomial and so is α , which is a contradiction.

Subcase 1.2.2. Suppose that h is a transcendental entire function.

Now from (2.21) we get

$$\left(\frac{g'}{g} - \frac{1}{2} \frac{h'}{h}\right)^2 \equiv \frac{1}{4} \left(\frac{h'}{h}\right)^2 - h^{-n}p_1^2. \quad (2.22)$$

Let

$$\alpha_2 = \frac{g'}{g} - \frac{1}{2} \frac{h'}{h}.$$

From (2.22) we get

$$\alpha_2^2 \equiv \frac{1}{4} \left(\frac{h'}{h}\right)^2 - h^{-n}p_1^2. \quad (2.23)$$

First we suppose $\alpha_2 \equiv 0$. Then we get $h^{-n}p_1^2 \equiv \frac{1}{4} \left(\frac{h'}{h}\right)^2$ and so $T(r, h) = S(r, h)$, which is impossible. Next we suppose that $\alpha_2 \not\equiv 0$. Differentiating (2.23) we get

$$2\alpha_2\alpha_2' \equiv \frac{1}{2} \frac{h'}{h} \left(\frac{h'}{h}\right)' + n h' h^{-n-1} p_1^2 - 2h^{-n} p_1 p_1'.$$

Applying (2.23) we obtain

$$h^{-n} \left(-n \frac{h'}{h} p_1^2 + 2p_1 p_1' - 2 \frac{\alpha_2'}{\alpha_2} p_1^2 \right) \equiv \frac{1}{2} \frac{h'}{h} \left(\left(\frac{h'}{h} \right)' - \frac{h'}{h} \frac{\alpha_2'}{\alpha_2} \right). \quad (2.24)$$

First we suppose

$$-n \frac{h'}{h} p_1^2 + 2p_1 p_1' - 2 \frac{\alpha_2'}{\alpha_2} p_1^2 \equiv 0.$$

Then there exist a non-zero constant c such that $\alpha_2^2 \equiv ch^{-n} p_1^2$ and so from (2.23) we get

$$(c+1)h^{-n} p_1^2 \equiv \frac{1}{4} \left(\frac{h'}{h} \right)^2.$$

If $c = -1$, then h will be a constant. If $c \neq -1$, then we have $T(r, h) = S(r, h)$, which is impossible. Next we suppose that

$$-n \frac{h'}{h} p_1^2 + 2p_1 p_1' - 2 \frac{\alpha_2'}{\alpha_2} p_1^2 \not\equiv 0.$$

Then by (2.24) we have

$$\begin{aligned} & n T(r, h) \\ &= n m(r, h) \\ &\leq m \left(r, h^n \frac{1}{2} \frac{h'}{h} \left(\left(\frac{h'}{h} \right)' - \frac{h'}{h} \frac{\alpha_2'}{\alpha_2} \right) \right) + m \left(r, \frac{1}{\frac{1}{2} \frac{h'}{h} \left(\left(\frac{h'}{h} \right)' - \frac{h'}{h} \frac{\alpha_2'}{\alpha_2} \right)} \right) + O(1) \\ &\leq T \left(r, \frac{1}{2} \frac{h'}{h} \left(\left(\frac{h'}{h} \right)' - \frac{h'}{h} \frac{\alpha_2'}{\alpha_2} \right) \right) + m \left(r, n \frac{h'}{h} p_1^2 - 2p_1 p_1' + 2 \frac{\alpha_2'}{\alpha_2} p_1^2 \right) \\ &\leq \overline{N}(r, 0; \alpha_2) + S(r, h) + S(r, \alpha_2). \end{aligned} \quad (2.25)$$

From (2.23) we get $T(r, \alpha_2) \leq \frac{1}{2} n T(r, h) + S(r, h)$. Now from (2.25) we get $\frac{1}{2} n T(r, h) \leq S(r, h)$, which is impossible.

Thus α and β are both polynomials. Also from (2.4) we can conclude that $\alpha(z) + \beta(z) \equiv C$ for a constant C and so $\alpha'(z) + \beta'(z) \equiv 0$. We deduce from (2.4) that

$$(f^n)' \equiv n[h_1^n \alpha' + h_1^{n-1} h_1'] e^{n\alpha} \equiv p(z) e^{n\alpha}, \quad (2.26)$$

and

$$(g^n)' = n[h_2^n \beta' + h_2^{n-1} h_2'] e^{n\beta} \equiv p(z) e^{n\beta}. \quad (2.27)$$

By virtue of polynomial p , from (2.26) and (2.27) we conclude that both h_1 and h_2 are nonzero constant.

So we can rewrite f and g as follows:

$$f = e^{\gamma_3}, \quad g = e^{\delta_3}. \quad (2.28)$$

Now from (2.4) we get

$$n^2 \gamma_3' \delta_3' e^{n(\gamma_3 + \delta_3)} \equiv p^2. \quad (2.29)$$

Also from (2.29) we can conclude that $\gamma_3(z) + \delta_3(z) \equiv C$ for a constant C and so $\gamma_3'(z) + \delta_3'(z) \equiv 0$. Thus from (2.29) we get $n^2 e^{nC} \gamma_3' \delta_3' \equiv p^2(z)$. By computation we get $\gamma_3'(z) = cp(z)$ and $\delta_3'(z) = -cp(z)$. Hence $\gamma_3(z) = cQ(z) + b_1$ and $\delta_3(z) = -cQ(z) + b_2$, where $Q(z) = \int_0^z p(t)dt$ and $b_1, b_2 \in \mathbb{C}$. Finally we take f and g as $f(z) = c_1 e^{cQ(z)}$ and $g(z) = c_2 e^{-cQ(z)}$, where $c_1, c_2, c \in \mathbb{C}$ such that $(nc)^2 (c_1 c_2)^n = -1$.

Case 2. Let $p(z) = b \in \mathbb{C} \setminus \{0\}$. Since $n > 2k$, we have $f \neq 0$ and $g \neq 0$. Now using Case 1 we can prove that $f = e^\alpha$ and $g = e^\beta$, where α and β are non-constant entire functions.

We now consider the following two subcases:

Subcase 2.1. Let $k \geq 2$. We see that $N(r, 0; (f^n)^{(k)}) = 0$. Clearly

$$f^n(z)(f^n(z))^{(k)} \neq 0 \quad g^n(z)(g^n(z))^{(k)} \neq 0. \quad (2.30)$$

Then from (2.30) and Lemma 7 we must have $f(z) = e^{az+b}$, $g(z) = e^{cz+d}$, where $a \neq 0$, $b, c \neq 0$ and d are constants. From (2.4) it is clear that $a + c = 0$.

Subcase 2.1. Let $k = 1$. Considering Subcase 1.2 we get $f(z) = e^{az+b}$, $g(z) = e^{cz+d}$, where $a(\neq 0), b, c(\neq 0), d \in \mathbb{C}$. Finally we take f and g as $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, where $c_3, c_4, d \in \mathbb{C}$ such that $(-1)^k (c_3 c_4)^n (nd)^{2k} = b^2$. This completes the lemma. $\square$

Lemma 11. ([19], Lemma 6) *If $H \equiv 0$, then F, G share 1 CM. If further F, G share ∞ IM then F, G share ∞ CM.*

Lemma 12. *Let f, g be two transcendental meromorphic functions and let $F = (f^n)^{(k)}/p$, $G = (g^n)^{(k)}/p$, where $p(z)$ be a nonzero polynomial such that either $\deg(p) \leq n-1$ or zeros of $p(z)$ are of multiplicities atmost $n-k-1$ and $n, k \in \mathbb{N}$ such that $n > 3k+3$. If $H \equiv 0$, then one of following two conclusions holds:*

- (1) $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$,
- (2) if $p(z)$ is not a constant, then $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)dt$, $c_1, c_2, c \in \mathbb{C}$ such that $(nc)^2 (c_1 c_2)^n = -1$, if $p(z)$ is a nonzero constant b , then $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, where $c_3, c_4, d \in \mathbb{C}$ such that $(-1)^k (c_3 c_4)^n (nd)^{2k} = b^2$.

Proof. Since $H \equiv 0$, by Lemma 11 we get F and G share 1 CM. On integration we get

$$\frac{1}{F-1} \equiv \frac{bG+a-b}{G-1}, \quad (2.31)$$

where a, b are constants and $a \neq 0$. We now consider following cases.

Case 1. Let $b \neq 0$ and $a \neq b$.

If $b = -1$, then from (2.31) we have $F \equiv \frac{-a}{G-a-1}$. Therefore $\overline{N}(r, a+1; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f)$. So in view of Lemma 2 and the second fundamental theorem we get

$$\begin{aligned} & n T(r, g) \\ & \leq T(r, G) + N_{k+1}(r, 0; g^n) - \overline{N}(r, 0; G) \\ & \leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, a+1; G) + N_{k+1}(r, 0; g^n) \\ & \quad - \overline{N}(r, 0; G) + S(r, g) \\ & \leq \overline{N}(r, \infty; g) + N_{k+1}(r, 0; g^n) + \overline{N}(r, \infty; f) + S(r, g) \\ & \leq \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + N_{k+1}(r, 0; g^n) + S(r, g) \\ & \leq \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + (k+1)\overline{N}(r, 0; g) + S(r, g) \\ & \leq T(r, f) + (k+2) T(r, g) + S(r, f) + S(r, g). \end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r, f) \leq T(r, g)$ for $r \in I$.

So for $r \in I$ we have $(n-k-3) T(r, g) \leq S(r, g)$ which is a contradiction since $n > k+3$.

If $b \neq -1$, from (2.31) we obtain that $F - (1 + \frac{1}{b}) \equiv \frac{-a}{b^2[G+\frac{a-b}{b}]}$. So $\overline{N}(r, \frac{b-a}{b}; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f)$. Using Lemma 2 and the same argument as used in the case when $b = -1$ we can get a contradiction.

Case 2. Let $b \neq 0$ and $a = b$.

If $b = -1$, then from (2.31) we have $FG \equiv 1$, i.e., $(f^n)^{(k)}(g^n)^{(k)} \equiv p^2$, where $(f^n)^{(k)} - p(z)$ and $(g^n)^{(k)} - p(z)$ share 0 CM.

Then by Lemma 10 we get the conclusion (2).

If $b \neq -1$, from (2.31) we have $\frac{1}{F} \equiv \frac{bG}{(1+b)G-1}$. Therefore $\overline{N}(r, \frac{1}{1+b}; G) = \overline{N}(r, 0; F)$. So in view of Lemma 2 and the second fundamental theorem we get

$$\begin{aligned} & n T(r, g) \\ & \leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, \frac{1}{1+b}; G) + N_{k+1}(r, 0; g^n) \\ & \quad - \overline{N}(r, 0; G) + S(r, g) \end{aligned}$$

$$\begin{aligned}
&\leq \overline{N}(r, \infty; g) + (k+1)\overline{N}(r, 0; g) + \overline{N}(r, 0; F) + S(r, g) \\
&\leq \overline{N}(r, \infty; g) + (k+1)\overline{N}(r, 0; g) + (k+1)\overline{N}(r, 0; f) + k\overline{N}(r, \infty; f) \\
&\quad + S(r, f) + S(r, g) \\
&\leq (k+2) T(r, g) + (2k+1) T(r, f) + S(r, f) + S(r, g).
\end{aligned}$$

So for $r \in I$ we have $(n - 3k - 3) T(r, g) \leq S(r, g)$, which is a contradiction since $n > 3k + 3$.

Case 3. Let $b = 0$. From (2.31) we obtain

$$F \equiv \frac{G + a - 1}{a}. \quad (2.32)$$

If $a \neq 1$, then from (2.32) we obtain $\overline{N}(r, 1 - a; G) = \overline{N}(r, 0; F)$. We can similarly deduce a contradiction as in *Case 2*. Therefore $a = 1$ and from (2.32) we obtain $F \equiv G$, i.e., $(f^n)^{(k)} \equiv (g^n)^{(k)}$. Then by Lemma 4 we get $f \equiv tg$ for $t \in \mathbb{C}$ such that $t^n = 1$. This completes the proof. $\square$

3. PROOF OF THE THEOREM

Proof of Theorem 1. Let $F = \frac{(f^n)^{(k)}}{p}$ and $G = \frac{(g^n)^{(k)}}{p}$. It follows that F and G share $(1, 2)$ except for the zeros of p .

Case 1. Let $H \not\equiv 0$.

From (2.1) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G , (ii) those 1 points of F and G whose multiplicities are different, (iii) poles of F and G , (iv) zeros of $F'(G')$ which are not the zeros of $F(F-1)(G(G-1))$.

Since H has only simple poles we get

$$\begin{aligned}
N(r, \infty; H) &\leq \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) + \overline{N}_*(r, 1; F, G) \\
&\quad + \overline{N}(r, 0; F| \geq 2) + \overline{N}(r, 0; G| \geq 2) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G'), \quad (3.1)
\end{aligned}$$

where $\overline{N}_0(r, 0; F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r, 0; G')$ is similarly defined.

Here we see that

$$N(r, 1; F| = 1) \leq N(r, 0; H) \leq N(r, \infty; H) + S(r, F) + S(r, G). \quad (3.2)$$

Now in view of Lemma 3 we get

$$\begin{aligned}
&\overline{N}_0(r, 0; G') + \overline{N}(r, 1; F| \geq 2) + \overline{N}_*(r, 1; F, G) \\
&\leq \overline{N}_0(r, 0; G') + \overline{N}(r, 1; F| \geq 2) + \overline{N}(r, 1; F| \geq 3) \\
&= \overline{N}_0(r, 0; G') + \overline{N}(r, 1; G| \geq 2) + \overline{N}(r, 1; G| \geq 3) \\
&\leq \overline{N}_0(r, 0; G') + N(r, 1; G) - \overline{N}(r, 1; G)
\end{aligned}$$

$$\leq N(r, 0; G' \mid G \neq 0) \leq \overline{N}(r, 0; G) + \overline{N}(r, \infty; g) + S(r, g). \quad (3.3)$$

Hence using (3.1), (3.2), (3.3), Lemmas 1 and 2 we get from second fundamental theorem that

$$\begin{aligned} & n T(r, f) \\ & \leq T(r, F) + N_{k+2}(r, 0; f^n) - N_2(r, 0; F) + S(r, f) \\ & \leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + \overline{N}(r, 1; F) + N_{k+2}(r, 0; f^n) \\ & \quad - N_2(r, 0; F) - N_0(r, 0; F') \\ & \leq \overline{N}(r, 0; F) + \overline{N}(r, \infty; F) + N(r, 1; |F| = 1) + \overline{N}(r, 1; |F| \geq 2) \\ & \quad + N_{k+2}(r, 0; f^n) - N_2(r, 0; F) - N_0(r, 0; F') + S(r, f) \\ & \leq 2\overline{N}(r, \infty, f) + 2\overline{N}(r, \infty; g) + N_2(r, 0; F) + N_2(r, 0; G) + N_{k+2}(r, 0; f^n) \\ & \quad - N_2(r, 0; F) + S(r, f) + S(r, g) \\ & \leq 2\overline{N}(r, \infty; f) + 2\overline{N}(r, \infty; g) + N_{k+2}(r, 0; f^n) + k \overline{N}(r, \infty; g) \\ & \quad + N_{k+2}(r, 0; g^n) + S(r, f) + S(r, g) \\ & \leq (k+4) \overline{N}(r, \infty; f) + (2k+4) \overline{N}(r, \infty; g) + S(r, f) + S(r, g) \\ & \leq (3k+8) T(r) + S(r). \end{aligned} \quad (3.4)$$

In a similar way we can obtain

$$n T(r, g) \leq (3k+8) T(r) + S(r). \quad (3.5)$$

Combining (3.4) and (3.6) we see that

$$(n - 3k - 8) T(r) \leq S(r). \quad (3.6)$$

Since $n \geq 3k + 9$, (3.7) leads to a contradiction.

Case 2. Let $H \equiv 0$.

Then by Lemma 12 we get the theorem. This completes the proof. $\square$

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