Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 15, No. 1-2/2023

SOLVING MULTIPLE-SETS SPLIT MONOTONE VARIATIONAL INCLUSION PROBLEM IN REAL HILBERT SPACES.*

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DOI https://doi.org/10.56082/annalsarscimath.2023.1-2.535

Abstract

In this paper, we study and introduce a self adaptive method together with a Halpern iterative algorithm for approximating solutions of multiple-sets split monotone variational inclusion problem which includes the multiple-sets split feasibility problem, split feasibility problem, split monotone variational inclusion problem and split variational inclusion problem, to mention a few. Using our iterative algorithm, we prove a strong convergence result for approximating the solution of the aforementioned problems. Numerical examples on finite-dimensional and infinite-dimensional spaces are displayed to illustrate the performance of our iterative method. The result discussed in this article extends and complements many related results in literature.

MSC: 47H09; 47H10; 47J05; 47J25.

keywords: Multiple-sets split monotone variational problem, Halpern method, fixed point problems, Iterative method.

*Accepted for publication on November 18-th, 2022

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1 Introduction

The Multiple-Set Split Feasibility Problem (MSSFP) is known to be a generalization of Convex feasibility problem and two-sets Split Feasibility Problem (SFP). The MSSFP is to find a point

$$x \in C = \bigcap_{i=1}^{N} C_i \ such\ that\ Ax \in Q = \bigcap_{j=1}^{M} Q_j, \tag{1}$$

where N and M are positive integers, $C_i \subset \mathcal{R}^B, i = 1, 2, \cdots N$ and $Q_j \subset \mathcal{R}^K,\ j = 1, 2, \cdots M$ are closed convex, A is a $B \times K$ real matrix. When $N = M = 1$, problem (1) becomes the SFP which is to find

$$x \in C \ such\ that\ Ax \in Q. \tag{2}$$

The MSSFP was introduced by Censor et al. [11] to resolve the intensity modulated radiation therapy treatment planning [10]. Several numerical algorithms have been developed to solve MSSFP and its generalization, see [8, 4, 6, 11, 17, 29]. Recently, Kim and Dinh [17] introduced the Multiple-set Split Equilibrium Problem (MSSEP) which is an extension of the MSSFP and proposed two new parallel extragradient algorithms for solving MSSEP when the equilibrium bifunctions are Lipschitz-type continuous and pseudomonotone with respect to their solution sets. They proved a weak and a strong convergence result to a solution of MSSEP.
Very recently, Nghia and Thuy [25] proposed a self adaptive method for solving the Multiple-set Split Variational inequality problem (MSSVIP) in real Hilbert spaces. They proved a strong convergence result for approximating the solution of MSSVIP.
Let H_1 and H_2 be real Hilbert spaces, $f : H_1 \to H_1,\ g : H_2 \to H_2$ be inverse strongly monotone mappings and $A : H_1 \to 2^{H_1},\ B : H_2 \to 2^{H_2}$ be maximal monotone mappings. Let $F : H_1 \to H_2$ be a bounded linear operator. The Split Monotone Variational Inclusion Problem (SMVIP) is to find $x^* \in H_1$ such that

$$0 \in f(x^*) + A(x^*) \tag{3}$$

and

$$y^* = Fx^* \in H_2 \ such\ that\ 0 \in g(y^*) + B(y^*). \tag{4}$$

Recently, Moudafi [21] introduced the SMVIP which is a generalization of split feasibility problem, split common fixed point problem, split variational inequality and split zero problem to mention a few, see [1, 6, 5, 8, 16, 18, 19, 22, 23] which have been studied extensively by many authors and applied to solving many real life problems such as modelling of inverse problems arising from phase retrievals and sensor networks in computerised tomography and data compression. We denote by Γ_A the solution set of (3)-(4).

Remark 1. *Suppose $f \equiv 0$ and $g \equiv 0$ in (3)-(4), then we obtain the Split Variational Inclusion Problem (SVIP), which is to find $x^* \in H_1$ such that*

$$0 \in A(x^*), \tag{5}$$

and

$$y^* = Fx^* \in H_2 \text{ such that } 0 \in B(y^*). \tag{6}$$

Let $C_i, i = 1, 2, \cdots, N$ and $Q_j, j = 1, 2, \cdots M$ be nonempty, closed and convex subsets of H_1 and H_2 respectively. In this article, we extend the SMVIP (3)-(4) to Multiple-Sets Split Monotone Variational Inclusion Problem, which is to

$$\begin{aligned} &find\ x^* \in C = \bigcap_{i=1}^{N} C_i \ such\ that\ (A_i + f_i)^{-1}(0), \\ &for\ all\ i = 1, 2, \cdots N,\ and\ such\ that\ the\ point \end{aligned} \tag{7}$$

$$y^* = Fx^* \in Q = \bigcap_{j=1}^{M} Q_j \ solves\ (B_j + g_j)^{-1}(0),\ for\ all\ j = 1, 2, \cdots M. \tag{8}$$

Remark 2. *(i) If $M = N = 1$, $A := A_1$, $f := f_1$, $B := B_1$ and $g := g_1$, then MSSMVIP (7)-(8) reduces to SMVIP (3)-(4).*
It is well-known that the SMVIP can be applied to solve split minimization problem, split saddle-point problem and split equilibrium problem, to mention a few, see [21].

(ii) If $A_i x = f_i x = 0$ for all $x \in C_i, i = 1, 2, \cdots, N$ and $B_j y = g_j y = 0$ for all $y \in Q_j, j = 1, 2, \cdots, M$, then MSSMVIP (7)-(8) reduces to MSSFP (1).

(iii) It can be seen from (1) and (ii) above that MSSMVIP (7)-(8) can be reduced to (2).

Remark 3. *We state our contributions as follows:*

1. *We considered approximating the solution of MSSMVIP (7)-(8) in real Hilbert spaces which is more general than the results of [1, 8, 5, 18, 19, 21, 24].*

2. *Our method uses self-adaptive stepsizes and the implementation of our method does not require the prior knowledge of the norm of the bounded linear operator F, (see [17]).*

3. *The sequences generated by our proposed method converges strongly to the solution of the problem (7)-(8) which is desirable to the weak convergence result obtained in [17].*

Motivated by the results of [17], [21] and [25], we introduced the multiple-set split monotone variational inclusion problem which includes the multiple-set split feasibility problem, split feasibilty problem and split monotone variational inclusion problem as special cases. Using a Halpern iterative iterative algorithm together with a self adaptive method, we prove a strong convergence result for solving the aforementioned problems. We displayed some numerical examples to illustrate the performance of our method. The result discussed in this article extends and complements many related results on SFP, SMVIP and MSSFP in literature.

2 Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\rightarrow$" and "$\rightharpoonup$" respectively.
Let C be a nonempty closed and convex subset of a real Hilbert space H. Recall that a mapping $T : H \rightarrow H$ is said to be

1. nonexpansive, if

$$||Tx - Ty|| \leq ||x - y||, \ \forall \ x, y \in H,$$

2. quasi-nonexpansive, if $Fix(T) \neq \emptyset$ and

$$||Tx - q|| \leq ||x - q||, \ \forall \ x \in H, \ q \in Fix(T),$$

where $Fix(T) = \{x \in C : Tx = x\}$.
A mapping $M : C \rightarrow C$ is said to be

(i) monotone, if
$$\langle Mx - My, x - y\rangle \geq 0,\ \forall\ x, y \in C,$$

(ii) $\alpha-$ inverse strongly monotone (ism), if there exists a constant $\alpha > 0$ such that
$$\langle Mx - My, x - y\rangle \geq \alpha\|Mx - My\|^2,\ \forall\ x, y \in C,$$

(iii) firmly nonexpansive, if
$$\langle Mx - My, x - y\rangle \geq \|Mx - My\|^2,\ \forall\ x, y \in C,$$

.

If M is a multi-valued mapping, i.e $M : H \to 2^H$, then M is called monotone, if
$$\langle x - y, u - v\rangle \geq 0\ \forall\ x, y \in H, u \in M(x), v \in M(y)$$
and M is maximal monotone, if the graph $G(M)$ of M defined by
$$G(M) =: \{(x, y) \in H \times H : y \in M(x)\}$$

is not properly contained in the graph of any other monotone mapping. It is generally known that M is maximal if and only if for $(x, u) \in H \times H, \langle x - y, u - v\rangle \geq 0$ for all $(y, v) \in G(M)$ implies $u \in M(x)$. It is well-known that $I + \lambda M$ is onto, and the resolvent operator J_λ^M associated with M and λ is the mapping $J_\lambda^M : H \to H$ defined by
$$J_\lambda^M(x) := (I + \lambda M)^{-1}x,\ x \in H, \lambda > 0. \tag{9}$$
It is well known that the resolvent operator J_λ^M is single valued, nonexpansive and 1-inverse strongly monotone and the solution of (3) is equivalent to fixed point of $J_\lambda^M(I - \lambda f),\ \forall\ \lambda > 0$, see [7].
The metric projection P_C is a map defined on H onto C which assign to each $x \in H$, the unique point in C, denoted by $P_C x$ such that
$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

It is well known that P_Cx is characterized by the inequality $\langle x - P_Cx, z - P_Cx\rangle \leq 0,\ \forall\ z \in C$ and P_C is a firmly nonexpansive mapping. For more information on metric projections, (see [14]) and the references therein.

Lemma 1. *[12] Let H be a real Hilbert space, then $\forall\ x, y \in H$ and $\alpha \in (0,1)$, we have*

(i) $2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x-y||^2 = ||x+y||^2 - ||x||^2 - ||y||^2$,

(ii) $||\alpha x + (1-\alpha)y||^2 = \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x-y||^2$,

(iii) $||x+y||^2 \leq ||x||^2 + 2\langle y, x+y\rangle$

Lemma 2. *[21] Let $A_i, i = 1,2$ be maximal monotone mappings and $f_i, i = 1,2$ be η_i-inverse strongly monotone mappings. Suppose constants $\rho_i > 0, i = 1,2$ and $\rho_i \in (0, 2\eta_i)$, then*

$$x \text{ solves } (3)-(4) \Leftrightarrow x = J_{\rho_1}^{A_1}(1-\rho_1 f_1)x \text{ and } Fx = J_{\rho_2}^{A_2}(I-\rho_2 f_2)Fx.$$

where F is a bounded linear operator.

Lemma 3. *[13] Let H be a real Hilbert space. Let $\{x_i, i = 1, \cdots m\} \subset H$. For $\alpha_i \in (0,1), i = 1, \cdots m$ such that $\sum\limits_{i=1}^{m} \alpha_i = 1$, the following identity holds:*

$$||\sum_{i=1}^{m} \alpha_i x_i||^2 = \sum_{i=1}^{m} \alpha_i ||x_i||^2 - \sum_{i,j=1, i\neq j}^{m} \alpha_i\alpha_j ||x_i - x_j||^2.$$

Definition 1. *Let H be a real Hilbert space and $T : H \to H$ be a mapping. Then, T is said to be demiclosed at 0 if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x^*$, and $||x_n - T(x_n)|| \to 0$, we have $x^* = Tx^*$.*

Lemma 4. *[15] Assume that T is a nonexpansive mapping of a closed and convex subset C of a Hilbert space H into H. Then the mapping $I^H - T$ is demiclosed on C, that is, whenever $\{x_n\}$ is a sequence in C which weakly converges to some point $x^* \in C$ and the sequence $\{(I^H - T)x_n\}$ strongly to some y, it follows that $(I^H - T)x^* = y$.*

Lemma 5. *[26] Let $\{a_n\}$ be a sequence of positive real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ such that $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Suppose that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n d_n, n \geq 1.$$

If $\limsup_{k\to\infty} d_{n_k} \leq 0$ *for all subsequences* $\{a_{n_k}\}$ *of* $\{a_n\}$ *satisfying the condition*

$$\liminf_{k\to\infty}\{a_{n_k+1} - a_{n_k}\} \geq 0,$$

then, $\lim_{n\to\infty} a_n = 0.$

3 Main Results

In this section, we introduce a Halpern iterative algorithm for approximating a solution of multiple-sets split monotone variational inclusion problem and prove its strong convergence in the framework of real Hilbert spaces. We state the following assumptions that are needed in our result.

Assumption 1. *1. Let* C *and* Q *be nonempty closed convex subsets of* H_1 *and* H_2*, respectively. Suppose* $f_i : H_1 \to H_1, i = 1, 2, \cdots, N$ *and* $g_j : H_2 \to H_2, j = 1, 2, \cdots M$ *are* σ_{f_i} *and* $\sigma_{g_j}-$ *inverse strongly monotone mappings respectively.*

2. *Let* $F : H_1 \to H_2$ *be a bounded linear operator with adjoint* F^**. Let* $A_i : H_1 \to 2^{H_1}, i = 1, 2, \cdots N$ *and* $B_j : H_2 \to 2^{H_2}, j = 1, 2, \cdots M$ *be multi-valued maximal monotone mappings respectively.*

3. *Assume that the solution set of MSSMVIP (7) denoted* $\Omega \neq \emptyset$.

4. *The sequence* $\{\alpha_{n,i}\}, \{\beta_{n,j}\}, \{\gamma_{n,i}\}, \{\mu_n\}, \{\eta_n\}$ *and* $\{\phi_n\}$ *are sequences in* $(0, 1)$ *and the parameter* λ *satisfies the following conditions:*

   (i) $\lim_{n\to\infty} \phi_n = 0, \ \sum_{n=1}^{\infty} \phi_n = \infty;$

   (ii) $\{\alpha_{n,i}\}, \{\beta_{n,j}\} \subset [\underline{\rho}, \overline{\rho}]$, *with* $0 < \underline{\rho} \leq \overline{\rho} \leq 1$ *and* $\sum_{i=1}^{N} \alpha_{n,i} = \sum_{j=1}^{M} \beta_{n,j} = 1,$

   (iii) $\{\eta_n\} \in (0, k)$ *for* $k > 0,$

   (iv) $0 < \lambda \leq 2\min\{\sigma_{f_i}, \sigma_{g_j} : i = 1, 2, \cdots N, j = 1, 2, \cdots M\},$

   (v) $0 < \liminf_{n\to\infty} \gamma_{n,i} \leq \limsup_{n\to\infty} \gamma_{n,i} < 1.$

Algorithm 1. Halpern Iterative Method for Solving Multiple-Sets Split Monotone Variational Inclusion Problem.

Define a sequence $\{x_n\}_{n=0}^{\infty}$ generated iteratively by chosen $u, x_1 \in H_1$, such that

Step 1: Compute

$$u_{n,i} = \gamma_{n,i}x_n + (1-\gamma_{n,i})J_\lambda^{A_i}(I^{H_1} - \lambda f_i)x_n,\ i = 1, 2, \cdots, N.$$

Step 2: Set

$$\overline{u_n} = \sum_{i=1}^{N} \alpha_{n,i}u_{n,i}.$$

Step 3: Compute

$$z_{n,j} = J_\lambda^{B_j}(I^{H_2} - \lambda g_j)F\overline{u_n},\ j = 1, 2, \cdots M.$$

Step 4: Set

$$\overline{z_n} = \sum_{j=1}^{m} \beta_{n,j}z_{n,j}.$$

Step 5: Take

$$t_n = \overline{u_n} + \xi_n F^*(\overline{z_n} - F\overline{u_n}),$$

where the step size ξ_n is denoted by

$$\xi_n = \mu_n \frac{||\overline{z_n} - F\overline{u_n}||^2}{||F^*(\overline{z_n} - F\overline{u_n})||^2 + \eta_n}. \tag{10}$$

Step 6: Compute

$$x_{n+1} = \phi_n u + (1-\phi_n)t_n,\ \textit{set } n := n+1 \textit{ and go to step} 1.$$

Lemma 6. *Suppose that Assumptions 1 holds, then $\{x_n\}$ is bounded.*

Proof Let $p \in \Omega$, then using the nonexpansive property of $J_\lambda^{A_i}(I^{H_1} - \lambda f_i)$, we have from Lemma 1 (ii) that

$$\begin{aligned}
\left\|u_{n,i} - p\right\|^2 &= \left\|\gamma_{n,i}x_n + (1-\gamma_{n,i})\left[J_\lambda^{A_i}(I^{H_1} - \lambda f_i)x_n - p\right]\right\|^2 \\
&\leq \gamma_{n,i}||x_n - p||^2 + (1-\gamma_{n,i})\|J_\lambda^{A_i}(I^{H_1} - \lambda f_i)x_n - p\|^2 \\
&- \gamma_{n,i}(1-\gamma_{n,i})\|x_n - J_\lambda^{A_i}(I^{H_1} - \lambda f_i)x_n\|^2 \\
&= \|x_n - p\|^2 - \gamma_{n,i}(1-\gamma_{n,i})\|x_n - J_\lambda^{A_i}(I^{H_1} - \lambda f_i)x_n\|^2 \qquad (11) \\
&\leq \|x_n - p\|^2. \qquad (12)
\end{aligned}$$

By applying step 2 of (1) and Lemma (3), we get

$$
\begin{aligned}
\|\overline{u_n} - p\|^2 &= \|\sum_{i=1}^{N} \alpha_{n_i} u_{n,i} - p\|^2 = \|\sum_{i=1}^{N} \alpha_{n,i}(u_{n,i} - p)\|^2 \\
&= \sum_{i=1}^{N} \alpha_{n,i}\|u_{n,i} - p\|^2 - \sum_{i,j=1, i\neq j}^{N} \alpha_{n,i}\alpha_{n,j}\|u_{n,i} - u_{n,j}\|^2 \\
&\leq \|x_n - p\|^2 - \sum_{i=1}^{N} \alpha_{n,i}\gamma_{n,i}(1-\gamma_{n,i})\|x_n - J_\lambda^{A_i}(I^{H_1} - \lambda f_i)x_n\|^2 \\
&\leq ||x_n - p||^2. \qquad (13)
\end{aligned}
$$

From step 3, step 4 and Lemma 3, one has

$$
\begin{aligned}
\|\overline{z_n} - Fp\|^2 &= \|\sum_{j=1}^{M} \beta_{n,j}(z_{n,j} - Fp)\|^2 \\
&= \sum_{j=1}^{M} \beta_{n,j}\|z_{n,j} - Fp\|^2 - \sum_{j,k=1, j\neq k}^{M} \beta_{n,j}\beta_{n,k}\|z_{n,j} - z_{n,k}\|^2 \\
&\leq \sum_{j=1}^{M} \beta_{n,j}\|J_\lambda^{B_j}(I^{H_2} - \lambda g_j)F\overline{u_n} - Fp\|^2. \qquad (14)
\end{aligned}
$$

By definition of t_n in step 5 of (1) and (14), we have

$$
\begin{aligned}
\|t_n - p\|^2 &= \|\overline{u_n} + \xi_n F^*(\overline{z_n} - F\overline{u_n}) - p\|^2 \\
&= \|\overline{u_n} - p\|^2 + \xi_n^2\|F^*(\overline{z_n} - F\overline{u_n})\|^2 + 2\xi_n\langle \overline{u_n} - p, F^*(\overline{z_n} - F\overline{u_n})\rangle \\
&= \|\overline{u_n} - p\|^2 + \xi_n^2\|F^*(\overline{z_n} - F\overline{u_n})\|^2 + 2\xi_n\langle F\overline{u_n} - Fp, (\overline{z_n} - F\overline{u_n})\rangle \\
&= ||\overline{u_n} - p||^2 + \xi_n^2\|F^*(\overline{z_n} - F\overline{u_n})\|^2 + \xi_n\Big[\sum_{j=1}^{M} \beta_{n,j}\big(\|J_\lambda^{B_j}(I^{H_2} - \lambda g_j)F\overline{u_n} \\
&\quad - J_\lambda^{B_j}(I^{H_2} - \lambda g_j)Fp\|^2\big) - \|F\overline{u_n} - Fp\|^2 - \|\overline{z_n} - F\overline{u_n}\|^2\Big] \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
&\le \|\overline{u_n} - p\|^2 + \xi_n^2 \|F^*(\overline{z_n} - F\overline{u_n})\|^2 \\
&+ \xi_n \Big(\|F\overline{u_n} - Fp\|^2 - \|F\overline{u_n} - Fp\|^2 - \|\overline{z_n} - F\overline{u_n}\|^2 \Big) \\
&= \|\overline{u_n} - p\|^2 + \xi_n^2 \|F^*(\overline{z_n} - F\overline{u_n})\|^2 - \xi_n \|\overline{z_n} - F\overline{u_n}\|^2 \\
&\le \|\overline{u_n} - p\|^2 + \mu_n^2 \frac{\|\overline{z_n} - F\overline{u_n}\|^4}{(\|F^*(\overline{z_n} - F\overline{u_n}\|^2 + \eta_n)^2} \times ((\|F^*(\overline{z_n} - F\overline{u_n}\|^2 + \eta_n)) \\
&- \mu_n \frac{\|\overline{z_n} - F\overline{u_n}\|^4}{\|F^*(\overline{z_n} - F\overline{u_n}\|^2 + \eta_n)} \\
&= \|\overline{u_n} - p\|^2 - \mu_n(1 - \mu_n) \frac{\|\overline{z_n} - F\overline{u_n}\|^4}{\|F^*(\overline{z_n} - F\overline{u_n}\|^2 + \eta_n)} \le \|\overline{u_n} - p\|^2.
\end{aligned}
$$

Applying step 6 of (1), (13) and (15), we obtain that

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|\phi_n u + (1 - \phi_n) t_n - p\| \\
&\le \phi_n \|u_n - p\| + (1 - \phi_n)\|t_n - p\| \\
&\le \phi_n \|u_n - p\| + (1 - \phi_n)\|\overline{u_n} - p\| \\
&\le \phi_n \|u_n - p\| + (1 - \phi_n)\|x_n - p\| \\
&\le \max\{\|u - p\|, \|x_1 - p\|\} \\
&\vdots \\
&\le \max\{\|u - p\|, \|x_n - p\|\}.
\end{aligned}
$$

Hence, the sequence $\{x_n\}$ is bounded. Consequently, it follows from (11)-(15) that $\{\overline{u_n}\}, \{t_n\}$ and $\{\overline{z_n}\}$ are also bounded.

Theorem 1. *Suppose that Assumption 1 holds, then the sequence $\{x_n\}$ generated by (1) converges strongly to $z = P_\Omega u$, where P_Ω is the metric projection of H_1 onto Ω.*

Proof From Algorithm 1, we obtain

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \langle \phi_n u + (1 - \phi_n) t_n - p, x_{n+1} - p \rangle \\
&= (1 - \phi_n)\langle t_n - p, x_{n+1} - p \rangle + \phi_n \langle u - p, x_{n+1} - p \rangle \\
&\le \frac{(1 - \phi_n)}{2} \big(\|x_{n+1} - p\|^2 + \|t_n - p\|^2 \big) + \phi_n \langle u - p, x_{n+1} - p \rangle.
\end{aligned}
$$

This implies from (11) and (15) that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\phi_n)\|\overline{u_n}-p\|^2-\mu_n(1-\mu_n)\frac{\|\overline{z_n}-F\overline{u_n}\|^4}{\|F^*(\overline{z_n}-F\overline{u_n}\|^2+\eta_n)}\\
&+2\phi_n\langle u-p,x_{n+1}-p\rangle\\
&\le (1-\phi_n)\|x_n-p\|^2-(1-\phi_n)\sum_{i=1}^{N}\alpha_{n,i}\gamma_{n,i}(1-\gamma_{n,i})\\
&\cdot\|x_n-J_\lambda^{A_i}(I^{H_1}-\lambda f_i)x_n\|^2\\
&-\mu_n(1-\mu_n)\frac{\|\overline{z_n}-F\overline{u_n}\|^4}{\|F^*(\overline{z_n}-F\overline{u_n}\|^2+\eta_n)}+2\phi_n\langle u-p,x_{n+1}-p\rangle\\
&\le (1-\phi_n)\|x_n-p\|^2+\phi_n\big(2\langle u-p,x_{n+1}-p\rangle\big)\\
&=(1-\phi_n)\|x_n-p\|^2+\phi_n d_n,
\end{aligned}\tag{16}$$

where $d_n=2\langle u-p,x_{n+1}-p\rangle$. According to Lemma 5, to conclude our proof. It sufficies to show that $\limsup\limits_{k\to\infty} d_{n_k}\le 0$ for every subsequence $\{\|x_{n_k}-p\|\}$ satisfies the condition

$$\liminf_{k\to\infty}\Big(\|x_{n_{k+1}}-p\|-\|x_{n_k}-p\|\Big)\ge 0.\tag{17}$$

To prove this, suppose that $\{||x_{n_k}-x^*||\}$ is a subsequence of $\{||x_n-x^*||\}$ such that (17) holds. Then

$$\begin{aligned}
&\liminf_{k\to\infty}\Big(||x_{n_k+1}-x^*||^2-||x_{n_k}-x^*||^2\Big)\\
&=\liminf_{k\to\infty}\Big((||x_{n_k+1}-x^*||-||x_{n_k}-x^*||)(||x_{n_k+1}-x^*||+||x_{n_k}-x^*||)\Big)\ge 0.
\end{aligned}\tag{18}$$

From (16), we obtain that

$$
\begin{aligned}
&\limsup_{k\to\infty}\Bigg(-(1-\phi_{n_k})\sum_{i=1}^{N}\alpha_{n_k,i}\gamma_{n_k,i}(1-\gamma_{n_k,i})\|x_{n_k}-J_\lambda^{A_i}(I^{H_1}-\lambda f_i)x_{n_k}\|^2\\
&-\mu_{n_k}(1-\mu_{n_k})\frac{\|\overline{z_{n_k}}-F\overline{u_{n_k}}\|^4}{\|F^*(\overline{z_{n_k}}-F\overline{u_{n_k}}\|^2+\eta_{n_k})}\Bigg)\\
&\leq \limsup_{k\to\infty}\Bigg((1-\phi_{n_k})||x_{n_k}-x^*||^2-||x_{n_k+1}-x^*||^2\Bigg)\\
&+\limsup_{k\to\infty}\Bigg(2\phi_{n_k}\langle u-x^*,x_{n_k+1}-x^*\rangle\Bigg)\\
&\leq \limsup_{k\to\infty}\Bigg(||x_{n_k}-x^*||^2-||x_{n_k+1}-x^*||^2\Bigg)\\
&+\limsup_{k\to\infty}\Bigg(2\phi_{n_k}\langle u-x^*,x_{n_k+1}-x^*\rangle\Bigg)\\
&=-\liminf_{k\to\infty}\Bigg(||x_{n_k+1}-x^*||^2-||x_{n_k}-x^*||^2\Bigg)\leq 0. \qquad (19)
\end{aligned}
$$

Thus, by applying conditions (i) and (iii) of Algorithm 1, we get

$$\lim_{k\to\infty}\|(I^{H_1}-J_\lambda^{A_i}(I-\lambda f_i)x_{n_k})\|=0, i=1,2,\cdots N. \qquad (20)$$

and

$$\lim_{k\to\infty}\|\overline{z_{n_k}}-F\overline{u_{n_k}}\|=0. \qquad (21)$$

From step 1 of Algorithm 1, we obtain

$$\|u_{n_k,i}-x_{n_k}\|=(1-\gamma_{n_k,i})\Bigg\|\Bigg(I^{H_1}-J_\lambda^{A_i}(I^{H_1}-\lambda f_i)\Bigg)x_{n_k}\Bigg\|=0,\ \ l\to\infty, \quad (22)$$

which also implies that

$$\lim_{k\to\infty}\|\overline{u_{n_k}}-x_{n_k}\|=\lim_{k\to\infty}\sum_{i=1}^{N}\alpha_{n_k,i}\|u_{n_{k_j}}-x_{n_{k_j}}\|=0. \qquad (23)$$

By applying step 3, step 4 of Algorithm 1, one get

$$\lim_{k\to\infty}\|(I^{H_2}-J_\lambda^{B_j}(I^{H_2}-\lambda g_j))F\overline{u_{n_k}}\|=0, j=1,2,\cdots M. \qquad (24)$$

Using step 5 of (1), one get that

$$\|t_{n_k}-\overline{u_{n_k}}\|=\xi_{n_k}\|F^*(\overline{z_{n_k}}-F\overline{u_{n_k}})\|.$$

By applying (21) and the property of the adjoint operator F^*, we obtain

$$\lim_{k\to\infty} \|t_{n_k} - \overline{u_{n_k}}\| = 0. \tag{25}$$

From (23) and (25), one get

$$\lim_{k\to\infty} \|t_{n_k} - x_{n_k}\| = 0. \tag{26}$$

We obtain from the step 6 of Algorithm 1 and (26) that

$$\lim_{k\to\infty} \|x_{n_{k+1}} - x_{n_k}\| = \lim_{k\to\infty} (1-\phi_{n_k})\|t_{n_k} - x_{n_k}\| = 0. \tag{27}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ which converges weakly to x^*. Also, from (23) and (26), there exist subsequences $\{\overline{u_{n_{k_l}}}\}$ of $\{\overline{u_{n_k}}\}$ and $\{t_{n_{k_l}}\}$ of $\{t_{n_k}\}$ which converge weakly to x^*, respectively. From (20), Lemma 2 and Lemma 4, we obtain that $0 \in \bigcap\limits_{i=1}^{N} (f_i(x^*) + A_i(x^*))$. Using the fact that F is a bounded linear operator, we have that $Fx_{n_{k_l}} \rightharpoonup Fx^*$. Also, by applying Lemma 2, Lemma 4 and (24), we obtain that $0 \in \bigcap\limits_{j=1}^{M} (g_j(Fx^*) + B_j(Fx^*))$. Hence, we conclude that $x^* \in \Omega$.
Let $z = P_\Omega u$, suppose that $\{x_{n_{k_l}}\}$ is a subsequence of $\{x_{n_k}\}$ such that $\{x_{n_{k_l}}\} \rightharpoonup x^* \in \Omega$, then we obtain

$$\begin{aligned}\limsup_{k\to\infty} \langle u - z, x_{n_k} - z\rangle &= \lim_{l\to\infty} \langle u - z, x_{n_{k_l}} - z\rangle \\ &= \langle u - z, x^* - z\rangle \le 0. \end{aligned} \tag{28}$$

On substituting (28) into (16), we obtain that $\limsup\limits_{k\to\infty} d_{n_k} \le 0$. Thus by applying Lemma 5 to (16), we conclude that $||x_n - z|| \to 0$ as $n \to \infty$. Therefore, $\{x_n\}$ converges strongly to $z = P_\Omega u$.
If $M = N = 1$ in Algorithm 1, we have the following result.

Corollary 1.

Algorithm 2. Halpern Iterative Method for Solving Split Monotone Variational Inclusion Problem.

Define a sequence $\{x_n\}_{n=0}^{\infty}$ *generated iteratively by chosen* $u, x_1 \in H_1$, *such that*
Step 1: Compute $u_n = \gamma_n x_n + (1-\gamma_n) J_\lambda^A (I^{H_1} - \lambda f) x_n$,
Step 2: Compute $z_n = J_\lambda^B (I^{H_2} - \lambda g) F u_n$,

Step 3: Take $t_n = u_n + \xi_n F^*(z_n - Fu_n)$, where the step size ξ_n is denoted by

$$\xi_n = \mu_n \frac{||z_n - Fu_n||^2}{||F^*(z_n - Fu_n)||^2 + \eta_n}. \tag{29}$$

Step 4: Compute $x_{n+1} = \phi_n u + (1-\phi_n)t_n$, set $k := k+1$ and go to step 1. Suppose that Assumptions 1-4 and conditions (i)-(iv) of (refAlg) holds for $i = j = 1$, then $\{x_n\}$ converges strongly to $z = P_\Omega u$.

4 Numerical Example

Example 1. *Let $H_1 = H_2 = \mathcal{R}$. For every $i = 1, 2, \cdots N$ and $j = 1, 2, \cdots M$, let $f_i : \mathcal{R} \to \mathcal{R}$ be defined by $f_i x = \frac{ix}{4} + (i+1)$ for all $x \in H_1$ and $g_j : \ell_2 \to \ell_2$ be defined by $g_j y = \frac{jy}{2}$ for all $y \in H_2$. It is easy to observe that f_i is $\frac{4}{i}-$ inverse strongly monotone mapping and g_j is $\frac{2}{j}-$ inverse strongly monotone mapping. Let $A_i : \mathcal{R} \to 2^{\mathcal{R}}$ be defined by $A_i x = \frac{ix}{4}$ for all $x \in \mathcal{R}$ and $B_j : \mathcal{R} \to 2^{\mathcal{R}}$ be defined by $B_j y = \frac{jy}{2}$ for all $y \in \mathcal{R}$. Let $Fx = x$, for all $x \in \mathcal{R}$. In our method, we take $\phi_n = \frac{1}{n+5}, \gamma_{n,i} = 0.5, \eta_n = 0.1, \mu_n = 0.9, \alpha_{n,i} = \frac{1}{M}, \beta_{n,j} = \frac{1}{N}$ and $\lambda = 1$.It is easy to see that $J_\lambda^{A_i} x = \frac{4x}{4+\lambda i}$ and $J_\lambda^{B_j}(x) = \frac{x}{1+3\lambda j}$ respectively. Let $E_n = \|x_{n+1} - x_n\|^2 = 10^{-4}$, be the stropping criterion and checking the cases for different values of N and M with the initial points $x_1 = 0.5$ for*

Case 1 $N = 5$ and $M = 10$;

Case 2 $N = 10$ and $M = 10$;

Case 3 $N = 15$ and $M = 20$;

Case 4 $N = 20$ and $M = 10$;

The results of this experiment are reported in Figure 1.

Example 2. *Let $H_1 = H_2 = \ell_2$ be the linear space whose elements consist of all 2-summable sequence $(x_1, x_2, \cdots, x_t, \cdots)$ of scalars, i.e. $\ell_2 = \{x : x = (x_1, x_2, \cdots, x_t, \cdots)$ and $\sum\limits_{t=1}^{\infty} |x_t|^2 < \infty\}$ with an inner product $\langle ., . \rangle : \ell_2 \times \ell_2 \to \mathcal{R}$ defined by $\langle x, y \rangle = \sum\limits_{t=1}^{\infty} x_t y_t$ where $x = \{x_t\}_{t=1}^{\infty} \in \ell_2$ and $y = \{y_t\}_{t=1}^{\infty} \in \ell_2$, and a norm $\|.\| : \ell_2 \to \mathcal{R}$ defined by $\|x\|_2 =$*

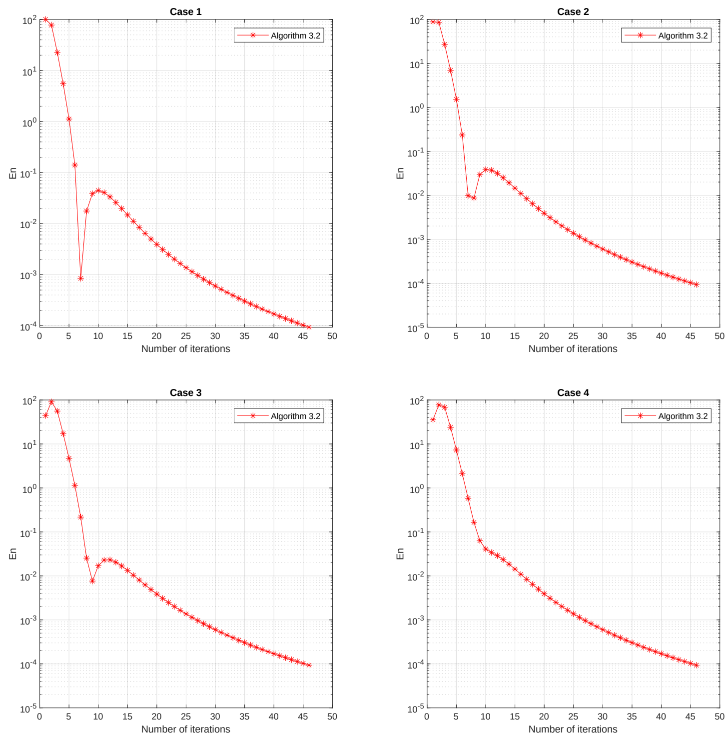

Figure 1: Example 1. Top left: Case 1, Top right: Case 2, Bottom left: Case 3, Bottom right: Case 4.

$(\sum\limits_{t=1}^{\infty}|x_t|^2)^{\frac{1}{2}}$, *where* $x = \{x_t\}_{t=1}^{\infty} \in \ell_2$. *Let* $F : \ell_2 \to \ell_2$ *be defined by* $Fx = (\frac{x_1}{2}, \frac{x_2}{2}, \cdots, \frac{x_t}{2}, \cdots)$ *for all* $x = \{x_t\}_{t=1}^{\infty} \in \ell_2$, *and* $F^* : \ell_2 \to \ell_2$ *be defined by* $F^*x = (\frac{e_1}{2}, \frac{e_2}{2}, \cdots, \frac{e_t}{2}, \cdots)$ *for all* $x = \{e_t\}_{t=1}^{\infty} \in \ell_2$. *For every* $i = 1, 2, \cdots N$, *let the mapping* $f_i : \ell_2 \to \ell_2$ *be defined by* $f_i x = (\frac{2x_1}{3i}, \frac{2x_2}{3i}, \cdots \frac{2x_t}{3i}, \cdots)$ *and for every* $j = 1, 2, \cdots M$, *let the mapping* $g_j : \ell_2 \to \ell_2$ *be defined by* $g_j y = (\frac{2y_1-1}{4j}, \frac{2y_2-1}{4j}, \cdots, \frac{2y_t-1}{4j}, \cdots)$, *respectively for all* $x = \{x_t\}_{t=1}^{\infty} \in \ell_2$ *and* $y = \{y_t\}_{t=1}^{\infty} \in \ell_2$. *For every* $i = 1, 2, \cdots, N$ *and* $j = 1, 2, \cdots, M,$, *let* $A_i : \ell_2 \to \ell_2$ *be defined by* $A_i x = (\frac{ix_1}{4}, \frac{ix_2}{4}, \cdots, \frac{ix_t}{4}, \cdots)$ *and* $B_j : \ell_2 \to \ell_2$ *be defined by* $B_j y = (3jy_1, 3jy_2, \cdots 3jy_t, \cdots)$, *for all* $x = \{x_t\}_{t=1}^{\infty} \in \ell_2$ *and* $y = \{y_t\}_{t=1}^{\infty} \in \ell_2$, *respectively. In this experiment, we take* $\phi_n = \frac{1}{n+5}, \gamma_{n,i} = 0.5, \eta_n = 0.1, \mu_n = 0.9, \alpha_{n,i} = \frac{1}{M}, \beta_{n,j} = \frac{1}{N}$ *and* $\lambda = 1$. *Checking the cases for different values of* x_1 *and fix the values of* N *and* M.

Let $E_n = \|x_{n+1} - x_n\|^2 = 10^{-4}$, *be the stropping criterion, then we consider the following cases for initial values of* x_0 *and* x_1 :

Case 1 $x_1 = (1.5, 0.5, 1.5, \cdots);$

Case 2 $x_1 = (0.75, 0.896, \cdots);$

Case 3 $x_1 = (-1.50, -2.09, \cdots);$

Case 4 $x_1 = (-3.6, -1.9, \cdots).$

The results of this experiment are reported in Figure 2. Stopping.

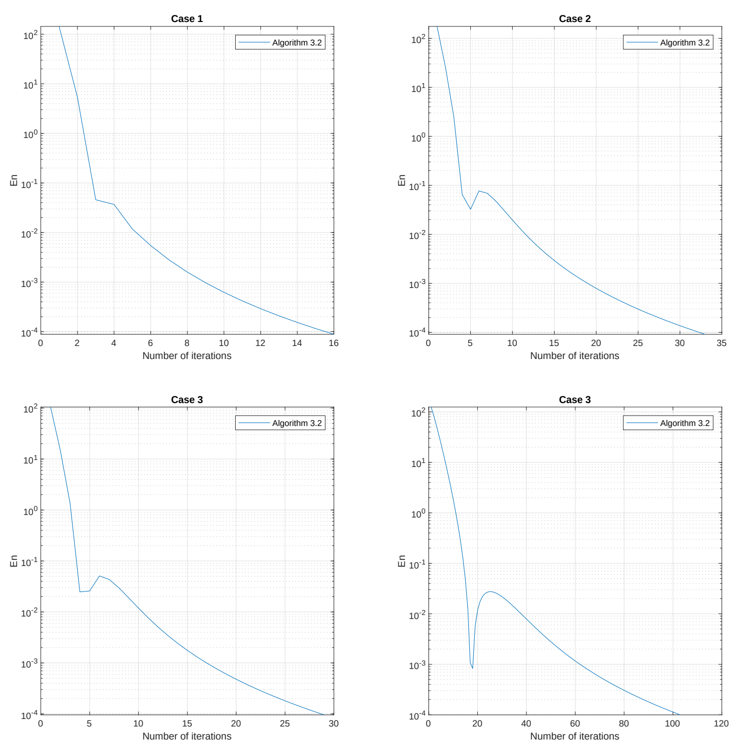

Figure 2: Example 2. Top left: Case 1, Top right: Case 2, Bottom left: Case 3, Bottom right: Case 4.

Acknowledgement: The author acknowledge with thanks the bursary and financial support from Department of Science and Technology and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DSI-NRF COE-MaSS) Post-Doctoral Bursary. Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS.

Declaration

The authors declare that they have no competing interests.

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