Ann. Acad. Rom. Sci.
Ser. Math. Appl.
ISSN 2066-6594 Vol. 15, No. 1-2/2023

FULL DESCRIPTION OF THE SPECTRUM OF A STEKLOV-LIKE EIGENVALUE PROBLEM INVOLVING THE (p, q)-LAPLACIAN*

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DOI https://doi.org/10.56082/annalsarscimath.2023.1-2.30

Dedicated to Dr. Dan Tiba on the occasion of his 70$^{\text{th}}$ anniversary

Abstract

In this paper we consider in a bounded domain $\Omega \subset \mathbb{R}^N$ a Steklov-like eigenvalue problem involving the (p, q)-Laplacian plus some potentials. Under suitable assumptions, using the Nehari manifold method and a variational approach, we are able to determine the full eigenvalue set of this problem as being an open interval $(\lambda_*, +\infty)$ with $\lambda_* > 0$.

MSC: 35J60, 35J92, 35P30

keywords: Eigenvalues, (p, q)-Laplacian, Sobolev spaces, Nehari manifold, variational method.

*Accepted for publication on November 4-th, 2022

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with smooth boundary $\partial\Omega$. Consider the eigenvalue problem

$$\begin{cases} -\Delta_p u - \Delta_q u + \rho_1(x) \mid u \mid^{p-2} u + \rho_2(x) \mid u \mid^{q-2} u = 0, \;\; x \in \Omega, \\ \frac{\partial u}{\partial \nu_{pq}} + \gamma_1(x) \mid u \mid^{p-2} u + \gamma_2(x) \mid u \mid^{q-2} u = \lambda \mid u \mid^{q-2} u, \;\; x \in \partial\Omega. \end{cases} \tag{1}$$

Recall that, for $\theta \in (1, \infty)$, Δ_θ denotes the θ-Laplacian, $\Delta_\theta u = \operatorname{div}(\mid \nabla u \mid^{\theta-2} \nabla u)$. In the above boundary condition we have used the notation

$$\frac{\partial u}{\partial \nu_{pq}} := \big(\mid \nabla u \mid^{p-2} + \mid \nabla u \mid^{q-2} \big) \frac{\partial u}{\partial \nu},$$

where ν is the outward unit normal to $\partial\Omega$.

The following hypotheses will be assumed throughout this paper.
(h_{pq}) $p,\ q \in (1, \infty),\ p \neq q;$

$(h_{\rho_1\gamma_1})$ $\rho_1 \in L^\infty(\Omega)$ and $\gamma_1 \in L^\infty(\partial\Omega)$, $\rho_1,\ \gamma_1$ are nonnegative functions such that

$$\int_\Omega \rho_1 \, dx + \int_{\partial\Omega} \gamma_1 \, d\sigma > 0; \tag{2}$$

$(h_{\rho_2\gamma_2})$ $\rho_2 \in L^\infty(\Omega)$, $\gamma_2 \in L^\infty(\partial\Omega)$ and ρ_2 is a nonnegative function.

It is worth pointing out that the potential function γ_2 is allowed to be sign changing.

The operator $\big(\Delta_p + \Delta_q\big)$, called (p, q)-Laplacian, occurs in many applications that include models of elementary particles ([3], [7]), elasticity theory ([17]), reaction-diffusion equations ([5]).

The solution u of (1) is understood as an element of the Sobolev space $W := W^{1,\max\{p,q\}}(\Omega)$ satisfying equation $(1)_1$ in the sense of distributions and $(1)_2$ in the sense of traces.

Definition 1. *A scalar $\lambda \in \mathbb{R}$ is said to be an eigenvalue of the problem* (1) *if there exists* $u_\lambda \in W \setminus \{0\}$ *such that for all* $w \in W$

$$\begin{aligned}
&\int_\Omega (\mid \nabla u_\lambda \mid^{p-2} + \mid \nabla u_\lambda \mid^{q-2}) \nabla u_\lambda \cdot \nabla w \, dx \\
&\quad + \int_\Omega (\rho_1 \mid u_\lambda \mid^{p-2} + \rho_2 \mid u_\lambda \mid^{q-2}) u_\lambda w \, dx \\
&\quad + \int_{\partial\Omega} (\gamma_1 \mid u_\lambda \mid^{p-2} + \gamma_2 \mid u_\lambda \mid^{q-2}) u_\lambda w \, d\sigma = \lambda \int_{\partial\Omega} \mid u_\lambda \mid^{q-2} u_\lambda w \, d\sigma.
\end{aligned} \tag{3}$$

This u_λ *is called an eigenfunction of the problem* (1) *(corresponding to the eigenvalue* λ*).*

According to a Green type formula (see [4], p. 71), $u \in W \setminus \{0\}$ is a solution of (1) if and only if it satisfies (3).

Now, let us introduce the notations

$$\begin{aligned}
K_p(u) &:= \int_\Omega (\mid \nabla u \mid^p + \rho_1 \mid u \mid^p) dx + \int_{\partial\Omega} \gamma_1 \mid u \mid^p \, d\sigma, \\
K_q(u) &:= \int_\Omega (\mid \nabla u \mid^q + \rho_2 \mid u \mid^q) dx + \int_{\partial\Omega} \gamma_2 \mid u \mid^q \, d\sigma \quad \text{for all} \quad u \in W.
\end{aligned} \tag{4}$$

For $\theta > 1$, the Lebesgue norms of the spaces $L^\theta(\Omega)$ and $L^\theta(\partial\Omega)$ will be denoted by $\| \cdot \|_\theta$ and $\| \cdot \|_{\partial\Omega,\theta}$, respectively. Also, in the Sobolev space $W^{1,\theta}(\Omega)$ we will consider the norm

$$\| w \| := \Big(\int_\Omega \mid \nabla w \mid^\theta dx \Big)^{1/\theta} + \Big(\int_{\partial\Omega} \mid w \mid^\theta d\sigma \Big)^{1/\theta} \text{ for all } w \in W^{1,\theta}(\Omega) \tag{5}$$

which is equivalent to the usual norm of $W^{1,\theta}(\Omega)$.

In order to state our main results, we define

$$\Lambda_q := \inf_{w \in W \setminus \{0\}} \frac{K_q(w)}{\| w \|^q_{\partial\Omega,q}}. \tag{6}$$

Let us now state the main result of this paper.

Theorem 1. *Assume that* (h_{pq}), $(h_{\rho_i\gamma_i})$, $i = 1, 2$ *are fulfilled. Then, the set of eigenvalues of problem* (1) *is precisely* (Λ_q, ∞).

If $p = q = 2$, $\gamma_1 \equiv 0$ (hence, according to (2), $\int_\Omega \rho_1\, dx > 0$), and $\gamma_2 \equiv 0$, then $(1)_2$ is precisely the classic Steklov boundary condition. That is why we call our problem (1) a *Steklov-like eigenvalue problem.* Even if the case $p = q$ is here excluded, this name still seems apropriate.

Eigenvalue problems for the $(p, q)-$Laplacian have been extensively investigated in recent years. For the case of the Dirichlet boundary condition we refer to Cherfils-Il'yasov [5], Faria-Miyagaki-Motreanu [8], Marano-Mosconi-Papageorgiou [12], Bobkov-Tanaka [2] and references therein.

The case of the (p, q)-Laplacian (unaccompanied by any potential) with a Robin boundary condition was investigated by Gyulov-Moroşanu [11]. Let us also mention the recent paper by Papageorgiou-Vetro-Vetro [13] concerning the case $\rho_1 \equiv 0,\ \gamma_1 \equiv 0,\ \gamma_2 \equiv \text{const.} > 0$, with the potential function ρ_2 being sign changing.

While in the previous papers [11] and [13] only subsets of the corresponding spectra were determined, in this paper the presence of the potential functions $\rho_i,\ \gamma_i$ satisfying assumptions $(h_{\rho_i \gamma_i}),\ i = 1, 2$, allows the full description of the spectrum.

2 Preliminary results

In this section we state some auxiliary results which will be used in the proofs of our main results.

Let $\theta,\ r \in (1, \infty)$ and $r < \theta(N-1)/(N-\theta)$ if $\theta < N$. Let $\alpha \in L^\infty(\Omega),\ \beta \in L^\infty(\partial\Omega)$ be nonnegative functions such that $\int\limits_\Omega \alpha\, dx + \int\limits_{\partial\Omega} \beta\, d\sigma > 0$ and define

$$k_r(u) := \int\limits_\Omega \alpha \mid u \mid^r dx + \int\limits_{\partial\Omega} \beta \mid u \mid^r d\sigma \ \forall\ u \in W^{1,\theta}(\Omega).$$

Note that $u \to \big(k_r(u)\big)^{\frac{1}{r}}$ is a seminorm on $W^{1,\theta}(\Omega)$ which satisfies
(i) $\exists d > 0$ such that $k_r(u)^{\frac{1}{r}} \leq d \parallel u \parallel_{W^{1,\theta}(\Omega)} \quad \forall u \in W^{1,\theta}(\Omega)$, and
(ii) if $u =$ constant, then $k_r(u) = 0$ implies $u \equiv 0$.
Hence, from [6, Proposition 3.9.55] we obtain the following result

Lemma 1. *Under the assumptions mentioned above on $r,\ \theta,\ \alpha$ and β, the norm $\parallel u \parallel_{\theta,r} := \parallel \nabla u \parallel_\theta + \big(k_r(u)\big)^{\frac{1}{r}}\ \forall\ u \in W^{1,\theta}(\Omega)$ is equivalent to the usual norm of the Sobolev space $W^{1,\theta}(\Omega)$.*

Remark 1. *As a consequence of Lemma 1 we obtain that under assumptions $(h_{\rho_1\gamma_1})$, $K_p^{1/p}(\cdot)$ is a norm equivalent to the usual norm of the Sobolev space $W^{1,p}(\Omega)$.*

Next, for $\theta > 1$, we consider the eigenvalue problem

$$\begin{cases} -\Delta_\theta u + \rho(x) \mid u \mid^{\theta-2} u = 0 \text{ in } \Omega, \\ \mid \nabla u \mid^{\theta-2} \frac{\partial u}{\partial \nu} + \gamma(x) \mid u \mid^{\theta-2} u = \lambda \mid u \mid^{\theta-2} u \text{ on } \partial\Omega, \end{cases} \tag{7}$$

where $\rho \in L^\infty(\Omega)$ and $\gamma \in L^\infty(\partial\Omega)$ are given functions, with $\rho \geq 0$ a.e. on Ω.

As usual, the number $\lambda \in \mathbb{R}$ is said to be an eigenvalue of problem (7) if there exists a function $u_\lambda \in W^{1,\theta}(\Omega) \setminus \{0\}$ such that

$$\begin{aligned} &\int_\Omega \mid \nabla u_\lambda \mid^{\theta-2} \nabla u_\lambda \cdot \nabla w \, dx + \int_\Omega \rho \mid u_\lambda \mid^{\theta-2} u_\lambda w \, dx \\ &\quad + \int_{\partial\Omega} \gamma \mid u_\lambda \mid^{\theta-2} u_\lambda w \, d\sigma = \lambda \int_{\partial\Omega} \mid u_\lambda \mid^{\theta-2} u_\lambda w \, d\sigma \ \forall \ w \in W^{1,\theta}(\Omega). \end{aligned} \tag{8}$$

Define the C^1 functional

$$\Theta_\theta : W^{1,\theta}(\Omega) \setminus \{0\} \to \mathbb{R}, \ \Theta_\theta(v) := \frac{K_\theta(v)}{\parallel v \parallel_{\partial\Omega,\theta}^\theta} \quad \forall \ v \in W^{1,\theta}(\Omega) \setminus \{0\},$$

where $K_\theta(v) := \int_\Omega \big(\mid \nabla v \mid^\theta + \rho \mid v \mid^\theta \big) dx + \int_{\partial\Omega} \gamma \mid v \mid^\theta d\sigma$.

Lemma 2. *If $\rho \in L^\infty(\Omega)$, $\gamma \in L^\infty(\partial\Omega)$ and $\rho \geq 0$ a.e. on Ω then, there exists $u_* \in W^{1,\theta}(\Omega) \setminus \{0\}$ such that*

$$\Theta_\theta(u_*) = \lambda_\theta := \inf_{w \in W^{1,\theta}(\Omega) \setminus \{0\}} \Theta_\theta(w).$$

In addition, λ_θ is the smallest eigenvalue of the problem (8) and u_* is an eigenfunction corresponding to λ_θ.

Proof. First of all, note that functional Θ_θ is positively homogeneous of degree zero. Therefore, we can find a minimizing sequence $\big(u_n\big)_n \subset W^{1,\theta}(\Omega) \setminus \{0\}$ for

$$\lambda_\theta := \inf_{w \in W^{1,\theta}(\Omega) \setminus \{0\}} \Theta_\theta(w),$$

such that $\parallel u_n \parallel_{\partial\Omega,\theta} = 1 \ \ \forall \ n \geq 1$, i. e.,

$$\Theta_\theta(u_n) = K_\theta(u_n) \to \inf_{w \in W^{1,\theta} \setminus \{0\}} \Theta_\theta(w) = \lambda_\theta. \tag{9}$$

In particular, as $\rho \geq 0$ a.e. on Ω, we have that $\lambda_\theta \geq - \parallel \gamma \parallel_{\partial\Omega,\infty}$ thus, $\lambda_\theta \neq -\infty$. Obviously, the sequence $\big(u_n\big)_n$ is bounded in $W^{1,\theta}(\Omega)$ and so,

we may assume that there exist $u_* \in W^{1,\theta}(\Omega)$ and a subsequence of $\big(u_n\big)_n$, again denoted $\big(u_n\big)_n$, such that $u_n \rightharpoonup u_*$ in $W^{1,\theta}(\Omega)$ and $u_n \to u_*$ in $L^\theta(\Omega)$ as well as in $L^\theta(\partial\Omega)$. As $\| u_n \|_{\partial\Omega,\theta} = 1 \ \forall \ n \geq 1$, we have $\| u_* \|_{\partial\Omega,\theta} = 1$, thus $u_* \neq 0$.
Also, we have

$$\| \nabla u_* \|_\theta^\theta \leq \liminf_{n\to\infty} \| \nabla u_n \|_\theta^\theta,$$

$$\lim_{n\to\infty} \int_\Omega \rho \mid u_n \mid^\theta dx = \int_\Omega \rho \mid u \mid^\theta dx,$$

$$\lim_{n\to\infty} \int_{\partial\Omega} \gamma \mid u_n \mid^\theta d\sigma = \int_\Omega \gamma \mid u_* \mid^\theta d\sigma \Rightarrow$$

$$K_\theta(u_*) \leq \liminf_{n\to\infty} K_\theta(u_n).$$

Consequently, as $\| u_* \|_{\partial\Omega,\theta} = \| u_n \|_{\partial\Omega,\theta} = 1 \ \forall \ n \geq 1$, it follows that

$$\Theta_\theta(u_*) = K_\theta(u_*) \leq \liminf_{n\to\infty} K_\theta(u_n) = \lambda_\theta, \tag{10}$$

thus, we have $\Theta_\theta(u_*) = \lambda_\theta$.

We claim that $u_* \in W^{1,\theta}(\Omega) \setminus \{0\}$ is an eigenfunction of problem (7) corresponding to the eigenvalue λ_θ. Obviously, Θ_θ is a C^1 functional on $W^{1,\theta}(\Omega) \setminus \{0\}$, and for every $w \in W^{1,\theta}(\Omega)$ we have

$$\begin{aligned} 0 = \langle \Theta'_\theta(u_*), w\rangle = \Bigg[\Big(& \int_\Omega \mid \nabla u_* \mid^{\theta-2} \nabla u_* \cdot \nabla w \, dx \\ & + \int_\Omega \rho \mid u_* \mid^{\theta-2} u_* w \, dx + \int_{\partial\Omega} \gamma \mid u_* \mid^{\theta-2} u_* w \, d\sigma \Big) \\ & - \Big(\int_{\partial\Omega} \mid u_* \mid^{\theta-2} u_* w \, d\sigma \Big) \Theta_\theta(u_*) \Bigg] \frac{\theta}{\| u_* \|_{\partial\Omega,\theta}^\theta}. \end{aligned} \tag{11}$$

It follows from $\lambda_\theta = \Theta_\theta(u_*)$ that identity (8) is satisfied. Therefore, u_* is indeed an eigenfunction of problem (7) corresponding to the eigenvalue λ_θ.

Finally, let us suppose by way of contradiction that there exists another eigenfunction of problem (7), say $u_\mu \in W^{1,\theta}(\Omega) \setminus \{0\}$, corresponding to the eigenvalue μ, $0 < \mu < \lambda_\theta$. But then, taking $\lambda = \mu$ and $w = u_\lambda = u_\mu$ in (8) we obtain that $\mu = \Theta_\theta(u_\mu) < \lambda_\theta = \Theta_\theta(u_*)$. Obviously, this contradicts the definition of λ_θ. $\square$

We conclude this section by recalling a result which is known as the Lagrange multiplier rule (see, e.g., [9, Theorem 5.5.26, p. 701])

Lemma 3. *Let X, Y be real Banach spaces and let $f : D \to \mathbb{R}$ be Fréchet differentiable, $g \in C^1(D, Y)$, where $D \subseteq X$ is a nonempty open set. If v_0 is a local minimizer of the constraint problem*

$$\min\ f(v), \quad g(v) = 0,$$

and $\mathcal{R}(g'(v_0))$ (the range of $g'(v_0)$) is closed, then there exist $\lambda^* \in \mathbb{R}$, $y^* \in Y^*$ not both equal to zero such that $\lambda^* f'(v_0) + y^* \circ g'(v_0) = 0$, where Y^* stands for the dual of Y.

3 Proof of Theorem 1

Throughout this section we assume that the hypotheses (h_{pq}) and $(h_{\rho_i \gamma_i})$, $i = 1, 2$, are fulfilled and will be used without mentioning them in the statements below.
Now, for $\lambda \in \mathbb{R}$ define the C^1 energy functional for problem (1),

$$\mathcal{J}_\lambda : W \to \mathbb{R},\ \mathcal{J}_\lambda(u) = \frac{1}{p} K_p(u) + \frac{1}{q} K_q(u) - \frac{\lambda}{q} \parallel u \parallel_{\partial\Omega, q}^q . \tag{12}$$

Its derivative is given by

$$\begin{aligned} \langle \mathcal{J}_\lambda'(u), v \rangle &= \int_\Omega \big(\mid \nabla u \mid^{p-2} \mid \nabla u \mid^{q-2} \big) \nabla u \cdot \nabla v \, dx \\ &+ \int_\Omega \big(\rho_1 \mid u \mid^{p-2} + \rho_2 \mid u \mid^{q-2} \big) uv \, dx + \int_{\partial\Omega} \big(\gamma_1 \mid u_\lambda \mid^{q-2} + \gamma_2 \mid u_\lambda \mid^{q-2} \big) uv \, d\sigma \\ &- \lambda \int_{\partial\Omega} \mid u \mid^{q-2} uv \, d\sigma \quad \forall u, v \in W. \end{aligned} \tag{13}$$

So, according to Definition 1, λ is an eigenvalue of problem (1) if and only if there exists a critical point $u_\lambda \in W \setminus \{0\}$ of $\mathcal{J}_\lambda$, i. e. $\mathcal{J}_\lambda'(u_\lambda) = 0$.

The proof of Theorem 1 is based on some lemmas, as follows.

Lemma 4. *There is no eigenvalue of problem* (1) *inside the interval* $(-\infty, \Lambda_q]$. *Moreover, we have the equality*

$$\widetilde{\lambda}_q := \inf_{w \in W \setminus \{0\}} \frac{\frac{1}{q} K_q(w) + \frac{1}{p} K_p(w)}{\frac{1}{q} \parallel w \parallel_{\partial\Omega, q}^q} = \Lambda_q. \tag{14}$$

Proof. First, we deduce from Lemma 2 with $\theta = q$ that $\Lambda_q \geq \Theta_q(u^*)$. More exactly, if $q > p$ we have $\Lambda_q = \lambda_q$. Otherwise, if $q < p$ then $\Lambda_q \geq \lambda_q$, as $W = W^{1,p}(\Omega) \subset W^{1,q}(\Omega)$. In particular, Λ_q is a finite real number.

Now, let us check that there is no eigenvalue of problem (1) in $(-\infty, \Lambda_q]$. Assume the contrary, that there is an eigenpair $(\lambda, u_\lambda) \in (-\infty, \Lambda_q] \times (W \setminus \{0\})$. Then (3) with $w = u_\lambda$ will imply

$$\lambda = \frac{K_q(u_\lambda) + K_p(u_\lambda)}{\| u_\lambda \|^q_{\partial\Omega,q}} \leq \Lambda_q. \tag{15}$$

If $\lambda < \Lambda_q$, we have a contradiction with the definition of Λ_q. On the other hand, if $\lambda = \Lambda_q$ we have $K_p(u_\lambda) = 0$ which implies $u_\lambda \equiv 0$ (see Remark 1). This is impossible since u_λ was assumed to be an eigenfunction.

Finally, let us check the equality (14). Note that the infimum on $W \setminus \{0\}$ of the Rayleigh-type quotient associated to the eigenvalue problem (1) is given by $\widetilde{\lambda}_q$. The estimate $\Lambda_q \leq \widetilde{\lambda}_q$ is obvious. On the other hand, for each $v \in W \setminus \{0\}$ and $t > 0$, we have

$$\widetilde{\lambda}_q = \inf_{w \in W \setminus \{0\}} \frac{K_q(w) + \frac{q}{p} K_p(w)}{\| w \|^q_{\partial\Omega,q}} \leq \frac{K_q(v)}{\| v \|^q_{\partial\Omega,q}} + t^{p-q} \frac{q K_p(v)}{p \| v \|^q_{\partial\Omega,q}}.$$

Now letting $t \to \infty$ if $p < q$ and $t \to 0_+$ if $p > q$, then passing to infimum over all $v \in W \setminus \{0\}$, we get $\tilde{\lambda}_q \leq \Lambda_q$, which concludes the proof. □

In what follows we shall prove that every $\lambda \in (\Lambda_q, \infty)$ is an eigenvalue of problem (1). We distinguish two cases which are complementary to each other.

3.1 Case 1: $q < p$

In this case we have $W = W^{1,p}(\Omega)$.
The following lemma shows, essentially, that the functional defined in (12) is coercive for $q < p$.

Lemma 5. *If $q < p$ then, the functional $\mathcal{J}_\lambda$ is coercive on W, i.e.,*

$$\lim_{\|u\|_W \to \infty} \mathcal{J}_\lambda(u) = \infty.$$

Proof. Assume by way of contradiction that functional $\mathcal{J}_\lambda$ is not coercive. So, there exist a positive constant C and a sequence $\big(u_n\big)_n \subset W$ such that $\| u_n \| \to \infty$ as $n \to \infty$ and $\mathcal{J}_\lambda(u_n) \leq C$. Therefore

$$\frac{1}{p} K_p(u_n) + \frac{1}{q} K_q(u_n) - \frac{\lambda}{q} \| u_n \|^q_{\partial\Omega,q} \leq C \quad \forall\, n \geq 1. \tag{16}$$

In particular,

$$0 \leq \frac{1}{p} K_p(u_n) \leq \frac{\lambda}{q} \parallel u_n \parallel_{\partial\Omega,q}^q + \parallel \gamma_2 \parallel_{\partial\Omega,\infty} \parallel u_n \parallel_{\partial\Omega,q}^q + C \ \forall \ n \geq 1. \tag{17}$$

It follows from estimate (17) and Lemma 1 with $\theta = p, r = q, \alpha \equiv 0, \beta \equiv 1$ that $\parallel u_n \parallel_{\partial\Omega,q} \to \infty$ as $n \to \infty$.
Define $v_n := u_n / \parallel u_n \parallel_{\partial\Omega,q} \ \forall \ n \geq 1$ and divide inequality (17) by $\parallel u_n \parallel_{\partial\Omega,q}^p$. As $q < p$, we obtain that $K_p(v_n) \to 0$ as $n \to \infty$. Hence, $v_n \to 0$ in W (see Remark 1) as well as in $L^q(\partial\Omega)$. In particular, $\parallel v_n \parallel_{\partial\Omega,q} \to 0$, as $n \to \infty$, but this contradicts the fact that $\parallel v_n \parallel_{\partial\Omega,q} = 1$ for all $n \geq 1$. So, $\mathcal{J}_\lambda$ is coercive on W. □

Lemma 6. *If $q < p$ then, every $\lambda > \Lambda_q$ is an eigenvalue of problem* (1).

Proof. Let $\lambda > \Lambda_q$ be fixed. Taking into account Lemma 5, the functional $\mathcal{J}_\lambda$ is coercive. Since in a Banach space the norm functionals are weakly lower semicontinuous, using a similar reasoning as in the proof of Lemma 2 we obtain that $\mathcal{J}_\lambda$ is also weakly lower semicontinuous on W. So there exists a global minimizer $u_* \in W$ for $\mathcal{J}_\lambda$, i.e., $\mathcal{J}_\lambda(u_*) = \min_W \mathcal{J}_\lambda$ (see, e.g., [15, Theorem 1.2]).

On the other hand, from Lemma 4 we have $\Lambda_q = \widetilde{\lambda}_q$ hence, as $\lambda > \Lambda_q$, there is some $u_{0\lambda} \in W \setminus \{0\}$ such that $\mathcal{J}_\lambda(u_{0\lambda}) < 0$.

We note that $\mathcal{J}_\lambda(u_*) \leq \mathcal{J}_\lambda(u_{0\lambda}) < 0$, which implies $u_* \neq 0$. In addition, $\mathcal{J}_\lambda'(u_*) = 0$. Consequently, u_* is an eigenfunction of problem (1) corresponding to the eigenvalue λ. □

3.2 Case 2: $q > p$

In this case, $W = W^{1,q}(\Omega)$. If $q > p$ we cannot expect coercivity on W of the functional $\mathcal{J}_\lambda$. So, we need to use another approach. Consider the Nehari type manifold (see [16]) defined by

$$\mathcal{N}_\lambda = \{v \in W \setminus \{0\}; \langle \mathcal{J}_\lambda'(v), v \rangle = 0\}$$

$$= \{v \in W \setminus \{0\}; K_p(v) + K_q(v) = \lambda \parallel v \parallel_{\partial\Omega,q}^q\}.$$

We shall consider the restriction of $\mathcal{J}_\lambda$ to $\mathcal{N}_\lambda$ since any possible eigenfunction corresponding to λ belongs to $\mathcal{N}_\lambda$. Note that on $\mathcal{N}_\lambda$ functional $\mathcal{J}_\lambda$ has the form

$$\mathcal{J}_\lambda(u) = \frac{q-p}{qp} K_p(u) > 0 \ \forall \ u \in \mathcal{N}_\lambda. \tag{18}$$

In what follows, $\lambda > \Lambda_q$ will be a fixed real number.

Lemma 7. *If $q > p$, then there exists a point $u_* \in \mathcal{N}_\lambda$ where $\mathcal{J}_\lambda$ attains its minimal value, $m_\lambda := \inf_{w \in \mathcal{N}_\lambda} \mathcal{J}_\lambda(w) > 0$.*

Proof. We shall follow an argument similar to that used in Barbu-Moroşanu [1, Case 2, Steps 1-4], so, we split the proof into four steps.

Step 1. $\mathcal{N}_\lambda \neq \emptyset$.

In fact, from $\lambda > \Lambda_q$ and the definition of Λ_q (see (6)) there exists $v_0 \in W \setminus \{0\}$ such that $K_q(v_0) < \lambda \parallel v_0 \parallel^q_{\partial\Omega,q}$. In addition, taking into account Remark 1 we have $K_p(v_0) > 0$.

We claim that for a convenient $\tau > 0$, $\tau v_0 \in \mathcal{N}_\lambda$. Indeed, the condition $\tau v_0 \in \mathcal{N}_\lambda$, $\tau > 0$, reads $\tau^p K_p(v_0) + \tau^q K_q(v_0) = \lambda \tau^q \parallel v_0 \parallel^q_{\partial\Omega,q}$, and this equation can be solved for τ, more exactly,

$$\tau = \left(\frac{K_p(v_0)}{\lambda \parallel v_0 \parallel^q_{\partial\Omega,q} - K_q(v_0)} \right)^{\frac{1}{q-p}}$$

and hence, for this τ we have $\tau v_0 \in \mathcal{N}_\lambda$.

Step 2. Every minimizing sequence $\big(u_n\big)_n \subset \mathcal{N}_\lambda$ for $\mathcal{J}_\lambda$ restricted to $\mathcal{N}_\lambda$ is bounded in W.

Let $\big(u_n\big)_n \subset \mathcal{N}_\lambda$ be such a minimizing sequence for $\mathcal{J}_\lambda$. Assume by contradiction that $\big(u_n\big)_n$ is unbounded in W hence, on a subsequence, again denoted $\big(u_n\big)_n$, we have $\|u_n\| \to \infty$. Since $\big(u_n\big)_n \subset \mathcal{N}_\lambda$, we have (see equality (18))

$$\mathcal{J}_\lambda(u_n) = \frac{q-p}{qp} K_p(u_n) \to m_\lambda \geq 0 \text{ as } n \to \infty, \tag{19}$$

and

$$0 \leq K_p(u_n) = \lambda \parallel u_n \parallel^q_{\partial\Omega,q} - K_q(u_n) \ \forall\ n \geq 1. \tag{20}$$

Set $v_n = u_n / \parallel u_n \parallel$, $n \geq 1$ (where $\parallel \cdot \parallel$ is that defined by (5) with $\theta = q$). Obviously, $\parallel v_n \parallel = 1 \ \forall\ n \geq 1$, so $\big(v_n\big)_n$ is bounded in W. Therefore, there exists $v_0 \in W$ such that $v_n \rightharpoonup v_0$ in W (hence also in $W^{1,p}(\Omega)$ to the same v_0) and $v_n \to v_0$ in $L^q(\Omega)$ as well as in $L^q(\partial\Omega)$. In addition, we also have $\parallel v_0 \parallel = 1$.

Now, dividing (19) by $\parallel u_n \parallel^p$ and making use of $\parallel u_n \parallel \to \infty$ in W, we deduce $K_p(v_n) \to 0$, and so $v_0 \equiv 0$ (see Remark 1). This contradicts the fact that $\parallel v_0 \parallel = 1$. Therefore, $\big(u_n\big)_n$ is bounded in W.

Step 3. $m_\lambda := \inf_{w \in \mathcal{N}_\lambda} \mathcal{J}_\lambda(w) > 0$.

Suppose the contrary, that $m_\lambda = 0$ and let $\big(u_n\big)_n \subset \mathcal{N}_\lambda$ be a minimizing sequence for $\mathcal{J}_\lambda$. By Step 2, $\big(u_n\big)_n$ is bounded in W, so for some $u_0 \in W$,

$u_n \rightharpoonup u_0$ (on a subsequence) in W (and also weakly in $W^{1,p}(\Omega)$ to the same u_0), and $u_n \to u_0$ in both $L^q(\Omega)$ and $L^q(\partial\Omega)$. We have (see (19)) $K_p(u_n) \to 0$, hence $u_0 \equiv 0$ (see Remark 1).

Define $w_n = u_n / \parallel u_n \parallel_{\partial\Omega,q}$, $n \geq 1$. Next, we are going to check that $\big(w_n\big)_n$ is bounded in W.

Indeed, let $u \in W^{1,q}(\Omega)$ be fixed. Clearly, we have

$$\int_{\partial\Omega} \gamma_2 \mid u \mid^q d\sigma \leq \parallel \gamma_2 \parallel_{\partial\Omega,\infty} \parallel u \parallel^q_{\partial\Omega,q} . \tag{21}$$

Now, taking into account (21), we have for every $\varepsilon > 0$

$$\begin{aligned} \parallel u \parallel^q - \int_{\partial\Omega} \gamma_2 \mid u \mid^q d\sigma &= \parallel u \parallel^q_{\partial\Omega,q} + \parallel \nabla u \parallel^q_q - \int_{\partial\Omega} \gamma_2 \mid u \mid^q d\sigma \\ &\leq \parallel u \parallel^q_{\partial\Omega,q} + \parallel \nabla u \parallel^q_q + \varepsilon \parallel \gamma_2 \parallel_{\partial\Omega,\infty} \parallel u \parallel^q + \parallel \gamma_2 \parallel_{\partial\Omega,\infty} \parallel u \parallel^q_{\partial\Omega,q}, \end{aligned}$$

which implies

$$\begin{aligned} &(1 - \varepsilon \parallel \gamma_2 \parallel_{\partial\Omega,\infty}) \parallel u \parallel^q \\ &\quad \leq \parallel \nabla u \parallel^q_q + \int_{\partial\Omega} \gamma_2 \mid u \mid^q d\sigma + (\parallel \gamma_2 \parallel_{\partial\Omega,\infty} + 1) \parallel u \parallel^q_{\partial\Omega,q} \\ &\quad \leq K_q(u) + (\parallel \gamma_2 \parallel_{\partial\Omega,\infty} + 1) \parallel u \parallel^q_{\partial\Omega,q}, \end{aligned} \tag{22}$$

where we have used the assumption $\rho_2 \geq 0$ a.e. on Ω.

Consequently, choosing $\varepsilon < 1/ \parallel \gamma_2 \parallel_{\partial\Omega,\infty}$ we obtain

$$\parallel u \parallel^q \leq C_1 K_q(u) + C_2 \parallel u \parallel^q_{\partial\Omega,q}, \tag{23}$$

where $C_1 = (1 - \varepsilon \parallel \gamma_2 \parallel_{\partial\Omega,\infty})^{-1}$, $C_2 = C_1(1 + \parallel \gamma_2 \parallel_{\partial\Omega,\infty})$ are positive constants independent of u.

Dividing (20) by $\parallel u_n \parallel^q_{\partial\Omega,q}$ we get

$$K_q(w_n) \leq \lambda \ \text{ for all } \ n \geq 1. \tag{24}$$

Now, from (24) and (23), taking into account that $\parallel w_n \parallel_{\partial\Omega,q} = 1$ for all $n \geq 1$, it follows that

$$\parallel w_n \parallel^q \leq C_1 \lambda + C_2 \ \text{ for all } \ n \geq 1. \tag{25}$$

Hence, the sequence $\big(w_n\big)_n$ is bounded in W and therefore, on a subsequence, $w_n \rightharpoonup w_0$ in W for some $w_0 \in W$ and strongly in both $L^q(\Omega)$ and $L^q(\partial\Omega)$, to the same w_0. and respectively to the trace of w_0 on $\partial\Omega$.

Now, we divide (20) by $\| u_n \|^p_{\partial\Omega,q}$ and taking into account (24), (25) and $u_n \to 0$ in both $L^q(\Omega)$ and $L^q(\partial\Omega)$, we get

$$K_p(w_n) = \| u_n \|^{q-p}_{\partial\Omega,q} \left[\lambda - K_q(w_n)\right] \to 0. \tag{26}$$

This implies $w_n \to 0$ in $W^{1,p}(\Omega)$, thus $w_0 \equiv 0$. In particular, $w_n \to 0$ in $L^q(\partial\Omega)$ which contradicts the fact that $\| w_n \|_{\partial\Omega,q} = 1$ for all $n \geq 1$. This contradiction shows that $m_\lambda > 0$.

Step 4. There exists $u_* \in \mathcal{N}_\lambda$ such that $\mathcal{J}_\lambda(u_*) = m_\lambda$.

Let $\big(u_n\big)_n \subset \mathcal{N}_\lambda$ be a minimizing sequence, i.e., $\mathcal{J}_\lambda(u_n) \to m_\lambda$. In particular, the sequence $\big(u_n\big)_n$ satisfies (20) and is bounded in W (by Step 2) thus, on a subsequence, $u_n \rightharpoonup u_* \in W$ and strongly in $L^q(\Omega)$ and $L^q(\partial\Omega)$ (to the same u_*).

We claim that, $u_* \not\equiv 0$. First, (20) and (23) imply that

$$\| u_n \|^q \leq C_1 K_q(u_n) + C_2 \| u_n \|^q_{\partial\Omega,q} \leq \lambda C_1 \| u_n \|^q_{\partial\Omega,q} + C_2 \| u_n \|^q_{\partial\Omega,q}$$

thus,

$$0 \leq \| u_n \|^q \leq (C_1 + \lambda C_2) \| u_n \|^q_{\partial\Omega,q} \quad \text{for all} \ \ n \geq 1. \tag{27}$$

If $u_* \equiv 0$, we get from (27) that $\| u_n \| \to 0$ in W and also in $W^{1,p}(\Omega)$. Hence, (19) will give $m_\lambda = 0$ thus, contradicting the statement of Step 3. Using a reasoning similar to one used in the proof of Lemma 2, by passing to limit as $n \to \infty$ in (20), we find

$$K_p(u_*) + K_q(u_*) \leq \lambda \| u_* \|^q_{\partial\Omega,q} . \tag{28}$$

If we have equality in (28) then $u_* \in \mathcal{N}_\lambda$ and the proof is complete since in this case $\mathcal{J}_\lambda(u_*) = m_\lambda$. In what follows we show that the strict inequality

$$K_p(u_*) + K_q(u_*) < \lambda \| u_* \|^q_{\partial\Omega,q} \tag{29}$$

is impossible. Let us assume by contradiction that (29) holds true. Let us check that there exists $\tau \in (0,1)$ such that $\tau u_* \in \mathcal{N}_\lambda$. For this purpose, we consider the function

$$f : (0,\infty) \to \mathbb{R}, \ f(t) := t^{p-q} K_p(u_*) + K_q(u_*) - \lambda \| u_* \|^q_{\partial\Omega,q} .$$

As $K_p(u_*) > 0$, we have $f(t) \to \infty$ as $t \to 0_+$. Since $f(1) < 0$ (see (29)), there exists $\tau \in (0,1)$ such that $f(\tau) = 0$ which implies $\tau u_* \in \mathcal{N}_\lambda$. But then,

$$0 < m_\lambda \leq \mathcal{J}_\lambda(\tau u_*) = \tau^p \frac{q-p}{qp} K_p(u_*) \leq \tau^p \lim_{n\to\infty} \mathcal{J}_\lambda(u_n) = \tau^p m_\lambda < m_\lambda,$$

which is impossible.

□

Lemma 8. *If $p < q$ then, every $\lambda \in (\Lambda_q, \infty)$ is an eigenvalue of problem* (1).

Proof. We claim that the minimizer $u_* \in \mathcal{N}_\lambda$ from Lemma 7 is an eigenfunction of problem (1) with corresponding eigenvalue λ.

Clearly, u_* is a solution of the constraint minimization problem

$$\min_{v \in W \setminus \{0\}} \mathcal{J}_\lambda(v), \quad g_q(v) := K_p(v) + K_q(v) - \lambda \parallel v \parallel_{\partial\Omega,q}^q = 0.$$

We can use Lemma 3, with $X = W$, $D = W \setminus \{0\}$, $Y = \mathbb{R}$, $f = \mathcal{J}_\lambda$. Note that all the assumptions of Lemma 3 are satisfied in our case, including the surjectivity of $g_q'(u_*)$, i.e. for all $\xi \in \mathbb{R}$ there exists a $w \in W \setminus \{0\}$ such that $\langle g_q'(u_*), w \rangle = \xi$. Indeed, if we choose in the above equations w of the form $w = \chi u_*$, $\chi \in \mathbb{R}$, and use $u_* \in \mathcal{N}_\lambda$, we obtain

$$\chi\Big(pK_p(u_*) + q\big(K_q(u_*) - \lambda \parallel u_* \parallel_{\partial\Omega,q}^q\big)\Big) = \xi \;\Leftrightarrow\; \chi K_p(u_*)(p - q) = \xi$$

which has a unique solution χ (by Remark 1). Thus $g_q'(u_*)$ is indeed surjective and so Lemma 3 is applicable to the above constraint minimization problem. Therefore there exist $\lambda^*, \mu \in \mathbb{R}$, not both equal to zero, such that

$$\lambda^* \langle \mathcal{J}_\lambda'(u_*), v \rangle + \mu \langle g_q'(u_*), v \rangle = 0, \quad \forall\, v \in W.$$

Testing with $v = u_*$ and using the fact that $u_* \in \mathcal{N}_\lambda$, we derive

$$\mu(p - q)K_p(u_*) = 0,$$

which implies $\mu = 0$. Therefore, $\lambda^* \neq 0$, hence

$$\langle \mathcal{J}_\lambda'(u_*), v \rangle = 0 \;\forall\; v \in W,$$

i. e. λ is an eigenvalue of problem (1). □

Finally, we can see that Theorem 1 follows from Lemmas 6 and 8 above.

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