A note on maximal matchings in rooted products of paths and short cycles

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Abstract

A matching M in a graph G is maximal if it cannot be extended to a larger matching in G . The enumerative properties of maximal matchings are much less known and researched than for maximum and perfect matchings. In this paper we present the recurrences and generating functions for the sequences enumerating maximal matchings in rooted products of paths and short cycles. We also analyze the asymptotic behavior of those sequences.

Keywords: maximal matching, rooted product of graphs

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1 Introduction

A matching in a graph is a collection of its edges such that no two edges in this collection have a vertex in common. Many problems in natural, technical and social sciences can be successfully modeled by matchings in graphs. Today the matching theory is a well developed part of graph theory, strongly influenced by chemical applications. Particularly well researched are perfect and maximum matchings. For a general background on matching theory and terminology we refer the reader to the classical monograph by Lóvasz and Plummer [4].

A matching is perfect if its edges are collectively incident to all vertices of G . It is clear that perfect matchings are as large as possible and that no other matching in G

can be “larger” than a perfect one. In all other applications we are also interested mostly in large matchings. If we are using the number of edges as the measure of “largeness”, we get the maximum matchings. For them we have well developed structural theory and many enumerative results. No such theory, however, exists for another class of large matchings - maximal matchings. A matching M is **maximal** if it cannot be extended to a larger matching in G . Obviously, every maximum matching is also maximal, but the opposite is generally not true. Maximal matchings are much less researched, but still very useful. That goes both for their structural and their enumerative aspects. The crucial difference is non-locality. That means that we cannot split the set of all maximal matchings into those containing an edge e and those not containing it without taking into account the edge-neighborhood of e . The main goal of this work is to increase our knowledge about the enumerative aspects of maximal matchings.

We begin by establishing the principal terminology and notation which we will use throughout the article. It is taken from [7]. All graphs G considered in this paper will be finite and simple, with vertex set $V(G)$ and set of edges $E(G)$. We will denote by $n = |V(G)|$ the number of vertices and by $m = |E(G)|$ the number of edges in G . As usual, the cycle and complete graph on n vertices are denoted by C_n and K_n , respectively. However, by P_n we denote the path of length n , i.e., on $n + 1$ vertices. Let $\Psi(G)$ denote the number of maximal matchings in graph G .

2 Rooted product of graphs

Many interesting graphs arise from simpler building blocks via some binary operations known as graph products. In 1978, Godsil and McKay [2] introduced a new product of two graphs G_1 and G_2 , called the rooted product, and denoted by $G_1 \odot G_2$. The rooted product graphs are used, for example, in internet networking for connecting local networks to a wider frame. One can view such a situation as a combination of many local networks (copies of graph H) having a server (the root vertex of graph H). These servers are themselves connected through a global network (the graph G). So, one motivation to count the maximal matchings can be securing networks with as few sensors as possible.

The main goal of this paper is to study the enumerative aspects of maximal matchings in some classes of rooted products. We establish recurrences for the enumerating sequences and, in some cases, we use generating functions to determine

their asymptotic behavior. We consider operations of rooted product of paths and cycles, and enumerate maximal matchings in the resulting graphs.

Let $V(G) = \{u_i | 1 \leq i \leq n\}$ and $v \in V(H)$. **The rooted product** $G \odot H$ of a graph G and a rooted graph H with respect to the root v is defined as follows: take $n = |V(G)|$ copies of H , and for every vertex u_i of G , identify u_i with the root v of the i -th copy of H . Obviously,

$$\begin{aligned} |V(G \odot H)| &= |V(G)| |V(H)| \\ |E(G \odot H)| &= |E(G)| + |V(G)| |E(H)|. \end{aligned}$$

Hence, we take as many copies of graph H as we have vertices in graph G . If H or G is the singleton graph, then $G \odot H$ is equal to G or H , respectively. In this sense, to obtain the rooted product $G \odot H$, hereafter we will only consider graphs G and H of orders greater than or equal to two. We note that the rooted product is not commutative.

The rooted product is especially relevant for trees, as a rooted product of two trees is another tree. For instance, [3] used rooted products to find graceful numberings for a wide family of trees. Also, the rooted product of two graphs is a subgraph of the Cartesian product of those two graphs.

3 Rooted product graph $P_n \odot P_m$

We first consider the case when we have a rooted product of two paths, always taking a leaf as a root. In all cases we obtain a tree. An example is shown in Figure 1.

Figure 1: Example of $P_n \odot P_2$ rooted product

The next lemma is a direct consequence of Proposition 6.1. from article [1].

Lemma 1. *The number of maximal matchings in rooted product graph $P_n \odot P_1$ is equal to the value of the $(n + 2)$ -nd Fibonacci number, i.e. $\Psi(P_n \odot P_1) = F_{n+2}$.*

Proof. It is clear that every vertex of the P_n must be covered by an edge of a maximal matching. If vertex n is covered by the edge $v_{n-1}v_n$, the remaining edges of a maximal matching must form a valid maximal matching in $P_{n-2} \odot P_1$, and hence are counted by $\Psi(P_{n-2} \odot P_1)$. If v_n is covered by the pendent edge of P_1 , the remaining maximal matchings are counted by $\Psi(P_{n-1} \odot P_1)$. Hence, the number of maximal matchings in $P_n \odot P_1$ satisfies the recurrence

$$\Psi(P_n \odot P_1) = \Psi(P_{n-1} \odot P_1) + \Psi(P_{n-2} \odot P_1),$$

with initial conditions $\Psi(P_0 \odot P_1) = 1, \Psi(P_1 \odot P_1) = 2$.

The sequence F_n of Fibonacci numbers is defined by the same recurrence relation and the same initial conditions, so the claim follows. $\square$

We now give another result for a rooted product of two paths, this time for the case $P_n \odot P_2$ shown in Figure 1. We will denote the number of maximal matchings in the graph $P_n \odot P_2$ with t_n .

Proposition 1. *The sequence t_n satisfies the recurrence $t_n = t_{n-1} + 2t_{n-2} + t_{n-3}$ with the initial conditions $t_0 = 2, t_1 = 4$.*

In the proof we will repeat argument from paper [6] – we split graph into two parts: counting the number of matchings containing the root and counting the matchings which do not contain the root. We must take into account the edge-neighborhood of e .

Proof. Each maximal matching in $P_n \odot P_2$ either covers the rightmost root or does not cover it. In the first case, the remaining edges must form either a valid matching in $P_{n-1} \odot P_2$ (if is root covered by edge of P_2) or a valid maximal matching in $P_{n-2} \odot P_2$ (if the root is covered by edge of P_n).

Maximal matchings that do not cover that root must cover both its neighbors. The neighbor in P_n can be covered in two ways, one of them counted by the number of maximal matchings in $P_{n-2} \odot P_2$, another by the number of maximal matchings in $P_{n-3} \odot P_2$. The claim now follows by adding the two contributions. $\square$

The following proposition and corollary give the generating function and asymptotic behavior of t_n .

Proposition 2. *The generating function $T(x)$ for the sequence t_n is given by*

$$T(x) = \frac{x^2 + 2x + 2}{1 - x - 2x^2 - x^3}.$$

Corollary 1. *The asymptotic behavior of t_n is given by $t_n \sim 1.92485 \cdot 0.51952^n$.*

The sequence t_n provides a new combinatorial interpretation of sequence A141015 from the OEIS [5].

4 Rooted product graph $P_n \odot C_m$

In this section we will show some results about rooted product graph $P_n \odot C_m$. Figure 2 shows the first case - the rooted product $P_n \odot C_3$.

Figure 2: Example of $P_n \odot C_3$ rooted product

The next lemma is a consequence of Proposition 7.3. from article [1]. We will denote the number of maximal matchings in the rooted product graph $P_n \odot C_3$ with p_n .

Lemma 2. *The sequence p_n satisfies the recurrence*

$$p_n = 2p_{n-1} + 3p_{n-2} + p_{n-3},$$

with the initial conditions $p_0 = 3, p_1 = 9, p_2 = 28$.

The sequence p_n appears as A084084 in the OEIS.

An example for the next case, the rooted product $P_n \odot C_4$, is shown in Figure 3. We will denote the number of maximal matchings in this case with $\Psi(P_n \odot C_4) = r_n$.

Proposition 3. *The sequence r_n satisfies the recurrence*

$$r_n = 2r_{n-1} + 4r_{n-2},$$

with the initial conditions $r_0 = 2, r_1 = 8$.

Proof. Each maximal matching in $P_n \odot C_4$ must cover the root of the copy of C_4 rooted at the rightmost vertex of P_n . If the root is covered by one of the two edges of C_4 incident with it, the remaining edges of this maximal matching must form a valid maximal matching in $P_{n-1} \odot C_4$. If the root is covered by the edge incident with it in P_n , the remaining edges must form a valid maximal matching in $P_{n-2} \odot C_4$. In that case, there are four ways to form maximal matchings in the rest of the two rightmost copies of C_4 . The claim now follows by adding the two contributions. $\square$

Proposition 4. *The generating function $R(x)$ for the sequence r_n is given by*

$$R(x) = \frac{1}{1 - 2x - 4x^2}.$$

Corollary 2. *The asymptotic behavior of r_n is given by $r_n \sim 1 + \sqrt{5}$, twice the golden ratio.*

Figure 3: Example of $P_n \odot C_4$ rooted product

The sequence r_n is known as the Horadam sequence $a_n = s \cdot a_{n-1} + r \cdot a_{n-2}$ for $s = 2, r = 4$. It appears as A085449 in the OEIS and it counts, among other things, the number of ways to tile an n -board with two types of colored squares and four types of colored dominoes.

5 Concluding remarks

In this paper we have counted maximal matchings in rooted products of some graphs. We have addressed only the combination of paths and short cycles. It would be probably too ambitious to hope for general enumerative results for rooted products. However, many interesting cases should be within the reach. Another interesting thing in such graphs would be to consider their saturation number and to compute their maximal matching polynomials.

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