Verification and Strengthening of the Atiyah–Sutcliffe Conjectures for Several Types of Almost Collinear Configurations in Euclidean and Hyperbolic Plane

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Abstract

In 2001 Sir M. F. Atiyah formulated a conjecture C1 and later with P. Sutcliffe two stronger conjectures C2 and C3. These conjectures, inspired by physics (spin-statistics theorem of quantum mechanics), are geometrically defined for any configuration of points in the Euclidean three space. The conjecture C1 is proved for $n = 3, 4$ and for general n only for some special configurations (M. F. Atiyah, M. Eastwood and P. Norbury, D.Đoković). In [11] and [12] we have verified the conjectures C2 and C3 for parallelograms, cyclic quadrilaterals and some infinite families of tetrahedra, and have proposed a strengthening of conjecture C3 for configurations of four points (Four Points Conjectures). All three Euclidean four-point conjectures have been proved in [14] (2010.) (see also [15] for more information) then independently four years later in [10].

For almost collinear configurations of type A (with all but one point on a line) we propose in [12] several new conjectures (some for symmetric functions) which imply C2 and C3. By using computations with multi-Schur functions we can do verifications up to $n = 9$ of our conjectures. For a type B (and some special dihedral configurations) we verify a stronger conjecture of Đoković which implies C2 for his nonplanar configurations with dihedral symmetry. Recently

we have observed that our generalizations from [12] are capable to imply hyperbolic C2 for type A and B configurations and we make these connections clear in Sections 1.2 and 1.3.

Finally we mention that by minimizing a geometrically defined energy, figuring in these conjectures, one gets a connection to some complicated physical theories, such as Skyrmions and Fullerenes.

Keywords: Atiyahs conjecture, Configuration of N points in the three-dimensional Euclidean space, Configuration of N points in the Hyperbolic plane

MSC: 51M04, 51M16, 70G25

1 Almost collinear configurations. Đoković's approach

1.1 Type A configurations

By a type A configurations of N points $x_1, \dots, x_N$ we shall mean the case when $N - 1$ of the points $x_1, \dots, x_N$ are collinear. Set $n = N - 1$. In ([6]) Đoković has proved, for configurations of type A, both the Atiyah conjecture (Theorem 2.1) and the first Atiyah–Sutcliffe conjecture (Theorem 3.1). By using Cartesian coordinates, with $x_i = (a_i, 0)$, $a_1 < a_2 < \dots < a_n$ and $x_N = x_{n+1} = (0, b)$ (with $b = 1$), the normalized Atiyah matrix $M_{n+1} = M_{n+1}(\lambda_1, \dots, \lambda_n)$ (denoted by P in [6] when $b = -1$) is given by

$$M_{n+1} = \begin{bmatrix} 1 & \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda_2 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & & 1 & \lambda_n \\ (-1)^n e_n & (-1)^{n-1} e_{n-1} & \cdots & \cdots & -e_1 & 1 \end{bmatrix}$$

where $\lambda_1 = a_1 + \sqrt{a_1^2 + b^2} < \lambda_2 = a_2 + \sqrt{a_2^2 + b^2} < \dots < \lambda_n = a_n + \sqrt{a_n^2 + b^2}$ (with $b = 1$) are positive real numbers and where $e_k = e_k(\lambda_1, \dots, \lambda_n)$, $1 \leq k \leq n$, is the k -th elementary symmetric function of $\lambda_1, \lambda_2, \dots, \lambda_n$. Its determinant satisfies the inequality

$$\begin{aligned} D_n &= 1 + \lambda_n e_1 + \lambda_n \lambda_{n-1} e_2 + \cdots + \lambda_n \lambda_{n-1} \cdots \lambda_1 e_n \\ &\geq 1 + e_1(\lambda_1^2, \dots, \lambda_n^2) + e_2(\lambda_1^2, \dots, \lambda_n^2) + \cdots + e_n(\lambda_1^2, \dots, \lambda_n^2) \\ &= \prod_{i=1}^n (1 + \lambda_i^2) \end{aligned}$$

equivalent to the first Atiyah–Sutcliffe conjecture ([4], Conjecture 2). The second Atiyah–Sutcliffe conjecture ([4], Conjecture 3) for configurations of type A is equivalent to the following inequality

$$[D_{n+1}(\lambda_1, \dots, \lambda_n)]^{n-1} \geq \prod_{k=1}^n D_n(\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_n) \quad (1.1)$$

For $n = 2$ this inequality takes the form

$$1 + \lambda_2 e_1(\lambda_1, \lambda_2) + \lambda_1 \lambda_2 e_2(\lambda_1, \lambda_2) \geq (1 + \lambda_2 e_1(\lambda_2))(1 + \lambda_1 e_1(\lambda_1))$$

i.e.

$$1 + \lambda_2 e_1(\lambda_1, \lambda_2) + \lambda_1 \lambda_2 e_2(\lambda_1, \lambda_2) \geq (1 + \lambda_2^2)(1 + \lambda_1^2). \quad (1.2)$$

This reduces to $(\lambda_2 - \lambda_1)\lambda_1 \geq 0$, so it is true.

1.2 Type A configurations - hyperbolic case

Let $H^2 = \{(x, y) \in \mathbb{R}^2 | y > 0\}$ be the upper half plane model of hyperbolic plane. A type A hyperbolic configuration denoted by $A_{n,1} = A(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n; \mathbf{x}_{n+1})$ consists of $N = n + 1$ points in H^2 where n points are collinear, $(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n)$, $\mathbf{x}_i = (0, b_i)$, $b_1 > b_2 > \dots > b_n > 0$ and $\mathbf{x}_{n+1} = (a, b)$.

Let us abbreviate the directions t_{ij} from $\mathbf{x}_i$ to $\mathbf{x}_j$ (viewed as points on the absolute $\mathbb{R} \cup \{\infty\}$)

$$\begin{aligned} t_{i,n+1} &= X_i, 1 \leq i \leq n & t_{ij} &= 0 \text{ for } 1 \leq i < j \leq n, \\ t_{n+1,i} &= -\xi_i, 1 \leq i \leq n & t_{ji} &= \infty \text{ for } 1 \leq i < j \leq n. \end{aligned}$$

Then we have $t_{i,n+1} \cdot t_{n+1,i} = -b_i^2$ i.e. $\xi_i X_i = b_i^2$.

The Atiyah polynomials associated to points $\mathbf{x}_i$ ($1 \leq i \leq n + 1$) defined by

$$p_i = \prod_{j \neq i} (z - t_{ij})$$

(with $z - t_{ij}$ interpreted as 1 if $t_{ij} = \infty$) are given explicitly as follows

$$\begin{aligned} p_1 &= z^{n-1}(z - X_1) \\ p_2 &= z^{n-2}(z - X_2) \\ &\vdots \\ p_{n-1} &= z(z - X_{n-1}) \\ p_n &= z - X_n \\ p_{n+1} &= (z + \xi_1)(z + \xi_2) \cdots (z + \xi_n) = z^n + e_1 z^{n-1} + \cdots + e_n, \end{aligned}$$

where $e_1, \dots, e_n$ denotes the elementary symmetric functions of $\xi_1, \dots, \xi_n$.

The Atiyah determinant (the determinant of the matrix of coefficients of p_i 's) $D_{n,1}^{\text{hyp}} = D_{n,1}^{\text{hyp}}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n; \mathbf{x}_{n+1})$ is then equal to

$$D_{n,1}^{\text{hyp}} = \begin{vmatrix} 1 & -X_1 & & & & \\ 0 & 1 & -X_2 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ & & & & 1 & -X_n \\ 1 & e_1 & e_2 & \dots & e_{n-1} & e_n \end{vmatrix} = X_1 \cdots X_n + X_2 \cdots X_n e_1 + \cdots + e_n = \sum_{i=0}^n e_i X_{i+1} \cdots X_n .$$

$t_{i,j} = t_{ij}$ = direction from $\mathbf{x}_i$ to $\mathbf{x}_j$; $0 < a < X_1 < X_2 < \cdots < X_n$

Note that

$$X_1 \cdots X_n \cdot D_{n,1}^{\text{hyp}} \Big|_{X_i \rightarrow X_i^{-1}}$$

is a specialization of the polynomial

$$\Psi_{12\dots n}^{12\dots n} \in Q[X_1, \dots, X_n, \xi_1, \dots, \xi_n]$$

defined in the section 1.4 which is symmetric in $\xi_1, \dots, \xi_n$. The properties of polynomials Ψ_j^I will enable us to study simultaneously both the euclidean and the hyperbolic configurations of type A (and B, defined later). Since

$$\begin{vmatrix} 1 & -t_{j,n+1} \\ 1 & -t_{n+1,j} \end{vmatrix} = t_{j,n+1} - t_{n+1,j} = X_j + \xi_j, 1 \leq j \leq n,$$

we obtain for the normalized Atiyah determinant the following formula

$$D_{n,1} = \frac{\sum_{i=0}^n e_i X_{i+1} \cdots X_n}{\prod_{j=1}^n (X_j + \xi_j)} \quad (\text{with } \xi_i X_i = b_i^2) .$$

Lemma 1.1 *For $0 < X_1 \leq X_2 \leq \dots \leq X_n$ and $\xi_1, \dots, \xi_n$ commuting indeterminates, the inequality*

$$\prod_{j=1}^n (X_j + \xi_j) \leq \sum_{i=0}^n e_i(\xi_1, \dots, \xi_n) X_{i+1} \cdots X_n$$

holds true coefficientwise.

Proof .

$$\begin{aligned} \prod_{j=1}^n (X_j + \xi_j) &= \sum_{S \subseteq [n]} \prod_{j \in S} \xi_j \cdot \prod_{k \in S^C} X_k \leq \sum_{S \subseteq [n]} \prod_{j \in S} \xi_j \cdot X_{|S|+1} \cdots X_n \\ &= \sum_{i=0}^n e_i \cdot X_{i+1} \cdots X_n \quad \blacksquare \end{aligned}$$

Corollary 1.2 *For any n points on a line and one point outside it in a hyperbolic plane (or space) we have that the second Atiyah–Sutcliffe conjecture holds true:*

$$D_{n,1}^{hyp}(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{x}_{n+1}) \geq \prod_{1 \leq i < j \leq n+1} D_2(\mathbf{x}_i, \mathbf{x}_j) = \prod_{j=1}^n D_2(\mathbf{x}_j, \mathbf{x}_{n+1}) .$$

Proof .

We have that l.h.s. is equal to $\sum e_i X_{i+1} \cdots X_n$ and the r.h.s. reduces to the product $\prod_{j=1}^n (X_j + \xi_j)$. $\blacksquare$

1.3 Type B configurations - hyperbolic case

Let $\mathbf{x}_i = (0, b_i)$, $b_1 > b_2 > \cdots > b_r > c > b_{r+1} > \cdots > b_n > 0$, ($c = \sqrt{a^2 + b^2}$), $\mathbf{x}_{n+1} = (a, b)$ and $\mathbf{x}_{n+2} = (-a, b)$ be a type B hyperbolic configuration of $N = n + 2$ points in H^2 denoted by $B_{n,2} = B(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n; \mathbf{x}_{n+1}, \mathbf{x}_{n+2})$.

The hyperbolic directions from x_i to x_j we abbreviate as follows

$$\begin{aligned} t_{i,n+1} &= X_i, 1 \leq i \leq n & t_{ij} &= 0 \text{ for } 1 \leq i < j \leq n , \\ t_{n+1,i} &= -\xi_i, 1 \leq i \leq n & t_{ji} &= \infty \text{ for } 1 \leq i < j \leq n , \\ t_{i,n+2} &= -X_i, 1 \leq i \leq n & t_{n+1,n+2} &= -c, \\ t_{n+2,i} &= \xi_i, 1 \leq i \leq n & t_{n+2,n+1} &= c. \end{aligned}$$

The Atiyah polynomials associated to points $\mathbf{x}_i$ ($1 \leq i \leq n + 2$) defined (in [1] - [4]) by

$$p_i = \prod_{j \neq i} (z - t_{ij})$$

(with $z - t_{ij}$ interpreted as 1 if $t_{ij} = \infty$) associated to points $\mathbf{x}_i$ ($1 \leq i \leq n + 2$) are then

$$\begin{aligned} p_1 &= z^{n-1}(z - X_1)(z + X_1) \\ p_2 &= z^{n-2}(z - X_2)(z + X_2) \\ &\vdots \\ p_{n-1} &= z(z - X_{n-1})(z + X_{n-1}) \\ p_n &= (z - X_n)(z + X_n) \\ p_{n+1} &= (z + \xi_1)(z + \xi_2) \cdots (z + \xi_n)(z + c) = z^{n+1} + e_1 z^n + \cdots + e_{n+1} \\ p_{n+2} &= (z - \xi_1)(z - \xi_2) \cdots (z - \xi_n)(z - c) = z^{n+1} - e_1 z^n + \cdots + (-1)^{n+1} e_{n+1} . \end{aligned}$$

$$(-1)^{n+1} D_{n,2}^{\text{hyp}} = (-1)^{n+1} \begin{vmatrix} 1 & 0 & -X_1^2 & 0 & & 0 \\ 0 & 1 & 0 & -X_2^2 & & 0 \\ \vdots & & & & \ddots & \\ 0 & 0 & 0 & & 1 & 0 & -X_n^2 \\ 1 & e_1 & e_2 & e_3 & \cdots & e_n & e_{n+1} \\ 1 & -e_1 & e_2 & -e_3 & \cdots & & (-1)^{n+1} e_{n+1} \end{vmatrix} =$$

We first add $n+1$ -st row to the $n+2$ -nd row and then subtract a half of the $n+2$ -nd row from the $n+1$ -st row.

$$= (-1)^{n+1} 2 \begin{vmatrix} 1 & 0 & -X_1^2 & 0 & & 0 \\ 0 & 1 & 0 & -X_2^2 & & 0 \\ \vdots & & & & \ddots & \\ 0 & 0 & 0 & & 1 & 0 & -X_n^2 \\ 0 & e_1 & 0 & e_3 & \cdots & 0 & e_{n+1} \\ 1 & 0 & e_2 & 0 & \cdots & e_n & 0 \end{vmatrix} \quad (n \text{ even}) = \dots$$

Then we add $-1 \cdot 1^{\text{st}}$ row $-(X_1^2 + e_2) \cdot 3^{\text{rd}}$ row $+X_3^2(X_1^2 + e_2) \cdot 5^{\text{th}}$ row $+\dots$ to the $n+2$ -nd row and we also add $-e_1 \cdot 2^{\text{nd}}$ row $+(-(X_2^2 e_1 + e_3)) \cdot 4^{\text{th}}$ row $+\dots$ to the $n+1$ -st row.

$$= 2 \begin{vmatrix} 1 & 0 & -X_1^2 & 0 & 0 & & 0 \\ 0 & 1 & 0 & -X_2^2 & 0 & & 0 \\ 0 & 0 & 1 & 0 & -X_3^2 & & 0 \\ 0 & 0 & 0 & 1 & 0 & -X_4^2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -X_5^2 & 0 \\ \vdots & & & & \vdots & & \vdots \\ 0 & 0 & 0 & X_2^2 e_1 + e_3 & 0 & e_5 & \cdots \\ 0 & 0 & X_1^2 + e_2 & 0 & e_4 & \cdots \end{vmatrix} = \dots$$

At the end we get a triangular matrix whose determinant is equal to $D_{n,2}^{\text{hyp}} = 2f_0 f_1$ where we have for n even

$$f_{n,2}^0 = X_1^2 X_3^2 \cdots X_{n-1}^2 + X_3^2 \cdots X_{n-1}^2 e_2 + \cdots + X_{n-1}^2 e_{n-2} + e_n$$

$$f_{n,2}^1 = X_2^2 X_4^2 \cdots X_n^2 e_1 + X_4^2 \cdots X_n^2 e_3 + \cdots + X_n^2 e_{n-1} + e_{n+1},$$

and for n odd

$$f_{n,2}^0 = X_1^2 X_3^2 \cdots X_n^2 + X_3^2 \cdots X_n^2 e_2 + \cdots + X_n^2 e_{n-1} + e_{n+1}$$

$$f_{n,2}^1 = X_2^2 X_4^2 \cdots X_{n-1}^2 e_1 + X_4^2 \cdots X_{n-1}^2 e_3 + \cdots + X_{n-1}^2 e_{n-2} + e_n.$$

For example

$$\begin{aligned} D_{1,2}^{\text{hyp}} &= 2(X_1^2 + e_2)e_1 \\ D_{2,2}^{\text{hyp}} &= 2(X_1^2 + e_2)(X_2^2 e_1 + e_3) \\ D_{3,2}^{\text{hyp}} &= 2(X_1^2 X_3^2 + X_3^2 e_2 + e_4)(X_2^2 e_1 + e_3) \\ D_{4,2}^{\text{hyp}} &= 2(X_1^2 X_3^2 + X_3^2 e_2 + e_4)(X_2^2 X_4^2 e_1 + X_4^2 e_3 + e_5) \end{aligned}$$

Now we shall verify Atiyah–Sutcliffe Conjecture 2 for the hyperbolic configuration $B_{n,2}$ of the type B which reads as follows

$$D_{n,2}^{\text{hyp}} \geq (X_1 + \xi_1)^2 (X_2 + \xi_2)^2 \cdots (X_n + \xi_n)^2 2c \quad (\star)$$

Case 1. (n even) We first rewrite $f_{n,2}^0$ and $f_{n,2}^1$ in terms of c and elementary symmetric functions e'_k ($1 \leq k \leq n$) of $\xi_1, \dots, \xi_n$:

$$\begin{aligned} f_{n,2}^0 &= X_1^2 X_3^2 \cdots X_{n-1}^2 + X_3^2 \cdots X_{n-1}^2 e'_2 + \cdots + X_{n-1}^2 e'_{n-2} + e'_n \\ &\quad + X_3^2 \cdots X_{n-1}^2 e'_1 c + \cdots + X_{n-1}^2 e'_{n-3} c + e'_{n-1} c \\ f_{n,2}^1 &= X_2^2 X_4^2 \cdots X_n^2 e'_1 + X_4^2 \cdots X_n^2 e'_3 + \cdots + X_n^2 e'_{n-1} + e'_{n+1} \\ &\quad X_2^2 X_4^2 \cdots X_n^2 c + X_4^2 \cdots X_n^2 e'_2 c + \cdots + X_n^2 e'_{n-2} c + e'_n c \end{aligned}$$

Now by Cauchy–Schwartz

$$\begin{aligned} D_{n,2}^{\text{hyp}} &= 2f_{n,2}^0 f_{n,2}^1 \geq \\ &\geq 2(X_1 X_3 \cdots X_{n-1} \cdot X_2 X_4 \cdots X_n \sqrt{c} + X_3 X_5 \cdots X_{n-1} \cdot X_2 X_4 \cdots X_n e'_1 \sqrt{c} + \cdots + e'_n \sqrt{c})^2 = \\ &= (X_1 X_2 \cdots X_n + X_2 X_3 \cdots X_n e'_1 + \cdots + X_n e'_{n-1} + e'_n)^2 \cdot 2c = \\ &= (D_{n,1}^{\text{hyp}})^2 \cdot 2c \end{aligned}$$

Now by Corollary 1.2 for $D_{n,1}^{\text{hyp}}$ the inequality $(\star)$ follows.

Case 2. (n odd) Is similar to Case 1.

This concludes the verification of the Atiyah–Sutcliffe Conjecture C2 for hyperbolic type B configurations.

Now we state the strongest Atiyah–Sutcliffe conjecture C3 for type A and type B hyperbolic configurations.

Conjecture 3.

- (i) $D_{n,1}^{\text{hyp}}(\mathbf{x}_1, \dots, \mathbf{x}_n; \mathbf{x}_{n+1})^{n-1} \geq \prod_{k=1}^n D_{n-1,1}^{\text{hyp}}(\mathbf{x}_1, \dots, \widehat{\mathbf{x}}_k, \dots, \mathbf{x}_n; \mathbf{x}_{n+1})$
 (a hyperbolic analogue of formula (1.1))
- (ii) $D_{n,2}^{\text{hyp}}(\mathbf{x}_1, \dots, \mathbf{x}_n; \mathbf{x}_{n+1}, \mathbf{x}_{n+2})^n \geq D_{n,1}^{\text{hyp}}(\mathbf{x}_1, \dots, \mathbf{x}_n; \mathbf{x}_{n+1}) \cdot D_{n,1}^{\text{hyp}}(\mathbf{x}_1, \dots, \mathbf{x}_n; \mathbf{x}_{n+2}) \prod_{k=1}^n D_{n-1,2}^{\text{hyp}}(\mathbf{x}_1, \dots, \widehat{\mathbf{x}}_k, \dots, \mathbf{x}_n; \mathbf{x}_{n+1}, \mathbf{x}_{n+2})$

For example for $n = 2$ we have

$$\begin{aligned} \left(D_{2,2}^{\text{hyp}}\right)^2 &= 4 \left[X_1^2 + \xi_1 \xi_2 + (\xi_1 + \xi_2)c\right]^2 \left[X_2^2(\xi_1 + \xi_2) + (X_2^2 + \xi_1 \xi_2)c\right]^2 \geq \\ &\geq (X_1 X_2 + X_2(\xi_1 + \xi_2) + \xi_1 \xi_2)^2 \cdot (X_1^2 + \xi_1 c)(c + \xi_2) \cdot (X_2^2 + \xi_2 c)(c + \xi_1) \end{aligned}$$

which can be proved by considering the following three cases:

Case 1: $0 < X_1 < X_2 < c < \xi_2 < \xi_1$

Case 2: $0 < \xi_2 < X_1 < c < X_2 < \xi_1$

Case 3: $0 < \xi_2 < \xi_1 < c < X_1 < X_2$

and we get coefficient-wise inequality in terms of variables representing increments e.g. in Case 1: $h_1 = X_2 - X_1$, $h_2 = c - X_2$, $h_3 = \xi_2 - c$, $h_4 = \xi_1 - \xi_2$ and X_1 .

We expect that the general proof will not be so elementary.

For hyperbolic planar four points case the conjectures C1 and C2 were verified and reported in [16] and the conjecture C1 for non planar case of four points in H^3 were treated in [9] therefore conjecture C1 was verified for any four points in H^3 .

$t_{i,j} = t_{ij}$ = direction from $\mathbf{x}_i$ to $\mathbf{x}_j$; $0 < a < X_1 < X_2 < \cdots < X_n$

1.4 Type (A) configurations (cont.)

Even for $n = 3$ the inequality (1.1) is quite messy thanks to nonsymmetric character of both sides.

Knowing that sometimes it is easier to solve a more general problem we followed that path (although we didn't solve the problem in full generality). So let us start with the case $n = 2$. If we look at the following inequality

$$1 + X_1(\xi_1 + \xi_2) + X_1X_2\xi_1\xi_2 \geq (1 + X_1\xi_1)(1 + X_2\xi_2)$$

which is clearly true if $X_1 \geq X_2 \geq 0$ and $\xi_1, \xi_2 \geq 0$ we obtain the inequality (1.2) simply by a specialization $X_1 = \xi_1 = \lambda_2$, $X_2 = \xi_2 = \lambda_1$. So we proceed as follows:

Let $\xi_1, \dots, \xi_n, X_1, \dots, X_n, n \geq 1$ be two sets of commuting indeterminates. For any $l, 1 \leq l \leq n$ and any sequences $1 \leq i_1 \leq \dots \leq i_l \leq n, 1 \leq j_1, \dots, j_l \leq n$ we define polynomials $\Psi_J^I = \Psi_{j_1 \dots j_l}^{i_1 \dots i_l} \in \mathbb{Q}[\xi_1, \dots, \xi_n, X_1, \dots, X_n]$ as follows:

$$\Psi_J^I := \sum_{k=0}^l e_k(\xi_{j_1}, \xi_{j_2}, \dots, \xi_{j_l}) X_{i_1} X_{i_2} \dots X_{i_k}, \quad (l \geq 1), \quad \Psi_\emptyset^\emptyset := 1 \quad (j = 0)$$

where e_k is the k -th elementary symmetric function.

In particular we have

$$\begin{aligned} \Psi_j^i &= 1 + \xi_j X_i, \\ \Psi_{j_1 j_2}^{i_1 i_2} &= 1 + (\xi_{j_1} + \xi_{j_2}) X_{i_1} + \xi_{j_1} \xi_{j_2} X_{i_1} X_{i_2}, \\ \Psi_{j_1 j_2 j_3}^{i_1 i_2 i_3} &= 1 + (\xi_{j_1} + \xi_{j_2} + \xi_{j_3}) X_{i_1} + (\xi_{j_1} \xi_{j_2} + \xi_{j_1} \xi_{j_3} + \xi_{j_2} \xi_{j_3}) X_{i_1} X_{i_2} + \\ &\quad + \xi_{j_1} \xi_{j_2} \xi_{j_3} X_{i_1} X_{i_2} X_{i_3}, \\ &\text{etc.} \end{aligned}$$

The polynomials Ψ_J^I are symmetric w.r.t. $\xi_{j_1}, \xi_{j_2}, \dots, \xi_{j_l}$, but nonsymmetric w.r.t. $X_{i_1}, X_{i_2}, \dots, X_{i_l}$. By specializing X_i 's to assume real values such that $X_{i_1} \geq X_{i_2} \geq \dots \geq X_{i_l} \geq 0$ then we obtain polynomials in ξ_j 's satisfying the following simple but important property.

Proposition 1.3 (*Partition property*)

Let $(I_1, \dots, I_s)$ and $(J_1, \dots, J_s)$ be ordered set partitions of respective sets $I = \bigcup_{p=1}^s I_p$ and $J = \bigcup_{p=1}^s J_p$ such that $|I_p| = |J_p|$, $1 \leq p \leq s$. Then the inequality

$$\Psi_J^I \geq \prod_{p=1}^s \Psi_{J_p}^{I_p}$$

holds coefficientwise w.r.t. ξ_j 's.

Proof .

Proof is evident from the definition of Ψ_J^I and the monotonicity of X_i 's. ■

For the powers $(\Psi_J^I)^m$ we have the following conjecture.

Conjecture 1.4 (*Weighted Multiset Partition Conjecture*)

For given natural number m and sets I and J , $|I| = |J|$, of natural numbers let $(I_1, \dots, I_s)$ and $(J_1, \dots, J_s)$ be the partitions of the multiset I^m consisting of m copies of all elements of I and similarly for J^m .

(i) Then the inequality

$$(\Psi_J^I)^m \geq \prod_{p=1}^s \Psi_{J_p}^{I_p}$$

holds coefficientwise w.r.t. ξ_j 's.

(ii) The difference

$$(\Psi_J^I)^m - \prod_{p=1}^s \Psi_{J_p}^{I_p}$$

is multi-Schur positive with respect to partial alphabets corresponding to the atoms of the intersection lattice of the set system $\{J_1, \dots, J_s\}$.

For example, by Partition property, we have the following inequalities

$$\Psi_{1\dots n}^{1\dots n} \geq \Psi_k^k \Psi_{1..\hat{k}..n}^{1..\hat{k}..n}, \quad (1 \leq k \leq n)$$

which imply the following inequality

$$(\Psi_{1\dots n}^{1\dots n})^n \geq \prod_{k=1}^n \Psi_k^k \prod_{k=1}^n \Psi_{1..\hat{k}..n}^{1..\hat{k}..n}$$

By Partition property we also have the following inequality

$$\Psi_{1\dots n}^{1\dots n} \geq \prod_{k=1}^n \Psi_k^k$$

The last two inequalities suggest the validity of the following inequality

$$(\Psi_{1\dots n}^{1\dots n})^{n-1} \geq \prod_{k=1}^n \Psi_{1\dots \hat{k} \dots n}^{1\dots \hat{k} \dots n}$$

which is far from obvious (see Conjecture 1.5 below) although it would be a simple consequence of our Weighted Multiset Partition Conjecture.

This last conjectural inequality is interesting because it generalizes some special cases of not yet proven conjectures of Atiyah and Sutcliffe on configurations of points in three dimensional Euclidean space.

Our conjecture reads as follows:

Conjecture 1.5 *For any $n \geq 1$, let $X_1 \geq X_2 \geq \dots \geq X_n \geq 0$, $\xi_1, \xi_2, \dots, \xi_n \geq 0$, be nonnegative real numbers. Then we have coefficientwise (w.r.t. $\xi_1, \xi_2, \dots, \xi_n$) inequality*

$$(\Psi_{12\dots n}^{12\dots n})^{n-1} \geq \prod_{k=1}^n \Psi_{12\dots \hat{k} \dots n}^{12\dots \hat{k} \dots n}$$

where $12\dots \hat{k} \dots n$ denotes the sequence $12\dots(k-1)(k+1)\dots n$. The equality obviously holds true iff $X_1 = X_2 = \dots = X_n$.

This Conjecture implies the strongest Atiyah–Sutcliffe’s conjecture for almost collinear configurations of points (all but one point are collinear, called type(A) in [6]).

To illustrate the Conjecture (1.5) we consider first the cases $n = 2$ and $n = 3$.

Case $n = 2$: We have

$$\begin{aligned} \Psi_{12}^{12} &= 1 + (\xi_1 + \xi_2)X_1 + \xi_1\xi_2X_1X_2 = \\ &= 1 + \xi_1X_1 + \xi_2X_2 + \xi_1\xi_2X_1X_2 + (X_1 - X_2)\xi_2 = \\ &= (1 + \xi_1X_1)(1 + \xi_2X_2) + \xi_2(X_1 - X_2) \geq \\ &\geq (1 + \xi_1X_1)(1 + \xi_2X_2) = \Psi_1^1\Psi_2^2. \end{aligned}$$

Case $n = 3$: We first write Ψ_{123}^{123} in two different ways:

$$\Psi_{123}^{123} = \xi_2(X_1 - X_2) + \widehat{\Psi}_{123}^{123} \quad \text{and} \quad \Psi_{123}^{123} = \xi_3(X_1 - X_2) + \widehat{\Psi}_{123}^{123}.$$

Note that $\widehat{\Psi}_{123}^{123}$ is obtained from Ψ_{123}^{123} by replacing the linear term $\xi_2 X_1$ by $\xi_2 X_2$, hence all its coefficients are nonnegative.

The left hand side of the Conjecture (1.5) L_3 can be rewritten as follows:

$$\begin{aligned} L_3 &= (\Psi_{123}^{123})^2 = (\xi_2(X_1 - X_2) + \widehat{\Psi}_{123}^{123})\Psi_{123}^{123} \\ &= \xi_2(X_1 - X_2)\Psi_{123}^{123} + \widehat{\Psi}_{123}^{123}\Psi_{123}^{123} \\ &= \xi_2(X_1 - X_2)\Psi_{123}^{123} + \widehat{\Psi}_{123}^{123}(\xi_3(X_1 - X_2) + \widehat{\Psi}_{123}^{123}) \\ &= L'_3(X_1 - X_2) + \widehat{\Psi}_{123}^{123}\widehat{\Psi}_{123}^{123} \end{aligned}$$

where $L'_3 = \xi_2\Psi_{123}^{123} + \xi_3\widehat{\Psi}_{123}^{123}$ is a positive polynomial.

Now we have

$$L_3 \geq \widehat{L}_3 := \widehat{\Psi}_{123}^{123}\widehat{\Psi}_{123}^{123}.$$

By using the formula

$$\widehat{\Psi}_{123}^{123} = \Psi_{13}^{12} + \xi_2 X_2 \Psi_{13}^{13} = (\Psi_2^2 - 1)\Psi_{13}^{13} + \Psi_{13}^{12}$$

we can rewrite $\widehat{L}_3$ as

$$\begin{aligned} \widehat{L}_3 &= [(\Psi_{13}^{12} - \Psi_{13}^{13}) + \Psi_2^2 \Psi_{13}^{13}] \widehat{\Psi}_{123}^{123} \\ &= \xi_1 \xi_3 X_1 (X_2 - X_3) \widehat{\Psi}_{123}^{123} + \Psi_{13}^{13} (\Psi_2^2 \widehat{\Psi}_{123}^{123}) \end{aligned}$$

The last term in parenthesis can be written as

$$\begin{aligned} \Psi_2^2 \widehat{\Psi}_{123}^{123} &= \Psi_{12}^{12} \Psi_{23}^{23} + \Psi_2^1 (\Psi_{23}^{22} - \Psi_{23}^{23}) \\ &= \Psi_{12}^{12} \Psi_{23}^{23} + \xi_2 \xi_3 X_2 (X_2 - X_3) \Psi_2^1, \end{aligned}$$

so we get

$$\widehat{L}_3 = L''_3(X_2 - X_3) + \Psi_{12}^{12} \Psi_{13}^{13} \Psi_{23}^{23}$$

where L''_3 denotes the positive polynomial

$$L''_3 = \xi_1 \xi_3 X_1 \widehat{\Psi}_{123}^{123} + \xi_2 \xi_3 X_2 \Psi_2^1 \Psi_{13}^{13}.$$

We now have an explicit formula for L_3 :

$$L_3 = L'_3(X_1 - X_2) + L''_3(X_2 - X_3) + \Psi_{12}^{12} \Psi_{13}^{13} \Psi_{23}^{23}$$

with L'_3, L''_3 positive polynomials, which together with $X_1 \geq X_2 \geq X_3 (\geq 0)$ implies that

$$L_3 \geq R_3 := \Psi_{12}^{12} \Psi_{13}^{13} \Psi_{23}^{23}$$

and the Conjecture (1.5) ($n = 3$) is proved.

In fact we have proven an instance $n = 3$ $\widehat{L}_3 \geq R_3$ of a stronger conjecture which we are going to formulate now. Let $2 \leq k \leq n$. We define the modified polynomials $\widehat{\Psi}_{12 \dots \underline{k} \dots n}^{12 \dots k \dots n}$ as follows:

$$\widehat{\Psi}_{12 \dots \underline{k} \dots n}^{12 \dots k \dots n} := \xi_k (X_2 - X_1) + \Psi_{12 \dots n}^{12 \dots n}$$

obtained from $\Psi_{12 \dots n}^{12 \dots n}$ by replacing only one term $\xi_k X_1$ by $\xi_k X_2$, hence $\widehat{\Psi}_{12 \dots \underline{k} \dots n}^{12 \dots k \dots n}$ are still positive. Let us introduce the following notation:

$$\widehat{L}_n := \prod_{k=2}^n \widehat{\Psi}_{12 \dots \underline{k} \dots n}^{12 \dots k \dots n}; \quad R_n := \prod_{k=1}^n \Psi_{12 \dots \hat{k} \dots n}^{12 \dots \hat{k} \dots n}.$$

Then clearly $L_n := (\Psi_{12 \dots n}^{12 \dots n})^{n-1} \geq \widehat{L}_n$. Now our stronger conjecture reads as

Conjecture 1.6

$$\widehat{L}_n \geq R_n \quad (n \geq 1)$$

with equality iff $X_2 = X_3 = \dots = X_n$.

More generally, we conjecture that the difference $\widehat{L}_n - R_n$ is a polynomial in the differences $X_2 - X_3, X_3 - X_4, \dots, X_{n-1} - X_n$ with coefficients in $\mathbb{Z}_{\geq 0}[X_1, \dots, X_n, \xi_1, \dots, \xi_n]$.

Proposition 1.7

$$L_n = L'_n (X_1 - X_2) + \widehat{L}_n$$

for some positive polynomial L'_n .

Proof of Proposition 1.7.

$$\begin{aligned}
L_n &= (\Psi_{12\dots n}^{12\dots n})^{n-1} = (\xi_2(X_1 - X_2) + \widehat{\Psi}_{12\dots n}^{12\dots n})(\Psi_{12\dots n}^{12\dots n})^{n-2} \\
&= \xi_2(X_1 - X_2)(\Psi_{12\dots n}^{12\dots n})^{n-2} + \widehat{\Psi}_{12\dots n}^{12\dots n}(\xi_3(X_1 - X_2) + \widehat{\Psi}_{123\dots n}^{123\dots n})(\Psi_{12\dots n}^{12\dots n})^{n-3} \\
&= \xi_2(X_1 - X_2)(\Psi_{12\dots n}^{12\dots n})^{n-2} + \xi_3(X_1 - X_2)\widehat{\Psi}_{12\dots n}^{12\dots n}(\Psi_{12\dots n}^{12\dots n})^{n-3} + \\
&\quad + \widehat{\Psi}_{12\dots n}^{12\dots n}\widehat{\Psi}_{123\dots n}^{123\dots n}(\Psi_{12\dots n}^{12\dots n})^{n-3} \\
&\vdots \\
&= (\sum_{k=1}^{n-1} \xi_{k+1}(\prod_{j=2}^k \widehat{\Psi}_{12\dots j\dots n}^{12\dots j\dots n})(\Psi_{12\dots n}^{12\dots n})^{n-k})(X_1 - X_2) + \prod_{j=2}^n \widehat{\Psi}_{12\dots j\dots n}^{12\dots j\dots n}.
\end{aligned}$$

■

Now we turn to study the quotient

$$\frac{L_n}{R_n} = \frac{(\Psi_{1\dots n}^{1\dots n})^{n-1}}{\prod_{k=1}^n \Psi_{1\dots \widehat{k}\dots n}^{1\dots \widehat{k}\dots n}}$$

by studying the growth behaviour of quotients of its factors $\Psi_{1\dots n}^{1\dots n}/\Psi_{1\dots \widehat{k}\dots n}^{1\dots \widehat{k}\dots n}$ w.r.t. any of its arguments X_r , $1 \leq r \leq n$.

In the following theorem we obtain an explicit formula for the numerators of the derivatives w.r.t. X_r , ($1 \leq r \leq n, r \neq k$) of the quantities $\Psi_{1\dots n}^{1\dots n}/\Psi_{1\dots \widehat{k}\dots n}^{1\dots \widehat{k}\dots n}$. From this formulas we get some monotonicity properties which enable us to state some new (refined) conjectures later on.

Theorem 1.8 *Let*

$$\Delta_r := \partial_{X_r} \Psi_{1\dots n}^{1\dots n} \cdot \Psi_{1\dots \widehat{k}\dots n}^{1\dots \widehat{k}\dots n} - \Psi_{1\dots n}^{1\dots n} \cdot \partial_{X_r} \Psi_{1\dots \widehat{k}\dots n}^{1\dots \widehat{k}\dots n}, \quad (1 \leq r \leq n). \quad (1.3)$$

Then we have the following explicit formulas

(i) *for any r , $1 \leq r < k(\leq n)$ we have*

$$\begin{aligned}
\Delta_r &= \xi_k \sum_{0 \leq i < r \leq j \leq n} s_{(j-1,i)}^{(k)} X_1^2 \cdots X_i^2 X_{i+1} \cdots \widehat{X}_k \cdots X_j + \\
&\quad + \sum_{0 \leq i < r, k \leq j < n} e_i e_j^{(k)} X_1^2 \cdots X_i^2 X_{i+1} \cdots \widehat{X}_r \cdots \widehat{X}_k \cdots X_j (X_k - X_{j+1})
\end{aligned}$$

(ii) for any r , $(1 \leq)k < r \leq n$ we have

$$\Delta_r = - \left(\sum_{0 \leq i < r \leq j \leq n} s_{(j-1,i)}^{(k)} X_1^2 \cdots X_i^2 X_{i+1} \cdots \widehat{X}_k \cdots \widehat{X}_r \cdots X_j + \sum_{0 \leq i < k, r \leq j < n} e_i^{(k)} e_j X_1^2 \cdots X_i^2 X_{i+1} \cdots \widehat{X}_k \cdots \widehat{X}_r \cdots X_j (X_{j+1} - X_k) \right)$$

where $s_{\lambda}^{(k)}$ denotes the λ -th Schur function of $\xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_n$ (ξ_k omitted).

Proof of Theorem 1.8.

(i) For any r , $1 \leq r < k (\leq n)$ we find explicitly a formula as follows. We shall use notations $X_{1..i} := X_1 X_2 \cdots X_i$, for multilinear monomials and $e_i := e_i(\xi_1, \dots, \xi_n)$, $e_i^{(k)} = e_i(\xi_1, \dots, \widehat{\xi}_k, \dots, \xi_n)$ for the elementary symmetric functions (here k is fixed). Then we can rewrite our basic quantities

$$\Psi_{1..n}^{1..n} := \sum_{i=0}^n e_i X_{1..i} \quad (1.4)$$

$$\begin{aligned} \Psi_{1..\widehat{k}..n}^{1..\widehat{k}..n} &:= \sum_{i=0}^{k-1} e_i^{(k)} X_{1..i} + \frac{1}{X_k} \sum_{i=k}^{n-1} e_i^{(k)} X_{1..i+1} = \\ &= \sum_{i=0}^{n-1} e_i^{(k)} X_{1..i} + \frac{1}{X_k} \sum_{i=k}^{n-1} e_i^{(k)} X_{1..i} (X_{i+1} - X_k) \end{aligned} \quad (1.5)$$

For the derivatives we get immediately

$$\partial_{X_r} \Psi_{1..n}^{1..n} = \frac{1}{X_r} \sum_{i=r}^n e_i X_{1..i} = \frac{1}{X_r} \left(\Psi_{1..n}^{1..n} - \sum_{i=0}^{r-1} e_i X_{1..i} \right) \quad (1.6)$$

$$\partial_{X_r} \Psi_{1..\widehat{k}..n}^{1..\widehat{k}..n} = \frac{1}{X_r} \sum_{i=r}^{n-1} e_i^{(k)} X_{1..i} + \frac{1}{X_k X_r} \sum_{i=k}^{n-1} e_i^{(k)} X_{1..i} (X_{i+1} - X_k) \quad (1.7)$$

$$= \frac{1}{X_r} \left(\Psi_{1..\widehat{k}..n}^{1..\widehat{k}..n} - \sum_{i=0}^{r-1} e_i^{(k)} X_{1..i} \right) \quad (1.8)$$

By plugging (1.6) and (1.8) into (1.3) we obtain

$$X_r \Delta_r = \Psi_{1\dots n}^{1\dots n} \left(\sum_{i=0}^{r-1} e_i^{(k)} X_{1..i} \right) - \Psi_{1\dots \widehat{k} \dots n}^{1\dots \widehat{k} \dots n} \left(\sum_{i=0}^{r-1} e_i X_{1..i} \right) =$$

and after simple cancelation, by invoking (1.5) we get

$$= \left(\sum_{j=r}^n e_j X_{1..j} \right) \left(\sum_{i=0}^{r-1} e_i^{(k)} X_{1..i} \right) - \left(\sum_{j=r}^{n-1} e_j^{(k)} X_{1..j} + \frac{1}{X_k} \sum_{j=k}^{n-1} e_j^{(k)} X_{1..j} (X_{j+1} - X_k) \right) \left(\sum_{i=0}^{r-1} e_i X_{1..i} \right)$$

i.e.

$$X_r \Delta_r = \sum_{0 \leq i < r \leq j \leq n} (e_j e_i^{(k)} - e_i e_j^{(k)}) X_{1..i} X_{1..j} + \frac{1}{X_k} \sum_{0 \leq i < r, k \leq j < n} e_i e_j^{(k)} X_{1..i} X_{1..j} (X_k - X_{j+1})$$

If we use a simple identity $e_j = e_j^{(k)} + \xi_k e_{j-1}^{(k)}$, we can identify the quantity

$$\begin{aligned} e_j e_i^{(k)} - e_i e_j^{(k)} &= (e_j^{(k)} + \xi_k e_{j-1}^{(k)}) e_i^{(k)} - (e_i^{(k)} + \xi_k e_{i-1}^{(k)}) e_j^{(k)} = \\ &= \begin{vmatrix} e_{j-1}^{(k)} & e_j^{(k)} \\ e_{i-1}^{(k)} & e_i^{(k)} \end{vmatrix} \xi_k = s_{2^i 1^{j-i-1}}^{(k)} \xi_k \end{aligned}$$

Thus in this case ($1 \leq r < k$) we obtain a formula

$$\begin{aligned} \Delta_r &= \xi_k \sum_{0 \leq i < r \leq j \leq n} s_{(j-1, i)'}^{(k)} X_1^2 \cdots X_i^2 X_{i+1} \cdots \widehat{X}_r \cdots X_j + \\ &\quad + \sum_{0 \leq i < r, k \leq j < n} e_i e_j^{(k)} X_1^2 \cdots X_i^2 X_{i+1} \cdots \widehat{X}_r \cdots \widehat{X}_k \cdots X_j (X_k - X_{j+1}) \end{aligned}$$

(where $e_j^{(k)} = e_j(\xi_1, \dots, \widehat{\xi}_k, \dots, \xi_n)$) in terms of Schur functions (of arguments $\xi_1, \dots, \widehat{\xi}_k, \dots, \xi_n$) corresponding to a transpose $(j-1, i)' = (2^i 1^{j-i-1})$ of a partition $(j-1, i)$ (cf. Jacobi–Trudi formula, I 3.5 in [8]).

(ii) For any r , $(1 \leq) k < r \leq n$. In this case we use

$$\partial_{X_r} \Psi_{1\dots \widehat{k} \dots n}^{1\dots \widehat{k} \dots n} = \frac{1}{X_k X_r} \sum_{j=r-1}^{n-1} e_j^{(k)} X_{1..j+1}$$

$$\begin{aligned}\Psi_{1\ldots\widehat{k}\ldots n}^{1\ldots\widehat{k}\ldots n} &= \sum_{i=0}^{k-1} e_i^{(k)} X_{1..i} + \frac{1}{X_k} \sum_{i=k}^{n-1} e_i^{(k)} X_{1..i+1} = \\ &= \frac{1}{X_k} \left(\sum_{i=0}^{k-1} X_{1..i} (X_k - X_{i+1}) + \sum_{i=0}^{n-1} e_i^{(k)} X_{1..i} \right)\end{aligned}$$

By plugging this into (1.3) we get

$$\begin{aligned}X_k X_r \Delta_r &= \left(\sum_{j=r}^n e_j X_{1..j} \right) \left(\sum_{i=0}^{k-1} e_i^{(k)} X_{1..i} (X_k - X_{i+1}) + \sum_{i=0}^{n-1} e_i^{(k)} X_{1..i+1} \right) - \\ &\quad - \left(\sum_{j=0}^{r-1} e_j X_{1..j} + \sum_{j=r}^n e_j X_{1..j} \right) \left(\sum_{i=r-1}^{n-1} e_i^{(k)} X_{1..i+1} \right) \\ &= \left(\sum_{i=0}^{r-2} e_i^{(k)} X_{1..i+1} \right) \left(\sum_{j=r}^n e_j X_{1..j} \right) - \left(\sum_{i=0}^{r-1} e_i X_{1..i} \right) \left(\sum_{j=r-1}^{n-1} e_j^{(k)} X_{1..j+1} \right) + \\ &\quad + \sum_{i=0}^{k-1} \sum_{j=r}^n e_i^{(k)} e_j X_{1..i} X_{1..j} (X_k - X_{i+1}) \\ &= \left(\sum_{i=1}^{r-1} e_{i-1}^{(k)} X_{1..i} \right) \left(\sum_{j=r}^n e_j X_{1..j} \right) - \left(\sum_{i=0}^{r-1} e_i X_{1..i} \right) \left(\sum_{j=r}^n e_{j-1}^{(k)} X_{1..j} \right) + \\ &\quad + \sum_{i=0}^{k-1} \sum_{j=r}^n e_i^{(k)} e_j X_{1..i} X_{1..j} (X_k - X_{i+1})\end{aligned}$$

By using a formula for elementary symmetric functions ($e_i = e_i^{(k)} + \xi_k e_{i-1}^{(k)}$) we can write in terms of Schur functions (of arguments $\xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_n$), where λ' is a conjugate of λ .

$$e_{i-1}^{(k)} e_j - e_i e_{j-1}^{(k)} = e_{i-1}^{(k)} e_j^{(k)} - e_i^{(k)} e_{j-1}^{(k)} = - \begin{vmatrix} e_{j-1}^{(k)} & e_j^{(k)} \\ e_{i-1}^{(k)} & e_i^{(k)} \end{vmatrix} = -s_{2^i 1^{j-i-1}}^{(k)} = -s_{(j-1, i)}^{(k)}$$

Thus we obtain a formula

$$\begin{aligned}\Delta_r &= - \left(\sum_{0 \leq i < r \leq j \leq n} s_{(j-1, i)}^{(k)}, X_1^2 \cdots X_i^2 X_{i+1} \cdots \widehat{X}_k \cdots \widehat{X}_r \cdots X_j + \right. \\ &\quad \left. + \sum_{0 \leq i < k, r \leq j < n} e_i^{(k)} e_j X_1^2 \cdots X_i^2 X_{i+1} \cdots \widehat{X}_k \cdots \widehat{X}_r \cdots X_j (X_{j+1} - X_k) \right)\end{aligned}$$

Corollary 1.9 (X_r -monotonicity)

Let $X_1 \geq \cdots \geq X_n \geq 0$, $\xi_1, \dots, \xi_n \geq 0$ be as before. Then

(i) for any r , $1 \leq r < k$ ($\leq n$) we have

$$\frac{\Psi_{1\dots n}^{1\dots n}}{\Psi_{1\dots \widehat{k}\dots n}^{1\dots \widehat{k}\dots n}} \geq \frac{\Psi_{1\dots r}^{1\dots r+1} \Psi_{r+1}^{r+1} \dots \Psi_n^{n}}{\Psi_{1\dots r}^{1\dots r+1} \Psi_{r+1}^{r+1} \dots \Psi_n^{n}}$$

(ii) for any r , $(1 \leq) k < r$ ($\leq n$) we have

$$\frac{\Psi_{1\dots n}^{1\dots n}}{\Psi_{1\dots \widehat{k}\dots n}^{1\dots \widehat{k}\dots n}} \geq \frac{\Psi_{1\dots r-1}^{1\dots r-1} \Psi_r^{r-1} \dots \Psi_n^{n}}{\Psi_{1\dots \widehat{k}\dots r-1}^{1\dots \widehat{k}\dots r-1} \Psi_r^{r-1} \dots \Psi_n^{n}}$$

Now we illustrate how to use Corollary 1.9 to prove our Conjecture 1.5 for $n = 2, 3, 4$ and 5.

Case $n = 2$

$$Q_2 := \frac{\Psi_{12}^{12}}{\Psi_1^1 \Psi_2^2} \geq \frac{\Psi_{12}^{22}}{\Psi_1^2 \Psi_2^2} = 1 \text{ (by (i))}$$

Case $n = 3$

$$\begin{aligned} Q_3 &:= \frac{\Psi_{123}^{123} \Psi_{123}^{123}}{\Psi_{12}^{12} \Psi_{13}^{13} \Psi_{23}^{23}} \geq \frac{\Psi_{123}^{223} \Psi_{123}^{123}}{\Psi_{12}^{22} \Psi_{13}^{13} \Psi_{23}^{23}} \geq \frac{\Psi_{123}^{223} \Psi_{123}^{223}}{\Psi_{12}^{22} \Psi_{13}^{23} \Psi_{23}^{23}} \text{ (by (i))} \\ &\geq \frac{\Psi_{123}^{222} \Psi_{123}^{223}}{\Psi_{12}^{22} \Psi_{13}^{22} \Psi_{23}^{23}} \geq \frac{\Psi_{123}^{222} \Psi_{123}^{222}}{\Psi_{12}^{22} \Psi_{13}^{22} \Psi_{23}^{22}} = 1 \text{ (by (ii))} \end{aligned}$$

Case $n = 4$

$$Q_4 := \frac{(\Psi_{1234}^{1234})^3}{\Psi_{123}^{123} \Psi_{124}^{124} \Psi_{134}^{134} \Psi_{234}^{234}} \geq \cdots \geq \frac{\Psi_{1234}^{2244} (\Psi_{1234}^{2244})^2}{\Psi_{123}^{224} \Psi_{124}^{224} \Psi_{134}^{224} \Psi_{234}^{224}} \quad (\geq 1)$$

This last inequality follows from the following symmetric function identity:

$$\begin{aligned} &\Psi_{1234}^{2244} (\Psi_{1234}^{2244})^2 - \Psi_{123}^{224} \Psi_{124}^{224} \Psi_{134}^{224} \Psi_{234}^{224} = \\ &X_2^2 X_4^4 m_{2222} + 2X_2^2 X_4^3 m_{2221} + X_2^2 X_4^2 m_{222} + 3X_2^2 X_4^2 m_{2211} + X_2^2 X_4 m_{221} \\ &+ 4X_2^2 X_4 m_{2111} + X_2^2 m_{211} + X_2(3X_2 + 2X_4) m_{1111} + X_2 m_{111} \end{aligned}$$

where $m_\lambda = m_\lambda(\xi_1, \xi_2, \xi_3, \xi_4)$ are the monomial symmetric functions.

Case $n = 5$

$$Q_5 := \frac{(\Psi_{1\dots 5}^{1\dots 5})^4}{\prod_{k=1}^5 \Psi_{1\dots \widehat{k} \dots 5}^{1\dots \widehat{k} \dots 5}} \geq \dots \geq \frac{(\Psi_{12345}^{22244} \Psi_{12345}^{22444})^2}{\Psi_{1234}^{2244} \Psi_{1235}^{2244} \Psi_{1245}^{2244} \Psi_{1345}^{2244} \Psi_{2345}^{2244}} \quad (\geq 1)$$

The last inequality is equivalent to an explicit symmetric function identity with all coefficients (w.r.t. monomial basis) positive.

Now we state our stronger conjecture.

Conjecture 1.10 (for symmetric functions)

Let $X_1 \geq X_2 \geq \dots \geq X_n \geq 0$ and $\xi_1, \dots, \xi_n \geq 0$. Then the inequalities

(a) For n even

$$\Psi_{1 \ 2 \ 3 \dots n-1 \ n}^{2 \ 2 \ 4 \ 4 \dots n \ n} \left(\prod_{k=1}^{n/2} \Psi_{1 \ 2 \ 3 \ 4 \dots 2k \ 2k \ 2k \dots n-2 \ n-2 \ n}^{2 \ 2 \ 4 \ 4 \dots 2k \ 2k \ 2k \dots n-2 \ n-2 \ n} \right)^2 \geq \prod_{k=1}^n \Psi_{1 \ 2 \dots \widehat{k} \dots n-1 \ n}^{2 \ 2 \ 4 \ 4 \dots n-2 \ n-2 \ n}$$

(b) For n odd

$$\left(\prod_{k=1}^{\lfloor n/2 \rfloor} \Psi_{1 \ 2 \ 3 \ 4 \dots 2k \ 2k \ 2k \dots n-1 \ n-1}^{2 \ 2 \ 4 \ 4 \dots 2k \ 2k \ 2k \dots n-1 \ n-1} \right)^2 \geq \prod_{k=1}^n \Psi_{1 \ 2 \dots \widehat{k} \dots n}^{2 \ 2 \ 4 \ 4 \dots n-1 \ n-1}$$

hold true coefficientwise (m -positivity).

Now we motivate another inequalities for symmetric functions which also refine the strongest Atiyah–Sutcliffe conjecture for configurations of type (A). Let $n = 3$. We apply Corollary 1.9 by using steps (ii) only.

$$Q_3 := \frac{\Psi_{123}^{123} \Psi_{123}^{123}}{\Psi_{12}^{12} \Psi_{13}^{12} \Psi_{23}^{12}} \geq \frac{\Psi_{123}^{113} \Psi_{123}^{123}}{\Psi_{12}^{12} \Psi_{13}^{12} \Psi_{23}^{13}} \geq \frac{\Psi_{123}^{112} \Psi_{123}^{123}}{\Psi_{12}^{12} \Psi_{13}^{12} \Psi_{23}^{13}} \geq \frac{\Psi_{123}^{112} \Psi_{123}^{122}}{\Psi_{12}^{12} \Psi_{13}^{12} \Psi_{23}^{12}} \geq 1$$

The last inequality is equivalent to nonnegativity of the expression

$$\Psi_{123}^{112} \Psi_{123}^{122} - \Psi_{12}^{12} \Psi_{13}^{12} \Psi_{23}^{12} \quad (= X_1(X_1 - X_2)^2 \xi_1 \xi_2 \xi_3 \geq 0).$$

Similarly, for $n = 4$, the symmetric function inequality stronger than $Q_4 \geq 1$ would be the following

$$\Psi_{1234}^{1123} \Psi_{1234}^{1223} \Psi_{1234}^{1233} \geq \Psi_{123}^{123} \Psi_{124}^{123} \Psi_{134}^{123} \Psi_{234}^{123}$$

Now we state a general conjecture for symmetric functions which imply the strongest Atiyah–Sutcliffe conjecture for almost collinear type (A) configurations.

Conjecture 1.11 *Let $X_1 \geq \dots \geq X_n \geq 0$, $\xi_1, \dots, \xi_n \geq 0$. Then the following inequality for symmetric functions in $\xi_1, \dots, \xi_n$*

$$\Psi_{123\dots n}^{112\dots n-1} \Psi_{1234\dots n}^{1223\dots n-1} \dots \Psi_{12\dots n-2}^{12\dots n-2} \frac{n-1}{n-1} \frac{n-1}{n} \geq \Psi_1^{1 \ 2\dots n-1} \Psi_1^{1 \ 2\dots n-1} \Psi_1^{1 \ 2\dots n-1} \dots \Psi_2^{1 \ 2\dots n-1}$$

i.e.

$$\prod_{k=1}^{n-1} \Psi_1^{1 \ 2\dots k \ k \dots n} \geq \prod_{k=1}^n \Psi_1^{1 \ 2 \dots n-1}$$

holds true coefficientwise (m -positivity, even s -positivity).

Remark 1.12 *Conjectures 1.10 and 1.11 seems to hold also for the Schur basis of symmetric functions in $\xi_1, \dots, \xi_n$.*

We have checked this Conjecture 1.11 up to $n = 5$ by using **Maple** and symmetric function package **SF** of J. Stembridge. For n bigger than five the computations are extremely intensive and hopefully in the near future would be possible by using more powerful computers.

Note that the right hand side of the Conjecture 1.11 involves symmetric functions of partial alphabets $\xi_1, \xi_2, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_n$. But the left hand side doesn't have this "defect". Our objective now is to give explicit formula for the right hand side in terms of the elementary symmetric functions of the full alphabet $\xi_1, \xi_2, \dots, \xi_n$. This we are going to achieve by using resultants as follows.

Lemma 1.13 *For any k , ($1 \leq k \leq n$), we have*

$$\Psi_{1\dots k\dots n}^{1\dots k\dots n-1} = \sum_{j=0}^{n-1} a_j \xi_k^{n-1-j}$$

where

$$\begin{aligned} a_{n-1} &= 1 + X_1 e_1 + X_1 X_2 e_2 + \dots + X_1 \dots X_{n-1} e_{n-1}, \\ a_{n-2} &= -X_1 - X_1 X_2 e_1 - \dots - X_1 \dots X_{n-1} e_{n-2}, \\ &\dots \\ a_0 &= (-1)^{n-1} X_1 \dots X_{n-1} \end{aligned}$$

i.e.

$$a_{n-1-j} = (-1)^j \sum_{i=j}^{n-1} X_1 \dots X_i e_{i-j}$$

Proof of Lemma 1.13.

By definition we have

$$\Psi_{1 \dots \widehat{k} \dots n}^{1 \dots n-1} = \sum_{i=0}^{n-1} X_1 \cdots X_i e_i^{(k)} \quad (1.9)$$

where $e_i^{(k)}$ is the i -th elementary function of $\xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_n$. Now from the decomposition

$$(1 + \xi_k t)^{-1} \prod_{j=1}^n (1 + \xi_j t) = \prod_{j \neq k} (1 + \xi_j t) = \sum_{i=0}^{n-1} e_i^{(k)} t^i$$

we get

$$e_i^{(k)} = e_i - e_{i-1} \xi_k + e_{i-2} \xi_k^2 - \cdots + (-1)^i \xi_k^i$$

By substituting this into equation (1.9) the Lemma 1.13 follows. ■

Then, by Lemma 1.13, the right hand side of the Conjecture 1.11

$$R_n = \prod_{k=1}^n \Psi_{1 \ 2 \ \dots \ \widehat{k} \ \dots \ n}^{1 \ 2 \ \dots \ k \ \dots \ n-1} = \prod_{k=1}^n \left(\sum_{j=0}^{n-1} a_j \xi_k^{n-1-j} \right)$$

can be interpreted as a resultant $R_n = \text{Resultant}(f, g)$ of the following two polynomials

$$\begin{aligned} f(x) &= \sum_{j=0}^{n-1} a_j x^{n-1-j} \\ g(x) &= \prod_{i=1}^n (x - \xi_i) = \sum_{j=0}^n (-1)^j e_j x^{n-j} \end{aligned}$$

Then Sylvester formula

$$R_n = \left| \begin{array}{cccccc} 1 & -e_1 & e_2 & -e_3 & \cdots & (-1)^n e_n \\ & 1 & -e_1 & e_2 & -e_3 & \cdots \\ & & \ddots & & & \\ & & & 1 & -e_1 & \cdots \\ a_0 & a_1 & a_2 & \cdots & a_n & \\ & a_0 & a_1 & a_2 & \cdots & a_n \\ & & \ddots & & & \\ & & & a_0 & a_1 & a_2 & \cdots & a_n \end{array} \right| \quad \left(=: \left| \begin{array}{cc} A & B \\ C & D \end{array} \right| \right)$$

can be simplified as

$$= |A| \cdot |D - CA^{-1}B| = |D - CA^{-1}B|.$$

The entries of the $n \times n$ matrix $\Delta := D - CA^{-1}B$ are given by

$$\delta_{ij} = \begin{cases} (-1)^{j-i-1} \sum_{k=j+1}^n X_1 \cdots X_{k+i-j} e_k, & 0 \leq i < j \leq n-1 \\ (-1)^{j-i} \sum_{k=0}^j X_1 \cdots X_{k+i-j} e_k, & 0 \leq j \leq i \leq n-1 \end{cases}$$

For example, for $n = 3$

$$\Delta_3 = \left| \begin{array}{ccc} 1 & X_1 e_2 + X_1 X_2 e_3 & -X_1 e_3 \\ -X_1 & 1 + X_1 e_1 & X_1 X_2 e_3 \\ X_1 X_2 & -X_1 - X_1 X_2 e_1 & 1 + X_1 e_1 + X_1 X_2 e_2 \end{array} \right|$$

By elementary operations (including multiplication of 2^{nd} row and column by -1) we get

$$\begin{aligned} \Delta_3 &= \left| \begin{array}{ccc} 1 & * & * \\ 0 & \Psi_{123}^{112} & X_1(X_2 - X_1)e_3 \\ 0 & X_2 - X_1 & \Psi_{123}^{122} \end{array} \right| = \left| \begin{array}{cc} \Psi_{123}^{112} & X_1(X_2 - X_1)e_3 \\ X_2 - X_1 & \Psi_{123}^{122} \end{array} \right| = \\ &= \left| \begin{array}{cc} \Psi_{123}^{112} & X_1(X_1 - X_2)e_3 \\ X_1 - X_2 & \Psi_{123}^{122} \end{array} \right|. \end{aligned}$$

Similarly, for $n = 4$ we obtain

$$\begin{aligned} \Delta_4 &= \begin{vmatrix} \Psi_{1234}^{1123} & -X_1(X_1 - X_2)e_3 - X_1X_2(X_1 - X_3)e_4 & X_1(X_1 - X_2)e_4 \\ -(X_1 - X_2) & \Psi_{1234}^{1223} & -X_1X_2(X_2 - X_3)e_4 \\ X_1(X_2 - X_3) & -(X_1 - X_3) - X_1(X_2 - X_3)e_1 & \Psi_{1234}^{1233} \end{vmatrix} = \\ &= \begin{vmatrix} \Psi_{1234}^{1123} & X_1(X_1 - X_2)e_3 + X_1X_2(X_1 - X_3)e_4 & X_1(X_1 - X_2)e_4 \\ (X_1 - X_2) & \Psi_{1234}^{1223} & X_1X_2(X_2 - X_3)e_4 \\ X_1(X_2 - X_3) & (X_1 - X_3) + X_1(X_2 - X_3)e_1 & \Psi_{1234}^{1233} \end{vmatrix}. \end{aligned}$$

By using abbreviations $X_{ij} = X_i - X_j$ it can be rewritten as

$$\Delta_4 = \begin{vmatrix} \Psi_{1234}^{1123} & X_1X_{12}e_3 + X_1X_2X_{13}e_4 & X_1X_{12}e_4 \\ X_{12} & \Psi_{1234}^{1223} & X_1X_2X_{23}e_4 \\ X_1X_{23} & X_{13} + X_1X_{23}e_1 & \Psi_{1234}^{1233} \end{vmatrix}.$$

In general

$$\Delta_n = \det(\delta'_{ij})_{1 \leq i, j \leq n-1}$$

where

$$\delta'_{ij} = \begin{cases} \sum_{k=j+1}^n X_1 \cdots X_{k+i-j-1} (X_i - X_{k+i-j}) e_k, & 1 \leq i < j \leq n-1 \\ \Psi_{1 \ 2 \ \dots \ i \ i \ \dots \ n}^1, & i = j \\ \sum_{k=0}^j X_1 \cdots X_{k+i-j-1} (X_{k+i-j} - X_i) e_k, & 1 \leq j < i \leq n-1 \end{cases}$$

Note that all δ'_{ij} are symmetric polynomials with non negative coefficients.

Corollary 1.14 *The conjecture 1.11 is equivalent to a Hadamard type inequality, holding coefficientwise (m -positivity, even s -positivity), for the (non Hermitian) matrix $(\delta'_{ij})_{1 \leq i, j \leq n-1}$, i.e.*

$$\prod_{i=1}^{n-1} \delta'_{ii} \geq \det(\delta'_{ij}).$$

2 Verification of the Đoković’s strengthening of the Atiyah–Sutcliffe Conjecture (C2) for some nonplanar configurations with dihedral symmetry

Here we basically follow Đoković’s [7], where he considered the following dihedral configurations $C_{m,n}$ (type D).

Let $N = m + n$ points be such that

1. The first m points $x_1, \dots, x_m$ lie on a line L .
2. The remaining n points $y_j = x_{m+j+1}$ ($j = 0, 1, \dots, n-1$) are the vertices of a regular n -gon whose plane is perpendicular to L and whose centroid lies on L .

He only proved Atiyah conjecture C1. We make some additional refinements including a proof of Atiyah–Sutcliffe conjecture C2 for such configurations.

We may assume $L = \mathbb{R} \times \{0\} \subset \mathbb{R} \times \mathbb{C} = \mathbb{R}^3$ and write $x_i = (a_i, 0)$, $1 \leq i \leq m$, $a_1 \leq \dots \leq a_m$ and $y_j = (0, b_j)$, $b_j = -\xi^j$, $\xi = e^{2\pi i/n}$, $0 \leq j \leq n-1$.

We set

$$\lambda_i = a_i + \sqrt{1 + a_i^2}$$

Recall that $a_1 < \dots < a_m$ and, consequently $0 < \lambda_1 < \dots < \lambda_m$. Then the associated polynomials p_i (up to scalar factors) are given by

$$\begin{aligned} p_i(x, y) &= x^{m-i} y^{i-1} (x^n - \lambda_i^n y^n), \quad 1 \leq i \leq m \\ p_{m+j+1}(x, y) &= \prod_{s \neq j} \left(x + \frac{\overline{b_s} - \overline{b_j}}{|b_s - b_j|} y \right) \cdot \prod_{i=1}^m (y - \lambda_i b_j x), \quad 0 \leq j < n \end{aligned}$$

By noting that

$$b_s - b_j = 2i\xi^{\frac{j+s}{2}} \sin \frac{\pi(j-s)}{n}$$

(in Đoković ξ^{j+s} should be replaced by $\xi^{\frac{j+s}{2}}$) we obtain

$$x + \frac{\overline{b_s} - \overline{b_j}}{|b_s - b_j|} y = \left(-\overline{b_j} y - i\xi^{\frac{s-j}{2}} \operatorname{sgn}(s-j) \right) \frac{1 - \overline{b_s} b_j}{|b_s - b_j|}$$

and

$$y - \lambda_i b_j x = -b_j(-\overline{b_j}y + \lambda_i x)$$

Note also that

$$\{\xi^{\frac{s-j}{2}} \operatorname{sgn}(s-j) | s=1, \dots, j-1, j+1, \dots, n\} = \{e^{\pi i k/n} | k=1, \dots, n-1\}$$

Thus, after dehomogenizing the polynomials p_i by setting $x=1$, we obtain (up to scalar factors) the following polynomials:

$$\begin{aligned} \tilde{P}_i(y) &= y^{i-1}(1 - \lambda_i^n y^n), \quad 1 \leq i \leq m; \\ \tilde{P}_{m+j+1}(y) &= f(\xi^{-1}y), \quad 0 \leq j < n \end{aligned}$$

where

$$f(y) = \prod_{s=1}^{n-1} (y - ie^{\pi i s/n}) \prod_{i=1}^m (y + \lambda_i)$$

(in Đoković the last n polynomials are reordered)

The main result of Đoković is the Theorem 3.1 where he proved Atiyah conjecture for configurations described above, by explicitly computing the determinant of the coefficients matrix $\tilde{P}$ of the polynomials $\{\tilde{p}_k(y) | k=1, \dots, \underbrace{m+n}_N\}$ in terms of the coefficients of

$$f(y) = \sum_{k=0}^{N-1} \tilde{E}_k y^{N-1-k}$$

His formula reads as follows:

$$\left| \det(\tilde{P}) \right| = n^{n/2} \prod_{k=0}^{n-1} f_k$$

where

$$f_k = \sum_{s \geq 0} \left(\prod_{j=1}^s \lambda_{N-jn-k}^n \right) \tilde{E}_{k+sn}, \quad 0 \leq k < n.$$

We shall now present an amazingly simple formula for coefficients of the polynomial

$$h(y) := \prod_{s=1}^{n-1} (y - ie^{\pi i s/n}) = \sum_{j=0}^{n-1} c_j y^{n-1-j}$$

Proposition 2.1 *let $\gamma_k := \cot\left(\frac{k\pi}{2n}\right)$. Then*

$$c_0 = 1, \quad c_j = \prod_{k=1}^j \gamma_k \quad (1 \leq j \leq n-1)$$

Proof .

Put $\xi_k = -ie^{\pi i k/n}$, $k = 1, \dots, n-1$. Then

$$\begin{aligned} c_j &= \text{the } j\text{-th elementary symmetric function of } \xi_1, \dots, \xi_{n-1} \\ &= e_j(\xi_1, \dots, \xi_{n-1}) \end{aligned}$$

Let us first compute the power sums

$$\begin{aligned} p_s &= \sum_{k=1}^{n-1} \xi_k^s = (-i)^s \sum_{k=1}^{n-1} e^{\pi i s k/n} = (-i)^s (e^{\pi i s/n} - e^{\pi i s}) / (1 - e^{\pi i s}) \\ &= \begin{cases} (-1)^{\frac{s}{2}-1}, s \text{ even} \\ (-1)^{\frac{s-1}{2}} \cot\left(\frac{s\pi}{2n}\right) = (-1)^{\frac{s-1}{2}} \gamma_s, s \text{ odd} \end{cases} \end{aligned}$$

The proof will be by induction. For $j = 1$ we have $c_1 = \xi_1 + \dots + \xi_{n-1} = p_1 = \gamma_1$. Suppose that the proposition is true for all $k < i$. Then by Newton formula for symmetric functions

$$j e_j = \sum_{k=1}^j (-1)^{k-1} p_k e_{j-k} = \sum_{l=1}^{\lfloor j/2 \rfloor} (p_{2l-1} e_{j-2l+1} - p_{2l} e_{j-2l})$$

we obtain by writing $c_{j-2l+1} = c_{j-2l}\gamma_{j-2l+1}$

$$\begin{aligned}
je_j &= \sum_{l=1}^{\lceil j/2 \rceil} \left((-1)^{l-1} \gamma_{2l-1} \gamma_{j-2l+1} - (-1)^{l-1} \right) c_{j-2l} \\
&= \sum_{l=1}^{\lceil j/2 \rceil} (-1)^{l-1} (\gamma_{2l-1} \gamma_{j-2l+1} - 1) c_{j-2l} \\
&\stackrel{*}{=} \sum_{l=1}^{\lceil j/2 \rceil} (-1)^{l-1} (\gamma_{2l-1} + \gamma_{j-2l+1}) \gamma_j c_{j-2l} \\
&= \sum_{l=1}^{\lceil j/2 \rceil} (p_{2l-1} c_{j-2l} - p_{2l-2} \gamma_{j-2l+1} c_{j-2l}) \gamma_j \quad (\text{here } p_0 := -1) \\
&= \sum_{l=1}^{\lceil j/2 \rceil} (p_{2l-1} c_{j-2l} - p_{2l-2} c_{j-2l+1}) \gamma_j \\
&= \sum_{l=1}^{\lceil j/2 \rceil} (p_{2l-1} c_{j-1-(2l-1)} - p_{2l-2} c_{j-1-(2l-2)}) \gamma_j \\
&= (-p_0 c_{j-1} + \sum_{l=1}^{\lceil (j-1)/2 \rceil} (p_{2l-1} c_{j-1-(2l-1)} - p_{2l} c_{j-1-2l})) \gamma_j \\
&\stackrel{**}{=} (c_{j-1} + (j-1) c_{j-1}) \gamma_j \\
&= j c_{j-1} \gamma_j = j c_j
\end{aligned}$$

Here in (*) we have used the cotangent addition formula $\cot(\alpha) \cot(\beta) - 1 = (\cot \alpha + \cot \beta) \cot(\alpha + \beta)$ and in (**) Newton formula for $i - 1$ which holds by induction hypothesis. The proposition is thus proved. $\blacksquare$

For our dihedral configurations we can state the stronger conjecture of Atiyah and Sutcliffe ([7], Conjecture 2.) as follows

$$n^{\frac{n}{2}} \prod_{k=0}^{n-1} f_k \geq 2^{\binom{n}{2}} \prod_{i=0}^n (1 + \lambda_i^2)^n \quad (2.10)$$

where

$$f_k = \sum_{s \geq 0} \left(\prod_{j=1}^s \lambda_{N-jn-k}^n \tilde{E}_{k+sn}, \quad (0 \leq k < n) \right) \quad (2.11)$$

From the factorization

$$f(y) = h(y) \prod_{i=1}^m (y + \lambda_i)$$

we can write

$$\tilde{E}_k = \sum_{i=0}^{n-1} c_i E_{k-i}$$

in terms of elementary symmetric functions $E_k = e_k(\lambda_1, \dots, \lambda_m)$ of our positive quantities $0 < \lambda_1 < \dots < \lambda_m$ with coefficients c_i given in Proposition 2.1 (note that $c_0 = 1 \leq c_1 \leq \dots \leq c_{\lfloor \frac{n-1}{2} \rfloor} \geq \dots \geq c_{n-1} = 1$ (unimodality) and $c_i = c_{n-1-i}$ (symmetry)).

Now we shall prove a generalization of the Đoković's conjecture which apparently strengthens (2.10).

Theorem 2.2 *We have:*

$$\begin{aligned} 1. \quad & \prod_{k=0}^{n-1} f_k \geq \prod_{k=0}^{n-1} c_k \left(\sum_{l=0}^m \left(\prod_{j=0}^{l-1} \lambda_{m-j} E_l \right) \right)^n \\ 2. \quad & \prod_{k=0}^{n-1} f_k \geq \prod_{k=0}^{n-1} c_k \prod_{i=1}^m (1 + \lambda_i^2)^n \end{aligned}$$

Proof .

Let us write

$$f_k = \sum_{l=0}^m \varphi_{kl} E_l$$

Let us substitute $\tilde{E}_{k+sn} = \sum_{i=0}^{n-1} c_i E_{k-i+sn}$ into (2.11). Then for fixed k ($0 \leq k < n-1$) and given l ($0 \leq l \leq m$) we seek $s \geq 0$ and i , $0 \leq i < n$ such that $l = k - 1 + sn$, i.e. $l - k = sn - i$, $0 \leq i < n$. We conclude that s and i are uniquely determined by a division algorithm (with nonpositive remainder):

$$s_k := \left\lceil \frac{l - k}{n} \right\rceil, \quad i_k = s_k n - l - k.$$

Hence

$$\varphi_{kl} = \prod_{j=1}^{s_k} \lambda_{N-jn-k}^n c_{i_k}$$

with s_k and i_k just defined. It is easy to see that

$$s_k = s_0 \left(= \left\lceil \frac{l}{n} \right\rceil \right) \text{ and } i_k = i_0 + k \text{ for } 0 \leq k \leq n - i_0 - 1$$

and

$$s_k = s_0 - 1 \text{ and } i_k = i_0 + k - n \text{ for } n - i_0 \leq k \leq n - 1.$$

Lemma 2.3 *For each l , $0 \leq l \leq m$, we have*

$$\prod_{k=0}^{n-1} \varphi_{kl} = \prod_{j=0}^{l-1} \lambda_{m-j}^n \prod_{j=0}^{n-1} c_j$$

Proof (of Lemma).

$$\begin{aligned} \prod_{k=0}^{n-1} \varphi_{kl} &= \prod_{k=0}^{n-i_0-1} \left(\prod_{j=1}^{s_0} \lambda_{N-jn-k}^n \prod_{k=i_0}^{n-1} c_k \right) \prod_{k=n-i_0}^{n-1} \prod_{j=1}^{s_0-1} \lambda_{N-jn-k}^n \prod_{k=0}^{i_0-1} c_k \\ &= \prod_{k=0}^{n-1} \prod_{j=1}^{s_0-1} \lambda_{N-jn-k}^n \prod_{k=0}^{n-i_0-1} \lambda_{N-s_0n-k}^n \prod_{k=0}^{n-1} c_k \end{aligned}$$

We put now $N = n + m$

$$\begin{aligned} &= \lambda_m^n \lambda_{m-1}^n \cdots \lambda_{m+n-s_0n-(n-i_0-1)}^n \prod_{k=0}^{n-1} c_k \\ &= \lambda_m^n \lambda_{m-1}^n \cdots \lambda_{m-l+1}^n \prod_{k=0}^{n-1} c_k \end{aligned}$$

■

Proof (of Theorem).

We shall use the Hölder inequality

$$\begin{aligned}
\prod_{k=0}^{n-1} f_k &= \prod_{k=0}^{n-1} \left(\sum_{l=0}^m \varphi_{kl} E_l \right) \geq \left(\sum_{l=0}^m \left(\prod_{k=1}^{n-1} \varphi_{kl} E_l \right)^{\frac{1}{n}} \right)^n \\
&= \left(\sum_{l=0}^m \prod_{j=0}^{l-1} \lambda_{m-j} \left(\prod_{j=0}^{n-1} c_j \right)^{\frac{1}{n}} E_l \right)^n \quad (\text{by lemma}) \\
&= \left(\prod_{j=0}^{n-1} c_j \right) \left(\sum_{l=0}^m \prod_{j=0}^{l-1} \lambda_{m-j} E_l \right)^n
\end{aligned}$$

Thus 1. is proved. To obtain 2. we apply Đoković proof of Atiyah conjecture for type A configurations

$$\sum_{l=0}^m \prod_{j=0}^{l-1} \lambda_{m-j} E_l \geq \prod_{i=1}^m (1 + \lambda_i^2)$$

(c.f. section 3.) ■

3 Appendix

After the first version of this paper was finished, in the meantime, we have discovered formulas for the partial derivatives, of the quantities $\Psi_{1\dots n}^{1\dots n} / \Psi_{1\dots \hat{k} \dots n}^{1\dots \hat{k} \dots n}$, with respect to variables ξ_r (Note that in Theorem 1.8 we have given formulas w.r.t. variables X_r !).

Lemma 3.1 *For $2 \leq r \leq n$ the partial derivative w.r.t. ξ_r of the quotient $\Psi_{1\dots n}^{1\dots n} / \Psi_{2\dots n}^{2\dots n}$ is given by*

$$(\Psi_{2\dots n}^{2\dots n})^2 \partial_{\xi_r} \left(\frac{\Psi_{1\dots n}^{1\dots n}}{\Psi_{2\dots n}^{2\dots n}} \right) = \sum_{i \geq j} s'_{ij} X_1 (X_2 \cdots X_j)^2 X_{j+1} \cdots X_{i+1} (X_{j+1} - X_{i+2})$$

where s'_{ij} is the conjugated Schur function $s_{ij} = s_{ij}(\xi_2, \dots, \xi_{r-1}, \xi_{r+1}, \dots, \xi_n)$ corresponding to a two-rowed partition $\lambda = (i \geq j)$.

In particular for $X_1 \geq \cdots \geq X_n > 0$ the function $\Psi_{1\dots n}^{1\dots n} / \Psi_{2\dots n}^{2\dots n}$ is monotonically increasing w.r.t. the variable ξ_r (for $r = 1$, too).

Proof .

By using the formula $\Psi_{1\dots n}^{1\dots n} = \Psi_{1\dots \widehat{r}\dots n}^{1\dots n-1} + X_1 \xi_r \Psi_{1\dots \widehat{r}\dots n}^{2\dots n}$ we get

$$\begin{aligned}
& \partial_{\xi_r}(\Psi_{1\dots n}^{1\dots n} \Psi_{2\dots n}^{2\dots n} - \Psi_{1\dots n}^{1\dots n} \partial_{\xi_r}(\Psi_{2\dots n}^{2\dots n})) = \\
& = X_1 \Psi_{1\dots \widehat{r}\dots n}^{2\dots n} (\Psi_{2\dots \widehat{r}\dots n}^{2\dots n-1} + X_2 \xi_r \Psi_{2\dots \widehat{r}\dots n}^{3\dots n}) - (\Psi_{1\dots \widehat{r}\dots n}^{1\dots n-1} + X_1 \xi_r \Psi_{1\dots \widehat{r}\dots n}^{2\dots n}) X_2 \Psi_{2\dots \widehat{r}\dots n}^{3\dots n} \\
& = X_1 \Psi_{1\dots \widehat{r}\dots n}^{2\dots n} \Psi_{2\dots \widehat{r}\dots n}^{2\dots n-1} - X_2 \Psi_{1\dots \widehat{r}\dots n}^{1\dots n-1} \Psi_{2\dots \widehat{r}\dots n}^{3\dots n} \\
& = X_1 (\Psi_{2\dots \widehat{r}\dots n}^{2\dots n-1} + X_2 \xi_1 \Psi_{2\dots \widehat{r}\dots n}^{3\dots n}) \Psi_{2\dots \widehat{r}\dots n}^{2\dots n-1} - X_2 (\Psi_{2\dots \widehat{r}\dots n}^{1\dots n-2} + X_1 \xi_1 \Psi_{2\dots \widehat{r}\dots n}^{2\dots n-1}) \Psi_{2\dots \widehat{r}\dots n}^{3\dots n} \\
& = X_1 (\Psi_{2\dots \widehat{r}\dots n}^{2\dots n-2})^2 - X_2 \Psi_{2\dots \widehat{r}\dots n}^{1\dots n-2} \Psi_{2\dots \widehat{r}\dots n}^{3\dots n}
\end{aligned}$$

With $e_i = e_i^{(1r)} = e_i(\xi_2, \dots, \xi_{r-1}, \xi_{r+1}, \dots, \xi_n)$ denoting the i -th elementary symmetric function of the truncated alphabet $A^{(1r)} = \{\xi_2, \dots, \xi_{r-1}, \xi_{r+1}, \dots, \xi_n\}$ we have further

$$\begin{aligned}
& = X_1 \left(\sum_{i,j} e_i e_j X_{2\dots i+1} X_{2\dots j+1} \right) - X_2 \left(\sum_{i,j} e_i e_j X_{1\dots i} X_{3\dots j+2} \right) \\
& = \sum_{i,j} e_i e_j X_{1\dots i+1} X_{2\dots j+1} - \sum_{i,j} e_i e_j X_{1\dots i} X_{2\dots j+2} \\
& = \sum_{i,j} \begin{vmatrix} e_i & e_{i+1} \\ e_{j-1} & e_j \end{vmatrix} X_{1\dots i+1} X_{2\dots j+1} \\
& = \sum_{i \geq j} \begin{vmatrix} e_i & e_{i+1} \\ e_{j-1} & e_j \end{vmatrix} X_1 (X_{2\dots j})^2 X_{j+1} \cdots X_{i+1} (X_{j+1} - X_{i+2})
\end{aligned}$$

Now by Jacobi–Trudy formula we can write $\begin{vmatrix} e_i & e_{i+1} \\ e_{j-1} & e_j \end{vmatrix}$ as the conjugated Schur function $s'_{ij} = s'_{ij}^{(1r)}$ corresponding to a partition $(i \geq j)$. ■

Corollary 3.2 (ξ_n -monotonicity)

We have the following inequality:

$$\frac{\Psi_{1\dots n}^{1\dots n}}{\Psi_{2\dots n}^{2\dots n}} \geq \frac{\Psi_{1\dots n-1}^{1\dots n-1}}{\Psi_{2\dots n-1}^{2\dots n-1}}$$

Proof .

By Lemma 3.1 by letting $\xi_n \downarrow 0$ we get

$$\Psi_{1\dots n}^{1\dots n} / \Psi_{2\dots n}^{2\dots n} \geq \Psi_{1\dots n}^{1\dots n} / \Psi_{2\dots n}^{2\dots n} \Big|_{\xi_n=0} = \Psi_{1\dots n-1}^{1\dots n-1} / \Psi_{2\dots n-1}^{2\dots n-1}$$

■

By using this Corollary we state a strengthening of our Conjecture 1.5:

Conjecture 3.3

$$(\Psi_{1\dots n}^{1\dots n})^{n-2} \geq \Psi_{2\dots n-2}^{2\dots n-1} \prod_{k=2}^{n-1} \Psi_{1\dots \hat{k}\dots n}^{1\dots \hat{k}\dots n}$$

We also have formulas for partial derivative of the quotient $\Psi_{1\dots n}^{1\dots n}/\Psi_{1\dots \hat{k}\dots n}^{1\dots \hat{k}\dots n}$ w.r.t. variable ξ_r , $2 \leq r \leq n$, which are more complicated than for $k = 1$ (given in Lemma 3.1). Without loss of generality we take $r = n$ and proceed as follows:

$$\begin{aligned} & \partial_{\xi_n} (\Psi_{1\dots n}^{1\dots n}) \Psi_{1\dots \hat{k}\dots n}^{1\dots \hat{k}\dots n} - \Psi_{1\dots n}^{1\dots n} \partial_{\xi_n} (\Psi_{1\dots \hat{k}\dots n}^{1\dots \hat{k}\dots n}) = \\ &= X_1 \Psi_{1\dots n-1}^{2\dots n} \Psi_{1\dots \hat{k}\dots n}^{1\dots \hat{k}\dots n} - X_1 \Psi_{1\dots n}^{1\dots n} \Psi_{1\dots \hat{k}\dots n-1}^{2\dots \hat{k}\dots n} \\ &= X_1 \Psi_{1\dots n-1}^{2\dots n} \left(\Psi_{1\dots \hat{k}\dots n}^{1\dots \hat{k}\dots n} + X_1 \xi_n \Psi_{1\dots \hat{k}\dots n-1}^{2\dots \hat{k}\dots n} \right) - X_1 \left(\Psi_{1\dots n-1}^{1\dots n-1} + X_1 \xi_n \Psi_{1\dots n-1}^{2\dots n} \right) \Psi_{1\dots \hat{k}\dots n-1}^{2\dots \hat{k}\dots n} \\ &= X_1 \left(\Psi_{1\dots n-1}^{2\dots n} \Psi_{1\dots \hat{k}\dots n-1}^{1\dots \hat{k}\dots n-1} - \Psi_{1\dots n-1}^{1\dots n-1} \Psi_{1\dots \hat{k}\dots n-1}^{2\dots \hat{k}\dots n} \right) \\ &= X_1 \left[\left(\Psi_{1\dots \hat{k}\dots n-1}^{2\dots n-1} + X_2 \xi_k \Psi_{1\dots \hat{k}\dots n-1}^{3\dots n} \right) \Psi_{1\dots \hat{k}\dots n-1}^{1\dots \hat{k}\dots n-1} - \right. \\ &\quad \left. - \left(\Psi_{1\dots \hat{k}\dots n-1}^{1\dots n-2} + X_1 \xi_k \Psi_{1\dots \hat{k}\dots n-1}^{2\dots n-1} \right) \Psi_{1\dots \hat{k}\dots n-1}^{2\dots \hat{k}\dots n} \right] \\ &= X_1 \left[\Psi_{1\dots \hat{k}\dots n-1}^{2\dots n-1} \Psi_{1\dots \hat{k}\dots n-1}^{1\dots \hat{k}\dots n-1} - \Psi_{1\dots \hat{k}\dots n-1}^{1\dots n-2} \Psi_{1\dots \hat{k}\dots n-1}^{2\dots \hat{k}\dots n} + \right. \\ &\quad \left. + \xi_k \left(X_2 \Psi_{1\dots \hat{k}\dots n-1}^{3\dots n} \Psi_{1\dots \hat{k}\dots n-1}^{1\dots \hat{k}\dots n-1} - X_1 \Psi_{1\dots \hat{k}\dots n-1}^{2\dots n-1} \Psi_{1\dots \hat{k}\dots n-1}^{2\dots \hat{k}\dots n} \right) \right] \\ &= X_1 [I_1 - \xi_k I_2] \end{aligned}$$

Now we first compute

$$\begin{aligned}
I_1 &= \Psi_{1..\hat{k}..n-1}^{2...n-1} \Psi_{1..\hat{k}..n-1}^{1...\hat{k}...n-1} - \Psi_{1..\hat{k}..n-1}^{1...n-2} \Psi_{1..\hat{k}..n-1}^{2...\hat{k}...n} = \\
&\left(\sum_{i=0}^{k-2} e_i X_{2..i+1} + \sum_{i=k-1}^{n-2} e_i X_{2..i+1} \right) \left(\sum_{j=0}^{k-1} e_j X_{1..j} + \sum_{j=k}^{n-2} e_j X_{1..\hat{k}..j+1} \right) - \\
&- \left(\sum_{j=0}^{k-1} e_j X_{1..j} + \sum_{j=k}^{n-2} e_j X_{1..j} \right) \left(\sum_{i=0}^{k-2} e_i X_{2..i+1} + \sum_{i=k-1}^{n-2} e_i X_{2..\hat{k}..i+2} \right) = \\
&= \sum_{i=k-1}^{n-2} \sum_{j=0}^{k-1} e_i e_j \left(X_{2..i+1} X_{1..j} - X_{2..\hat{k}..i+1} X_{1..j} \right) + \\
&+ \sum_{j=k}^{n-2} \sum_{i=0}^{k-2} e_j e_i \left(X_{1..\hat{k}..j+1} X_{2..i+1} - X_{1..j} X_{2..i+1} \right) + \\
&+ \sum_{i=k-1}^{n-2} \sum_{j=k}^{n-2} e_i e_j \left(X_{2..i+1} X_{1..\hat{k}..j+1} - X_{1..j} X_{2..\hat{k}..i+2} \right)
\end{aligned}$$

By replacing, in the middle sum, j with $i+1$ and i with $j-1$, and observing that then $X_{1..\hat{k}..i+2} X_{2..j} - X_{1..i+1} X_{2..j} = -(X_{2..i+1} X_{1..j} - X_{2..\hat{k}..i+2} X_{1..j})$ the contribution of the first two sums is

$$\sum_{i=k-1}^{n-2} \sum_{j=0}^{k-1} \begin{vmatrix} e_i & e_{i+1} \\ e_{j-1} & e_j \end{vmatrix} X_{2..\hat{k}..i+1} (X_k - X_{i+2}) X_{1..j}$$

The third sum can similarly be transformed to the following form:

$$\sum_{k \leq j \leq i \leq n-2} \begin{vmatrix} e_i & e_{i+1} \\ e_{j-1} & e_j \end{vmatrix} X_{2..\hat{k}..i+1} (X_{j+1} - X_{i+2}) X_{1..j}$$

Hence

$$I_1 = \sum_{0 \leq j, \max\{j, k-1\} \leq i \leq n-2} s'_{ij} X_{2..\hat{k}..i+1} (X_{\max\{j+1, k\}} - X_{i+2}) X_{1..j} \quad (\geq 0)$$

By a similar manipulation we can obtain the expression for the quantity

$$\begin{aligned}
I_2 &= X_1 \Psi_{1..\hat{k}..n-1}^{2...n-1} \Psi_{1..\hat{k}..n-1}^{2...\hat{k}...n} - X_2 \Psi_{1..\hat{k}..n-1}^{3...n} \Psi_{1..\hat{k}..n-1}^{1...\hat{k}...n-1} = \\
&= X_1 - X_2 + \sum_{i=1}^{n-1} \sum_{j \leq \min\{k-1, i\}} s'_{ij} X_{2..\hat{k}..i+2} X_{1..j} (X_{j+1} - X_k) \geq 0
\end{aligned}$$

where s'_{ij} is conjugated Schur function $s'_{ij} = s'^{(kn)}_{ij}$. We see that

$$\left(\Psi_{1\dots\widehat{k}\dots n}^{1\dots\widehat{k}\dots n}\right)^2 \partial_{\xi_n} \left(\frac{\Psi_{1\dots\widehat{k}\dots n}^{1\dots\widehat{k}\dots n}}{\Psi_{1\dots\widehat{k}\dots n}^{1\dots\widehat{k}\dots n}}\right) = X_1 [I_1 - \xi_k I_2]$$

has both positive and negative terms. And we have not been able to apply it so far.

Now we illustrate use of ξ -monotonicity (in addition to X -monotonicity) for proving once more the case $n = 4$ of our Conjecture 1.5:

$$\begin{aligned} & \frac{(\Psi_{1234}^{1234})^3}{\Psi_{234}^{234} \Psi_{134}^{134} \Psi_{124}^{124} \Psi_{123}^{123}} = \frac{\Psi_{1234}^{1234}}{\Psi_{234}^{234} \Psi_{123}^{123}} \frac{\Psi_{1234}^{1234}}{\Psi_{134}^{134}} \frac{\Psi_{1234}^{1234}}{\Psi_{124}^{124}} \geq \text{(by } \xi_4\text{-monotonicity)} \\ & \geq \frac{1}{\Psi_{23}^{23}} \frac{\Psi_{1234}^{1234}}{\Psi_{134}^{134}} \frac{\Psi_{1234}^{1234}}{\Psi_{124}^{124}} \geq \text{(by } X_1\text{-monotonicity twice and } X_4\text{-monotonicity)} \\ & \geq \frac{1}{\Psi_{23}^{23}} \frac{\Psi_{1234}^{2234}}{\Psi_{143}^{143}} \frac{\Psi_{1234}^{2233}}{\Psi_{124}^{124}} \geq \text{(by } \xi_3\text{-monotonicity)} \\ & \geq \frac{1}{\Psi_{23}^{23}} \frac{\Psi_{124}^{223}}{\Psi_{14}^{14}} \frac{\Psi_{1234}^{2233}}{\Psi_{124}^{124}} = \frac{\Psi_{1234}^{2233}}{\Psi_{23}^{23} \Psi_{14}^{14}} \geq 1 \end{aligned}$$

Similarly the cases $n = 5, 6, 7$ of Conjecture 1.5 would be, by using ξ -monotonicity and X -monotonicity, consequences of the following inequalities

$$\tilde{Q}_n \geq 1$$

where

$$\begin{aligned} \tilde{Q}_5 &= \Psi_{12345}^{22344} \Psi_{12345}^{22344} / \Psi_{234}^{234} \Psi_{135}^{234} \Psi_{1245}^{2244} \\ \tilde{Q}_6 &= \Psi_{123456}^{223445} \Psi_{123456}^{223455} / \Psi_{2345}^{2345} \Psi_{1346}^{2345} \Psi_{1256}^{2345} \\ \tilde{Q}_7 &= \Psi_{1234567}^{2234556} \Psi_{1234567}^{2234566} \Psi_{1234567}^{22344566} / \Psi_{23456}^{23456} \Psi_{13457}^{23456} \Psi_{12467}^{23456} \Psi_{123567}^{234566} \end{aligned}$$

3.1 Computer verification of the Conjecture 1.5 (and hence of the Atiyah–Sutcliffe conjecture C3) for almost collinear $9 + 1$ configuration.

Let us now explain our computer verification of the inequality $\tilde{Q}_9 \geq 1$ where

$$\tilde{Q}_9 = \frac{\Psi_{123456789}^{223456778} \Psi_{123456789}^{223456788} \Psi_{123456789}^{223456678} \Psi_{123456789}^{2234456788}}{\Psi_{12345678}^{2345678} \Psi_{1345679}^{2345678} \Psi_{1245689}^{2345678} \Psi_{1235789}^{2345678} \Psi_{12346789}^{22346788}}$$

which refines the case $n = 9$ of the Conjecture 1.5. We have observed first that $\tilde{Q}_9$ is symmetric in partial alphabets

$$A_1 = \{\xi_1, \xi_2, \xi_8, \xi_9\}, \quad A_2 = \{\xi_3, \xi_4, \xi_6, \xi_7\}, \quad A_3 = \{\xi_5\}$$

then by introducing the elementary symmetric functions $\{e_1, e_2, e_3, e_4\}$ of A_1 and $\{f_1, f_2, f_3, f_4\}$ of A_2 we first computed the products

$$\Psi_{2345678}^{2345678} \Psi_{1345679}^{2345678} \text{ and } \Psi_{1245689}^{2345678} \Psi_{1235789}^{2345678}$$

in terms of $\{e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4, \xi_5\}$. Then by successive application of Stembridge’s **Maple SF** package we expressed the difference $\Delta := \text{numer}(\tilde{Q}_9) - \text{denom}(\tilde{Q}_9)$ in terms of the Schur functions of both alphabets A_1 and A_2 . Then we factored each coefficient in such a multi-Schur expansion and into non-monomial factors we substituted $X_2 = X_3 + h_2$, $X_3 = X_4 + h_3$, $\dots$, $X_7 = X_8 + h_7$. Then the computation showed that the coefficients of all monomials in $X_8, h_2, \dots, h_7$ were nonnegative. The factoring out the trivial monomial factors in $X_2, \dots, X_8$ (which are trivially nonnegative) was crucial because otherwise the expansion of multi-Schur function coefficients in terms of increments $h_2, \dots, h_7$ may not be feasible.

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