On the Bodlaender Sequence

Daniele Parisse
 Airbus Defence and Space GmbH
 Rechliner Str., 85077 Manching, Germany
 daniele.paris
 @t-online.de

Abstract

We will present two equivalent solutions of the Bodlaender sequence $g : \mathbb{N} \rightarrow \mathbb{Z}$ first introduced recursively by him et al. and used by them to determine the edge ranking number of the complete graphs. These solutions rely on the binary representation of $m \in \mathbb{N}$ either in the form $m = 2^{a_0} + \dots + 2^{a_l}$ with the strictly decreasing integer sequence $a_0 > a_1 > \dots > a_l \geq 0$, $l \geq 0$, or using the binary expansion of $m = \sum_{k=0}^n b_k 2^k$, where $n \in \mathbb{N}_0$ and $b_0, \dots, b_n \in \{0, 1\}$. In addition, we will determine sharp bounds of the solution and we will give some properties of related sequences such as $a(m) := \frac{1}{3}(m^2 + g(m))$, $m \in \mathbb{N}$, $d(m) := g(m + 1) + 1$, $m \in \mathbb{N}_0$, and the sequence of the partial sums of $g(m)$, $m \in \mathbb{N}$.

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1 Introduction

In order to determine the edge ranking number of complete graphs Hans L. Bodlaender et al. introduced in [1, Chapter 7] the following sequence defined recursively for any $m \in \mathbb{N}$ by the rules

$$\left. \begin{aligned} g(1) &= -1 \\ g(2m) &= g(m) \\ g(2m+1) &= g(m+1) + m \end{aligned} \right\} \quad (1.1)$$

with the first few values

$$(-1, -1, 0, -1, 2, 0, 2, -1, 6, 2, 5, 0, 8, 2, 6, -1, 14, 6, 11, 2, 15, 5, 11, 0, 20, \dots).$$

(This sequence is not available in the On-Line Encyclopedia of Integer Sequences (OEIS [®])[8].)

We briefly recall the definition of the edge ranking number of a graph G (for more details we refer to [1, Definition 2] or [7, p.1067].) Let $G = (V, E)$ be a (simple) graph and t a positive integer. An *edge- t -ranking* is an edge coloring $c' : E \rightarrow \{1, 2, \dots, t\}$ such that for any two edges of the same color, every path between them contains an intermediate edge with a larger color value. The *edge ranking number* denoted by $\chi'_r(G)$ is the smallest value of t such that the graph G has an edge- t -ranking.

Remark 1.1 In [4, p.1068] and [7, Corollary 4.9] the authors called it *Bodlaender function* g , but since (1.1) is defined only for all $m \in \mathbb{N}$, it is more appropriate to denote it *Bodlaender sequence*.

Note that for $m - 1$ instead of m we obtain from (1.1) $g(2(m - 1) + 1) = g(2m - 1) = g(m) + m - 1$ and therefore $g(2m - 1) - g(2m) = m - 1$ for all $m \in \mathbb{N}$.

The recurrence relation (1.1) is a special case of the general recurrence relation defined for all $m \in \mathbb{N}$

$$\left. \begin{aligned} f(1) &= \zeta \\ f(2m) &= \alpha f(m) + g(m) \\ f(2m + 1) &= \gamma f(m) + \delta f(m + 1) + h(m) \end{aligned} \right\} \quad (1.2)$$

with the parameters $\alpha, \gamma, \delta, \zeta \in \mathbb{Z}$ and the integer functions $g, h : \mathbb{N} \rightarrow \mathbb{Z}$. In our case it is $\alpha = 1, \gamma = 0, \delta = 1, \zeta = -1$ and $g(m) = 0, h(m) = m$ for all $m \in \mathbb{N}$.

Eq.(1.2) can also be written for all $m \in \mathbb{N}_2 := \{2, 3, 4, \dots\}$ as follows

$$f(m) = a(m)f(\lfloor m/2 \rfloor) + b(m)f(\lceil m/2 \rceil) + c(m), \quad f(1) = \zeta \quad (1.3)$$

with

$$a(m) := \frac{\alpha + \gamma}{2} + (-1)^m \frac{\alpha - \gamma}{2}, \quad b(m) := \frac{1 - (-1)^m}{2} \delta$$

and

$$c(m) := \frac{g(m) + h(m)}{2} + (-1)^m \frac{g(m) - h(m)}{2}.$$

Recurrence relations of this form are called (*binary*) *divide-and-conquer recurrences* and appear often in computer science, because algorithms based on the technique of *divide et impera* (divide and conquer) often reduce a problem of size m to

the solution of two problems of approximately equal sizes $\lfloor m/2 \rfloor$ and $\lceil m/2 \rceil$, where $m = \lfloor m/2 \rfloor + \lceil m/2 \rceil, m \in \mathbb{N}_0$. The solutions of the two subproblems are then used to solve the original problem.

A prominent example is given by the problem to sort m records, $m > 1$. One method is called *mergesort* [3, p.79] and consists in dividing the m records into two approximately equal parts, one of size $\lfloor m/2 \rfloor$ and the other of size $\lceil m/2 \rceil$. After each part has been sorted separately by the same method, the records are merged into their final order by doing at most $m - 1$ further comparisons. The total number of comparisons performed is at most $f(m)$, where $f(1) = 0$ and $f(m) = f(\lfloor m/2 \rfloor) + f(\lceil m/2 \rceil) + m - 1, m > 1$. This is a special case of (1.3) with $a(m) = b(m) = 1, c(m) = m - 1$ and $\zeta = 0$.

Another well-known example is given by Stern's diatomic sequence (this is the sequence [A002487](#) in the OEIS [8]), defined by $s(1) = 1, s(2m) = s(m)$ and $s(2m + 1) = s(m) + s(m + 1)$ for all $m \in \mathbb{N}$, that is $\alpha = \gamma = \delta = \zeta = 1$ and $g(m) = h(m) = 0$ for all $m \in \mathbb{N}$. Note that the value $s(0)$ has to be 0, since from the recurrence relation for odd indices for $m = 0$ we get $s(1) = s(0) + s(1)$, that is $s(0) = 0$.

Adding 1 and starting with the index 0 we get from $g(m)$ the sequence $d(m) := g(m + 1) + 1, m \in \mathbb{N}_0$, satisfying for all $m \geq 1$ the recurrence relation

$$\left. \begin{aligned} d(0) &= g(1) + 1 = -1 + 1 = 0, \quad d(1) = g(2) + 1 = 0 \\ d(2m) &= g(2m + 1) + 1 = g(m + 1) + m + 1 = d(m) + m \\ d(2m + 1) &= g(2m + 1 + 1) + 1 = g(m + 1) + 1 = d(m) \end{aligned} \right\} \quad (1.4)$$

This is the sequence [A233931](#) in the OEIS [8] with the first few values

$$(0, 0, 1, 0, 3, 1, 3, 0, 7, 3, 6, 1, 9, 3, 7, 0, 15, 7, 12, 3, 16, 6, 12, 1, 21, 9, 16, 3, \dots)$$

and a special case of (1.2) with $\alpha = 1, \gamma = 1, \delta = 0, \zeta = 0, g(m) = m$ and $h(m) = 0$.

In this paper we shall determine the solution of (1.1) by writing $m \in \mathbb{N}$ either in the form $m = 2^{a_0} + \dots + 2^{a_l}$ with the strictly decreasing integer sequence $a_0 > a_1 > \dots > a_l \geq 0, l \geq 0$, or using the binary expansion of $m = (b_n \dots b_0)_2 := \sum_{k=0}^n b_k 2^k$, where $n \in \mathbb{N}_0$ and $b_0, \dots, b_n \in \{0, 1\}$. Moreover, we shall also give sharp lower and upper bounds for $g(m)$.

Finally, we shall explore some properties of the sequence $a(m) := \frac{1}{3}(m^2 + g(m)), m \in \mathbb{N}$, giving the edge ranking number of complete graphs and investigate the sequence of the partial sums of $g(m), m \in \mathbb{N}$, and the sequence $d(m) := g(m + 1) + 1, m \in \mathbb{N}_0$.

2 Preliminaries

We start with a lemma which gives the values of the Bodlaender sequence for some special numbers.

Lemma 2.1 *Let $a, b, c \in \mathbb{N}_0$, then*

$$g(2^a) = -1, \quad a \geq 0 \quad (2.1)$$

$$g(2^a - 1) = 2^{a-1} - 2, \quad a \geq 1 \quad (2.2)$$

$$g(2^a + 1) = 2^a - 2, \quad a \geq 0 \quad (2.3)$$

$$g(2^a + 2^b) = 2^{a-b} - 2, \quad a > b \geq 0 \quad (2.4)$$

$$g(3 \cdot 2^a) = 0, \quad a \geq 0 \quad (2.5)$$

$$g(2^a + 2^b - 1) = 2^{a-1} + 2^{b-1} + 2^{a-b} - 3, \quad a > b \geq 1 \quad (2.6)$$

$$g(2^a + 2^b + 1) = 2^a + 2^b - 2^{a-b-1} - 3, \quad a > b \geq 0 \quad (2.7)$$

$$g(2^a + 2^b + 2^c) = 2^{a-c} + 2^{b-c} + 2^{a-b-1} - 3, \quad a > b > c \geq 0 \quad (2.8)$$

Proof. a) By repeated use of (1.1) we have $g(2^a) = g(1) = -1$.

b) By (1.1) and Eq.(2.1) we have $g(2^a - 1) = g(2 \cdot (2^{a-1} - 1) + 1) = g(2^{a-1} - 1 + 1) + 2^{a-1} - 1 = -1 + 2^{a-1} - 1 = 2^{a-1} - 2$.

c) Let $f(a) := g(2^a + 1)$, $a \geq 0$, then by (1.1) the sequence $(f(a))_{a \in \mathbb{N}_0}$ satisfies the recurrence relation

$$f(a+1) = g(2^{a+1} + 1) = g(2 \cdot 2^a + 1) = g(2^a + 1) + 2^a = f(a) + 2^a, \quad a \geq 0,$$

with $f(0) = g(2) = -1$. The solution of this linear first-order recurrence relation can be obtained by backward substitution. After a substitutions it follows $f(a) = f(a-1) + 2^{a-1} = f(a-2) + 2^{a-2} + 2^{a-1} = \dots = f(0) + 2^0 + 2^1 + \dots + 2^{a-1} = -1 + (2^a - 1) = 2^a - 2$ and this proves the formula (2.3).

d) By (1.1) we have $g(2^a + 2^b) = g(2^b(2^{a-b} + 1)) = g(2^{a-b} + 1) = 2^{a-b} - 2$, where the last equation follows from (2.3).

e) By Eq.(2.4) we have $g(3 \cdot 2^a) = g(2^{a+1} + 2^a) = 2^{a+1-a} - 2 = 0$.

f) By (1.1) and Eq.(2.4) we have $g(2^a + 2^b - 1) = g(2 \cdot (2^{a-1} + 2^{b-1} - 1) + 1) = g(2^{a-1} + 2^{b-1} - 1 + 1) + 2^{a-1} + 2^{b-1} - 1 = 2^{a-b} - 2 + 2^{a-1} + 2^{b-1} - 1$ and this is exactly formula (2.6).

g) By (1.1) we have $g(2^a + 2^b + 1) = g(2 \cdot (2^{a-1} + 2^{b-1}) + 1) = g(2^{a-1} + 2^{b-1} + 1) + 2^{a-1} + 2^{b-1}$. Let $f(a, b) := g(2^a + 2^b + 1)$, then the above equation means that

$f(a, b)$ satisfies the recurrence relation

$$f(a, b) = f(a - 1, b - 1) + 2^{a-1} + 2^{b-1},$$

which can be solved by backward substitution. After b substitutions we obtain $f(a, b) = f(a - b, 0) + 2^{a-b} + 2^0 + 2^{a-b+1} + 2^1 + \dots + 2^{a-1} + 2^{b-1}$. The first term is by (1.1) and (2.3) equal to $f(a - b, 0) = g(2^{a-b} + 2^0 + 1) = g(2 \cdot (2^{a-b-1} + 1)) = g(2^{a-b-1} + 1) = 2^{a-b-1} - 2$. The other term is the sum of three geometric series, namely $(1 + 2 + \dots + 2^{b-1}) + (1 + 2 + \dots + 2^{a-1}) - (1 + 2 + \dots + 2^{a-b-1}) = (2^b - 1) + (2^a - 1) - (2^{a-b} - 1) = 2^a + 2^b - 2^{a-b} - 1$. Summing up the two terms it follows $2^{a-b-1} - 2 + 2^a + 2^b - 2^{a-b} - 1 = 2^a + 2^b - 2^{a-b-1} - 3$, as asserted.

h) By (1.1) and Eq.(2.7) we have $g(2^a + 2^b + 2^c) = g(2^c \cdot (2^{a-c} + 2^{b-c} + 1)) = g(2^{a-c} + 2^{b-c} + 1) = 2^{a-c} + 2^{b-c} + 2^{a-b-1} - 3$. $\square$

The next lemma shows that it is sufficient to consider only the case $a_l = 0$, that is m is an odd number.

Lemma 2.2 *Let $m = 2^{a_0} + \dots + 2^{a_l}$, where $a_0 > a_1 > \dots > a_l \geq 0$, $l \geq 0$, is a strictly decreasing integer sequence, then for $a_l \neq 0$:*

$$g(m) = g(2^{-a_l}m) \quad (2.9)$$

and for $a_l = 0$:

$$g(m) = g(2^{-a_{l-1}}(m - 1) + 1) + (1 - 2^{-a_{l-1}})(m - 1) \quad (2.10)$$

Proof. a) Let $a_l \neq 0$, then by (1.1)

$$\begin{aligned} g(m) &= g(2^{a_0} + \dots + 2^{a_l}) = g(2^{a_l}(2^{a_0-a_l} + \dots + 2^{a_{l-1}-a_l} + 1)) \\ &= g(2^{a_0-a_l} + \dots + 2^{a_{l-1}-a_l} + 1) = g(2^{-a_l}(m - 2^{a_l}) + 1) = g(2^{-a_l}m) \end{aligned}$$

b) Now let $a_l = 0$, then by (1.1) and since $\sum_{k=0}^{l-1} 2^{a_k} = m - 1$ we obtain

$$\begin{aligned} g(m) &= g(2^{a_0} + \dots + 2^{a_{l-1}} + 1) = g(2 \cdot (2^{a_0-1} + \dots + 2^{a_{l-1}-1}) + 1) \\ &= g(2^{a_0-1} + \dots + 2^{a_{l-1}-1} + 1) + 2^{a_0-1} + \dots + 2^{a_{l-1}-1} \\ &= g(2 \cdot (2^{a_0-2} + \dots + 2^{a_{l-1}-2}) + 1) + \frac{1}{2}(m - 1) \\ &= g(2^{a_0-2} + \dots + 2^{a_{l-1}-2} + 1) + \frac{1}{4}(m - 1) + \frac{1}{2}(m - 1) \end{aligned}$$

Repeating this procedure a_{l-1} times and noting that $\sum_{k=1}^{a_{l-1}} 2^{-k} = \sum_{k=0}^{a_{l-1}-1} 2^{-k} - 1 = 2 \cdot (1 - 2^{-a_{l-1}-1}) - 1 = 1 - 2^{-a_{l-1}}$ we obtain

$$\begin{aligned} g(m) &= g(2^{a_0-a_{l-1}} + \dots + 2^{a_{l-1}-a_{l-1}} + 1) + (m-1) \sum_{k=1}^{a_{l-1}} 2^{-k} \\ &= g(2^{-a_{l-1}}(m-1) + 1) + (1 - 2^{-a_{l-1}})(m-1) \end{aligned}$$

and this proves Eq.(2.10). $\square$

Note that in Eq.(2.10) the argument of g on the left-hand side has $l+1$ summands, whereas on the right-hand side it has only l summands.

3 Main Result

We can now prove our main result.

Theorem 3.1 *Let $m = 2^{a_0} + \dots + 2^{a_l}$, where $a_0 > a_1 > \dots > a_l \geq 0$, $l \geq 0$, is a strictly decreasing integer sequence, then*

$$g(m) = 2^{-a_l}(m - 2^{a_l}) - (l+1) - \sum_{k=1}^l 2^{a_{l-k}} \left(\sum_{j=1}^{k-1} 2^{-a_{l-j}-1} \right) \quad (3.1)$$

In particular, for $a_l = 0$

$$g(m) = m - 1 - (l+1) - \sum_{k=1}^l 2^{a_{l-k}} \left(\sum_{j=1}^{k-1} 2^{-a_{l-j}-1} \right) \quad (3.2)$$

Proof. It is sufficient to prove Eq.(3.2), since for $a_l \neq 0$ we have $g(2^{a_0} + \dots + 2^{a_l}) = g(m) = g(2^{-a_l}m) = g(2^{a_0-a_l} + \dots + 2^{a_{l-1}-a_l} + 1)$ by Eq.(2.9). We obtain Eq.(3.1) by simply writing $a_k - a_l$ instead of a_k , $k = 0, 1, \dots, l$ in Eq.(3.2) and noting that $m - 1 = 2^{a_0-a_l} + \dots + 2^{a_{l-1}-a_l}$ and $a_{l-k} - a_l - (a_{l-j} - a_l) - 1 = a_{l-k} - a_{l-j} - 1$. Thus the double sum and the term $l+1$ in (3.1) do not change. For the first term we obtain $2^{a_0-a_l} + \dots + 2^{a_{l-1}-a_l} = 2^{-a_l}(2^{a_0} + \dots + 2^{a_{l-1}} + 2^{a_l} - 2^{a_l}) = 2^{-a_l}(m - 2^{a_l})$ thus obtaining Eq.(3.1).

We now prove the case $a_l = 0$.

By (2.10) we have

$$g\left(\sum_{k=0}^{l-1} 2^{a_k} + 1\right) = g\left(\sum_{k=0}^{l-2} 2^{a_k-a_{l-1}-1} + 1\right) + (1 - 2^{-a_{l-1}}) \sum_{k=1}^l 2^{a_{l-k}} \quad (3.3)$$

where the argument of g on the left-hand side consists of $l+1$ terms and that on the right-hand side of l terms. Setting $\beta_k := a_k - a_{l-1} - 1$ for any $k = 0, 1, \dots, l-2$, and applying again Eq.(2.10) to the first term in the above equation (3.3), we obtain

$$g\left(\sum_{k=0}^{l-2} 2^{\beta_k} + 1\right) = g\left(\sum_{k=0}^{l-3} 2^{\beta_k - \beta_{l-2} - 1} + 1\right) + (1 - 2^{-\beta_{l-2}}) \sum_{k=2}^l 2^{\beta_{l-k}}$$

or, in terms of a_k , noting that for $k = 0, 1, \dots, l-3$ we have

$$\beta_k - \beta_{l-2} - 1 = a_k - a_{l-1} - 1 - (a_{l-2} - a_{l-1} - 1) - 1 = a_k - a_{l-2} - 1$$

$$\begin{aligned} g\left(\sum_{k=0}^{l-2} 2^{a_k - a_{l-1} - 1} + 1\right) &= g\left(\sum_{k=0}^{l-3} 2^{a_k - a_{l-2} - 1} + 1\right) \\ &\quad + (1 - 2^{-(a_{l-2} - a_{l-1} - 1)}) \sum_{k=2}^l 2^{a_{l-k} - a_{l-1} - 1} \end{aligned}$$

or, substituting this equation into (3.3)

$$\begin{aligned} g\left(\sum_{k=0}^{l-1} 2^{a_k} + 1\right) &= g\left(\sum_{k=0}^{l-3} 2^{a_k - a_{l-2} - 1} + 1\right) + (1 - 2^{-a_{l-1}}) \sum_{k=1}^l 2^{a_{l-k}} \\ &\quad + (1 - 2^{-(a_{l-2} - a_{l-1} - 1)}) \sum_{k=2}^l 2^{a_{l-k} - a_{l-1} - 1} \end{aligned}$$

Note that the argument of g on the right-hand side has now $l-1$ terms. Repeating this procedure $l-1$ times we finally obtain

$$\begin{aligned} g(2^{a_0} + \dots + 2^{a_{l-1}} + 1) &= g(2^{a_0 - a_1 - 1} + 1) + (1 - 2^{-a_{l-1}}) \sum_{k=1}^l 2^{a_{l-k}} \\ &\quad + (1 - 2^{-(a_{l-2} - a_{l-1} - 1)}) \sum_{k=2}^l 2^{a_{l-k} - a_{l-1} - 1} \\ &\quad + \dots + (1 - 2^{-(a_1 - a_2 - 1)}) \sum_{k=l-1}^l 2^{a_{l-k} - a_2 - 1} \end{aligned}$$

Hence, by Eq.(2.3) and since $g(2^{a_0-a_1-1} + 1) = 2^{a_0-a_1-1} - 2 = (1 - 2^{-(a_0-a_1-1)}) \cdot 2^{a_0-a_1-1} - 1 = (1 - 2^{-(a_0-a_1-1)}) \sum_{k=l}^l 2^{a_{l-k}-a_1-1} - 1$ it follows

$$\begin{aligned} g(2^{a_0} + \dots + 2^{a_{l-1}} + 1) &= -1 + (1 - 2^{-a_{l-1}}) \sum_{k=1}^l 2^{a_{l-k}} \\ &\quad + (1 - 2^{-(a_{l-2}-a_{l-1}-1)}) \sum_{k=2}^l 2^{a_{l-k}-a_{l-1}-1} \\ &\quad + \dots + (1 - 2^{-(a_1-a_2-1)}) \sum_{k=l-1}^l 2^{a_{l-k}-a_2-1} \\ &\quad + (1 - 2^{-(a_0-a_1-1)}) \sum_{k=l}^l 2^{a_{l-k}-a_1-1} \end{aligned}$$

A further simplification of the right-hand side of this equation leads to

$$\begin{aligned} g\left(\sum_{k=0}^{l-1} 2^{a_k} + 1\right) &= -1 + \sum_{k=1}^l 2^{a_{l-k}} - 1 - \sum_{k=2}^l 2^{a_{l-k}-a_{l-1}} \\ &\quad + \sum_{k=2}^l 2^{a_{l-k}-a_{l-1}-1} - 1 - \sum_{k=3}^l 2^{a_{l-k}-a_{l-2}} \\ &\quad + \dots + \sum_{k=l-1}^l 2^{a_{l-k}-a_2-1} - 1 - 2^{a_0-a_1} + 2^{a_0-a_1-1} - 1 \\ &= \sum_{k=1}^l 2^{a_{l-k}} - \sum_{k=2}^l 2^{a_{l-k}-a_{l-1}-1} - \sum_{k=3}^l 2^{a_{l-k}-a_{l-2}} \\ &\quad - \dots - \sum_{k=l}^l 2^{a_{l-k}-a_1-1} - \sum_{k=1}^l 1 - 1 \end{aligned}$$

and finally

$$\begin{aligned}
 g\left(\sum_{k=0}^{l-1} 2^{a_k} + 1\right) &= 2^{a_{l-1}} + 2^{a_{l-2}}(1 - 2^{-a_{l-1}-1}) + 2^{a_{l-3}}\left(1 - \sum_{j=1}^2 2^{-a_{l-j}-1}\right) \\
 &\quad + \cdots + 2^{a_0}\left(1 - \sum_{j=1}^{l-1} 2^{-a_{l-j}-1}\right) - (l+1) \\
 &= \sum_{k=1}^l \left\{2^{a_{l-k}}\left(1 - \sum_{j=1}^{k-1} 2^{-a_{l-j}-1}\right) - 1\right\} - 1
 \end{aligned}$$

and this is Eq.(3.2), since $m - 1 = \sum_{k=1}^l 2^{a_{l-k}}$ and $\sum_{k=1}^l 1 = l$. $\square$

Remark 3.2 Note that $s_2(m) := l + 1$ is the number of nonzero digits of the binary expansion of m and $m - (l + 1)$ is the exponent of the highest power of 2 which divides $m!$ according to a theorem of Legendre [6, pp.10-12] (for the special case $p = 2$).

For example, let $m = 13 = 2^3 + 2^2 + 2^0$, that is $l = 2, a_0 = 3, a_1 = 2, a_2 = 0$, then by (3.2) we have $g(13) = 13 - 1 - (2 + 1) - 2^{a_0-a_1-1} = 9 - 2^{3-2-1} = 8$.

A consequence of Theorem 3.1 is the

Proposition 3.3 For all $m \in \mathbb{N}$ we have

$$-1 \leq g(m) \leq m - 1 \quad (3.4)$$

and, therefore, for all $m \in \mathbb{N}_0$

$$0 \leq d(m) \leq m + 1 \quad (3.5)$$

Proof. Let $m = 2^{a_0} + \cdots + 2^{a_l}$, where $a_0 > a_1 > \cdots > a_l \geq 0, l \geq 0$, is a strictly decreasing integer sequence, then $l + 1$ and the double sum in (3.1) are always ≥ 0 and therefore $g(m) \leq 2^{-a_l}(m - 2^{a_l}) = 2^{-a_l}m - 1 \leq m - 1$, since $2^{a_l} \geq 1$. This proves the upper bound.

In order to prove that -1 is a lower bound we note first that by Eq.(2.1) this value is attained for $m = 2^n, n \geq 0$, and secondly by Eq.(2.9) it is sufficient to consider only odd numbers m , that is $a_l = 0$. As shown in the proof of Theorem 3.1 the formula (3.2) can be written as

$$g(m) = \sum_{k=1}^l \left\{2^{a_{l-k}}\left(1 - \sum_{j=1}^{k-1} 2^{-a_{l-j}-1}\right) - 1\right\} - 1,$$

therefore we have to prove that

$$2^{a_{l-k}} \left(1 - \sum_{j=1}^{k-1} 2^{-a_{l-j}-1} \right) - 1 = 2^{a_{l-k}} - 1 - \sum_{j=1}^{k-1} 2^{a_{l-k}-a_{l-j}-1} \geq 0.$$

This can be proved as follows: By definition, $a_{l-j} \geq j$ for any $j = 0, 1, \dots, l$ or $-a_{l-j} \leq -j$ and, therefore, $\sum_{j=1}^{k-1} 2^{-a_{l-j}} \leq \sum_{j=1}^{k-1} 2^{-j} = 1 - 2^{-(k-1)}$. It follows $1 - \frac{1}{2} \sum_{j=1}^{k-1} 2^{-a_{l-j}} \geq 1 - \frac{1}{2} \cdot \left(1 - \frac{1}{2^{k-1}} \right) = \frac{1}{2} + \frac{1}{2^k} \geq \frac{1}{2^k}$ and, multiplying both sides by $2^{a_{l-k}}$ and adding to both sides -1 , we obtain $2^{a_{l-k}} \left(1 - \sum_{j=1}^{k-1} 2^{-a_{l-j}-1} \right) - 1 \geq 2^{a_{l-k}-k} - 1 \geq 0$, since $a_{l-k} - k \geq 0$ for any $k = 1, \dots, l$. This proves Eq.(3.4). Finally, the double inequality (3.5) follows immediately from the definition of the sequence $(d(m))_{m \in \mathbb{N}_0}$. $\square$

By means of $h(m) := g(m+1)$, $m \in \mathbb{N}_0$, $h(0) = g(1) = -1$, we can give another representation of the solution (3.1) which will lead to an improvement of Proposition 3.3.

Proposition 3.4 *The sequence $(h(m))_{m \in \mathbb{N}_0}$ satisfies for all $m \geq 1$ the recurrence relation*

$$\left. \begin{aligned} h(0) &= -1 \quad (h(1) = -1) \\ h(2m) &= h(m) + m \\ h(2m+1) &= h(m) \end{aligned} \right\} \quad (3.6)$$

Proof. By (1.1) it follows $h(0) = g(1) = -1$ and $h(1) = g(2) = -1$. Further

$$\begin{aligned} h(2m) &= g(2m+1) = g(m+1) + m = h(m) + m \\ h(2m+1) &= g(2m+2) = g(m+1) = h(m) \end{aligned}$$

and this proves the proposition. $\square$

Note that (3.6) is a special case of (1.2) with $\alpha = 1, \gamma = 1, \delta = 0, \zeta = -1$, $g(m) = m$ and $h(m) = 0$.

Using the binary expansion of $m = (b_n \dots b_0)_2$, where $b_0, \dots, b_n \in \{0, 1\}$, we can derive an alternative solution of (1.1).

Theorem 3.5 *Let $m = (b_n \dots b_0)_2 \in \mathbb{N}_0$, then the sequence $(h(m))_{m \in \mathbb{N}_0}$ satisfies the recurrence relation*

$$h((b_n \dots b_0)_2) = h((b_n \dots b_1)_2) + (1 - b_0) \cdot (b_n \dots b_1)_2 \quad (3.7)$$

with the solution

$$g((b_n \dots b_0)_2 + 1) = h((b_n \dots b_0)_2) = -1 + \sum_{k=0}^{n-1} (1 - b_k)(b_n \dots b_{k+1})_2 \quad (3.8)$$

and, therefore, for all $m \in \mathbb{N}_0$

$$d((b_n \dots b_0)_2) = g((b_n \dots b_0)_2 + 1) + 1 = \sum_{k=0}^{n-1} (1 - b_k)(b_n \dots b_{k+1})_2 \quad (3.9)$$

Proof. Let $m = (b_n \dots b_0)_2 \in \mathbb{N}_0$, then

$$\begin{aligned} h((b_n \dots b_0)_2) &= b_0 h((b_n \dots b_1)_2) + (1 - b_0) \left(h((b_n \dots b_1)_2) + (b_n \dots b_1)_2 \right) \\ &= h((b_n \dots b_1)_2) + (1 - b_0) \cdot (b_n \dots b_1)_2 \end{aligned}$$

since for $b_0 = 0$, that is m is even, we have $m/2 = (b_n \dots b_1)_2$ and by (3.6) $h(m) = h(m/2) + m/2 = h((b_n \dots b_1)_2) + (b_n \dots b_1)_2$ and for $b_0 = 1$, that is m is odd, we have $(m-1)/2 = (b_n \dots b_1)_2$ and by (3.6) $h(m) = h((m-1)/2) + m/2 = h((b_n \dots b_1)_2)$. This proves Eq.(3.7).

Repeating this procedure $n-1$ times we finally obtain $h((b_n \dots b_0)_2) = h(b_n) + \sum_{k=0}^{n-1} (1 - b_k)(b_n \dots b_{k+1})_2$ and this proves Eq.(3.8), since $b_n = 1$ and $h(1) = -1$. Finally, by definition it follows Eq.(3.9). $\square$

For example, let $m = 10 = (1010)_2$, that is $n = 3$, $b_0 = 0$, $b_1 = 1$, $b_2 = 0$ and $b_3 = 1$. Then

$$\begin{aligned} g(11) &= -1 + \sum_{k=0}^2 (1 - b_k)(b_3 \dots b_{k+1})_2 \\ &= -1 + (1 - b_0)(b_3 b_2 b_1)_2 + (1 - b_1)(b_3 b_2)_2 + (1 - b_2)(b_3)_2 \\ &= -1 + (101)_2 + 0 \cdot (10)_2 + 1 = -1 + 5 + 0 + 1 = 5. \end{aligned}$$

From Eq.(3.8) one can deduce that h attains its greatest value at $m = 2^n = (10 \dots 0)_2$. In this case we have $b_k = 0$ for all $k = 0, 1, \dots, n-1$. Hence $1 - b_k = 1$ for all $k = 0, 1, \dots, n-1$. Therefore $g(2^n + 1) = h(2^n) = -1 + \sum_{k=0}^{n-1} (b_n \dots b_{k+1})_2 = -1 + \sum_{k=0}^{n-1} 2^{n-1-k} = -1 + 2^n - 1 = 2^n - 2$ (see also Eq.(2.3)). Hence, $2^n + 1 - (2^n - 2) = 3$, that is for all $m \geq 2$ we have $m - g(m) \geq 3$.

Similarly, h attains its smallest value at $m = 2^n - 1 = (11 \dots 1)_2$. In this case we

have $b_k = 1$ for all $k = 0, 1, \dots, n-1$ and hence $1 - b_k = 0$ for all $k = 0, 1, \dots, n-1$. Therefore $g((1 \dots 1)_2 + 1) = g(2^n) = -1$ (see also Eq.(2.1)). Hence, $g(m) \geq -1$ for all $m \geq 1$.

Thus, we have shown the desired improvement of Proposition 3.3.

Proposition 3.6 *For all $m \in \mathbb{N}_2$ we have*

$$-1 \leq g(m) \leq m - 3, \quad (3.10)$$

where both bounds are sharp, since for $m = 2^n, n \geq 0$, it is $g(m) = -1$ and for $m = 2^n + 1, n \geq 0$, it is $g(m) = m - 3$.

Therefore, for all $m \in \mathbb{N}$ we have

$$0 \leq d(m) \leq m - 1 \quad (3.11)$$

4 Some Consequences

In [1, Theorem 25] Bodlaender et al. showed that the edge ranking number of the complete graphs on p vertices, $p \in \mathbb{N}$, is given by $\chi'_r(K_p) = a(p)$, where $a(p) := \frac{1}{3}(p^2 + g(p))$, and in [7, Theorem 7, Corollary 8] Lin, Juan and Wang showed that the edge ranking number of the Sierpiński graphs is given by $\chi'_r(S_p^n) = n\chi'_r(K_p) = \frac{n}{3}(p^2 + g(p))$, $n, p \in \mathbb{N}_2$. (For a definition of the Sierpiński graphs we refer the reader to the seminal paper of Klavžar and Milutinović [5] and to the survey paper on Sierpiński-type graphs by Hinz, Klavžar and Zemljčič [4].)

Proposition 4.1 *The sequence $a(m) := \frac{1}{3}(m^2 + g(m))$, $m \in \mathbb{N}$, satisfies for all $m \in \mathbb{N}$ the recurrence relation*

$$\left. \begin{aligned} a(1) &= 0 \\ a(2m) &= a(m) + m^2 \\ a(2m+1) &= a(m+1) + m(m+1) \end{aligned} \right\} \quad (4.1)$$

In particular,

$$3 \mid (m^2 + g(m)). \quad (4.2)$$

Note that (4.1) is a special case of (1.2) with $\alpha = 1, \gamma = 0, \delta = 1, \zeta = 0$, $g(m) = m^2$ and $h(m) = m(m+1)$.

Proof. By definition $a(1) = \frac{1}{3}(1^2 + g(1)) = \frac{1}{3}(1 - 1) = 0$ and by (1.1) it follows that

$$\begin{aligned} a(2m) &= \frac{4m^2 + g(2m)}{3} = \frac{m^2 + g(m) + 3m^2}{3} = a(m) + m^2 \\ a(2m+1) &= \frac{(2m+1)^2 + g(2m+1)}{3} = \frac{4m^2 + 4m + 1 + g(m+1) + m}{3} \\ &= \frac{(m+1)^2 + g(m+1) + 3m^2 + 3m}{3} = a(m+1) + m(m+1) \end{aligned}$$

This proves (4.1) and shows that $a(m) \in \mathbb{N}_0$, thus proving (4.2). $\square$

Note that for $m-1$ instead of m in Eq.(4.1) we obtain $a(2(m-1)+1) = a(2m-1) = a(m) + (m-1)m$ and therefore $a(2m) - a(2m-1) = m^2 - m(m-1) = m$ for all $m \in \mathbb{N}$.

The first few values of $(a(m))_{m \in \mathbb{N}}$ (not available in the OEIS [8]) are

$$(0, 1, 3, 5, 9, 12, 17, 21, 29, 34, 42, 48, 59, 66, 77, 85, 101, 110, 124, 134, \dots).$$

The next lemma gives the values of this sequence for some special numbers.

Lemma 4.2 *Let $n \in \mathbb{N}_0$, then*

$$a(2^n) = \frac{4^n - 1}{3} \tag{4.3}$$

$$a(2^n - 1) = \frac{4^n - 1}{3} - 2^{n-1}, \quad n \geq 1 \tag{4.4}$$

$$a(2^n + 1) = \frac{4^n - 1}{3} + 2^n \tag{4.5}$$

Proof. By definition and using Eqs.(2.1), (2.2) and (2.3), we have $a(2^n) = \frac{1}{3}(4^n + g(2^n)) = \frac{1}{3}(4^n - 1)$, $a(2^n - 1) = \frac{1}{3}((2^n - 1)^2 + g(2^n - 1)) = \frac{1}{3}(4^n - 1) - 2^{n-1} = a(2^n) - 2^{n-1}$ and $a(2^n + 1) = \frac{1}{3}((2^n + 1)^2 + g(2^n + 1)) = \frac{1}{3}(4^n - 1) + 2^n = a(2^n) + 2^n$. $\square$

We notice that (4.3) is the sequence [A002450](#), (4.5) is the sequence [A079319](#), whereas the sequence (4.4) is not available in the OEIS [8].

Note that for $m = 2^n$, $n \geq 0$, we have $a(m) = \frac{1}{3}(m^2 - 1)$ and for $m = 2^n + 1$, $n \geq 0$, we have $a(m) = \frac{1}{3}(m^2 + m - 3)$. By Proposition 3.6 we obtain in this way

Corollary 4.3 *For all $m \in \mathbb{N}_2$ we have*

$$\frac{m^2 - 1}{3} \leq a(m) \leq \frac{m^2 + m - 3}{3}, \tag{4.6}$$

where both bounds are sharp.

To conclude we mention some properties of the sequence of the partial sums of $g(m)$.

Proposition 4.4 *Let $t(m) := \sum_{k=1}^m g(k)$, $m \in \mathbb{N}$, then*

$$\left. \begin{aligned} t(1) &= -1 \\ t(2m) &= 2t(m) + \binom{m}{2} \\ t(2m+1) &= t(m) + t(m+1) + \binom{m+1}{2} \end{aligned} \right\} \quad (4.7)$$

Note that (4.7) is a special case of (1.2) with $\alpha = 2, \gamma = 1, \delta = 1, \zeta = -1, g(m) = \binom{n}{2}$ and $h(m) = \binom{m+1}{2}$.

Proof. It is $t(1) = g(1) = -1$ and by definition we have

$$\begin{aligned} t(2m) &= \sum_{k=1}^{2m} g(k) = \sum_{k=1}^m g(2k) + \sum_{k=1}^{m-1} g(2k+1) + g(1) \\ &= \sum_{k=1}^m g(k) + \sum_{k=1}^{m-1} (g(k+1) + k) + g(1) \\ &= t(m) + \sum_{k=1}^{m-1} g(k+1) + \sum_{k=1}^{m-1} k + g(1) \\ &= t(m) + t(m) - g(1) + \binom{m}{2} + g(1) = 2t(m) + \binom{m}{2} \end{aligned}$$

Similarly,

$$\begin{aligned} t(2m+1) &= \sum_{k=1}^{2m+1} g(k) = \sum_{k=1}^m g(2k) + \sum_{k=1}^m g(2k+1) + g(1) \\ &= \sum_{k=1}^m g(k) + \sum_{k=1}^m g(k+1) + \sum_{k=1}^m k + g(1) \\ &= t(m) + t(m+1) - g(1) + \binom{m+1}{2} + g(1) \\ &= 2t(m) + \binom{m+1}{2} \end{aligned}$$

and this proves (4.7). $\square$

The first few values of $(t(m))_{m \in \mathbb{N}}$ (not available in the OEIS [8]) are

$$(-1, -2, -2, -3, -1, -1, 1, 0, 6, 8, 13, 13, 21, 23, 29, 28, 42, 48, 59, 61, 76, \dots).$$

The next lemma gives the values of this sequence for some special numbers.

Lemma 4.5 *Let $n \in \mathbb{N}_0$, then*

$$t(2^n) = \frac{4^n - 5 \cdot 2^n - n \cdot 2^n}{4} \quad (4.8)$$

$$t(2^n - 1) = \frac{4^n - 5 \cdot 2^n - n \cdot 2^n}{4} + 1, \quad n \geq 1 \quad (4.9)$$

$$t(2^n + 1) = \frac{4^n - 5 \cdot 2^n - n \cdot 2^n}{4} + 2^n - 2, \quad (4.10)$$

Proof. a) Let $f(n) := t(2^n)$, $n \geq 0$, then by (4.7) the sequence $(f(n))_{n \in \mathbb{N}_0}$ satisfies the recurrence relation

$$f(n+1) = t(2^{n+1}) = t(2 \cdot 2^n) = 2t(2^n) + \binom{2^n}{2} = 2f(n) + 2^{n-1}(2^n - 1), \quad n \geq 0,$$

with $f(0) = t(1) = -1$. The solution of this linear first-order recurrence relation can be obtained again by backward substitution. After n substitutions it follows

$$\begin{aligned} f(n) &= 2f(n-1) + 2^{n-2}(2^{n-1} - 1) \\ &= 2(2f(n-2) + 2^{n-3}(2^{n-2} - 1)) + 2^{n-2}(2^{n-1} - 1) \\ &= 2^2 f(n-2) + 2 \cdot 2^{n-3}(2^{n-2} - 1) + 2^{n-2}(2^{n-1} - 1) \\ &= \dots\dots\dots \\ &= 2^n f(0) + \sum_{k=0}^{n-1} 2^k \cdot 2^{n-2-k} \cdot (2^{n-1-k} - 1) \\ &= -2^n + 2^{n-2}(2^n - 1 - n) = \frac{4^n - 5 \cdot 2^n - n \cdot 2^n}{4} \end{aligned}$$

and this proves (4.8).

b) By definition and using Eqs.(4.8) and (2.1) we get $t(2^n - 1) = \sum_{k=1}^{2^n-1} g(k) = \sum_{k=1}^{2^n} g(k) - g(2^n) = t(2^n) - (-1) = \frac{1}{4}(4^n - 5 \cdot 2^n - n \cdot 2^n) + 1$ and this proves (4.9).

c) By definition and using Eqs.(4.8) and (2.3) we get $t(2^n + 1) = \sum_{k=1}^{2^n+1} g(k) =$

$\sum_{k=1}^{2^n} g(k) + g(2^n + 1) = t(2^n) + 2^n - 2 = \frac{1}{4}(4^n - 5 \cdot 2^n - n \cdot 2^n) + 2^n - 2$ and this proves (4.10). $\square$

Note that for $m = 2^n$, $n \geq 0$, that is $n = \log_2(m)$, we obtain from (4.8) the sequence $\underline{t}(m) := \frac{m}{4}(m - 5 - \log_2(m))$, $m \in \mathbb{N}$. Numerical results suggest that $\underline{t}(m)$ is a lower bound of $t(m)$, that is $\underline{t}(m) \leq t(m)$, $m \in \mathbb{N}$.

An upper bound $\bar{t}(m)$ for $t(m)$ could be obtained from (3.10) as follows: It is $g(k) \leq k - 3$ for all $k \in \mathbb{N}_2$ and, therefore, $t(m) = g(1) + \sum_{k=2}^m g(k) \leq -1 + \sum_{k=2}^m (k - 3) = -1 + \sum_{k=2}^m k - 3 \sum_{k=2}^m 1 = -1 + \binom{m+1}{2} - 1 - 3(m-1) = \frac{1}{2}(m^2 - 5m + 2) =: \bar{t}(m)$. Since $\frac{1}{2}(1^2 - 5 \cdot 1 + 2) = -1 = t(1)$ we obtain

$$\underline{t}(m) \leq t(m) \leq \bar{t}(m), \quad m \in \mathbb{N}.$$

We notice that $\bar{t}(m+4) = \text{A034856}(m)$, $m \in \mathbb{N}$.

Remark 4.6 *All the sequences encountered so far, namely $g(m)$, $d(m) := g(m+1) + 1$, $h(m) := g(m+1)$, $a(m) := \frac{1}{3}(m^2 + g(m))$ and $t(m) := \sum_{k=1}^m g(k)$ are of divide-and-conquer type, that is their ordinary generating functions defined as the power series $G(s) := \sum_{m=1}^{\infty} g(m)s^m$, $D(s) := \sum_{m=0}^{\infty} d(m)s^m$, $H(s) := \sum_{m=0}^{\infty} h(m)s^m$, $A(s) := \sum_{m=1}^{\infty} a(m)s^m$ and $T(s) := \sum_{m=1}^{\infty} t(m)s^m$ satisfy a functional equation known as Mahlerian equation (cf.[2])*

$$a_0(s)F(s) + a_1(s)F(s^2) + \cdots + a_n(s)F(s^{2^n}) = f(s) \quad (4.11)$$

in which $n \in \mathbb{N}$, $f(s)$ is a formal series and $a_0(s), a_1(s), \dots, a_n(s)$ are polynomials not all zero. If $f(s) = 0$, then the solution of (4.11) is said to be a Mahlerian series.

Indeed, applying standard generating function techniques to the recurrence relations from Eqs.(1.1), (1.4), (3.6), (4.1) and (4.7) we have the functional equations

$$\begin{aligned} G(s) &= \left(1 + \frac{1}{s}\right)G(s^2) + \frac{s^3}{(1-s^2)^2} \\ D(s) &= (1+s)D(s^2) + \frac{s^2}{(1-s^2)^2} \\ H(s) &= (1+s)H(s^2) + \frac{s^2}{(1-s^2)^2} \\ A(s) &= \left(1 + \frac{1}{s}\right)A(s^2) + \frac{s^2}{(1+s)(1-s)^3} \\ T(s) &= s\left(1 + \frac{1}{s}\right)^2 T(s^2) + \frac{s^3}{(1-s)(1-s^2)^2} \end{aligned} \quad (4.12)$$

and the relations

$$D(s) = \frac{G(s)}{s} + \frac{1}{1-s}, H(s) = \frac{G(s)}{s}, A(s) = \frac{1}{3} \left(G(s) + \frac{t(1+s)}{(1-s)^3} \right), T(s) = \frac{G(s)}{1-s}.$$

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