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On extensions of the generalized cosine functions from some large sets

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Dedicated to Professor Zsolt Páles on the occasion of his 60th birthday

Abstract. Let $(G,+)$ be a commutative semigroup, τ be an endomorphism of G and involution, D be a nonempty subset of G, and P be a quadratically closed field with char$P \neq 2$. We show that if the set $D \setminus g^{-1}(\{0\})$ is 'sufficiently large', then each function $g : D \to P$, satisfying the condition: $g(x+y) + g(x+\tau(y)) = 2g(x)g(y)$ for $x, y \in D$ with $x+y, x+\tau(y) \in D$, can be extended to a unique solution $f : G \to P$ of the functional equation $f(x+y) + f(x+\tau(y)) = 2f(x)f(y)$ for $x, y \in G$.

Let P be a field that is quadratically closed (i.e., for each $x \in P$ there is $y \in P$ with $y^2 = x$), char$P \neq 2$, $(G,+)$ be a commutative semigroup, and D be a nonempty subset of G, unless explicitly stated otherwise. Let τ be an endomorphism of G and involution (cf. [4]), i.e., $\tau(x+y) = \tau x + \tau y$ and $\tau(\tau x) = x$ for $x, y \in G$, where $\tau x := \tau(x)$ for $x \in G$.

We say that a function $f : D \to P$ satisfies the functional equation

$$f_0(x+y) + f_0(x+\tau y) = 2f_0(x)f_0(y) \tag{1}$$

on the set D provided

$$f(x+y) + f(x+\tau y) = 2f(x)f(y), \quad x, y \in D, x+y, x+\tau y \in D. \tag{2}$$

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Note that if G is a group, then (1) is a natural generalization of the well-known d'Alembert (cosine) equation

$$f(x+y)+f(x-y)=2f(x)f(y), \quad x,y \in G, \tag{3}$$

with $\tau x \equiv -x$. Therefore, every solution to (1) can be called a generalized cosine function. For some information and references on the d'Alembert equation (3) and recent examples of results concerning its solutions, see [3], [9], [12], [14], [15], [18], [21], [22], [23], [24], [25], [26], [27], [28], [29].

Solutions of (1) have been determined in [22] in the case where G is a commutative group (under some additional assumptions). Later, it has been proved in [20] that a similar result is valid also in the case where G is 'only' a commutative semigroup.

In this paper we study possibilities of extensions of functions satisfying equation (1) on D to a solution $g_0 : G \to P$ of (1). It is obvious that such an extension does not need to exist if D is 'meagre' (i.e., not large enough). Therefore, we have to find an assumption on D that makes it 'sufficiently large' to guarantee the above-mentioned property of extension.

Clearly, it is natural to assume that a set $D \subset G$ is large provided it belongs to some filter of subsets of G that is proper (i.e., different from 2^G). So, we assume that $D \in \mathcal{L}$, where $\mathcal{L}$ is a family of subsets of G such that the following three conditions are valid:

$$\mathcal{L} \neq 2^G, \tag{4}$$

$$B \in \mathcal{L}, \quad B \in 2^G, 2^B \cap \mathcal{L} \neq \emptyset, \tag{5}$$

$$\tau(B), B-x, B+x, A\cap B \in \mathcal{L}, \quad A,B \in \mathcal{L}, x \in G, \tag{6}$$

where

$$T+a := \{a+x : x \in T\}, \quad T-a := \{x \in G : x+a \in T\}$$

for $a \in G$ and $T \in 2^G$. Note that (4) and (5) imply that $\emptyset \notin \mathcal{L}$.

We show that then, for each function $g : D \to P$ satisfying (2), there exists a solution $g_0 : G \to P$ of (1) with $g(x) = g_0(x)$ for $x \in D$. This outcome corresponds in particular to [8, Theorems 1–4], [2, Theorem 1] and to some analogous results obtained for the equation of homomorphism in [1], [5], [6], [13], [16], [17] (see also [10, Theorem 4.1] or [19, Theorem 1.1, Ch. XVIII, p. 468]).

Remark 1. Let $\mathcal{I} \subset 2^G$ be an ideal (i.e., $2^B \subset \mathcal{I}$ and $B \cup C \in \mathcal{I}$ for every $B, C \in \mathcal{I}$) and

$$\mathcal{L} := \{A \subset G : G \setminus A \in \mathcal{I}\}.$$

Then it is easily seen that $\mathcal{L}$ is a filter, i.e., (5) holds and $A \cap B \in \mathcal{L}$ for every $A, B \in \mathcal{L}$. Moreover, if $\mathcal{I}$ has some additional suitable properties, then also conditions (4) and (6) are valid. Below we provide natural examples of such ideals $\mathcal{I} \subset 2^G$ having those suitable properties guaranteeing (4) and (6).

(a) G is cancellative and not of finite cardinality and $\mathcal{I} = \{A \subset G : \operatorname{card} A < \operatorname{card} G\}$.

(b) d is an invariant metric in G (i.e., $d(x+y, z+y) = d(x,z)$ for $x, y, z \in G$), $\sup_{x,y \in G} d(x,y) = \infty$, the set $\tau(B)$ is bounded (i.e., $\sup_{x,y \in \tau(B)} d(x,y) < \infty$) for each bounded set $B \in 2^G$, and $\mathcal{I}$ is the family of all bounded subsets of G.

(c) $G = \{z \in \mathbb{C} : \Re z > 0\}$ (with the usual addition of complex numbers), $\mathcal{I}$ is the family of all subsets A of G with $\sup_{z \in A} \Re z < \infty$ and $\tau z = \overline{z}$ for $z \in G$, where $\overline{z}$ is the complex conjugate and $\Re z$ denotes the real part of the complex number z.

(d) G is a topological group of the second category of Baire, $\mathcal{I}$ is the family of all first category subsets of G and τ is continuous (which actually means that τ is a homeomorphism, because $\tau^{-1} = \tau$).

(e) G is a locally compact topological group, μ is the Haar measure in G with $\mu(G) = \infty$, $\mathcal{I} = \{A \subset G : \mu(A) < \infty\}$ and τ is continuous.

(f) G is an abelian Polish group, $\mathcal{I}$ is the σ-ideal of Haar zero subsets of G (see [7]) and τ is continuous.

(g) G is an abelian Polish group, $\mathcal{I}$ is the σ-ideal of Christensen zero subsets of G (see [11]) and τ is continuous.

In this paper we also use the following notions:

$$T + S := \{x + y : x \in T, y \in S\}, \quad T - S := \{z \in G : (S + z) \cap T \neq \emptyset\},$$

for $S, T \subset G$. Clearly, if G is a group, then

$$T - S = \{x - y : x \in T, y \in S\}.$$

We start with some auxiliary lemmas. The first one is a very simple observation.

Lemma 1. *Assume that $S \in \mathcal{L}$. Then*

$$S - S = G. \tag{7}$$

Proof. Take $y \in G$. By (6),

$$S \cap (y + S) \neq \emptyset.$$

Hence there are $u, v \in S$ such that $u = y + v$, which means that $y \in S - S$. In this way we have shown that $G \subset S - S$, which completes the proof of (7). □

Lemma 2. *Assume that $(H, +)$ is an abelian group, $S \in \mathcal{L}$ and $h_0 : S \to H$ satisfies*

$$h_0(x + y) = h_0(x) + h_0(y), \quad x, y \in S, x + y, x + \tau y \in S. \tag{8}$$

Then there exists a unique solution $h : G \to H$ of the equation

$$h(x + y) = h(x) + h(y), \quad x, y \in G \tag{9}$$

such that $h(x) = h_0(x)$ for $x \in S$.

Proof. Take $a, b, c, d \in S$ with $a + d = b + c$ and write

$$\begin{aligned} S_1 &:= (S - a) \cap (S - \tau a) \cap (S - (a + d)) \cap (S - (a + \tau d)), \\ S_2 &:= (S - \tau b) \cap (S - b) \cap (S - (b + c))) \cap (S - (b + \tau c)). \end{aligned}$$

Clearly, by (6), $S_1, S_2 \in \mathcal{L}$, whence $S_0 := S \cap S_1 \cap S_2 \neq \emptyset$.

Let $v \in S_0$. Then

$$v, v + a, v + \tau a, v + a + d, v + a + \tau d \in S,$$

$$v + b, v + \tau b, v + b + c, v + b + \tau c \in S.$$

Consequently, by (8),

$$\begin{aligned} h_0(v) + h_0(b) + h_0(c) &= h_0(v + b) + h_0(c) = h_0(v + b + c) \\ &= h_0(v + a + d) = h_0(v + a) + h_0(d) \\ &= h_0(v) + h_0(a) + h_0(d). \end{aligned}$$

Thus we have proved that

$$h_0(a) - h_0(b) = h_0(c) - h_0(d), \quad a, b, c, d \in S, a + d = c + b. \tag{10}$$

Note that if $z \in G$ and $z + b = a$, $z + d = c$ for some $a, b, c, d \in S$, then

$$a + d = z + b + d = b + z + d = b + c.$$

Therefore, in view of (7) and (10), we may define $h : G \to H$ by

$$h(z) := h_0(a) - h_0(b)$$

for every $z \in G$ and $a, b \in S$ such that $z + b = a$.

First, we show that $h(z) = h_0(z)$ for $z \in S$. To this end, take $z \in S$ and $u \in S \cap (S - \tau z) \cap (S - z)$. Then $u + z, u + \tau z \in S$ and, according to the definition of h and (8),

$$h(z) = h_0(z + u) - h_0(u) = h_0(z) + h_0(u) - h_0(u) = h_0(z).$$

Next, we prove that (9) holds. Let $z, w \in G$. According to (7), there exist $a, b, c, d \in S$ with $z + b = a$ and $w + d = c$ and, in view of the definition of h, $h(z) = h_0(a) - h_0(b)$ and $h(w) = h_0(c) - h_0(d)$. Write

$$S_a := (S - a) \cap (S - \tau a) \cap (S - (a + c)) \cap (S - (a + \tau c)),$$

$$S_b := (S - b) \cap (S - \tau b) \cap (S - (b + d)) \cap (S - (b + \tau d)).$$

Further, there is $u \in S \cap S_a \cap S_b \in \mathcal{L}$. Hence

$$\begin{aligned}
h(z + w) &= h_0(u + a + c) - h_0(u + b + d) \\
&= h_0(u + a) + h_0(c) - (h_0(u + b) + h_0(d)) \\
&= h_0(u) + h_0(a) + h_0(c) - (h_0(u) + h_0(b) + h_0(d)) \\
&= h_0(a) - h_0(b) + h_0(c) - h_0(d) = h(z) + h(w).
\end{aligned}$$

It remains to show the uniqueness of h. So, let $h_1 : G \to H$ be such that $h_1(x) = h_0(x)$ for $x \in S$ and

$$h_1(x + y) = h_1(x) + h_1(y), \quad x, y \in G.$$

Take $z \in G$ and $a, b \in S$ with $z + b = a$. Then

$$h_1(z) = h_1(a) - h_1(b) = h_0(a) - h_0(b) = h(z). \qquad \square$$

It is easily seen that Lemma 2 implies the following corollary (cf. [1], [5], [6], [13], [16], [17]; see also [10, Theorem 4.1] or [19, Theorem 1.1, Ch. XVIII, p. 468]).

Corollary 1. *Assume that $(H,+)$ is an abelian group, $S\in\mathcal{L}$ and $h_0:S\to H$ satisfies*

$$h_0(x+y)=h_0(x)+h_0(y),\quad x,y\in S, x+y\in S.$$

Then there is a unique solution $h:G\to H$ of (9) such that $h(x)=h_0(x)$ for $x\in S$.

Lemma 3. *Let $D\in\mathcal{L}$ and $g:D\to P$ fulfil (2). Then there is $\widehat{D}\in\mathcal{L}$ with $\widehat{D}\subset D$, $\tau(\widehat{D})=\widehat{D}$ and*

$$g(\tau x)=g(x),\quad x\in\widehat{D}. \tag{11}$$

Proof. If $g(x)=0$ for each $x\in D$, then it is enough to take $\widehat{D}:=D\cap\tau(D)$. So, assume now that there is $y\in D$ with $g(y)\neq 0$. Write

$$D_y:=(D-y)\cap D\in\mathcal{L},\quad \widehat{D}:=D_y\cap\tau(D_y).$$

Clearly, $\tau(\widehat{D})=\widehat{D}$ and $\widehat{D}\in\mathcal{L}$. Take $w\in\widehat{D}$. Then $\tau w\in\widehat{D}$. Moreover, $w+y,\tau w+y\in D$, and consequently,

$$\begin{aligned}2g(y)g(w)&=g(y+w)+g(y+\tau w)\\&=g(y+\tau w)+g(y+\tau(\tau w))=2g(y)g(\tau w),\end{aligned}$$

which yields $g(\tau w)=g(w)$. □

Lemma 4. *Let $D\in\mathcal{L}$, $g:D\to P$ fulfil (2) and $D_g:=g^{-1}(\{0\})\notin\mathcal{L}$. Then there exist $D_1\in\mathcal{L}$ and a function $m:G\to P$ such that $D_1\subset D$, $D_1=\tau(D_1)$,*

$$m(x+y)=m(x)m(y),\quad x,y\in G, \tag{12}$$

$$g(x)=\frac{m(x)+m(\tau x)}{2},\quad x\in D_1. \tag{13}$$

Proof. According to Lemma 3, there is $\widehat{D}\in\mathcal{L}$ such that (11) holds, $\widehat{D}\subset D$ and $\tau(\widehat{D})=\widehat{D}$.

First, consider the case where

$$g(x+y)=g(x+\tau y),\quad x,y\in\widehat{D}, x+y,x+\tau y\in\widehat{D}. \tag{14}$$

Then, by (2),

$$g(x+y)=g(x)g(y),\quad x,y\in\widehat{D}, x+y,x+\tau y\in\widehat{D}.$$

We show that $0 \notin g(\widehat{D})$. For the proof by contradiction suppose that there is $y \in \widehat{D}$ with $g(y) = 0$. Let

$$D_y := \widehat{D} \cap (\widehat{D} - y) \cap (\widehat{D} - \tau y) \in \mathcal{L}, \quad D_0 := (D_y + y) \cap \widehat{D} \in \mathcal{L}.$$

Take $z \in D_0$. Then $z = x + y$ with some $x \in D_y$ and $x + y, x + \tau y \in \widehat{D}$. Hence

$$g(z) = g(x + y) = g(x)g(y) = 0.$$

Thus we have shown that $g(D_0) = \{0\}$, which is a contradiction, because $D_g \notin \mathcal{L}$.

So, $0 \notin g(\widehat{D})$. Consequently, in view of Lemma 2, there is $m : G \to P$ such that $g(x) = m(x)$ for $x \in \widehat{D}$ and (12) holds. It is easily seen that, by (11),

$$g(x) = \frac{1}{2}(g(x) + g(\tau x)) = \frac{1}{2}(m(x) + m(\tau x)), \quad x \in \widehat{D},$$

which means that (13) holds with $D_1 = \widehat{D}$.

Now, let us study the case when there exist $x_0, y_0 \in \widehat{D}$ such that

$$x_0 + y_0, x_0 + \tau y_0 \in \widehat{D}$$

and $g(x_0 + y_0) \neq g(x_0 + \tau y_0)$. Write $D_2 := \widehat{D} \cap (\widehat{D} - y_0) \cap (\widehat{D} - \tau y_0)$ and

$$f(x) := g(x + y_0) - g(x + \tau y_0), \quad x \in D_2.$$

It is easily seen that $\tau(D_2) = D_2$ and $x_0 \in D_2$, whence

$$f(x_0) \neq 0. \tag{15}$$

Since $D_2 \subset \widehat{D}$, (11) implies

$$\begin{aligned} f(\tau x) &= g(\tau x + y_0) - g(\tau x + \tau y_0) = g(x + \tau y_0) - g(x + y_0) \\ &= -(g(x + y_0) - g(x + \tau y_0)) = -f(x), \quad x \in D_2. \end{aligned} \tag{16}$$

Take $x, y \in D_2$ with $x + y, x + \tau y \in D_2$. Then

$$x + y + y_0, x + y + \tau y_0, x + \tau y + y_0, x + \tau y + \tau y_0 \in \widehat{D},$$

and consequently,

$$\begin{aligned} f(x + y) &= g(x + y + y_0) - g(x + y + \tau y_0), \\ f(x + \tau y) &= g(x + \tau y + y_0) - g(x + \tau y + \tau y_0). \end{aligned}$$

Adding those two equalities, we get

$$\begin{aligned} f(x+y)+f(x+\tau y) &= g(x+y+y_0)+g(x+\tau y+y_0) \\ &\quad - g(x+y+\tau y_0)-g(x+\tau y+\tau y_0) \\ &= 2g(x+y_0)g(y)-2g(x+\tau y_0)g(y)=2f(x)g(y). \end{aligned} \tag{17}$$

Next, $y+\tau x\in D_2$, and consequently, $y+\tau x+y_0, y+\tau x+\tau y_0\in \widehat{D}$, because $\tau(D_2)=D_2$. So, analogously, we get

$$f(y+x)+f(y+\tau x)=2f(y)g(x). \tag{18}$$

Finally, conditions (16), (17) and (18) imply that

$$f(x+y)=f(x)g(y)+f(y)g(x), \quad x,y\in D_2, x+y, x+\tau y\in D_2. \tag{19}$$

Write

$$D_3 := D_2\cap(D_2-x_0)\cap(D_2-\tau x_0).$$

Clearly,

$$\tau(D_3)=D_3. \tag{20}$$

Let $x,y\in D_3$ and $x+y, x+\tau y\in D_3$. Then

$$y+x_0, y+\tau x_0, x+y+x_0, x+y+\tau x_0, x+\tau y+\tau x_0\in D_2,$$

and consequently, by (19),

$$\begin{aligned} f((x+y)+x_0) &= f(x+y)g(x_0)+f(x_0)g(x+y) \\ &= (f(x)g(y)+f(y)g(x))g(x_0)+f(x_0)g(x+y), \end{aligned}$$

$$\begin{aligned} f(x+(y+x_0)) &= f(x)g(y+x_0)+f(y+x_0)g(x) \\ &= f(x)g(y+x_0)+(f(y)g(x_0)+f(x_0)g(y))g(x), \end{aligned}$$

whence

$$(g(x+y)-g(x)g(y))f(x_0)=(g(y+x_0)-g(y)g(x_0))f(x),$$

which can be rewritten as

$$h(y)f(x)=g(x+y)-g(x)g(y),$$

where (see (15))

$$h(y) := \frac{g(y + x_0) - g(y)g(x_0)}{f(x_0)}.$$

Thus we have proved that

$$g(x + y) = h(y)f(x) + g(x)g(y), \quad x, y \in D_3, x + y, x + \tau y \in D_3. \tag{21}$$

If $f(x) = 0$ for every $x \in D_3$, then (21) yields

$$g(x + y) = g(x)g(y), \quad x, y \in D_3, x + y, x + \tau y \in D_3,$$

and we can argue as in the case of (14) (with $\widehat{D} = D_3$).

It remains to study the case where there is $x_1 \in D_3$ with $f(x_1) \neq 0$. It is easily seen that, by (20) and (21),

$$\begin{aligned} h(y)f(x) = g(x + y) - g(x)g(y) = h(x)f(y) \\ x, y \in D_3, x + y, x + \tau y \in D_3. \end{aligned} \tag{22}$$

Since P is quadratically closed, there is $a \in P$ such that

$$\frac{h(x_1)}{f(x_1)} = a^2,$$

and consequently,

$$h(y) = a^2 f(y), \quad y \in D_3, x_1 + y, x_1 + \tau y \in D_3.$$

This and (20) imply that

$$h(y) = a^2 f(y), \quad y \in D_4,$$

where $D_4 := D_3 \cap (D_3 - x_1) \cap (D_3 - \tau x_1) \in \mathcal{L}$. So, by (22), we can write that

$$g(x + y) = g(x)g(y) + a^2 f(x)f(y), \quad x, y \in D_4, x + y, x + \tau y \in D_4. \tag{23}$$

Clearly, (19) implies that

$$af(x + y) = af(x)g(y) + af(y)g(x), \quad x, y \in D_4, x + y, x + \tau y \in D_4. \tag{24}$$

Hence, adding (23) and (24), we obtain

$$\begin{aligned} g(x + y) + af(x + y) &= g(y)(g(x) + af(x)) + af(y)(g(x) + af(x)) \\ &= (g(x) + af(x))(g(y) + af(y)) \end{aligned} \tag{25}$$

for every $x, y \in D_4$ with $x + y, x + \tau y \in D_4$. Define $h_1, h_2 : D_4 \to P$ by

$$h_1(x) := g(x) + af(x), \quad h_2(x) := g(x) - af(x), \quad x \in D_4.$$

Then, by (11), (16) and the equality $\tau(D_4) = D_4$,

$$h_2(x) = g(x) - af(x) = g(\tau x) + af(\tau x) = h_1(\tau x), \quad x \in D_4,$$

whence

$$g(x) = \frac{h_1(x) + h_2(x)}{2} = \frac{h_2(\tau x) + h_2(x)}{2} = \frac{h_1(x) + h_1(\tau x)}{2}, \quad x \in D_4, \tag{26}$$

and (25) implies that

$$h_1(x + y) = h_1(x)h_1(y), \quad x, y \in D_4, x + y, x + \tau y \in D_4,$$

$$h_2(x + y) = h_1(\tau(x + y)) = h_1(\tau x)h_1(\tau y) = h_2(x)h_2(y),$$
$$x, y \in D_4, x + y, x + \tau y \in D_4.$$

Note that $h_1^{-1}(\{0\}) \notin \mathcal{L}$ or $h_2^{-1}(\{0\}) \notin \mathcal{L}$, because otherwise, by (26), we would get $D_g \in \mathcal{L}$. So, analogously, as in the case of (14), we show that $0 \notin h_1(D_4)$ or $0 \notin h_2(D_4)$. Hence, by Lemma 2, we deduce that there is $m : G \to P$ such that (12) holds and $h_1(x) = m(x)$ for $x \in D_4$ or $h_2(x) = m(x)$ for $x \in D_4$. This completes the proof (in view of (26)). $\square$

Lemma 5. *Let $f, g : D \to P$ be solutions to* (2),

$$S := \{x \in D : g(x) = f(x)\} \in \mathcal{L}$$

and $D_g \notin \mathcal{L}$. Then $g(x) = f(x)$ for $x \in D$.

Proof. Fix $w \in D$. Clearly,

$$A := S \cap (S - w) \cap (S - \tau w) \in \mathcal{L}.$$

Take $s \in A \setminus D_g$. Then $s + w, s + \tau w \in S$, whence

$$\begin{aligned} 2g(s)g(w) &= g(s + w) + g(s + \tau w) \\ &= f(s + w) + f(s + \tau w) = 2f(s)f(w) = 2g(s)f(w). \end{aligned}$$

Consequently, $g(w) = f(w)$. $\square$

Now we are in a position to prove the main result of this paper.

Theorem 1. *Let $D \in \mathcal{L}$, $g : D \to P$ satisfy* (2) *and $D_g \notin \mathcal{L}$. Then there exists a unique solution $f : G \to P$ of equation* (1) *such that $g(x) = f(x)$ for $x \in D$.*

Proof. On account of Lemma 4, there exist a set $D_1 \in \mathcal{L}$ and a function $m : G \to P$ such that $D_1 \subset D$, $D_1 = \tau(D_1)$, and conditions (12) and (13) are valid. Write

$$f(x) := \frac{m(x) + m(\tau x)}{2}, \quad x \in G.$$

It is easy to check that f is a solution to (1) and, by (13), $g(x) = f(x)$ for $x \in D_1$. Hence, by Lemma 5, $g(x) = f(x)$ for $x \in D$, because $D_g \notin \mathcal{L}$.

It remains to show the uniqueness of the function f. To this end, assume that $f_0 : G \to P$ is a solution to equation (1) with $g(x) = f_0(x)$ for $x \in D$. Let $F := \{y \in G : f(x) = 0\}$ and $F_0 := F \cap (G \setminus D)$. Clearly, $D \subset G \setminus F_0$, so $G \setminus F_0 \in \mathcal{L}$. Suppose that $F \in \mathcal{L}$. Then, by (6), $D_g = F \cap (G \setminus F_0) \in \mathcal{L}$, which is a contradiction.

Thus we have proved that $F \notin \mathcal{L}$. Consequently, from Lemma 5 (with $D := G$ and $g := f$) we deduce that $f_0 = f$. □

Remark 2. There arises a natural question if the assumption that $D_g \notin \mathcal{L}$ is really necessary in Theorem 1. In some particular cases this is not the case, e.g., when G is a group and $\tau x \equiv -x$. The following corollary shows this (we write $\frac{1}{2}T := \{y \in G : 2y \in T\}$ for $T \in 2^G$).

Corollary 2. *Let G be a group, $D \in \mathcal{L}$, $g : D \to P$ fulfil* (2) *with $\tau x \equiv -x$ and*

$$\frac{1}{2}T \in \mathcal{L}, \quad T \in \mathcal{L}. \tag{27}$$

Then there exists a unique solution $f : G \to P$ of equation (1) *such that $g(x) = f(x)$ for $x \in D$.*

Proof. First we show that $D_g \notin \mathcal{L}$ or $g(D) = \{0\}$. So, suppose that $D_g \in \mathcal{L}$ and take $v \in D$. Clearly, by (27),

$$D_v := D \cap (D_g - v) \cap \frac{1}{2}(D_g - v) \in \mathcal{L}.$$

Take $w \in D_v$. Then $2w + v, w + v \in D_g$, which means that

$$g(2w + v) = 0 = g(w + v),$$

whence

$$g(v) = g(v + w + w) + g(v + w - w) = 2g(w + v)g(w) = 0.$$

If $D_g \notin \mathcal{L}$, then we use Theorem 1. If $g(D) = \{0\}$, then it is enough to take $f(x) \equiv 0$.

It remains to show that $f(x) \equiv 0$ is the unique solution to (1) such that $f(D) = \{0\}$. So, take a solution $f_1 : G \to P$ of (1) with $f_1(D) = \{0\}$. Fix $y_0 \in G \setminus D$ and

$$z \in D \cap \frac{1}{2}(D - y_0) \in \mathcal{L}.$$

Then $z, y_0 + 2z \in D$, and consequently,

$$f_1(y_0) = f_1(y_0 + 2z) + f_1(y_0 + z - z) = 2f_1(y_0 + z)f_1(z) = 0. \qquad \square$$

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