# On normal approximations to strongly mixing random fields 

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#### Abstract

Let $\xi_{n}$ be a strongly mixing sequence of real random variables such that $\mathbb{E} \xi_{n}=0$. Write $S_{n}=\xi_{1}+\cdots+\xi_{n}$ and consider the normalized sums $Z_{n}=S_{n} / B_{n}$, where $B_{n}^{2}=\mathbb{E} S_{n}^{2}$. Assume that a thrice differentiable function $h: \mathbb{R} \rightarrow \mathbb{R}$ satisfies $\sup _{x \in \mathbb{R}}\left|h^{\prime \prime \prime}(x)\right|<\infty$. We obtain optimal (in a sense) bounds for $\Delta_{n}=\left|\mathbb{E} h\left(Z_{n}\right)-\mathbb{E} h(N)\right|$, where $N$ is a standard normal random variable. Namely, we show that $\Delta_{n}=O\left(n^{-1 / 2}\right)$, provided that the random variables $\xi_{n}$ are bounded by a constant, $B_{n}^{2} \geq c_{0} n$, where $c_{0}$ is a positive constant, and that the strong mixing coefficients $\alpha(r)$ satisfy $\sum_{r=1}^{\infty} r \alpha(r)<\infty$. The results extend to the case of random fields $\left\{\xi_{a}, a \in \mathbb{Z}^{d}\right\}$. To prove the results we apply a new method.


## 1. Introduction

Let $\left\{\xi_{a}, a \in \mathbb{Z}^{d}\right\}$ be a random field of real random variables (r.v.'s) $\xi_{a}$ indexed by points $a \in \mathbb{Z}^{d}$ of the standard lattice $\mathbb{Z}^{d} \subset \mathbb{R}^{d}$ ( $\mathbb{R}$ is the real line). For a subset $U \subset \mathbb{Z}^{d}$, let $\mathcal{F}_{U}$ be the minimal $\sigma$-algebra such that all r.v.'s $\xi_{a}$ with $a \in U$ are measurable.

Introduce the distance

$$
d(U, V)=\inf \{\|a-b\|: a \in U, b \in V\}
$$

between the subsets $U, V \subset \mathbb{Z}^{d}$, where $\|a\|=\max \left\{\left|a_{i}\right|: 1 \leq i \leq d\right\}$, for $a=$ $\left(a_{1}, \ldots, a_{d}\right) \in \mathbb{Z}^{d}$. Write $|U|=\operatorname{card} U$ for the number of elements in the set $U$.

Henceforth we assume that for $U, V \subset \mathbb{Z}^{d}$

$$
\begin{equation*}
\sup \left\{|\mathbb{P}(A B)-\mathbb{P}(A) \mathbb{P}(B)|: A \in \mathcal{F}_{U}, B \in \mathcal{F}_{V}\right\} \leq f(|U|+|V|) \alpha(d(U, V)) \tag{1}
\end{equation*}
$$

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with a nonnegative nondecreasing function $f(n)$ of integer argument $n \geq 2$ and nonnegative function $\alpha(r)$ of integer argument $r \geq 1$ such that $\alpha(r) \rightarrow 0$ as $r \rightarrow \infty$.

If condition (1) is satisfied, we say that the random field $\left\{\xi_{a}, a \in \mathbb{Z}^{d}\right\}$ satisfies the strong mixing condition.

In what follows, $N$ is a standard normal r.v.
Write

$$
S_{V}=\sum_{a \in V} \xi_{a}, \quad B_{V}^{2}=\mathbb{E} S_{V}^{2}, \quad Z_{V}=S_{V} / B_{V}
$$

for a fixed nonempty set $V \subset \mathbb{Z}^{d}$ with $|V|<\infty$ and $B_{V}>0$, and, for the functions $h: \mathbb{R} \rightarrow \mathbb{R}$,

$$
\Delta_{V}=\left|\mathbb{E} h\left(Z_{V}\right)-\mathbb{E} h(N)\right|
$$

Introduce the boundedness condition

$$
\begin{equation*}
\mathbb{P}\left\{\left|\xi_{a}\right| \leq M\right\}=1 \quad \text { for all } a \in V \tag{2}
\end{equation*}
$$

with a non-random constant $M>0$.
Write

$$
\lambda_{d}=\sum_{r=1}^{\infty} r^{2 d-1} \alpha(r) \quad \text { and } \quad\left\|h^{\prime \prime \prime}\right\|_{\infty}=\sup _{x \in \mathbb{R}}\left|h^{\prime \prime \prime}(x)\right|
$$

Here and in what follows, $h^{\prime}, h^{\prime \prime}$, and $h^{\prime \prime \prime}$ denote the first, second, and third derivative of the function $h$, respectively.

Our main result is the following Berry-Esseen (or Lyapunov) type bound for $\Delta_{V}$.

Theorem 1. Assume that a real random field $\left\{\xi_{a}, a \in \mathbb{Z}^{d}\right\}, d \geq 1$, with $\mathbb{E} \xi_{a}=0$ for all $a \in V$, satisfies the strong mixing condition (1) with $\lambda_{d}<\infty$ and boundedness condition (2). Let $h$ be a thrice differentiable function such that $\left\|h^{\prime \prime \prime}\right\|_{\infty}<\infty$. Then

$$
\begin{equation*}
\Delta_{V} \leq C\left\|h^{\prime \prime \prime}\right\|_{\infty} f(|V|) M^{3}|V| B_{V}^{-3} \tag{3}
\end{equation*}
$$

where $C=C\left(d, \lambda_{d}\right)$ is a positive finite factor depending only on the dimension $d$ and $\lambda_{d}$.

Bound (3) is optimal in the sense that for dependent fields (with an infinite range of dependence) it provides convergence rates in the CLT of the same order as in the case of i.i.d. summands with finite third absolute moment, i.e., in the special case $\left(\sup _{n} f(n)<\infty\right.$ in the definition of condition (1) and $B_{V}^{2} \geq c_{0}|V|$,
where $c_{0}$ is a positive constant), the obtained upper bound of $\Delta_{V}$ is of order $O\left(|V|^{-1 / 2}\right)$.

Note that the stationarity of an approached random field is not requested.
In the case $d=1$ one can use in (3) $f \equiv 1$ and $V=\{1, \ldots, n\}$ with $|V|=n$. Note that the strong mixing condition (1) for $d=1$ is not the same as the strong mixing condition introduced by Rosenblatt [26] between "the past" and "the future" (because one can use the order on the real line).

Write $\operatorname{cov}(\xi, \eta)=\mathbb{E} \xi \eta-\mathbb{E} \xi \mathbb{E} \eta$ for real r.v.'s $\xi$ and $\eta$.
Now we present an analogous version of Theorem 1, when the mixing coefficients are defined as follows: for $n \in \mathbb{N}, k, l \in \mathbb{N} \cup\{\infty\}$,

$$
\begin{aligned}
\alpha_{k, l}(n)=\sup \left\{|\mathbb{P}(A B)-\mathbb{P}(A) \mathbb{P}(B)|: A \in \mathcal{F}_{U}, B \in \mathcal{F}_{V}\right. \\
\left.U, V \subset \mathbb{Z}^{d},|U| \leq k,|V| \leq l, d(U, V) \geq n\right\}
\end{aligned}
$$

where $\mathbb{N}=\{1,2, \ldots\}$.
Denote

$$
\Lambda_{d}=\sum_{r=1}^{\infty} r^{2 d-1} \alpha_{2, \infty}(r)
$$

where $\alpha_{k, \infty}(n)=\sup _{l} \alpha_{k, l}(n)$.
The following statement is valid.
Theorem 2. Assume that a real random field $\left\{\xi_{a}, a \in \mathbb{Z}^{d}\right\}, d \geq 1$, with $\mathbb{E} \xi_{a}=0$ for all $a \in V$, satisfies the condition $\Lambda_{d}<\infty$ and boundedness condition (2). Let $h$ be a thrice differentiable function such that $\left\|h^{\prime \prime \prime}\right\|_{\infty}<\infty$. Then

$$
\begin{equation*}
\Delta_{V} \leq C\left\|h^{\prime \prime \prime}\right\|_{\infty} M^{3}|V| B_{V}^{-3} \tag{4}
\end{equation*}
$$

where $C=C\left(d, \Lambda_{d}\right)$ is a positive finite factor depending only on the dimension $d$ and $\Lambda_{d}$.

There is a rich literature to normal approximations of sums of weekly dependent sequences and fields, see papers [8], [14], [21], [24], [25], [28]-[36] and books [7], [13], and [18]. On the central limit theorem for random fields, see papers [3], [9], [19], [20], [22], [23]. About the mixing properties of random fields, see the books [11] and [13]. Equivalent mixing conditions for stationary random fields have been discussed in [4]. Let us discuss here only the results where bounds for errors of approximations are of the same order (as the sample size increases) as in the independent case. We omit the discussion where the dependence has a finite range, like for $m$-dependent sequences and fields. We only note that the case of
$m$-dependent random fields is well studied - see, for example, [5]-[7], [14], [16], [27], [30]-[33] and others.

For the sequence $\left\{\xi_{i}, i \geq 1\right\}$ with $\mathbb{E} \xi_{i}=0$, let $\left\{\zeta_{i}, i \geq 1\right\}$ with $\mathbb{E} \zeta_{i}=0$, be a Gaussian sequence such that the covariance structures of $\left\{\xi_{i}, i \geq 1\right\}$ and $\left\{\zeta_{i}, i \geq 1\right\}$ are identical, that is, $\mathbb{E} \xi_{i} \xi_{i+j}=\mathbb{E} \zeta_{i} \zeta_{i+j}$. As far as we are aware, UTEV (1991) was the first to obtain the optimal bound

$$
\left|\mathbb{E} h\left(n^{-1 / 2}\left(\xi_{1}+\cdots+\xi_{n}\right)\right)-\mathbb{E} h\left(n^{-1 / 2}\left(\zeta_{1}+\cdots+\zeta_{n}\right)\right)\right|=O\left(n^{-1 / 2}\right)
$$

assuming a weak stationarity of $\left\{\xi_{i}, i \geq 1\right\}$ and imposing some moment assumptions. His conditions include smootheness of $h$, as well as conditions on Ibragimov's (or uniformly strong) mixing. He used a nice and original modification of the classical Bernstein method and that of convolutions. This is quite unexpected since Bernstein's method (versus the Stein method [28], [29]) is commonly considered as a method which cannot produce bounds even close to optimal ones.

Rio (1996) established an optimal bound for the Kolmogorov metric

$$
\sup _{x \in \mathbb{R}}\left|\mathbb{P}\left\{n^{-1 / 2}\left(\xi_{1}+\cdots+\xi_{n}\right)<x\right\}-\Phi(x)\right|=O\left(n^{-1 / 2}\right)
$$

assuming that a strictly stationary sequence $\left\{\xi_{i}, i \geq 1\right\}$ satisfies the $\varphi$-mixing condition with $\sum_{r=1}^{\infty} r \varphi(r)<\infty$ and boundedness condition (2) for $V=\{1, \ldots, n\}$, where $\varphi(r)$ stand for Ibragimov's mixing coefficients, and $\Phi(x)$ is the standard normal distribution function. To prove this result, a modified version of the classical composition (or Lindeberg, Bergstrem, etc.) method was applied.

To prove Theorem 1, we extend a method introduced in Bentkus (2003). The idea is to consider an appropriate curve joining $Z_{V}$ and $N$ in the space of r.v.'s, and to apply the Newton-Leibnitz formula along the curve. Using this method, no assumptions on independence or equidistribution are needed, however an additive structure of the object is desirable (e.g., $Z_{V}$ is a sum). The method can be combined with the Fourier transform, i.e., it is applicable to characteristic functions. Some examples in the context of the CLT in $\mathbb{R}^{d}$ are provided in Bentkus (2003), the case of non-identically distributed summands is considered in Bentkus (2004). The present paper is just a pilot study of dependent sequences and fields. Our plan would be to obtain asymptotic (Edgeworth) expansions assuming sufficient smoothness, as well as to get Berry-Esseen bounds for Kolmogorov's and other metrics, each time looking for optimal combinations of smoothness, moment (or boundedness) and dependency conditions such that the bound is of the same order as in the independent case, as the sample size increases. We hope that one cannot replace the strong mixing condition in Theorem 1 by another (considerably) weaker dependency assumption.

## 2. Auxiliary Lemmas and Proposition 4

Let $\mathcal{G}$ and $\mathcal{H}$ be sub- $\sigma$-algebras of the $\sigma$-algebra $\mathcal{F}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote the strong mixing coefficient between $\mathcal{G}$ and $\mathcal{H}$ by

$$
\alpha(\mathcal{G}, \mathcal{H})=\sup \{|\mathbb{P}(A B)-\mathbb{P}(A) \mathbb{P}(B)|: A \in \mathcal{G}, B \in \mathcal{H}\}
$$

Lemma 3 (see [17, p. 388] and [15, p. 277]). Suppose that $\xi$ and $\eta$ are real r.v.'s which are $\mathcal{G}$-and $\mathcal{H}$-measurable, respectively, and that $\mathbb{P}\left\{|\xi| \leq C_{1}\right\}=$ $\mathbb{P}\left\{|\eta| \leq C_{2}\right\}=1$. Then

$$
\begin{equation*}
|\operatorname{cov}(\xi, \eta)| \leq 4 C_{1} C_{2} \alpha(\mathcal{G}, \mathcal{H}) \tag{5}
\end{equation*}
$$

Proposition 4. Let $\left\{A_{a}, a \in V\right\}$ and $\left\{Y_{a}, a \in V\right\}$ be two real random fields, defined on a fixed nonempty set $V$ with $|V|<\infty$ of standard lattice $\mathbb{Z}^{d}, d \geq 1$, such that $Z_{V}=\sum_{a \in V} A_{a}$ and $Y_{V}=\sum_{a \in V} Y_{a}$ are independent, $\mathbb{E} Z_{V}=\mathbb{E} Y_{V}=0, \mathbb{E} Z_{V}^{2}=\mathbb{E} Y_{V}^{2}, \mathbb{E}\left|A_{a}\right|^{3}<\infty$ and $\mathbb{E}\left|Y_{a}\right|^{3}<\infty$ for all $a \in V$. Let the r.v. $\tau$ be uniformly distributed in the interval $[0,1]$ and independent of all the other r.v.'s. Let $h: \mathbb{R} \rightarrow \mathbb{R}$ be a thrice differentiable function such that $\left\|h^{\prime \prime \prime}\right\|_{\infty}<\infty$. Introduce the following notation:

$$
\begin{gathered}
X_{a}=X_{a}(\gamma)=A_{a} \sin \gamma+Y_{a} \cos \gamma, \quad T_{V}=T_{V}(\gamma)=\sum_{a \in V} X_{a} \\
t_{a}^{(r)}=\sum_{\substack{b \in V:\|b-a\|=r, a-\text { fixed, } a \in V}} X_{b} \quad\left(t_{a}^{(0)}=X_{a}\right) \\
T_{a}^{(m)}=T_{V}-\sum_{r=0}^{m} t_{a}^{(r)}, \quad u_{a}^{(r)}=X_{a}^{\prime} t_{a}^{(r)}-\mathbb{E} X_{a}^{\prime} t_{a}^{(r)}
\end{gathered}
$$

where $0 \leq \gamma \leq \pi / 2$. Here and in what follows, $X_{a}^{\prime}$ and $T_{V}^{\prime}$ denote the first order derivatives with respect to $\gamma$ of $X_{a}$ and $T_{V}$, respectively.

Then

$$
\begin{equation*}
\mathbb{E} h\left(Z_{V}\right)-\mathbb{E} h\left(Y_{V}\right)=\int_{0}^{\pi / 2} \mathbb{E} T_{V}^{\prime} h^{\prime}\left(T_{V}\right) d \gamma \tag{6}
\end{equation*}
$$

and the following expansion is valid for $\mathbb{E} T_{V}^{\prime} h^{\prime}\left(T_{V}\right)$ :

$$
\begin{equation*}
\mathbb{E} T_{V}^{\prime} h^{\prime}\left(T_{V}\right)=\Sigma_{1}+\cdots+\Sigma_{6} \tag{7}
\end{equation*}
$$

where

$$
\begin{aligned}
& \Sigma_{1}=\sum_{a \in V} \sum_{r \geq 1} \sum_{q=r+1}^{2 r} \mathbb{E} u_{a}^{(r)} t_{a}^{(q)} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right), \\
& \Sigma_{2}=\sum_{a \in V} \sum_{r \geq 1} \sum_{q \geq 2 r+1} \mathbb{E} u_{a}^{(r)} t_{a}^{(q)} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right),
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_{3} & =\sum_{a \in V} \sum_{r \geq 1} \mathbb{E} u_{a}^{(0)} t_{a}^{(r)} h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right), \\
\Sigma_{4} & =\sum_{a \in V} \sum_{r \geq 1} \mathbb{E} X_{a}^{\prime}\left(t_{a}^{(r)}\right)^{2}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right), \\
\Sigma_{5} & =\sum_{a \in V} \mathbb{E}\left[X_{a}^{\prime} X_{a}(1-\tau)-\mathbb{E} X_{a}^{\prime} X_{a}\right] X_{a} h^{\prime \prime \prime}\left(T_{a}^{(0)}+\tau X_{a}\right), \\
\Sigma_{6} & =-\sum_{a \in V} \sum_{r \geq 1} \mathbb{E} X_{a}^{\prime} t_{a}^{(r)} \sum_{q=0}^{r} \mathbb{E} t_{a}^{(q)} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right) .
\end{aligned}
$$

Proof. The fundamental theorem of integral calculus yields

$$
h\left(Z_{V}\right)-h\left(Y_{V}\right)=\int_{0}^{\pi / 2} T_{V}^{\prime}(\gamma) h^{\prime}\left(T_{V}(\gamma)\right) \mathrm{d} \gamma
$$

Here and in the sequel, we omit the argument $\gamma$ in which it appears. Consider the integrand function $T_{V}^{\prime} h^{\prime}\left(T_{V}\right)$.

The Taylor expansion yields

$$
\begin{align*}
T_{V}^{\prime} h^{\prime}\left(T_{V}\right) & =\sum_{a \in V} X_{a}^{\prime} h^{\prime}\left(T_{a}^{(0)}+X_{a}\right) \\
& =\sum_{a \in V} X_{a}^{\prime}\left[h^{\prime}\left(T_{a}^{(0)}\right)+X_{a} h^{\prime \prime}\left(T_{a}^{(0)}\right)+X_{a}^{2} \mathbb{E}_{\tau}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(0)}+\tau X_{a}\right)\right] \tag{8}
\end{align*}
$$

where $\mathbb{E}_{\tau}$ is the expectation taken with respect to the r.v. $\tau$.
In what follows, the index $a(a \in V)$ is fixed.
Expansion of $\quad X_{a}^{\prime} X_{a} h^{\prime \prime}\left(T_{a}^{(0)}\right)$.
We rewrite the term $X_{a}^{\prime} X_{a} h^{\prime \prime}\left(T_{a}^{(0)}\right)$ in such a way:

$$
X_{a}^{\prime} X_{a} h^{\prime \prime}\left(T_{a}^{(0)}\right)=u_{a}^{(0)} h^{\prime \prime}\left(T_{a}^{(0)}\right)+h^{\prime \prime}\left(T_{a}^{(0)}\right) \mathbb{E} X_{a}^{\prime} X_{a}
$$

Note that $T_{a}^{(m-1)}=T_{a}^{(m)}+t_{a}^{(m)}$ for all $m=0,1,2, \ldots\left(T_{a}^{(-1)}=T_{V}\right)$. We shall use this fact without mentioning it in the sequel. Therefore, using the Taylor expansion, we eliminate the r.v. $t_{a}^{(1)}$ from the sum $T_{a}^{(0)}$ in the expression $u_{a}^{(0)} h^{\prime \prime}\left(T_{a}^{(0)}\right)$. From the remaining sum $T_{a}^{(1)}$, we eliminate the r.v. $t_{a}^{(2)}$ and so on, until zero appears instead of $T_{a}^{(0)}$. Step by step we obtain that

$$
\begin{equation*}
u_{a}^{(0)} h^{\prime \prime}\left(T_{a}^{(0)}\right)=u_{a}^{(0)} h^{\prime \prime}(0)+\sum_{r \geq 1} u_{a}^{(0)} t_{a}^{(r)} \mathbb{E}_{\tau} h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right) \tag{9}
\end{equation*}
$$

Now, adding and subtracting we pack away, one by one, the r.v.'s $t_{a}^{(r)}$, $t_{a}^{(r-1)}, \ldots$ into the sum $T_{a}^{(r)}$ up to the sum $T_{V}$ in the term $h^{\prime \prime}\left(T_{a}^{(r)}\right)$. Thus, using the Taylor expansion, we get that for all $r \geq 0$

$$
\begin{equation*}
h^{\prime \prime}\left(T_{a}^{(r)}\right)=h^{\prime \prime}\left(T_{V}\right)-\sum_{q=0}^{r} t_{a}^{(q)} \mathbb{E}_{\tau} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right) \tag{10}
\end{equation*}
$$

It follows from (9) and (10) with $r=0$ that

$$
\begin{align*}
X_{a}^{\prime} X_{a} h^{\prime \prime}\left(T_{a}^{(0)}\right)= & u_{a}^{(0)} h^{\prime \prime}(0)+\sum_{r \geq 1} u_{a}^{(0)} t_{a}^{(r)} \mathbb{E}_{\tau} h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right) \\
& +h^{\prime \prime}\left(T_{V}\right) \mathbb{E} X_{a}^{\prime} X_{a}-X_{a} \mathbb{E}_{\tau} h^{\prime \prime \prime}\left(T_{a}^{(0)}+\tau X_{a}\right) \mathbb{E} X_{a}^{\prime} X_{a} \tag{11}
\end{align*}
$$

Expansion of $\quad X_{a}^{\prime} h^{\prime}\left(T_{a}^{(0)}\right)$.
Further, using the Taylor expansion, we eliminate the r.v. $t_{a}^{(1)}$ from the sum $T_{a}^{(0)}$ in the expression $X_{a}^{\prime} h^{\prime}\left(T_{a}^{(0)}\right)$. From the remaining sum $T_{a}^{(1)}$, we eliminate the r.v. $t_{a}^{(2)}$ and so on, until zero appears instead of $T_{a}^{(0)}$. Step by step we obtain that

$$
\begin{align*}
X_{a}^{\prime} h^{\prime}\left(T_{a}^{(0)}\right)= & X_{a}^{\prime} h^{\prime}(0)+\sum_{r \geq 1} X_{a}^{\prime} t_{a}^{(r)} h^{\prime \prime}\left(T_{a}^{(r)}\right) \\
& +\sum_{r \geq 1} X_{a}^{\prime}\left(t_{a}^{(r)}\right)^{2} \mathbb{E}_{\tau}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right) \tag{12}
\end{align*}
$$

Next we rewrite the term $X_{a}^{\prime} t_{a}^{(r)} h^{\prime \prime}\left(T_{a}^{(r)}\right)$ in (12) (for all $r \geq 1$ ) in such a way:

$$
X_{a}^{\prime} t_{a}^{(r)} h^{\prime \prime}\left(T_{a}^{(r)}\right)=u_{a}^{(r)} h^{\prime \prime}\left(T_{a}^{(r)}\right)+h^{\prime \prime}\left(T_{a}^{(r)}\right) \mathbb{E} X_{a}^{\prime} t_{a}^{(r)}
$$

In the sum $T_{a}^{(r)}$ which appears in the term $u_{a}^{(r)} h^{\prime \prime}\left(T_{a}^{(r)}\right)$, we increase a "hole" up to the "hole" in the sum $T_{a}^{(2 r)}$ using the Taylor expansion. Later on, we eliminate, one by one, the r.v.'s $t_{a}^{(2 r+1)}, t_{a}^{(2 r+2)}, \ldots$, from the sum $T_{a}^{(2 r)}$ until zero appears instead of $T_{a}^{(2 r)}$. Then, using (10) we get that for all $r \geq 1$

$$
\begin{align*}
X_{a}^{\prime} t_{a}^{(r)} h^{\prime \prime}\left(T_{a}^{(r)}\right)= & u_{a}^{(r)} h^{\prime \prime}(0)+\sum_{q=r+1}^{2 r} u_{a}^{(r)} t_{a}^{(q)} \mathbb{E}_{\tau} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right) \\
& +\sum_{q \geq 2 r+1} u_{a}^{(r)} t_{a}^{(q)} \mathbb{E}_{\tau} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right) \\
& +h^{\prime \prime}\left(T_{V}\right) \mathbb{E} X_{a}^{\prime} t_{a}^{(r)}-\sum_{q=0}^{r} t_{a}^{(q)} \mathbb{E}_{\tau} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right) \mathbb{E} X_{a}^{\prime} t_{a}^{(r)} \tag{13}
\end{align*}
$$

Substituting (13) into (12) we obtain that

$$
\begin{align*}
& X_{a}^{\prime} h^{\prime}\left(T_{a}^{(0)}\right)=X_{a}^{\prime} h^{\prime}(0)+\sum_{r \geq 1} u_{a}^{(r)} h^{\prime \prime}(0)+\sum_{r \geq 1} \sum_{q=r+1}^{2 r} u_{a}^{(r)} t_{a}^{(q)} \mathbb{E}_{\tau} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right) \\
& \quad+\sum_{r \geq 1} \sum_{q \geq 2 r+1} u_{a}^{(r)} t_{a}^{(a)} \mathbb{E}_{\tau} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right) \\
& \quad+h^{\prime \prime}\left(T_{V}\right) \sum_{r \geq 1} \mathbb{E} X_{a}^{\prime} t_{a}^{(r)}-\sum_{r \geq 1} \sum_{q=0}^{r} t_{a}^{(a)} \mathbb{E}_{\tau} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right) \mathbb{E} X_{a}^{\prime} t_{a}^{(r)} \\
& \quad+\sum_{r \geq 1} X_{a}^{\prime}\left(t_{a}^{(r)}\right)^{2} \mathbb{E}_{\tau}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right) . \tag{14}
\end{align*}
$$

It follows from (8), (14), and (11) that

$$
\begin{align*}
& T_{V}^{\prime} h^{\prime}\left(T_{V}\right)=h^{\prime}(0) T_{V}^{\prime}+h^{\prime \prime}(0) \sum_{a \in V} \sum_{r \geq 0} u_{a}^{(r)} \\
& \quad+\sum_{a \in V} \sum_{r \geq 1} \sum_{q=r+1}^{2 r} u_{a}^{(r)} t_{a}^{(q)} \mathbb{E}_{\tau} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right) \\
& \quad+\sum_{a \in V} \sum_{r \geq 1} \sum_{q \geq 2 r+1} u_{a}^{(r)} t_{a}^{(q)} \mathbb{E}_{\tau} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right)+h^{\prime \prime}\left(T_{V}\right) \mathbb{E} T_{V}^{\prime} T_{V} \\
& \quad-\sum_{a \in V} \sum_{r \geq 1} \sum_{q=0}^{r} t_{a}^{(q)} \mathbb{E}_{\tau} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right) \mathbb{E} X_{a}^{\prime} t_{a}^{(r)} \\
& \quad+\sum_{a \in V} \sum_{r \geq 1} X_{a}^{\prime}\left(t_{a}^{(r)}\right)^{2} \mathbb{E}_{\tau}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right) \\
& \quad+\sum_{a \in V} \sum_{r \geq 1} u_{a}^{(0)} t_{a}^{(r)} \mathbb{E}_{\tau} h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)-\sum_{a \in V} X_{a} \mathbb{E}_{\tau} h^{\prime \prime \prime}\left(T_{a}^{(0)}+\tau X_{a}\right) \mathbb{E} X_{a}^{\prime} X_{a} \\
& \quad+\sum_{a \in V} X_{a}^{\prime} X_{a}^{2} \mathbb{E}_{\tau}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(0)}+\tau X_{a}\right) \tag{15}
\end{align*}
$$

Since the r.v. $Z_{V}$ is independent of $Y_{V}, \mathbb{E} Z_{V}=\mathbb{E} Y_{V}=0$ and $\mathbb{E} Z_{V}^{2}=\mathbb{E} Y_{V}^{2}$, one has that

$$
\begin{equation*}
\mathbb{E} T_{V}^{\prime}=0 \quad \text { and } \quad \mathbb{E} T_{V}^{\prime} T_{V}=0 \tag{16}
\end{equation*}
$$

From (15) and (16) it follows the proof of the basic identity (6)-(7).
In what follows, we denote the centered r.v. as $\widehat{\xi}=\xi-\mathbb{E} \xi$.
Lemma 5. Let $\left\{Y_{a}, a \in V\right\}$ be independent r.v.'s and $\left\{Y_{a}, a \in V\right\}$ be independent of the random field $\left\{A_{a}, a \in V\right\}, \mathbb{E} A_{a}=\mathbb{E} Y_{a}=0, \mathbb{E}\left|A_{a}\right|^{3}<\infty$ and
$\mathbb{E}\left|Y_{a}\right|^{3}<\infty$ for all $a \in V$, where $V$ is a fixed nonempty set $V$ with $|V|<\infty$ of standard lattice $\mathbb{Z}^{d}, d \geq 1$. Denote

$$
X_{a}=p A_{a}+q Y_{a}, \quad t_{a}^{(r)}=\sum_{\substack{b \in V:\|b-a\|=r, a-f i x e d, a \in V}} X_{b} \quad\left(t_{a}^{(0)}=X_{a}\right), \quad u_{a}^{(r)}=X_{a}^{\prime} t_{a}^{(r)}-\mathbb{E} X_{a}^{\prime} t_{a}^{(r)},
$$

where $p=\sin \gamma, q=\cos \gamma, 0 \leq \gamma \leq \pi / 2$. Here and in what follows, $X_{a}^{\prime}$ denote the first order derivative with respect to $\gamma$ of $X_{a}$.

The following connections are valid:
If $a \neq b$, then

$$
\begin{align*}
X_{a}^{\prime} X_{b}^{2}= & p^{2} q A_{a} A_{b}^{2}+2 p q^{2} A_{a} A_{b} Y_{b}+q^{3} A_{a} Y_{b}^{2} \\
& -p^{3} Y_{a} A_{b}^{2}-2 p^{2} q Y_{a} A_{b} Y_{b}-p q^{2} Y_{a} Y_{b}^{2} \tag{17}
\end{align*}
$$

If $a \neq b, a \neq c, b \neq c$, then

$$
\begin{align*}
X_{a}^{\prime} X_{b} X_{c}= & p^{2} q A_{a} A_{b} A_{c}+p q^{2} A_{a}\left(A_{b} Y_{c}+Y_{b} A_{c}\right)+q^{3} A_{a} Y_{b} Y_{c} \\
& -p^{3} Y_{a} A_{b} A_{c}-p^{2} q Y_{a}\left(A_{b} Y_{c}+Y_{b} A_{c}\right)-p q^{2} Y_{a} Y_{b} Y_{c} \tag{18}
\end{align*}
$$

If $a \neq b$, then

$$
\begin{align*}
u_{a}^{(0)} X_{b}= & p^{2} q \widehat{A_{a}^{2}} A_{b}-p^{2} q \widehat{Y_{a}^{2}} A_{b}+p q^{2} \widehat{A_{a}^{2}} Y_{b}-p q^{2} \widehat{Y_{a}^{2}} Y_{b} \\
& +p\left(q^{2}-p^{2}\right) A_{a} Y_{a} A_{b}+q\left(q^{2}-p^{2}\right) A_{a} Y_{a} Y_{b} \tag{19}
\end{align*}
$$

For all $r \geq 1$ :

$$
\begin{align*}
u_{a}^{(r)}= & p q \sum_{\substack{b \in V:\|b-a\|=r, a-\text {-fixed, } a \in V}} \widehat{A_{a} A_{b}}+q^{2} A_{a} \sum_{\substack{b \in V:\|b-a\|=r, a \text {-fixed, } a \in V}} Y_{b} \\
& -p^{2} Y_{a} \sum_{\substack{b \in V:\|b-a\|=r, a-\text {-fixed,a } a \in V}} A_{b}-p q Y_{a} \sum_{\substack{b \in V:\|b-a\|=r, a-\text { fixed, } a \in V}} Y_{b},  \tag{20}\\
X_{a}^{\prime} t_{a}^{(r)}= & p q A_{a} \sum_{\substack{b \in V:\|b-a\|=r, a-\text { fixed, } a \in V}} A_{b}+q^{2} A_{a} \sum_{\substack{b \in V:\|b-a\|=r, a-\text { fixed, } a \in V}} Y_{b} \\
& -p^{2} Y_{a} \sum_{\substack{b \in V:\|b-a\|=r, a-\text { fixed, } a \in V}} A_{b}-p q Y_{a} \sum_{\substack{b \in V:\|b-a\|=r, a \text {-fixed, } a \in V}} Y_{b} .  \tag{21}\\
\left|\mathbb{E} X_{a}^{\prime} X_{b}\right| \leq & \frac{1}{2}\left|\mathbb{E} A_{a} A_{b}\right|, \quad a \neq b, \tag{22}
\end{align*}
$$

$$
\begin{align*}
& \left|\mathbb{E} X_{a}^{\prime} X_{a}\right| \leq \frac{1}{2}\left|\mathbb{E} A_{a}^{2}-\mathbb{E} Y_{a}^{2}\right|  \tag{23}\\
& \mathbb{E}\left|X_{a}^{\prime}\right| X_{a}^{2} \leq \frac{1}{2} \mathbb{E}\left|A_{a}\right|^{3}+\mathbb{E}\left|A_{a}\right| \mathbb{E} Y_{a}^{2}+\mathbb{E}\left|Y_{a}\right| \mathbb{E} A_{a}^{2}+\frac{1}{2} \mathbb{E}\left|Y_{a}\right|^{3} \tag{24}
\end{align*}
$$

The proof of Lemma 5 is elementary, so we omit it.
Lemma 6. Let $\left\{\xi_{a}, a \in \mathbb{Z}^{d}\right\}, d \geq 1$, be a real random field satisfying condition (1), $\Lambda_{*}=\sum_{b \in \mathbb{Z}^{d} \backslash\{0\}} \alpha(\|b\|)<\infty, \mathbb{E} \xi_{a}=0$ and $\mathbb{P}\left\{\left|\xi_{a}\right| \leq M\right\}=1$ for all $a \in V$. Then

$$
\begin{equation*}
B_{V}^{2} \leq\left(1+4 f(2) \Lambda_{*}\right) M^{2}|V| \tag{25}
\end{equation*}
$$

The proof of Lemma 6 follows from (5) and (1).

## 3. Proof of Theorem 1

To estimate the difference $\mathbb{E} h\left(Z_{V}\right)-\mathbb{E} h(N)$, we use Proposition 4. Let $A_{a}$ and $Y_{a}$ in Proposition 4 be $A_{a}=\xi_{a} / B_{V}, Y_{a}=\eta_{a} / \sqrt{|V|}$ where $\left\{\eta_{a}, a \in V\right\}$ are independent standard normal r.v.'s, and $\left\{\eta_{a}, a \in V\right\}$ are independent of the random field $\left\{\xi_{a}, a \in V\right\}$. It is assumed that $B_{V}>0$. It is evident that $\sum_{a \in V}$ $Y_{a}=Y_{V}$ is also a standard normal r.v.

First of all note that, if $Y_{a} \sim \mathcal{N}\left(0, \frac{1}{|V|}\right)$ (i.e., if the r.v. $Y_{a}$ is normally distributed with $\mathbb{E} Y_{a}=0$ and $\left.\mathbb{E} Y_{a}^{2}=\frac{1}{|V|}\right)$, then it is easy to see that

$$
\begin{equation*}
\mathbb{E}\left|Y_{a}\right|=C_{1} \frac{1}{|V|^{1 / 2}}, \quad \mathbb{E}\left|Y_{a}\right|^{3}=C_{3} \frac{1}{|V|^{3 / 2}} \tag{26}
\end{equation*}
$$

where $C_{1}=\sqrt{2 / \pi}, C_{3}=\sqrt{8 / \pi}$.
If $\tau \sim U[0,1]$ (i.e., the r.v. $\tau$ is uniformly distributed in the interval $[0,1]$ ), then

$$
\begin{equation*}
\mathbb{E}|1-\tau|=\frac{1}{2} \tag{27}
\end{equation*}
$$

Moreover, for any fixed $a \in \mathbb{Z}^{d}, d \geq 1$,

$$
\begin{equation*}
\left|\left\{b \in \mathbb{Z}^{d}:\|a-b\|=r\right\}\right| \leq 2 d(2 r+1)^{d-1} \tag{28}
\end{equation*}
$$

We shall use relations (26)-(28) without mentioning them in the sequel.
We now estimate the terms on the right-hand side of (7).
Since for all $b \in V$

$$
\begin{equation*}
\mathbb{E}\left|X_{b}\right| \leq \frac{M}{B_{V}}+C_{1} \frac{1}{|V|^{1 / 2}} \tag{29}
\end{equation*}
$$

one has that, for all $a \in V$ and $q \geq 0$,

$$
\begin{equation*}
\mathbb{E}\left|t_{a}^{(q)}\right| \leq 2 d(2 q+1)^{d-1}\left(\frac{M}{B_{V}}+C_{1} \frac{1}{|V|^{1 / 2}}\right) \tag{30}
\end{equation*}
$$

Using (22), (5) from Lemma 3, and (1), we obtain that, for all $a \in V$ and $r \geq 1$,

$$
\begin{equation*}
\left|\mathbb{E} X_{a}^{\prime} t_{a}^{(r)}\right| \leq 4 d f(2) \frac{M^{2}}{B_{V}^{2}}(2 r+1)^{d-1} \alpha(r) \tag{31}
\end{equation*}
$$

From (31) and (30) it follows that

$$
\begin{align*}
\left|\Sigma_{6}\right| & \leq\left\|h^{\prime \prime \prime}\right\|_{\infty} \sum_{a \in V} \sum_{r \geq 1}\left|\mathbb{E} X_{a}^{\prime} t_{a}^{(r)}\right| \sum_{q=0}^{r} \mathbb{E}\left|t_{a}^{(q)}\right| \\
& \leq\left\|h^{\prime \prime \prime}\right\|_{\infty} 16 d^{2} 12^{d-1} f(2)\left(1+C_{1} \frac{B_{V}}{|V|^{1 / 2} M}\right) \frac{|V| M^{3}}{B_{V}^{3}} \sum_{r \geq 1} r^{2 d-1} \alpha(r) \tag{32}
\end{align*}
$$

Since the r.v. $\tau$ is independent of all the other r.v.'s, $\mathbb{P}\left\{\left|\xi_{a}\right| \leq M\right\}=1$ for all $a \in V$, using (24), (29), and (23) we obtain

$$
\begin{align*}
\left|\Sigma_{5}\right| \leq & \left\|h^{\prime \prime \prime}\right\|_{\infty} \sum_{a \in V}\left(\mathbb{E}\left|X_{a}^{\prime}\right| X_{a}^{2} \mathbb{E}|1-\tau|+\left|\mathbb{E} X_{a}^{\prime} X_{a}\right| \mathbb{E}\left|X_{a}\right|\right) \\
\leq & \left\|h^{\prime \prime \prime}\right\|_{\infty}\left[\frac{3}{4}+C_{1} \frac{B_{V}}{|V|^{1 / 2} M}+\frac{B_{V}^{2}}{|V| M^{2}}+\left(\frac{1}{2} C_{1}+\frac{1}{4} C_{3}\right) \frac{B_{V}^{3}}{|V|^{3 / 2} M^{3}}\right] \\
& \times \frac{|V| M^{3}}{B_{V}^{3}} \tag{33}
\end{align*}
$$

We now estimate $\left|\Sigma_{4}\right|$. We get

$$
\begin{align*}
\left|\Sigma_{4}\right| \leq & \sum_{a \in V} \sum_{r \geq 1} \sum_{b \in V:\|b-a\|=r}\left|\mathbb{E} X_{a}^{\prime} X_{b}^{2}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)\right| \\
& +\sum_{a \in V} \sum_{r \geq 1} \sum_{\substack{b \in V:\|b-a\|=r, c \in V:\|c-a\|=r, b \neq c}}\left|\mathbb{E} X_{a}^{\prime} X_{b} X_{c}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)\right| \tag{34}
\end{align*}
$$

Using the expression of $X_{a}^{\prime} X_{b}^{2}$ from (17), taking into account that $h^{\prime \prime \prime}\left(T_{a}^{(r)}+\right.$ $\left.\tau t_{a}^{(r)}\right)$ does not contain $Y_{a}$, and $\left\{Y_{a}, a \in V\right\}$ are independent of the random field $\left\{A_{a}, a \in V\right\}$ and the r.v. $\tau$, we obtain that, for all $a, b \in V:\|b-a\|=r \geq 1$,

$$
\begin{align*}
& \left|\mathbb{E} X_{a}^{\prime} X_{b}^{2}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)\right| \leq \frac{1}{2}\left|\mathbb{E} A_{a} A_{b}^{2}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)\right|  \tag{35}\\
+ & \left|\mathbb{E} A_{a} A_{b} Y_{b}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)\right|+\left|\mathbb{E} A_{a} Y_{b}^{2}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)\right| .
\end{align*}
$$

Note that

$$
T_{a}^{(r)}+\tau t_{a}^{(r)}=\sum_{c \in V:\|c-a\|>r} X_{c}+\tau \sum_{c \in V:\|c-a\|=r} X_{c}
$$

Denote the random vector $\mathcal{Y}$ and vector $y$ as

$$
\mathcal{Y}=\left(\tau, Y_{c}, c \in V:\|c-a\| \geq r\right), \quad y=\left(t, y_{c}, c \in V:\|c-a\| \geq r\right)
$$

In the sequel, $\mathbb{P}_{\xi}$ is the distribution of a random vector $\xi$. Denote

$$
s(r)=\sum_{c \in V:\|c-a\| \geq r} 1=|V|-\sum_{c \in V:\|c-a\| \leq r-1} 1
$$

Then, applying (5) of Lemma 3 and (1), we get that, for all $a, b \in V$ : $\|b-a\|=r \geq 1$,

$$
\begin{align*}
& \left|\mathbb{E} A_{a} A_{b}^{2}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)\right| \\
& = \\
& \quad \mid \int_{\mathbb{R}^{s(r)+1}}(1-t) \mathbb{E} A_{a} A_{b}^{2} h^{\prime \prime \prime}\left(\sin \gamma\left(\sum_{c \in V:\|c-a\|>r} A_{c}+t \sum_{c \in V:\|c-a\|=r} A_{c}\right)\right. \\
& \\
& \left.\quad+\cos \gamma\left(\sum_{c \in V:\|c-a\|>r} y_{c}+t \sum_{c \in V:\|c-a\|=r} y_{c}\right)\right) \mathbb{P}_{\mathcal{Y}}(\mathrm{d} y) \mid  \tag{36}\\
& \quad \leq 4\left\|h^{\prime \prime \prime}\right\|_{\infty} \frac{M^{3}}{B_{V}^{3}} f(s(r)+1) \alpha(r) \int_{\mathbb{R}^{s(r)+1}}|1-t| \mathbb{P}_{\mathcal{Y}}(\mathrm{d} y) \\
& =
\end{align*}\left\|h^{\prime \prime \prime}\right\|_{\infty} 2 \frac{M^{3}}{B_{V}^{3}} f(s(r)+1) \alpha(r) .
$$

Similarly to (36), we get that, for all $a, b \in V:\|b-a\|=r \geq 1$,

$$
\begin{align*}
& \left|\mathbb{E} A_{a} A_{b} Y_{b}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)\right| \\
& \quad \leq 4\left\|h^{\prime \prime \prime}\right\|_{\infty} \frac{M^{2}}{B_{V}^{2}} f(s(r)+1) \alpha(r) \int_{\mathbb{R}^{s(r)+1}}\left|y_{b}\right||1-t| \mathbb{P}_{\mathcal{Y}}(\mathrm{d} y) \\
& \quad=\left\|h^{\prime \prime \prime}\right\|_{\infty} 2 C_{1} \frac{M^{2}}{|V|^{1 / 2} B_{V}^{2}} f(s(r)+1) \alpha(r)  \tag{37}\\
& \left|\mathbb{E} A_{a} Y_{b}^{2}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)\right| \\
& \quad \leq 4\left\|h^{\prime \prime \prime}\right\|_{\infty} \frac{M}{B_{V}} f(s(r)+1) \alpha(r) \int_{\mathbb{R}^{s(r)+1}} y_{b}^{2}|1-t| \mathbb{P}_{\mathcal{Y}}(\mathrm{d} y) \\
& \quad=\left\|h^{\prime \prime \prime}\right\|_{\infty} 2 \frac{M}{|V| B_{V}} f(s(r)+1) \alpha(r) \tag{38}
\end{align*}
$$

Substituting (36)-(38) into (35), we get that for all $a, b \in V:\|b-a\|=r \geq 1$

$$
\begin{align*}
\mid \mathbb{E} X_{a}^{\prime} X_{b}^{2}(1 & -\tau) h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right) \mid \\
& \leq\left\|h^{\prime \prime \prime}\right\|_{\infty}\left(1+2 C_{1} \frac{B_{V}}{|V|^{1 / 2} M}+2 \frac{B_{V}^{2}}{|V| M^{2}}\right) \frac{M^{3}}{B_{V}^{3}} f(s(r)+1) \alpha(r) \tag{39}
\end{align*}
$$

It only remains to estimate an analogous term in (34), where instead of $X_{a}^{\prime} X_{b}^{2}$ we take $X_{a}^{\prime} X_{b} X_{c}$. Using the expression of $X_{a}^{\prime} X_{b} X_{c}$ from (18), taking into account that $h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)$ does not contain $Y_{a}$, and $\left\{Y_{a}, a \in V\right\}$ are independent of the random field $\left\{A_{a}, a \in V\right\}$ and the r.v. $\tau$, we obtain in the same way that, for all $a, b, c \in V:\|b-a\|=r \geq 1,\|c-a\|=r \geq 1, b \neq c$,

$$
\begin{align*}
& \left|\mathbb{E} X_{a}^{\prime} X_{b} X_{c}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)\right| \leq \frac{1}{2}\left|\mathbb{E} A_{a} A_{b} A_{c}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)\right| \\
& \quad+\frac{1}{2}\left|\mathbb{E} A_{a} A_{b} Y_{c}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)\right|+\frac{1}{2}\left|\mathbb{E} A_{a} Y_{b} A_{c}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)\right| \\
& \quad+\left|\mathbb{E} A_{a} Y_{b} Y_{c}(1-\tau) h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)\right| \\
& \leq 2\left\|h^{\prime \prime \prime}\right\|_{\infty} \frac{M^{3}}{B_{V}^{3}} f(s(r)+1) \alpha(r) \mathbb{E}|1-\tau| \\
& \quad+2\left\|h^{\prime \prime \prime}\right\|_{\infty} \frac{M^{2}}{B_{V}^{2}} f(s(r)+1) \alpha(r) \mathbb{E}\left|Y_{c}\right| \mathbb{E}|1-\tau| \\
& \quad+2\left\|h^{\prime \prime \prime}\right\|_{\infty} \frac{M^{2}}{B_{V}^{2}} f(s(r)+1) \alpha(r) \mathbb{E}\left|Y_{b}\right| \mathbb{E}|1-\tau| \\
& \quad+4\left\|h^{\prime \prime \prime}\right\|_{\infty} \frac{M}{B_{V}} f(s(r)+1) \alpha(r) \mathbb{E}\left|Y_{b}\right| \mathbb{E}\left|Y_{c}\right| \mathbb{E}|1-\tau| \\
& \leq\left\|h^{\prime \prime \prime}\right\|_{\infty}\left(1+2 C_{1} \frac{B_{V}}{|V|^{1 / 2} M}+2 C_{1}^{2} \frac{B_{V}^{2}}{|V| M^{2}}\right) \frac{M^{3}}{B_{V}^{3}} f(s(r)+1) \alpha(r) \tag{40}
\end{align*}
$$

Substituting (39) and (40) into (34), we have

$$
\begin{align*}
\left|\Sigma_{4}\right| \leq & \left\|h^{\prime \prime \prime}\right\|_{\infty} 8 d^{2} 3^{2(d-1)}\left(1+2 C_{1} \frac{B_{V}}{|V|^{1 / 2} M}+\left(1+C_{1}^{2}\right) \frac{B_{V}^{2}}{|V| M^{2}}\right) \\
& \times \frac{|V| M^{3}}{B_{V}^{3}} f(|V|) \sum_{r \geq 1} r^{2(d-1)} \alpha(r) \tag{41}
\end{align*}
$$

In the same manner as (35) and (40), using (19), (5) of Lemma 3, and (1), we obtain that, for all $a, b \in V:\|b-a\|=r \geq 1$,

$$
\left|\mathbb{E} u_{a}^{(0)} X_{b} h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)\right|
$$

$$
\begin{align*}
& \leq \frac{1}{2}\left|\mathbb{E} A_{b} \widehat{A_{a}^{2}} h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)\right|+\frac{1}{2}\left|\mathbb{E} Y_{b} \widehat{A_{a}^{2}} h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)\right| \\
& \leq\left\|h^{\prime \prime \prime}\right\|_{\infty} 4\left(1+C_{1} \frac{B_{V}}{|V|^{1 / 2} M}\right) \frac{M^{3}}{B_{V}^{3}} f(s(r)+1) \alpha(r) \tag{42}
\end{align*}
$$

Therefore

$$
\begin{align*}
\left|\Sigma_{3}\right| & \leq \sum_{a \in V} \sum_{r \geq 1} \sum_{b \in V:\|b-a\|=r}\left|\mathbb{E} u_{a}^{(0)} X_{b} h^{\prime \prime \prime}\left(T_{a}^{(r)}+\tau t_{a}^{(r)}\right)\right| \\
& \leq\left\|h^{\prime \prime \prime}\right\|_{\infty} 4\left(1+C_{1} \frac{B_{V}}{|V|^{1 / 2} M}\right) \frac{|V| M^{3}}{B_{V}^{3}} \sum_{r \geq 1} f(s(r)+1) \alpha(r) \sum_{b \in V:\|b-a\|=r} 1 \\
& \leq\left\|h^{\prime \prime \prime}\right\|_{\infty} 8 d 3^{d-1}\left(1+C_{1} \frac{B_{V}}{|V|^{1 / 2} M}\right) \frac{|V| M^{3}}{B_{V}^{3}} f(|V|) \sum_{r \geq 1} r^{d-1} \alpha(r) \tag{43}
\end{align*}
$$

We now estimate $\left|\Sigma_{2}\right|$. Analogously, using (20), we get that, for all $a \in V$, $r \geq 1, q \geq 2 r+1$,

$$
\begin{align*}
& \left|\mathbb{E} u_{a}^{(r)} t_{a}^{(q)} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right)\right| \\
& \quad \leq \frac{1}{2} \sum_{c \in V:\|c-a\|=q}\left|\mathbb{E} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right) A_{c} \sum_{b \in V:\|b-a\|=r} \widehat{A_{a} A_{b}}\right| \\
& \quad+\frac{1}{2} \sum_{c \in V:\|c-a\|=q}\left|\mathbb{E} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right) Y_{c} \sum_{b \in V:\|b-a\|=r} \widehat{A_{a} A_{b}}\right| \tag{44}
\end{align*}
$$

Note that

$$
\left|\sum_{b \in V:\|b-a\|=r} \widehat{A_{a} A_{b}}\right| \leq 4 d \frac{M^{2}}{B_{V}^{2}}(2 r+1)^{d-1}
$$

Other than the previous notation, in the sequel

$$
\mathcal{Y}=\left(\tau, Y_{e}, e \in V:\|e-a\| \geq q\right), \quad y=\left(t, y_{e}, e \in V:\|e-a\| \geq q\right)
$$

Denote

$$
s\left(t_{a}^{(r)}\right)=\sum_{b \in V:\|b-a\|=r} 1
$$

Then, applying (5) and (1) we have, for all $a \in V, r \geq 1, q \geq 2 r+1$,

$$
\left|\mathbb{E} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right) A_{c} \sum_{b \in V:\|b-a\|=r} \widehat{A_{a} A_{b}}\right|
$$

$$
\begin{align*}
&= \mid \int_{\mathbb{R}^{s}(q)+1} \mathbb{E} h^{\prime \prime \prime}\left(\sin \gamma\left(\sum_{e \in V:\|e-a\|>q} A_{e}+t \sum_{e \in V:\|e-a\|=q} A_{e}\right)\right. \\
&\left.+\cos \gamma\left(\sum_{e \in V:\|e-a\|>q} y_{e}+t \sum_{e \in V:\|e-a\|=q} y_{e}\right)\right) A_{c} \\
& \times \sum_{b \in V:\|b-a\|=r} \widehat{A_{a} A_{b} \mathbb{P}_{\mathcal{Y}}(\mathrm{d} y) \mid} \\
& \leq\left\|h^{\prime \prime \prime}\right\|_{\infty} 16 d \frac{M^{3}}{B_{V}^{3}} f\left(s(q)+s\left(t_{a}^{(r)}\right)\right)(2 r+1)^{d-1} \alpha(q-r),  \tag{45}\\
&\left|\mathbb{E} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right) Y_{c} \sum_{b \in V:\|b-a\|=r} \widehat{A_{a} A_{b}}\right| \\
& \leq\left\|h^{\prime \prime \prime}\right\|_{\infty} 16 d \frac{M^{2}}{B_{V}^{2}} f\left(s(q)+s\left(t_{a}^{(r)}\right)\right)(2 r+1)^{d-1} \alpha(q-r) \mathbb{E}\left|Y_{c}\right| \\
& \leq\left\|h^{\prime \prime \prime}\right\|_{\infty} 16 d C_{1} \frac{M^{2}}{|V|^{1 / 2} B_{V}^{2}} f\left(s(q)+s\left(t_{a}^{(r)}\right)\right)(2 r+1)^{d-1} \alpha(q-r) . \tag{46}
\end{align*}
$$

Therefore, from (44)-(46) we get that

$$
\begin{gathered}
\left|\Sigma_{2}\right| \leq \sum_{a \in V} \sum_{r \geq 1} \sum_{q \geq 2 r+1}\left|\mathbb{E} u_{a}^{(r)} t_{a}^{(q)} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right)\right| \\
\leq\left\|h^{\prime \prime \prime}\right\|_{\infty} 16 d^{2} 9^{d-1}\left(1+C_{1} \frac{B_{V}}{|V|^{1 / 2} M}\right) \frac{|V| M^{3}}{B_{V}^{3}} f(|V|) \sum_{r \geq 1} \sum_{q \geq 2 r+1}(r q)^{d-1} \alpha(q-r) .
\end{gathered}
$$

Since

$$
\begin{equation*}
\sum_{r \geq 1} \sum_{q \geq 2 r+1} q^{2(d-1)} \alpha(q-r)<4^{d-1} \sum_{r \geq 1} r^{2 d-1} \alpha(r) \tag{47}
\end{equation*}
$$

one has

$$
\begin{equation*}
\left|\Sigma_{2}\right| \leq\left\|h^{\prime \prime \prime}\right\|_{\infty} 16 d^{2} 18^{d-1}\left(1+C_{1} \frac{B_{V}}{|V|^{1 / 2} M}\right) \frac{|V| M^{3}}{B_{V}^{3}} f(|V|) \sum_{r \geq 1} r^{2 d-1} \alpha(r) \tag{48}
\end{equation*}
$$

It only remains now to estimate $\left|\Sigma_{1}\right|$. Recalling the definition of the r.v. $u_{a}^{(r)}$, we see that the largest "hole" of the two ones in the term $\mathbb{E} u_{a}^{(r)} t_{a}^{(q)} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right)$ is between the r.v.'s $t_{a}^{(r)}$ and $X_{a}^{\prime}$. Applying (21) we get that, for all $a \in V, r \geq 1$, $r+1 \leq q \leq 2 r$,

$$
\begin{aligned}
& \left|\mathbb{E} X_{a}^{\prime} t_{a}^{(r)} t_{a}^{(q)} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right)\right| \\
& \quad \leq \frac{1}{2}\left|\mathbb{E} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right) A_{a} \sum_{c \in V:\|c-a\|=q} A_{c} \sum_{b \in V:\|b-a\|=r} A_{b}\right|
\end{aligned}
$$

$$
\begin{align*}
& +\frac{1}{2}\left|\mathbb{E} h^{\prime \prime \prime}\left(T_{a}^{(q)}+\tau t_{a}^{(q)}\right) A_{a} \sum_{c \in V:\|c-a\|=q} Y_{c} \sum_{b \in V:\|b-a\|=r} A_{b}\right| \\
\leq & \left\|h^{\prime \prime \prime}\right\|_{\infty} 8 d^{2}(2 q+1)^{d-1}(2 r+1)^{d-1} \frac{M^{3}}{B_{V}^{3}} f(|V|) \alpha(r) \\
& \left.+\left.\left\|h^{\prime \prime \prime}\right\|_{\infty} 4 d(2 r+1)^{d-1} \frac{M^{2}}{B_{V}^{2}} f(|V|) \alpha(r) \mathbb{E}\right|_{c \in V:\|c-a\|=q} Y_{c} \right\rvert\, \\
\leq & \left\|h^{\prime \prime \prime}\right\|_{\infty} 8 d^{2} 15^{d-1}\left(1+C_{1} \frac{B_{V}}{|V|^{1 / 2} M}\right) \frac{M^{3}}{B_{V}^{3}} f(|V|) r^{2(d-1)} \alpha(r) . \tag{49}
\end{align*}
$$

Using (30) and (31), we get that, for all $a \in V, r \geq 1, r+1 \leq q \leq 2 r$,

$$
\left.\begin{array}{rl}
\mid \mathbb{E} t_{a}^{(q)} h^{\prime \prime \prime}\left(T_{a}^{(q)}\right. & \left.+\tau t_{a}^{(q)}\right) \mathbb{E} X_{a}^{\prime} t_{a}^{(r)} \mid
\end{array}\right) \leq\left\|h^{\prime \prime \prime}\right\|_{\infty} \mathbb{E}\left|t_{a}^{(q)}\right|\left|\mathbb{E} X_{a}^{\prime} t_{a}^{(r)}\right|, ~\left(1+C_{1} \frac{B_{V}}{|V|^{1 / 2} M}\right) \frac{M^{3}}{B_{V}^{3}} f(2) r^{2(d-1)} \alpha(r) .
$$

From (49) and (50) it follows the estimate of $\left|\Sigma_{1}\right|$ :

$$
\begin{equation*}
\left|\Sigma_{1}\right| \leq\left\|h^{\prime \prime \prime}\right\|_{\infty} 16 d^{2} 15^{d-1}\left(1+C_{1} \frac{B_{V}}{|V|^{1 / 2} M}\right) \frac{|V| M^{3}}{B_{V}^{3}} f(|V|) \sum_{r \geq 1} r^{2 d-1} \alpha(r) \tag{51}
\end{equation*}
$$

Substituting (51), (48), (43), (41), (33), and (32) into (7) and using (6), we have (3).

Theorem 1 is proved.
The proof of Theorem 2 is analogous to that of Theorem 1 , so we omit it.
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270 V. Bentkus and J. K. Sunklodas : On normal approximations to strongly...
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