

## Finiteness conditions and sums of rings

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**Abstract.** Let  $R$  be a ring, which is a sum of its additive subgroups  $R_s$ ,  $s \in S$ . Suppose that all rings among the  $R_s$  satisfy one of the following classical finiteness properties: right or left Artinian or Noetherian, right or left perfect, semilocal or semiprimary. We provide new conditions on the interaction of the  $R_s$  sufficient for the whole ring  $R$  to enjoy the same property.

This paper is devoted to several well-known finiteness conditions which play important roles in ring theory. We start with the following natural problem. Let  $\mathcal{K}$  be a class of associative rings,  $S$  a set,  $R$  a ring, and let  $R = \sum_{s \in S} R_s$  be a sum of additive subgroups  $R_s$  of  $R$ . Suppose that all rings among the  $R_s$  belong to  $\mathcal{K}$ . Find conditions sufficient for  $R$  to belong to  $\mathcal{K}$ .

In full generality this problem was first recorded in [7]. However, its many interesting special cases have been actively investigated by a number of authors including BAHTURIN, BEIDAR, BOKUT', CHICK, FERRERO, FUKSHANSKY, GARDNER, KEGEL, KELAREV, KEPCZYK, MCCONNELL, MIKHALEV, PETRAVCHUK, PUCZYŁOWSKI and SALWA (see [3]–[8] for references).

We use two restrictions on the interaction of the components  $R_s$  introduced in [7]. Let  $S$  be a semigroup,  $R = \sum_{s \in S} R_s$  a sum of additive subgroups  $R_s$  of  $R$ . If  $T \subseteq S$ , then we put  $R_T = \sum_{s \in T} R_s$ . We say that  $R$  is a *structural  $S$ -sum* if and only if, for each subsemigroup (left ideal, right ideal)  $T$  of  $S$ , the sum  $R_T$  is a subring (respectively, left ideal, right

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ideal) of  $R$ . For any  $s \in S$ , denote by  $\langle s \rangle$  the subsemigroup generated in  $S$  by  $s$ , and put  $R^s = R_{\langle s \rangle}$ . We say that  $R$  is an  $S$ -sum if  $R_s R_t \subseteq R^{st}$  for all  $s, t \in S$ . Many ring constructions are examples of structural  $S$ -sums and  $S$ -sums.

This paper investigates the question of when several classical finiteness conditions are preserved by  $S$ -sums and structural  $S$ -sums. First, we deal with left or right Artinian or Noetherian rings.

A semigroup entirely consisting of idempotents is called a *band*. A band is said to be a *semilattice* (*rectangular band*; *left zero band*; *right zero band*; *left regular band*; *right regular band*) if it satisfies the identity  $xy = yx$  ( $xyx = x$ ;  $xy = x$ ;  $xy = y$ ;  $xyx = xy$ ;  $xyx = yx$ ).

**Theorem 1.** *For any semigroup  $S$ , the following are equivalent:*

- (i) *for every structural  $S$ -sum  $R = \sum_{s \in S} R_s$ , if all rings among the  $R_s$  are right Artinian (Noetherian), then  $R$  is right Artinian (Noetherian), too;*
- (ii) *for every  $S$ -sum  $R = \sum_{s \in S} R_s$ , if all rings among the  $R_s$  are right Artinian (Noetherian), then  $R$  is right Artinian (Noetherian), too;*
- (iii)  *$S$  is a finite left regular band.*

Next, we look at semilocal, semiprimary, and left or right perfect rings. Let  $\mathcal{J}(R)$  be the Jacobson radical of  $R$ . A ring  $R$  is *semilocal* if  $R/\mathcal{J}(R)$  is Artinian. Recall that  $R$  is said to be *right (left)  $T$ -nilpotent* if, for every sequence of elements  $r_1, r_2, \dots$  of  $R$ , there exists  $n$  such that  $r_n \dots r_1 = 0$  (respectively,  $r_1 \dots r_n = 0$ ). A semilocal ring is *semiprimary* (*right perfect*, *left perfect*) if  $\mathcal{J}(R)$  is nilpotent (right  $T$ -nilpotent; left  $T$ -nilpotent). A semigroup is said to be *combinatorial* if all its subgroups are singletons. We obtain the following conditions sufficient for preservation of these properties by structural sums of rings.

**Theorem 2.** *Let  $S$  be a periodic combinatorial semigroup with a finite number of idempotents and locally nilpotent (nilpotent, right  $T$ -nilpotent, left  $T$ -nilpotent) nil factors, and let  $R = \sum_{s \in S} R_s$  be an  $S$ -sum. If all rings among the  $R_s$  are semilocal (respectively, semiprimary, right perfect, left perfect), then  $R$  is semilocal (respectively, semiprimary, right perfect, left perfect), too.*

A semigroup is said to be *semisimple* if all its principal factors are simple or 0-simple.

**Theorem 3.** *Let  $S$  be a semisimple periodic combinatorial semigroup with a finite number of idempotents, and let  $R = \sum_{s \in S} R_s$  be a structural  $S$ -sum. If all rings among the  $R_s$  are semilocal (or semiprimary, or right perfect, or left perfect), then  $R$  possesses the same property, too.*

PROOF of Theorem 1. The case where  $|S| = 1$  is trivial, and so throughout we assume that  $S$  is not a singleton. The implication (i) $\Rightarrow$ (ii) is obvious, because every  $S$ -sum is a structural  $S$ -sum.

(ii) $\Rightarrow$ (iii): Suppose that  $S$  contains an element  $t$  such that  $t \neq t^2$ . Take any finite field  $F$ . Denote by  $P$  the ring of all polynomials over  $F$  in commuting variables  $x_1, x_2, \dots$  without constant terms. Let  $I$  be the ideal generated in  $P$  by all polynomials  $x_i x_j - x_k x_\ell$ , for any positive integers  $i, j, k, \ell$ . Consider the ring  $R = P/(I + P^3)$  and its subrings  $R_t = \sum_{i=1}^{\infty} Fx_i$ ,  $R_{t^2} = Fx_1^2$ . Put  $R_s = 0$  for all  $s \in S \setminus \{t, t^2\}$ . Clearly,  $R = \sum_{s \in S} R_s$  is an  $S$ -sum. Moreover, it is an  $S$ -graded ring. The component  $R_t$  is not a subring. The component  $R_{t^2}$  is finite, and so it is right Artinian and Noetherian. All the other components are zero. However,  $R$  is neither Artinian, nor Noetherian. This contradiction shows that  $S$  must be a band.

If  $S$  is a band, then every  $S$ -sum is an  $S$ -graded ring, and therefore [3], Corollary 6.4, shows that (iii) holds.

(iii) $\Rightarrow$ (i): Suppose that  $S$  is a finite left regular band. Take any structural  $S$ -sum  $R = \sum_{s \in S} R_s$  such that all rings among the  $R_s$  are right Artinian (Noetherian). Clearly,  $R$  is an  $S$ -graded ring. Therefore [3], Corollary 6.4, tells us that  $R$  is right Artinian (Noetherian), as required.  $\square$

For any semigroup  $S$ , denote by  $S^0$  the semigroup  $S \cup \{0\}$  with zero adjoined. Let  $G$  be a group,  $I$  and  $\Lambda$  nonempty sets, and let  $P = (p_{\lambda i})$  be a  $\Lambda \times I$ -matrix with entries  $p_{\lambda i} \in G^0$  for all  $\lambda \in \Lambda$ ,  $i \in I$ . The *Rees matrix semigroup*  $M^0(G; I, \Lambda; P)$  over the group  $G$  with *sandwich-matrix*  $P$  consists of zero and all triples  $(g; i, \lambda)$ ,  $i \in I$ ,  $\lambda \in \Lambda$ , and  $g \in G$ , where all triples of the form  $(0; i, \lambda)$  are identified with zero, and multiplication is defined by the rule  $(g_1; i_1, \lambda_1)(g_2; i_2, \lambda_2) = (g_1 p_{\lambda_1 i_2} g_2; i_1, \lambda_2)$ . Lemma 3.1 of [3] immediately gives us the following

**Lemma 4.** *If  $S$  is a periodic combinatorial semigroup with a finite number of idempotents, then  $S^0$  has a finite ideal chain*

$$0 = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = S^0$$

*such that each factor  $S_{i+1}/S_i$ , where  $0 \leq i \leq n-1$ , is nil or is isomorphic to a finite Rees matrix semigroup  $M^0(\{e\}; I, \Lambda; P)$ .*

We also need the following well-known properties of semilocal, semiprimary, right and left perfect rings (see, for example, [3], Lemma 4.3).

**Lemma 5.** *The classes of semilocal, semiprimary, right perfect and left perfect rings are closed for ideal extensions, right and left ideals, homomorphic images and finite sums of one-sided ideals.*

PROOF of Theorem 2. Put  $R_0 = 0$ . Then  $R = \sum_{s \in S^0} R_s$  is an  $S^0$ -sum. Lemma 4 tells us that  $S^0$  has a finite ideal chain

$$0 = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = S^0$$

such that each factor  $S_{i+1}/S_i$ , where  $0 \leq i \leq n-1$ , is nil or is isomorphic to a finite Rees matrix semigroup  $M^0(\{e\}; I, \Lambda; P)$ . We show by induction on  $k = 0, 1, \dots, n$  that all rings  $R_{S_k}$  are semilocal (semiprimary, right perfect, left perfect).

The induction basis is trivial, as  $R_{S_0} = 0$ . Suppose that  $k > 0$  and  $R_{S_{k-1}}$  has the desired property. Since  $R_{S_k}$  is an extension of  $R_{S_{k-1}}$  by  $R_{S_k}/R_{S_{k-1}}$ , in view of Lemma 5 it suffices to show that  $R_{S_k}/R_{S_{k-1}}$  belongs to our class.

Clearly,  $Q = R_{S_k}/R_{S_{k-1}} = \sum_{s \in S_k \setminus S_{k-1}} R_s$  is a structural  $S_k/S_{k-1}$ -sum. By the hypothesis  $S_k/S_{k-1}$  is isomorphic to a finite Rees matrix semigroup  $M^0(\{e\}; I, \Lambda; P)$ . The definition of multiplication in  $M^0(\{e\}; I, \Lambda; P)$  implies that  $M^0(\{e\}; I, \Lambda; P)$  is a union of its left ideals  $L_\lambda = \{0\} \cup \{(e; i, \lambda) \mid i \in I\}$ , where  $\lambda \in \Lambda$ . Besides, each  $L_\lambda$  is a union of its right ideals  $I_i = \{0, (e; i, \lambda)\}$ , where  $i \in I$ . Given that  $Q$  is an  $S_k/S_{k-1}$ -sum, it follows that  $Q$  is a sum of its left ideals  $Q_{L_\lambda}$ ,  $\lambda \in \Lambda$ , and every  $Q_{L_\lambda}$  is a sum of its right ideals  $Q_{I_i} \cong R_{(e; i, \lambda)}$ . Since  $R_{(e; i, \lambda)}$  is a subring, we know that it is semilocal (semiprimary, right perfect, left perfect). Lemma 5 implies that all the  $Q_{L_\lambda}$  are semilocal (semiprimary, right perfect, left perfect), too, and so the same can be said of  $Q$ . This completes the proof.  $\square$

PROOF of Theorem 3. Set  $R_0 = 0$ . Evidently,  $R = \sum_{s \in S^0} R_s$  is a structural  $S^0$ -sum. Given that  $S$  is semisimple, Lemma 4 tells us that  $S^0$  has a finite ideal chain

$$0 = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = S^0$$

such that each factor  $S_{i+1}/S_i$ , where  $0 \leq i \leq n-1$ , is isomorphic to a finite Rees matrix semigroup  $M^0(\{e\}; I, \Lambda; P)$ . We show by induction on  $k = 0, 1, \dots, n$  that all rings  $R_{S_k}$  are semilocal (semiprimary, right perfect, left perfect).

The induction basis is trivial again. Suppose that  $k > 0$  and  $R_{S_{k-1}}$  has the desired property. As above in view of Lemma 5 it suffices to show that  $R_{S_k}/R_{S_{k-1}}$  belongs to our class.

Clearly,  $R_{S_k}/R_{S_{k-1}} = \sum_{s \in S_k \setminus S_{k-1}} R_s$  is a structural  $S_k/S_{k-1}$ -sum. Given that  $S_k/S_{k-1}$  is isomorphic to a finite Rees matrix semigroup, since every structural  $S_k/S_{k-1}$ -sum is an  $S_k/S_{k-1}$ -sum, it follows from Theorem 3 that  $R_{S_k}/R_{S_{k-1}}$  is semilocal (semiprimary, right perfect, left perfect). This completes our proof.  $\square$

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