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Some commutativity theorems for rings

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Abstract. In this paper we prove two results: (1) If, in a semiprime ring R, for $x, y \in R$, there exist fixed positive integers n, m greater than 1 such that either (a) $[x^n, y^m] \in Z(R)$ or (b) $(x^n \circ y^m) \in Z(R)$ then R is commutative. (2) If, in a division ring R, for $x, y \in R$ there exists a positive integer $n > 1$ such that $(xy)^n y^n - y^n (yx)^n$ commutes with y, then R is commutative.

1. Introduction

Throughout this paper R represents an associative ring, $Z(R)$ denotes the center of R; $[a, b]$ denotes the Lie product $ab - ba$ and $(a \circ b)$ denotes the Jordan product $ab + ba$.

In [5], QUADRI, KHAN and ASHRAF have shown, that if R is a prime ring and $n > 1$ a fixed positive integer, such that $[x^n, y]$ is central for all $x, y \in R$, then R is commutative. We generalise this result in the form of the following theorem:

Theorem 1.1. *Let R be a semiprime ring and n, m fixed positive integers larger than 1, such that R satisfies one of the conditions (a) $[x^n, y^m] \in Z(R)$, (b) $(x^n \circ y^m) \in Z(R)$, where $x, y \in R$, then R is commutative.*

ABU–KHUZAM and ADIL YAQUB [1] proved that if R is a division ring such that for all $x, y \in R$ there exists a positive integer $n = n(x, y)$ for which $(xy)^n - (yx)^n$ is in the center of R, then R is commutative. We prove the generalized version of this theorem as follows:

Theorem 1.2. *Let R be a division ring such that for all x, y in R there exists a positive integer $n = n(x, y)$ for which $(xy)^n y^n - y^n (yx)^n$ commutes with y, then R is commutative.*

2. Preliminaries

We mention a few Lemmas without proof, which are well known.

Lemma 2.1 (I.N. Herstein [2, Theorem]). *Let R be a ring having no nonzero nil ideals in which for every $x, y \in R$ there exist integers $m = m(x,y) \geq 1$, $n = n(x,y) \geq 1$ such that $[x^n, y^m] = 0$, then R is commutative.*

Lemma 2.2 (I. N. Herstein [3, Lemma 1.1]). *Let R be a ring and $(0) \neq A$ a right ideal of R. Suppose that given $a \in A$, $a^n = 0$ for a fixed integer n, then R has a nonzero nilpotent ideal.*

Lemma 2.3 E. C. Posner [4, Theorem 1]). *Let R be a prime ring of* Char $R \neq 2$ *and d_1, d_2 derivations of R such that the iterate $d_1 \cdot d_2$ is also a derivation. Then at least one of d_1, d_2 is zero.*

3. Preparatory results

To prove the above Theorem 1.1 we start with the following lemmas.

Lemma 3.1. *For all $x, y \in R$, if $[x^n, [x^n, y^m]] = 0$, then $2[x^n, y^m]^2 = [x^n, [x^n, y^{2m}]]$, where m and n are positive integers.*

Proof. We have to prove

$$
\begin{aligned}
& 2[x^n, y^m]^2 - [x^n, [x^n, y^{2m}]] = 0 \\
\text{L.H.S.} \; &= 2(x^n y^m)^2 + 2(y^m x^n)^2 - 2y^m x^{2n} y^m - x^{2n} y^{2m} - y^{2m} x^{2n} \\
&= 2x^n y^m x^n y^m + 2y^m x^n y^m x^n - y^m x^{2n} y^m - y^m x^{2n} y^m - \\
&\quad - x^{2n} y^{2m} - y^{2m} x^{2n} \\
&= (2x^n y^m x^n - y^m x^{2n} - x^{2n} y^m) y^m + y^m (2x^n y^m x^n - x^{2n} y^m - y^m x^{2n}) \\
&= -[x^n, [x^n, y^m]] y^m - y^m [x^n, [x^n, y^m]] \\
&= 0 = \text{R.H.S. by the given hypothesis.}
\end{aligned}
$$

Lemma 3.2. *Let m, n be fixed positive integers. If R is a prime ring satisfying one of the conditions $(a)\, [x^n, y^m] \in Z(R)$, $(b)\, (x^n \circ y^m) \in Z(R)$, then R contains no nonzero nilpotent elements.*

Proof. Suppose $0 \neq a \in R$ and $a^2 = 0$. Let R satisfy the condition $[x^n, y^m] \in Z(R)$ which implies

$$[[x^n, y^m], x] = 0 \quad \text{for all} \quad x, y \in R. \tag{1}$$

Putting ax for x and xa for y in (1), and using $a^2 = 0$, we get

$$(ax)^{n+1} (xa)^m = 0 \tag{2}$$

Replacing x by $xa + x$ in the above equation and again using $a^2 = 0$, we obtain

$$(axa + ax)^{n+1}(xa)^m = 0 \tag{3}$$

and so,

$$(axa + ax)^n[(axa)(xa)^m + (ax)(xa)^m] = 0$$

simplifying the above equation and using $a^2 = 0$, we get

$$\big\{(ax)^n a + (ax)^n\big\}\big\{(axa)(xa)^m + (ax)(xa)^m\big\} = 0$$

which implies $(ax)^n(axa)(xa)^m + (ax)^{n+1} + (xa)^m = 0$. Using (2) in the above equation, we get $(ax)^{n+1}a(xa)^m = 0$. This on simplifying gives $(ax)^{n+1}(ax)^m a = 0$ which implies that $(ax)^{n+m+2} = 0$ for all $x \in R$.

If $aR \neq 0$, then by the above argument aR is a nonzero nil right ideal, satisfying the identity $(z)^{n+m+2} = 0$ for all $z \in aR$. Thus by Lemma 2.2 we see that $aR = 0$ i.e. $aRa = (0)$. This by primeness of R forces that $a = 0$.

Similarly, we can prove the case when R satisfies the condition $(x^n \circ y^m) \in Z(R)$.

Lemma 3.3. *Suppose R is a division ring and m, n are fixed positive integers greater than 1. Let R satisfy one of the conditions*
(a) $[x^n, y^m] \in Z(R)$, (b) $(x^n \circ y^m) \in Z(R)$, for all $x, y \in R$, then R is commutative.

Proof. (a) First, we assume that

$$[x^n, y^m] \in Z(R) \quad \text{for all} \quad x, y \in R$$

This implies that $[x^n, y^m]$ commutes with x^k for any positive integer k, so that

$$[[x^n, y^m], x^n] = 0 \qquad \text{for all} \quad x, y \in R \text{ and} \tag{6}$$

n, m positive integers.
Now there arise two cases: (A) Char$R = 2$ and (B) Char$R \neq 2$.

Case (A): Simplifying (6), we obtain

$$x^{2n}y^m - 2x^n y^m x^n + y^m x^{2n} = 0\,,$$

which further yields $[x^{2n}, y^m] = 0$, because Char$R = 2$. This by lemma 2.1 gives that R is commutative.

Case (B): Putting y^2 for y in (6) we get

$$[x^n, [x^n, y^{2m}]] = 0 \tag{7}$$

but by lemma 3.1 we have

$$2[x^n, y^m]^2 = [x^n, [x^n, y^{2m}]]\,. \tag{8}$$

By equation (7) and (8) we get

$$2[x^n, y^m]^2 = 0\,.$$

This implies that $[x^n, y^m]^2 = 0$, since Char$R \neq 2$. This further yields by lemma 3.2 that $[x^n, y^m] = 0$. Now applying lemma 2.1 we get that R is commutative.

Proof (b). By hypothesis $(x^n \circ y^m) \in Z(R)$, which implies that $(x^n \circ y^m)$ commutes with x^k for any positive integer k, so that

$$[(x^n \circ y^m), x^n] = 0 \tag{9}$$

This gives $[x^{2n}, y^m] = 0$. Hence by lemma 2.1, R is commutative.

4. Proof of the theorems

We are now ready to complete the proof of our Theorem 1.1.

Proof of Theorem 1.1. In the beginning, we assume that R is a prime ring. Then, R contains no nonzero zero divisior [Lemma 3.2]. Thus, by a strengthening of Posner's theorem [7, Corollary 1] R can be embedded in a simple ring R', satisfying the same condition $[x^n, y^m] \in Z(R')$ or $(x^n \circ y^m] \in Z(R')$. Now for any simple ring, there arise two cases (1) either it is a division ring (2) or it is not a division ring. The case (1) gets through by lemma 3.3. For case (2) a subring of R is homomorphic to D_2, and 2×2 matrices over a division ring D satisfiy the given condition $[x^n, y^m] \in Z(R)$ or $(x^n \circ y^m) \in Z(R)$. But this contradicts the fact that $x = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $y = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right] \in D_2$ do not satisfy the foregoing conditions. Hence R is embeddeble in a division ring. Now using lemma 3.3 we obtain that R is commutative.

In the next phase, since semiprime rings are subdirect sums of prime rings. So the semiprime rings are also commutative under the hypothesis given in theorem 1.1.

Proof of Theorem 1.2. Let x, y be any nonzero elements of R. By hypothesis, there exists a positive integer $n = n(x, y) \geq 1$ such that

$$[(xy)^n y^n - y^n (yx)^n, y] = 0 \tag{10}$$

Replacing x by xy^{-1} in the equation (10) and simplifying, we get

$$[x^n y^n - y^{n+1}x^n y^{-1}, y] = 0 \tag{11}$$

or $(x^n y^n - y^{n+1}x^n y^{-1})y - y(x^n y^n - y^{n+1}x^n y^{-1}) = 0$ or

$$x^n y^{n+1} - y^{n+1}x^n - yx^n y^n + y^{n+2}x^n y^{-1} = 0 \tag{12}$$

Multiplying (12), by y on the right hand side, we obtain

$$x^n y^{n+2} - y^{n+1}x^n y - yx^n y^{n+1} + y^{n+2}x^n = 0$$

or $y(x^n y^{n+1} - y^{n+1}x^n) - (x^n y^{n+1} - y^{n+1}x^n)y = 0$ or

$$[y^{n+1}, [x^n, y]] = 0 \tag{13}$$

for all $x, y \in R$ and $n = n(x, y) \geq 1$.
Now, there exist two cases for the ring R.

Case (1): Characteristic of R is zero.
Substitute $(1 + y)$ for y in (13) and expand the resulting equation by binomial theorem then using (13), to obtain

$$(n+1)[y, [x^n, y]] + \frac{n(n+1)}{|\underline{2}}[y^2, [x^n, y]] + \cdots + (n+1)[y^n, [x^n, y]] = 0 \tag{14}$$

We observe that first replacement of y by $(1 + y)$ in (13), reduces the maximum power of y namely y^{n+1} to y^n. Similarly the second, same replacement, reduce y^n to y^{n-1} and so on. Using all the equation like (14), which are obtained by successive replacement of y by $(1 + y)$ in consequent equations, we ultimately obtain $N[y, [x^m, y]] = 0$, where N is some positive integer large enough. Since, characteristic of R is zero then the equation $N[y, [x^m, y]] = 0$ implies

$$[y, [x^n, y]] = 0 \tag{15}$$

Replace y by $x + y$ in (15) and using (15), we get

$$[x^n, [x, y]] = 0 \tag{16}$$

Let I_r denote the inner derivation by $I_r : x \to [r, x]$; then (16) becomes

$$I_x n\, I_x(y) = 0\,.$$

Thus, by lemma 2.3, we have either $I_x n = 0$ or $I_x = 0$. If $I_x n = 0$ then $I_x n(y) = 0$ for all $y \in R$ that is $[x^n, y] = 0$, which by lemma 2.1, gives that R is commutative. In the case when $I_x = 0$ then $I_x(y) = 0$ for all $y \in R$, which implies that $xy = yx$, thus, R is commutative.

Case (2) : Characteristic of $R = p > 0$.
By equation(13), we get

$$[y, [x^n, y^{n+1}]] = 0$$

Which yields

$$[y^{k(n+1)}, [x^n, y^{n+1}]] = 0 \tag{17}$$

where k is a positive integer.
By induction on k and using (17), we can prove

$$[x^n, y^{k(n+1)}] = ky^{(k-1)(n+1)}[x^n, y^{n+1}] \tag{18}$$

Let Char$R = p = k$ say, then

$$[x^n, y^{p(n+1)}] = py^{(p-1)(n+1)}[x^n, y^{n+1}]$$

But R is of characteristic p, so above equation yields

$$[x^n, y^{p(n+1)}] = 0 \qquad \text{for all } \; x, y \in R\,.$$

This by lemma 2.1, gives that R is commutative.

Remark 4.1. Theorem 1.2 can be further generalized for semi simple rings, proofs following the pattern of those given by Quadri and Ashraf [6].

Remark 4.2. The ring of 3×3 strictly upper triangular matrices over a ring provides a counter example to show that theorem 1.2 is not valid for arbitrary rings.

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