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On the maximal value of Dirichlet and Fejér kernels with respect to the Vilenkin-like space

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Abstract. The investigation of kernel functions is an important part of the Fourier analysis. The maximal values of the n-th Dirichlet and Fejér kernels for Walsh–Paley, Vilenkin and some other systems are n and $\frac{n-1}{2}$, respectively. In the present paper we will deal with a more general system; in this case the situation is different.

1. Introduction

The concept of the Vilenkin-like system was introduced by G. Gát in [4]. This orthonormal system is a generalisation of several well-known ones, see the list of examples later. In the case of every cited system, domains of the systems' functions were some special groups. The domain of the observed system in this paper is a set without any operation on it.

Let $m := (m_0, m_1, \dots)$ denote by a sequence of positive integers not less than 2. Denote by G_{m_j} a set, where the number of the elements is m_j $(j \in \mathbb{N})$. Define the measure on G_{m_k} as follows

$$\mu_k(\{j\}) := \frac{1}{m_k} \quad (j \in G_{m_k},\ k \in \mathbb{N}).$$

Let G_m be the complete direct product of the sets G_{m_j} (without any operation on it), with the product of the topologies and measures (denoted by μ). This product measure is a regular Borel one on G_m with $\mu(G_m) = 1$. If the sequence

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m is bounded, then G_m is called by bounded Vilenkin space, otherwise it is an unbounded one. The elements of G_m can be represented by sequences $x := (x_0, x_1, \dots)$ $(x_k \in G_{m_k})$. It is easy to give a neighbourhood base of G_m:

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}$$

for $x \in G_m$, $0 < n \in \mathbb{N}$. Define the well-known generalized number system in the usual way. If $M_0 := 1$, $M_{k+1} := m_k M_k$ $(k \in \mathbb{N})$, then every $n \in \mathbb{N}$ an be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in \{0, 1, \dots, m_j - 1\}$ $(j \in \mathbb{N})$, and only a finite number of n_j's differ from zero. Let $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ (that is, $M_{|n|} \leq n < M_{|n|+1}$) if $0 < n \in \mathbb{N}$, and $|0| := 0$. Let $n^{(k)} = \sum_{j=k}^{\infty} n_j M_j$.

Denote by $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)\,(x \in G_m,\ n \in \mathbb{N})$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n$ $(n \in \mathbb{N})$.

We introduce an orthonormal system on G_m, which will be called Vilenkin-like system. This system was defined by G. GÁT in his paper [4]. The complex valued functions $r_k^n : G_m \to \mathbb{C}$ $(k, n \in \mathbb{N})$ are called generalized Rademacher functions, if they have the following four properties.

i. r_k^n $(k, n \in \mathbb{N})$ is $\mathcal{A}_{k+1}$ measurable (i.e. $r_k^n(x)$ depends only on $x_0, \dots, x_k$ $(x \in G_m)$) and $r_k^0 = 1$.

ii. If M_k is a divisor of n, l and $n^{(k+1)} = l^{(k+1)}$ $(k, l, n \in \mathbb{N})$, then

$$E_k(r_k^n \bar{r}_k^l) = \begin{cases} 1 & \text{if } n_k = l_k, \\ 0 & \text{if } n_k \neq l_k \end{cases}$$

($\bar{z}$ is the complex conjugate of z).

iii. If M_k is a divisor of n (that is, $n = n_k M_k + n_{k+1} M_{k+1} + \dots + n_{|n|} M_{|n|}$), then

$$\sum_{n_k=0}^{m_k-1} |r_k^n(x)|^2 = m_k$$

for all $x \in G_m$.

iv. There exists a $\delta > 1$, for which $\|r_k^n\|_\infty \leq \sqrt{m_k/\delta}$ for all $k, n \in \mathbb{N}$.

Now define the Vilenkin-like system $\psi := (\psi_n : n \in \mathbb{N})$ as follows

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n^{(k)}} \quad (n \in \mathbb{N}).$$

(Since $r_k^0 = 1$, then $\psi_n = \prod_{k=0}^{|n|} r_k^{n^{(k)}}$.) The Vilenkin-like system ψ is orthonormal (see e.g. [4]).

Let us see some known examples to the Vilenkin-like system.

(1) The Walsh–Paley and Vilenkin systems. For more on these see e.g. [1], [12].
(2) The group of 2-adic integers and the group of characters [7], [11], [13].
(3) The product system of coordinate functions of unitary irreducible representation of non commutative Vilenkin groups (in this case the group G_m is the Cartesian product of any finite groups) [6], [14].
(4) A system in the field of number theory. The so-called $\psi\alpha$ Vilenkin-like system (on Vilenkin groups) was a new tool in order to investigate limit periodic arithmetical functions [5], [8].
(5) The UDMD product system (introduced by F. Schipp on the Walsh–Paley group) [10], [11].
(6) The universal contractive projections (UCP) (introduced by F. Schipp) [9].

For more on these examples and their proofs see e.g. [4].

We define the Dirichlet and Fejér kernels in this way

$$D_n(y,x) := \sum_{k=0}^{n-1} \psi_k(y)\overline{\psi}_k(x) \qquad (0 < n \in \mathbb{N}, D_0 := 0),$$

$$K_n(y,x) := \frac{1}{n}\sum_{k=0}^{n-1} D_k(y,x) \qquad (0 < n \in \mathbb{N}).$$

We notice that in most of restricted systems (denoted by ϑ now) Dirichlet and Fejér kernel functions depend only on one element of the domain. The "one way" connection between the two conceptions is $D_n(y,x) = D_n^{\vartheta}(y-x)$.

Finally, let us define the maximal value sequences of the Dirichlet and Fejér kernels in the following way

$$D_n := \sup_{x,y\in G_m} |D_n(y,x)| \quad (n \in \mathbb{N}), \qquad K_n := \sup_{x,y\in G_m} |K_n(y,x)| \quad (0 < n \in \mathbb{N}).$$

2. General results on Vilenkin space

Lemma 2.1. *If $R = D$ then let $n \in \mathbb{N}$, if $R = K$ then let $0 < n \in \mathbb{N}$. We have*

$$R_n = \sup_{x\in G_m} R_n(x,x).$$

Proof. On the one hand, it is trivial that

$$\sup_{x\in G_m} R_n(x,x) = \sup_{x\in G_m} |R_n(x,x)| \le \sup_{x,y\in G_m} |R_n(y,x)| = R_n.$$

On the other hand, from the Cauchy–Bunyakovsky–Schwarz inequality

$$|D_n(y,x)| = \left|\sum_{k=0}^{n-1} \psi_k(y)\overline{\psi}_k(x)\right| \leq \sqrt{\sum_{k=0}^{n-1} |\psi_k(y)|^2 \sum_{k=0}^{n-1} |\psi_k(x)|^2}$$
$$\leq \max\left(\sum_{k=0}^{n-1} |\psi_k(y)|^2, \sum_{k=0}^{n-1} |\psi_k(x)|^2\right) = \max\left(D_n(y,y), D_n(x,x)\right).$$

Thus,

$$D_n = \sup_{x,y\in G_m} |D_n(y,x)| \leq \sup_{x\in G_m} D_n(x,x).$$

From these facts we obtain $D_n = \sup_{x\in G_m} D_n(x,x)$ for arbitrary $n \in \mathbb{N}$, immediately.

We will prove for all $0 < n \in \mathbb{N}$, that

$$\sup_{x\in G_m} K_n(x,x) \geq \sup_{x,y\in G_m} |K_n(y,x)| = K_n$$

indirectly. Let us suppose that there exists $0 < n \in \mathbb{N}$ for which

$$\sup_{x\in G_m} K_n(x,x) < \sup_{x,y\in G_m} |K_n(y,x)|.$$

It means that there exists $0 < n \in \mathbb{N}$ and $x_0, y_0 \in G_m$ for which

$$\sup_{x\in G_m} K_n(x,x) < |K_n(y_0,x_0)|.$$

Let us observe that

$$K_n(x_0,x_0) \leq \sup_{x\in G_m} K_n(x,x) \quad \text{and} \quad K_n(y_0,y_0) \leq \sup_{x\in G_m} K_n(x,x)$$

are true. From these we obtain the following

$$\sum_{k=0}^{n-1}\sum_{j=0}^{k-1} |\psi_j(x_0)|^2 \leq n \sup_{x\in G_m} K_n(x,x) < \left|\sum_{k=0}^{n-1}\sum_{j=0}^{k-1} \psi_j(y_0)\overline{\psi}_j(x_0)\right|$$

and

$$\sum_{k=0}^{n-1}\sum_{j=0}^{k-1} |\psi_j(y_0)|^2 \leq n \sup_{x\in G_m} K_n(x,x) < \left|\sum_{k=0}^{n-1}\sum_{j=0}^{k-1} \psi_j(y_0)\overline{\psi}_j(x_0)\right|.$$

Multiplied them we get

$$\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}|\psi_j(y_0)|^2\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}|\psi_j(x_0)|^2<\left|\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}\psi_j(y_0)\overline{\psi}_j(x_0)\right|^2.$$

But using the Cauchy–Bunyakovsky–Schwarz inequality

$$\left|\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}\psi_j(y_0)\overline{\psi}_j(x_0)\right|^2\le\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}|\psi_j(y_0)|^2\sum_{k=0}^{n-1}\sum_{j=0}^{k-1}|\psi_j(x_0)|^2,$$

holds, which is a contradiction. It yields $K_n=\sup_{x\in G_m}K_n(x,x)$ for every $0<n\in\mathbb{N}$. □

In [14] R. Toledo confirmed the analogous statement in the case of representative product systems. He deals in his paper only with Dirichlet kernels.

Lemma 2.2. [3] *Let $x,y\in G_m,\ n\in\mathbb{N}$. Then*

$$D_{M_n}(y,x)=\begin{cases}M_n & \text{if } y\in I_n(x),\\ 0 & \text{if } y\notin I_n(x).\end{cases}$$

Corollary 2.3. *Let $n\in\mathbb{N}$. Then*

$$D_{M_n}=M_n.$$

Proof. This statement is a trivial consequence of Lemma 2.2. □

Theorem 2.4. *Let $n\in\mathbb{N}$. Then*

$$n\le D_n\le M_{|n|+1}.$$

Proof. From the orthonormality of the system ψ and $\mu(G_m)=1$ we have

$$\begin{aligned}n&=\sum_{k=0}^{n-1}\int_{G_m}|\psi_k(x)|^2d\mu(x)=\int_{G_m}\sum_{k=0}^{n-1}|\psi_k(x)|^2d\mu(x)\\&=\int_{G_m}D_n(x,x)d\mu(x)\le\int_{G_m}\sup_{x\in G_m}D_n(x,x)d\mu(x)\\&=\sup_{x\in G_m}D_n(x,x)=D_n.\end{aligned}$$

	e	(12)	(13)	(23)	(123)	(132)
r^0	1	1	1	1	1	1
r^1	$\sqrt{2}$	$\sqrt{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
r^2	$\sqrt{2}$	$-\sqrt{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
r^3	1	-1	-1	-1	1	1
r^4	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$
r^5	0	0	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$	$-\frac{\sqrt{6}}{2}$	$\frac{\sqrt{6}}{2}$

Table 1. A possible system for $\mathcal{S}_3$

For the other inequality let us use Lemma 2.1 and Lemma 2.2. It is clear that

$$\begin{aligned} D_n &= \sup_{x\in G_m} D_n(x,x) \\ &= \sup_{x\in G_m} \sum_{k=0}^{n-1} |\psi_k(x)|^2 \leq \sup_{x\in G_m} \sum_{k=0}^{M_{|n|+1}-1} |\psi_k(x)|^2 \\ &= \sup_{x\in G_m} D_{M_{|n|+1}}(x,x) = M_{|n|+1}. \end{aligned} \qquad \square$$

In the cases of the original (commutative) Vilenkin system (and Walsh–Paley system) $n = D_n$ and $\frac{n-1}{2} = K_n$ hold for every $n \in \mathbb{N}$, because of $n = D_n(0) \geq |D_n(x)|$ for all $x \in G_m$ and $n \in \mathbb{N}$ in those systems.

Table 1 contains the values of a possible system for the symmetric group $\mathcal{S}_3$ (for details see [14]). You can see a part of D_n sequence from this system in Figure 1, and a part of K_n sequence in Figure 2. This non-commutative system is not our main topic in this article, we just present these figures because they are good examples to the nontrivial cases of our theorems.

Corollary 2.5. *Let $0 < n \in \mathbb{N}$. Then*

$$1 \leq \frac{D_n}{n} \leq m_{|n|}.$$

Proof. Since $M_{|n|} \leq n$, so from Theorem 2.4

$$n \leq D_n \leq M_{|n|+1} = m_{|n|}M_{|n|} \leq m_{|n|}n. \qquad \square$$

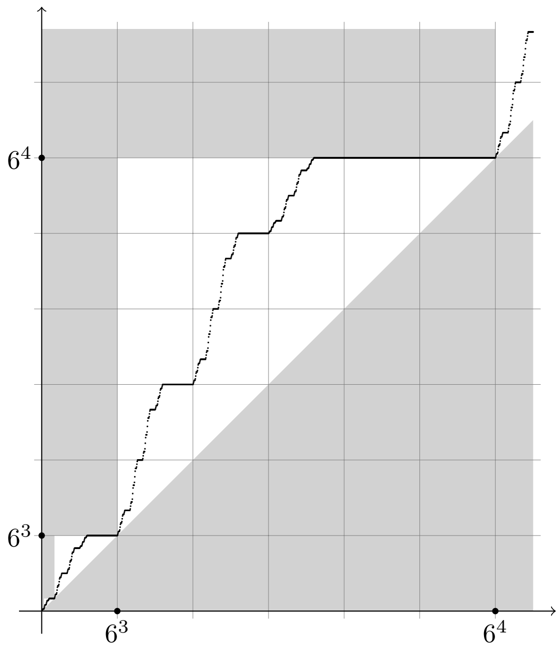

Figure 1. $n \le D_n \le 6^{|n|+1}$ on the complete product of $\mathcal{S}_3$.

Theorem 2.6. *Let* $0 < n \in \mathbb{N}$. *Then*

$$\frac{n-1}{2} \le K_n \le \frac{1}{n} \sum_{k=0}^{n-1} M_{|k|+1}.$$

Proof. From Theorem 2.4 we have

$$\begin{aligned} K_n = \sup_{x,y \in G_m} |K_n(y,x)| &= \sup_{x,y \in G_m} \left| \frac{1}{n} \sum_{k=0}^{n-1} D_k(y,x) \right| \\ &\le \frac{1}{n} \sum_{k=0}^{n-1} \sup_{x,y \in G_m} |D_k(y,x)| = \frac{1}{n} \sum_{k=0}^{n-1} D_k \\ &\le \frac{1}{n} \sum_{k=0}^{n-1} M_{|k|+1}. \end{aligned}$$

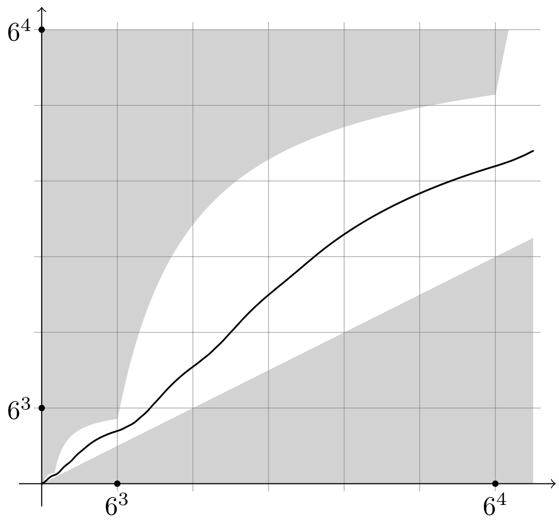

Figure 2. $\frac{n-1}{2} \leq K_n \leq \sum_{k=0}^{n} 6^{|n|+1}$ on the complete product of $\mathcal{S}_3$.

Using $1 = \mu(G_m) = \int_{G_m} |\psi_n(x)|^2 d\mu(x)$ and Lemma 2.1 we get

$$\begin{aligned} K_n &= \sup_{x \in G_m} K_n(x,x) = \int_{G_m} \sup_{x \in G_m} K_n(x,x) d\mu(x) \geq \int_{G_m} K_n(x,x) d\mu(x) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \int_{G_m} D_k(x,x) d\mu(x) = \frac{1}{n} \sum_{k=0}^{n-1} \int_{G_m} \sum_{j=0}^{k-1} |\psi_j(x)|^2 d\mu(x) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} \int_{G_m} |\psi_j(x)|^2 d\mu(x) = \frac{1}{n} \sum_{k=0}^{n-1} k = \frac{n-1}{2}. \qquad \square \end{aligned}$$

Corollary 2.7. *Let* $1 < n \in \mathbb{N}$. *Then*

$$1 \leq \frac{2}{n-1} K_n \leq \max_{1 \leq k < n} m_{|k|}.$$

Proof. It comes from Theorem 2.6. Similarly to the proof of the Corollary 2.5

$$\frac{n-1}{2} \leq K_n \leq \frac{1}{n} \sum_{k=0}^{n-1} M_{|k|+1} \leq \frac{1}{n} \sum_{k=0}^{n-1} m_{|k|} k \leq \frac{n-1}{2} \max_{1 \leq k < n} m_{|k|}. \qquad \square$$

Theorem 2.8. *Let $0<n\in\mathbb{N}$. Equality $D_k=k$ holds for all $k\in\{0,\dots,n-1\}$ if and only if*

$$K_n = \frac{n-1}{2}.$$

Proof. Let us suppose that $D_k = k$ holds for all $k \in \{0,\dots,n-1\}$. From Theorem 2.6 and from

$$\begin{aligned} K_n &= \sup_{x,y\in G_m} |K_n(y,x)| = \sup_{x,y\in G_m} \left| \frac{1}{n}\sum_{k=0}^{n-1} D_k(y,x)\right| \\ &\le \frac{1}{n}\sum_{k=0}^{n-1} \sup_{x,y\in G_m} |D_k(y,x)| = \frac{1}{n}\sum_{k=0}^{n-1} D_k \\ &= \frac{1}{n}\sum_{k=0}^{n-1} k = \frac{n-1}{2} \end{aligned}$$

we obtain the proof immediately.

On the other hand, let us suppose that $K_n = \frac{n-1}{2}$ for some $0 < n \in \mathbb{N}$ and indirectly there exists $j \in \{0,\dots,n-1\}$, for which $D_j \ne j$. In this situation $D_j > j$ comes from Theorem 2.4. It means that there exists $x_0 \in G_m$ for which $D_j(x_0,x_0) > j$. From this we get

$$K_n = \sup_{x\in G_m} \frac{1}{n}\sum_{k=0}^{n-1} D_n(x,x) \ge \frac{1}{n}\sum_{k=0}^{n-1} D_k(x_0,x_0) > \frac{n-1}{2},$$

and it is a contradiction. □

Theorem 2.9. *Sequence D_n is monotonically increasing, sequence K_n is strictly increasing.*

Proof. Let $x \in G_m$ and $n \in \mathbb{N}$ be arbitrary. It is easy to confirm that

$$D_n(x,x) \le D_n(x,x) + |\psi_n(x)|^2 = D_{n+1}(x,x).$$

This inequality and Lemma 2.1 imply

$$D_n = \sup_{x\in G_m} D_n(x,x) \le \sup_{x\in G_m} D_{n+1}(x,x) = D_{n+1}$$

hence sequence D_n is monotonically increasing.

Let us consider that $D_0(x,x)=0$ for all $x\in G_m$ by definition, and $D_1(x,x)=1$ for all $x \in G_m$ because of Lemma 2.2. On the other hand, D_n is monotonically increasing, so for all $x \in G_m$ and $0 < n \in \mathbb{N}$

$$D_0(x,x) + 1 \le D_n(x,x),$$

$$D_1(x,x) \leq D_n(x,x),$$
$$\vdots$$
$$D_{n-1}(x,x) \leq D_n(x,x).$$

Summarizing inequalities we have

$$1 + \sum_{k=0}^{n-1} D_k(x,x) \leq nD_n(x,x)$$

for all $x \in G_m$ and $0 < n \in \mathbb{N}$. Simple calculation shows that the last inequality is equivalent with the inequality

$$\begin{aligned}\frac{1}{n(n+1)} + K_n(x,x) &= \frac{1}{n(n+1)} + \frac{1}{n}\sum_{k=0}^{n-1} D_k(x,x) \\ &\leq \frac{1}{n+1}\sum_{k=0}^{n} D_k(x,x) = K_{n+1}(x,x),\end{aligned}$$

consequently, using Lemma 2.1

$$K_n = \sup_{x \in G_m} K_n(x,x) < \sup_{x \in G_m} K_{n+1}(x,x) = K_{n+1}$$

for all $0 < n \in \mathbb{N}$.

This completes the proof of the Theorem 2.9. $\square$

From the proof of the Theorem 2.9 it is easy to see that if $\inf_{x \in G_m} |\psi_n(x)| > 0$ then $D_n < D_{n+1}$. This holds in the most "classical" situation (for example for Walsh–Paley, original Vilenkin and $\psi\alpha$ Vilenkin-like system), but for representative product systems we can find examples, where $D_n = D_{n+1}$ for some $n \in \mathbb{N}$ (see e.g. Figure 1 or [14]).

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